THE GAMES AND PUZZLES JOURNAL

Published by

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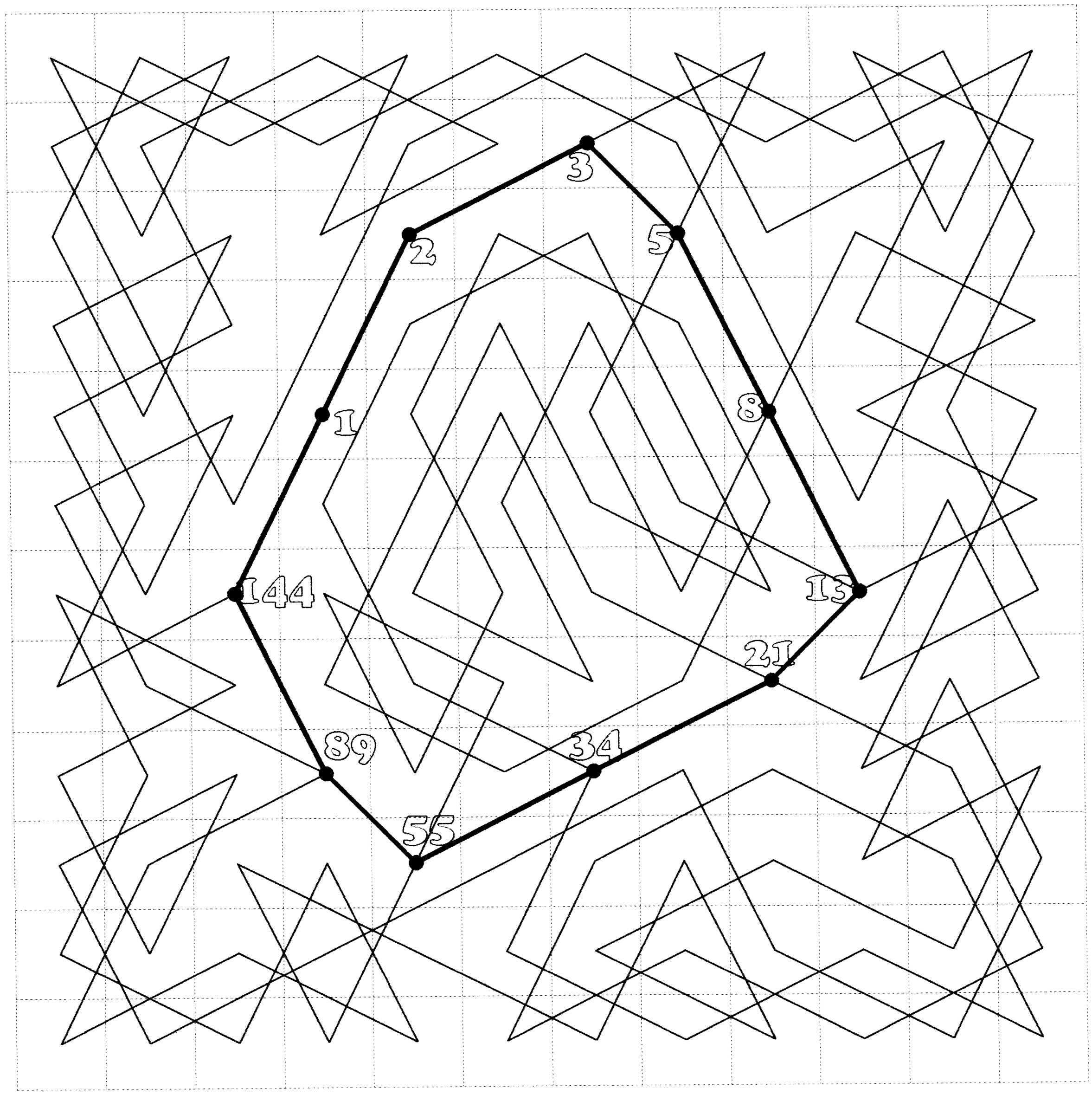
Top Floor
63 Eversfield Place
St Leonards on Sea
East Sussex
TN37 6DB

ISSN 0267 369X

Our cover illustration, intended as an Easter greeting but now a little late, is a figured, reentrant knight's tour of the 12×12 board, showing the first few Fibonacci numbers in a convex circuit. Each number is of course the sum of the two preceding numbers, beginning here with 1 and 2. All moves are either internal or external to the oval. I believe one other oval arrangement is possible, but it may not be usable in a tour.

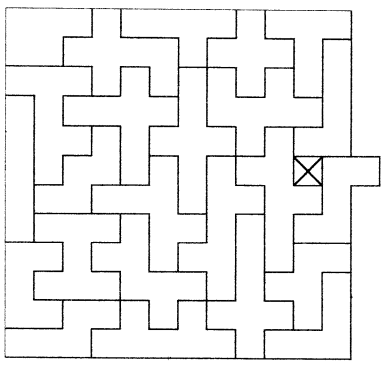


Editorial Meanderings

Thanks to early subscribers for their patience in waiting for this first issue to appear. There's rather less on games and more on knight's tours than intended, but it is probably best for Ernest Bergholt's *Memoranda* 1–4 to appear all together; 5–6 will appear next time; they've certainly waited long enough to gain this wider circulation. A geometrical item has been held over until I've had more practice on doing the diagrams. This entire issue, including the figures, has been produced as a single file (amounting to 3129664 bits at the last count, including this self-referential comment) within the Lotus word processor AmiPro, which came with my new computer, which is a 486DX-100 multimedia 'Time Machine' package. The computer came last year, but it's only recently that I've begun to get anywhere with it. We begin with a look at what's going on elsewhere, in various of our contemporary journals.

WordsWorth: *A Magazine for the PC Program User.*
Issue 10 of this interesting magazine came out at the beginning of May. It is published by Ted Clarke, Menanhyl, Trenance, Newquay, Cornwall, TR8 4DA, but no subscription rate is mentioned. The publisher also produces a computer program called WordsWorth designed to assist with word puzzles, which are a regular feature. For instance, earlier issues included 'pangrams', to which I contributed the following: WRY JUMP-FANS BLITZ DOCK, VEX GHQ, which I'm particularly pleased with because it uses no obscure words (in fact they all appear in my 1952 edition of the Pocket Oxford Dictionary; though admittedly 'blitz' and this sense of 'fan' had only just made it among the Addenda). It seems impossible to make a pangram without at least one abbreviation, obscure word or made-up name. Ted Clarke's own offering was MEG SCHWARZKOPF QUIT JYNX BLVD and the oldest example he records is from the *Guinness Book of Records* 1971 (originator not recorded): CWM FJORD-BANK GLYPHS VEXT QUIZ. A striking aspect of the magazine is its use of computer-printed colour on almost every page. Mathematical questions are also included: this issue has a four-page article on Tiling.

Games & Puzzles Magazine
This famous title from the 70s was revived in April 1994 by Paul Lamford, with a strong team of contributors, including R. C. Bell, David Pritchard, David Parlett and David Singmaster and ran to 16 issues, ending July 1995. There was an article in #10 on Heptominoes by Brian Caswell, but the problems proposed to my way of thinking allowed rather too free a selection of pieces. A problem that I proposed but was not followed up is to arrange the 12 five-square pieces and 12 seven-square pieces to form a 12×12 square. The 12 heptominoes are to be chosen from the 108 by some strict criterion. For example, the following is the best I was able to achieve using the 12 'four-ended' heptominoes (an 'end' being a square attached by only one side).



Classification of heptominoes by ends and edge-length

ends	longest edge length 7	6	5	4	3	2	1	
0	-	-	-	-	-	1	-	1
1	-	-	1	2	3	2	-	8
2	1	1	4	10	17	10	-	43
3	-	2	4	8	12	18	-	44
4	-	-	1	-	2	2	7	12
	1	3	10	20	34	33	7	108

Games Games Games
Formerly known as *The Small Furry Creatures Press*, this is now the only UK publication, apart from trade journals, that tries to cover the whole field of commercially produced games, though their traditional expertise is in role-playing games. It is edited by Theo Clarke and Paul Evans. The April issue, #100, has a supplement, *Ludology*, listing over 200 games published in 1995/6. Subscription rate is £18 for 10 issues per year. Write to SFC Press, 42 Wynndale Road, London, E18 1DX.

World Game Review

The fact that number 12 of this magazine bears the subtitle of 'tenth anniversary issue' indicates its rate of publication. By the time number 13 comes out it will have become an annual, but it is always worth waiting for. #12 is priced at $4 and contains a cumulative index to #7–12. The publisher's address is: Michael Keller, 1747 Little Creek Drive, Baltimore, MD 21207-5230, USA. The following specially interesting 'Castawords' puzzle was sent to *WGR* by Eduard Riekstins, editor of a puzzle column in the Latvian newspaper *CM Cevodnya*. "A puzzle-lover has found in his attic five toy letter-blocks from his childhood. Each block contains six letters. With these cubes he can produce the following words (mostly related to games): BLACK BOARD CHESS CRAZE FACET JOKER LOTTO NORTH PAWNS POINT POKER QUEEN SEVEN SHOGI SIXTY TRUMP VIXEN WHIST WHITE [identify the letters on each cube]."

HPCC Datafile: *Journal of the Handheld and Portable Computer Club*

The HP in HPCC also stands for Hewlett Packard, and the club is a user group for this make of machine, though apparently independent of the manufacturers. Much of the content, such as meeting reports and a history of HP calculators, by club chairman Wlodek Mier-Jedrzejowicz, is so 'in' as to be incomprehensible to the 'out'. But there are items of wider interest, such as programs for card-shuffling by Joseph P. Horn, and for the period of a recurring decimal by Jeremy Hawdon. Annual subscription, UK, is £25 (plus £3 joining fee): enquiries to Membership Secretary, David Hodges, 8 Stratford Court, Salisbury Road, Farnborough, Hampshire, GU14 7AJ.

This stimulated me to look at the calculator provided with 'Windows' on my 486. It seems to have all the features of a normal 'scientific' calculator and a bit more, registering up to 13 digits and offering calculation in bases 2, 8 and 16 (but not, alas, 6 or 12).

The Journal of Recreational Mathematics

My article on 'Generalized Knights and Hamiltonian Tours', submitted in October 1992 has at last appeared in the latest issue of this journal (vol.27, no.3, pp.191–200). This volume is dated 1995 but is a year behind schedule, though the rapidity of appearance of the last three issues suggest it may soon catch up, especially now that the founder-editor, Joseph Madachy, has recovered from an eye operation. Besides that long article, I've also had a series of short items on knight's tours appearing in the Problems and Conjectures department, with more to come. There is a rather macabre polycube construction problem by Allen Prentice in this issue, in which the 12 five-cube planar pieces are to be used to make a 'grave' 2×3×9 surmounted by a cross in various positions (the cross being made of the cross-shaped piece standing on a single-cube plinth). The journal is published by Baywood Publishing Company Inc, 26 Austin Avenue, P.O. Box 337, Amityville, N.Y. 11701, U.S.A., individual subscription costs $31.20, including postage from the UK, but must be drawn on a US bank.

Mathematical Spectrum: *A Magazine for Students and Teachers of Mathematics.*

This journal changed format from A5 to A4 in 1994. The subscription rate is £8.50 per three-issue volume; enquiries to: The Editor, *Math. Spectrum*, Hicks Building, The University, Sheffield, S3 7RH. Each issue now usually begins with a biography of a well-known mathematician. The last three featured G. Cantor, T. P. Kirkman and G. H. Hardy.

Thomas Penyngton Kirkman (1806–1895), best known for the combinatorial problem of the 'Fifteen Schoolgirls' (a chapter in W. W. Rouse Ball's *Mathematical Recreations and Essays*), was an amateur mathematician, his 'day job' being Rector of Croft near Warrington for nearly 50 years (not to be confused with Croft, Yorkshire, where the Rector was Charles Dodgson, father of C. L.). It seems that he submitted a study on polyhedra to the Royal Society in 1861, only 2 of the 21 sections of which were ever published. Part of this deals with closed tours of the edges, in a more general manner than his more famous contemporary William Rowan Hamilton (1805–1865), whose name is now associated with this topic. Kirkman studied at Trinity College Dublin in 1833, as what would now be called a 'mature student'. Hamilton, who had studied there in 1827, was already Professor of Astronomy. One wonders if they ever met, or discussed the problem. The first known reference to the tour problem on the dodecahedron is in a letter from Hamilton to his friend John T. Graves in 1856.

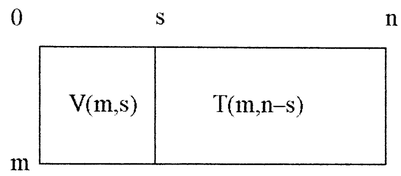
Dominizing the Chessboard

by G. P. Jelliss (including notes by R. J. Chapman)

The subject of enumerating the dissections of a rectangular board $m \times n$ into two-celled pieces (dominoes) was briefly discussed in *Chessics* (1986, vol.1, no.28, p.138), the issue on chessboard dissections, where (in different notation) the following recurrence relation was given:

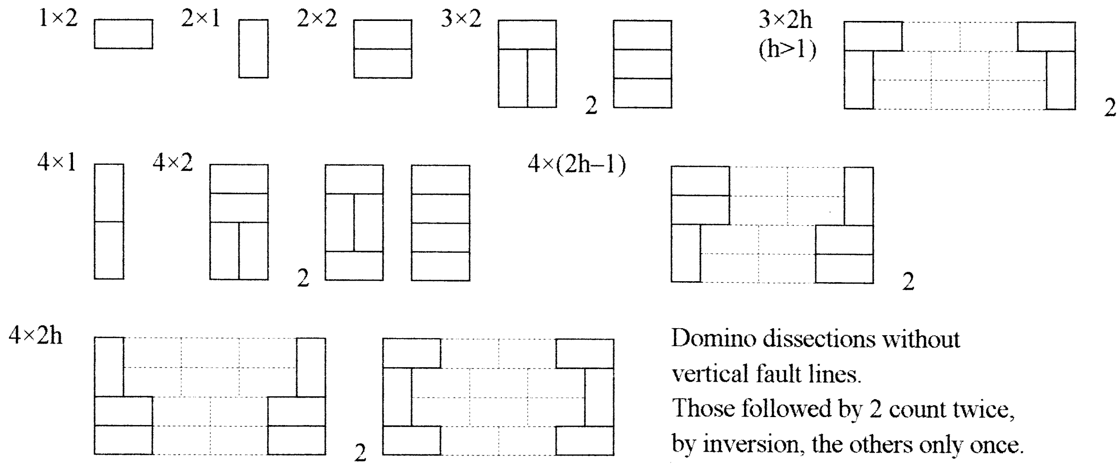
$$T_{m,n} = \sum_{s=1}^{n} V_{m,s} T_{m,n-s}$$

where $V_{m,s}$ is the number of $m \times s$ dissections without vertical fault lines, and $T_{m,0} = 1$.



Proof of the formula:
V(m,s) is the number of ways of dissecting the part of the rectangle before the first vertical fault line (or the whole rectangle when there are no fault lines) and T(m,n–s) is the nmber of ways of dissecting the rest of the rectangle (becoming T(m,0) = 1 when there are no fault lines).

This method is practical for calculating the totals when $m \leq 4$ since the values of $V_{m,s}$ are then easily found by direct construction, mostly being zero; the non-zero cases are illustrated here.



Domino dissections without vertical fault lines. Those followed by 2 count twice, by inversion, the others only once.

From these diagrams we find the values: $V_{1,2} = 1$, $V_{2,1} = 1$, $V_{2,2} = 1$, $V_{3,2} = 3$, $V_{3,2h} = 2$, $V_{4,1} = 1$, $V_{4,2} = 4$, $V_{4,2h-1} = 2$, $V_{4,2h} = 3$, and using the above summation we can deduce the following recurrence relations to calculate $T_{2,n}$, $T_{3,n}$ and $T_{4,n}$.

W. L. Patton had shown, in *American Mathematical Monthly* 1961, that the $2 \times n$ case is solved by the Fibonacci sequence, given by the recurrence: $T_{2,n} = T_{2,n-1} + T_{2,n-2}$. The subject was raised again recently in an article by Chris Holt in *Mathematical Spectrum* (1994/5 vol.27 no.3 p.62), where he gave a recurrence for the $3 \times 2k$ case, in our notation: $T_{3,2k} = 4T_{3,2k-2} - T_{3,2k-4}$. In a letter to *Mathematical Spectrum* (1995/6 vol.28 no.2 p.44) I was able to supply a recurrence for the $4 \times n$ case, as follows: $T_{4,n} = T_{4,n-1} + 5T_{4,n-2} + T_{4,n-3} - T_{4,n-4}$. This gave the total of 2245 domino dissections of the 4×8 board. So that for the 8×8 board the total could be estimated at more than 5 million.

However, a letter to me (24/i/1996) from Robin J. Chapman, Department of Mathematics, University of Exeter, has now pointed out that the problem, on boards of all sizes, was completely solved as long ago as 1961! The relevant paper by P. W. Kasteleyn, Shell-Laboratorium, Amsterdam, has the title 'The statistics of dimers on a lattice, I: The number of dimer arrangements on a quadratic lattice', and appeared in the journal *Physica* (vol.27, 1961, pp.1209–25).

This reference seems to have escaped the notice of the recreational mathematics fraternity, presumably because it was published in a physics journal and its title makes no mention of dominoes, dissections or chessboards; also it gives no explicit numerical results.

The formula for the number $T_{m,n}$ of dissections $m \times n$ can be expressed in the form:

$$T_{m,n} = \prod_{j=1}^{m} \prod_{k=1}^{n} (4\cos^2 \tfrac{j\pi}{m+1} + 4\cos^2 \tfrac{k\pi}{n+1})^{1/4}$$

[I must admit to being baffled as to how cos and pi, get into the act, but Professor Cranium assures me that this is just a bit of flim-flam and that they cancel out each other's effect (rather like the White Knight's plan to dye his whiskers green and always carry so large a fan that they could not be seen).]

An unpublished paper by James Propp, 'Dimers and dominoes', Massachusetts Institute of Technology, dated 24/ix/1992, based on Kasteleyn, notes that the number of domino tilings of an 8-by-8 checkerboard is 12,988,816 = 3604^2. Robin Chapman has kindly evaluated the formula for all cases up to 20×20 using the computer algebra system MAPLE. Here are the results up to 72 cells:

4	2×2	2	34	2×17	2584	56	2×28	514229
6	2×3	3	36	2×18	4181		4×14	1174500
8	2×4	5		3×12	2131		7×8	1292697
10	2×5	8		4×9	6336	58	2×29	832040
12	2×6	13		6×6	6728	60	2×30	1346269
	3×4	11	38	2×19	6765		3×20	413403
14	2×7	21	40	2×20	10946		4×15	3335651
16	2×8	34		4×10	18061		5×12	2332097
	4×4	36		5×8	14824		6×10	4213133
18	2×9	55	42	2×21	17711	62	2×31	2178309
	3×6	41		3×14	7953	64	2×32	3524578
20	2×10	89	44	2×22	28657		4×16	9475901
	4×5	95		4×11	51205		8×8	12988816
22	2×11	144	46	2×23	46368	66	2×33	5702887
24	2×12	233	48	2×24	75025		3×22	1542841
	3×8	153		3×16	29681		6×11	21001799
	4×6	281		4×12	145601	68	2×34	9227465
26	2×13	377		6×8	167089		4×17	26915305
28	2×14	610	50	2×25	121393	70	2×35	14930352
	4×7	781		5×10	185921		5×14	29253160
30	2×15	987	52	2×26	196418		7×10	53175517
	3×10	571		4×13	413351	72	2×36	24157817
	5×6	1183	54	2×27	317811		3×24	5757961
32	2×16	1597		3×18	110771		4×18	76455961
	4×8	2245		6×9	817991		6×12	106912793

Propp also notes that the total is a square on square boards of side $4k$, and twice a square on square boards of side $4k+2$. For the 6×6 board 6728 = 2×58^2. The total for the 10×10 board is 258584046368 = 2×359572^2 and for the 12×12 board is 53060477521960000 = 230348600^2

Mr Chapman notes that Kasteleyn's formula also gives the number of tilings of an $m \times n$ rectangle with a given number of horizontal tiles. He has computed these for an 8×8 square. The results are listed below. If A_n is the number of tilings with n horizontal tiles (n even of course) then $A_n = A_{32-n}$.

$A_0 = A_{32} = 1$, $A_2 = A_{30} = 70$, $A_4 = A_{28} = 1785$, $A_6 = A_{26} = 21656$, $A_8 = A_{24} = 144092$
$A_{10} = A_{22} = 580620$, $A_{12} = A_{20} = 1511368$, $A_{14} = A_{18} = 2644858$, $A_{16} = 3179916$

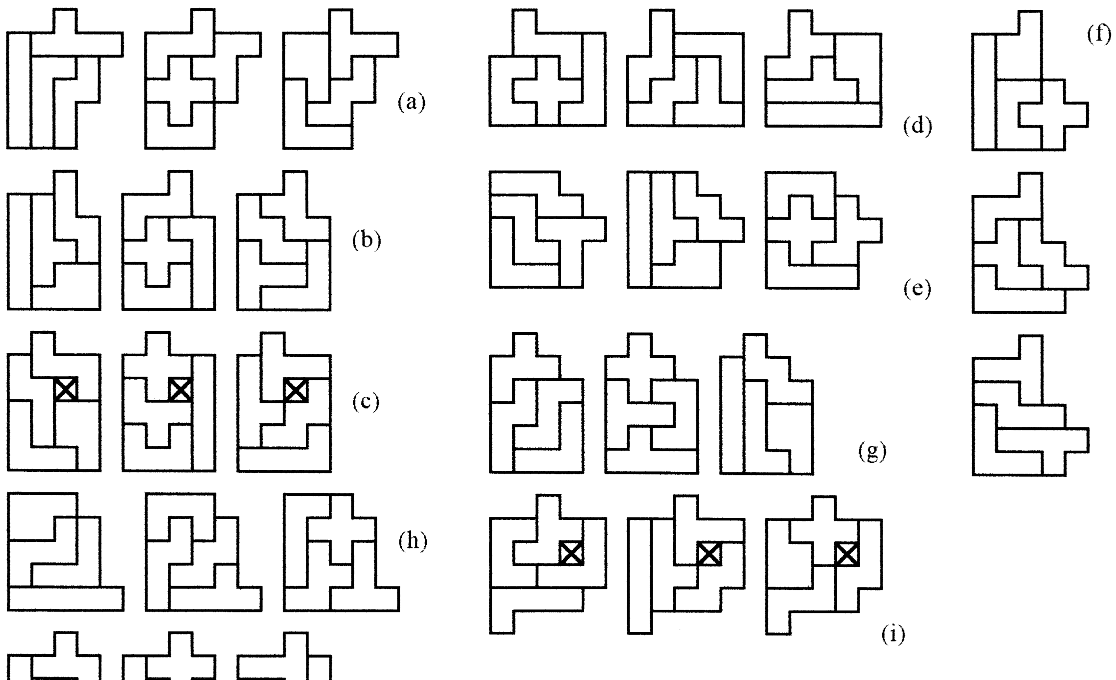
The problem remains of finding a way of calculating $V_{m,n}$ for larger values of m, and also of determining the number $F_{m,n}$ of fault-free dissections (with neither vertical nor horizontal fault lines).

Polycube Constructions

from notes by Walter Stead

In *Chessics* #28, the special issue on chessboard dissections, acknowledgement was given, on p.138, to the notebooks of Walter Stead on 'Problems and Puzzles' and, on p.147, I mentioned that these manuscripts contain some 3D constructions that would be worth a chapter to themselves sometime. The time has now arrived. The following notes are roughly in the sequence in which they are given in the notebooks. This is not a subject I have myself studied very closely before, so I am unable to say how far the results are new. Most have probably appeared before in the *Fairy Chess Review* and some may have been independently rediscovered since. Perhaps experts on these types of constructions will provide comments on the results.

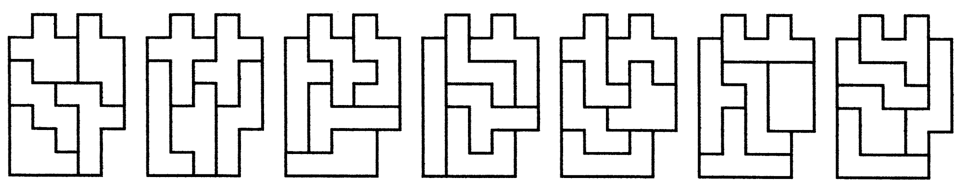
The first 3D construction (1) in the notebooks is the problem of arranging the twelve 5-cube planar pieces in three similar layers, and the following solutions are diagrammed.



(e) R.J.French (f) W.H.Reilly 1945 (g-h) F. Hansson 1954
others W. Stead 1954

In (b), (c), (d) and (e) it is noted that there are pairs of pieces that can be interchanged. In (c) and (i) there is a hole through the construction. In (j) the P-piece is completely surrounded (and can thus be completely buried within the 3D structure if used in the middle layer).

The next construction noted (2) shows seven simultaneous decks each of five 6-cube planar pieces, thus using all 35 pieces. This noted as due to Maurice J. Povah of Blackburn (9 Nov. 1965).



(2)

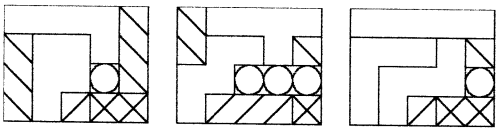
The next item on 3D constructions is a list of all the pieces formed with 1–5 cubes, 20 of which are non-planar, making 41 in all. We will return to this subject in the next issue, when I've worked out the best way of doing the three-dimensional diagrams.

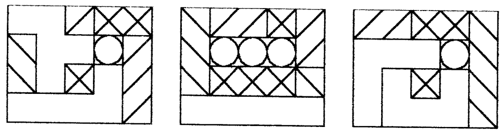
Later in the notebook are some further constructions using flat pieces only, but not in separate layers. These logically come next. I show the different pieces by markings in place of numbering.

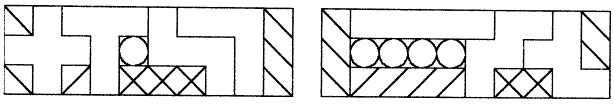
(3) Block 3×4×5 using the 12 flat 5-pieces. Two solutions are given. It is noted that in 3a the P- and L- pieces in the first layer can 'interchange', while in 3b the pieces on Aa4 and Ba4 can 'rotate'.

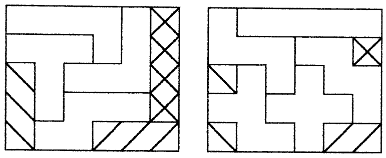
(4) Block 2×3×10 using all the 12 flat 5-pieces. Proposed by F. Hansson *FCR* June 1948, solved by W. S. Here the end-pieces, on Bj, will 'reverse' (as in 3b).

(5) Block 2×5×6. This was also proposed by F. Hansson *FCR* June 1948, and this solution is noted as 'WS & DN's Sol^{n}.' presumably working together (with Dennison Nixon).

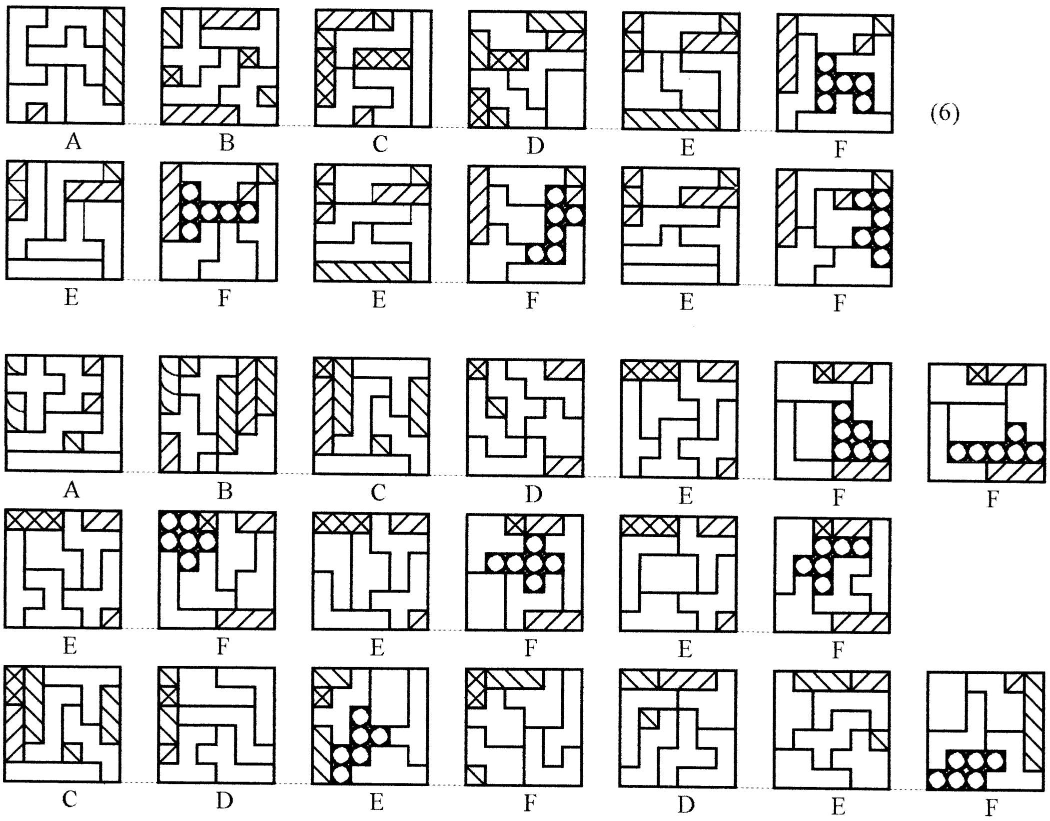

(3a)


(3b)

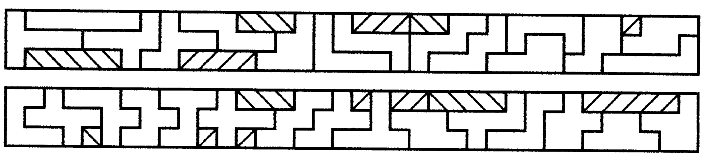

(4)


(5)

The next problems use all the 35 flat 6-pieces, plus one duplicate piece, to make the total 6^3. (6) Cube 6×6×6. Solutions are given "doubling all 11 oddly chequered pieces", shown by circled shading in the diagrams below where I have lettered the layers A–F instead of I–VI as used by W. S. (7) Column 2×3×36. Problem (8) is given for solution among the Puzzle Questions on p.224.



(6)


(7)

to be continued

Knight`s Tour News

Readers of *Chessics* and of volume 1 of this journal will know that I have been working on a comprehensive book on knight`s tours, and related questions since 1985. This work is at last nearing completion and is to have the title *Knight`s Tour Notes*. In order to contain the subject within a single volume it has been necessary to summarise many topics that I would like to have treated in fuller manner. Accordingly this *Knight`s Tour News* department will initially serve as an outlet for the overflow, and once the book is published, will continue to keep readers up-to-date with the latest developments, and provide an opening for new results to be published.

One of the chapters in *Knight`s Tour Notes* will reproduce three memoirs written by Ernest Bergholt in 1918 on the subject of 'Mixed Quaternary Symmetry' but never previously published. The only previous account of this subject, which was invented by Bergholt, occurs in a brief series of knight`s tour problems published in the *British Chess Magazine* in 1918 (pages 7-8, 48, 74, 104, 195), in which he also gives the first ever examples of closed tours on 3-rank boards and of tours in 180° rotational symmetry in which the diametrally opposite numbers have constant sum instead of constant difference. These three remarkable innovations by one author at one time deserve greater recognition.

The existence of the memoirs written by Bergholt was mentioned at the end of the *BCM* series, where he wrote: "I need scarcely say that I do not construct such tours as these at random, but on demonstrable mathematical principles, which I have explained in a series of manuscript memoirs. They are in the custody of Mr H. J. R. Murray." Since I was unable to find these memoirs among Murray`s papers at the Bodleian Library, Oxford, in 1985, I had presumed them lost. However, in June 1991 the existence of further Murray manuscripts was brought to my notice by Dr Irving Finkel of the Department of Western Asiatic Antiquities at the British Museum, who had learnt of them as a result of the board games exhibition held at the museum in September 1990. This resulted in my being able to visit Dr Elizabeth Murray on 3rd July 1991 to see these manuscripts, and to study them, with a view to publication, before they were placed in the archives of the Bodleian Library on 7th February 1992.

Among these manuscripts I was delighted to find eight of the nine Bergholt memoirs. Murray intended to include an account of Bergholt`s work in a book on *The Knight's Problem*, a manuscript for which was completed in 1942, but this was never published. Also among the Murray papers were 'Miscellaneous Articles and Notes by Ernest Bergholt' which include copies of a short series of articles that he published in the magazine *Queen* in 1915-16. Since these will not be readily obtainable by readers, and wishing to present a complete account of Bergholt`s work on tours, these are also reproduced here, preceding the Memoirs 1–4. Memoirs 5–6 are to appear in the next issue.

Very little biographical detail about Bergholt has been established, not even his dates of birth and death. He was active 1910–20, with articles on magic squares in *Nature* (1910, vol.83, pp.368–9) and *Educational Times Reprints* (1913, ser.2, vol.23, pp.99–101) and on tours in *Queen* (1915–16) and *BCM* (1918). From the British Library catalogue Prof. D. E. Knuth notes that he published about a dozen books, about card games, dominoes and solitaire (1920), and also *Translations into Latin Elegiac Verse* (1918) where his full name is given as Ernest George Bincknes Bergholt. Currently there are no Bergholts in the entire English telephone directory, although judging by East Bergholt in Suffolk it would seem to be an English name, but perhaps of Dutch origin.

Notes in square brackets [...] are by the editor.

Some Original Knight's Tours

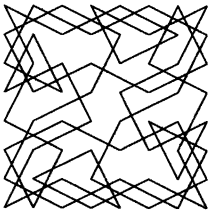
by Ernest Bergholt

***Queen* 25/xii/1915.** The methods that have been hitherto published for the construction of tours of the knight at chess over the ordinary board of sixty-four squares have been very imperfect in principle and greatly restricted in result. The total number of possible tours that can be made is so vast that it is safe to predict that no mathematician will ever succeed in counting up the total.

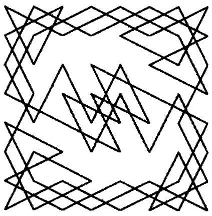
I may perhaps on some other occasion publish a new and general method which I have perfected, enabling any kind of symmetrical tour to be constructed rapidly and with ease. Meanwhile, it may interest the readers of the *Queen* to see a few specimen designs. In both the designs [1] and [2] the knight returns, at the end of his peregrinations, to the square from which he set out. Both exhibit complete bilateral symmetry; that is to say, every one of the moves has a counterpart move, similarly situated with respect to the central point of the board.

As the ideas of writers have in some cases been not at all clear on the question of symmetry, it may be stated that it is not possible to attain a perfect quadrilateral symmetry, although many very pleasing approximations thereto have been published. [A handwritten marginal note states: "I have since proved that it is possible!"] Example [2] is an improvement upon a similar design published in the *Twentieth Century Standard Puzzle Book* by the Rev. A. Cyril Pearson. [A handwritten footnote states that [2] is a "Solution of the problem: In a diametrally symmetrical tour, to make the greatest possible number of consecutive moves within the central 4^2."]

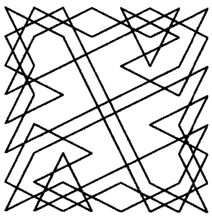
1. "Arabesque"



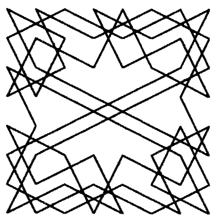
2. "The Hour-Glass"



3. "St. George"



4. "St. Andrew"



***Queen* 1/i/1916**. This week I will submit two little studies in crosses, [3] and [4]. Mr Henry E. Dudeney, who is a great authority on puzzles, but by no means an authority on the meaning and use of words, objects to my qualifying these two tours, and the two preceding ones, as 'symmetrical'. He writes: "I cannot agree with you that your 'bi-lateral symmetry' is symmetrical at all. Symmetry is a matter for the eye; this is a sort of intellectual symmetry." It would appear that Mr Dudeney's eye is differently constituted from the eye of the ordinary person. To me, indeed, these tours convey a most pleasingly symmetrical effect, and I have little doubt that the vast majority of my readers will take equal delight in contemplating them. To Mr Dudeney, Hogarth's well-known 'line of beauty' would be quite devoid of symmetry: he would admit that a capital X might be drawn symmetrically, but not a capital S. Such are the eccentricities of genius. [The reference is to *The Analysis of Beauty* (1753) by the painter William Hogarth (1697–1764).]

Mr Dudeney also warns me against the habit of "constructing new crack-jaw terminology" for everything on which I treat; whence I gather that he is unaware that the term *bi-lateral symmetry* is in common use among scientific men (see, for instance, the explanations under 'Symmetry' in *Webster's New International Dictionary*). On reconsideration, however, I should prefer to describe the tours I have printed as 'centrosymmetrical', which is another well-known term, and which, I think, more clearly defines the regularity of the designs. If we wished to be pedantically accurate, we might say that they are 'centrosymmetrical by contrary flexure' (terms not invented by me); or, to use a simpler but less euphonious term, we might call them 'S-symmetrical'. The test of the exact symmetry of my tours is that <u>any</u> straight line through the centre of the board divides the whole design into two equal and similar parts.

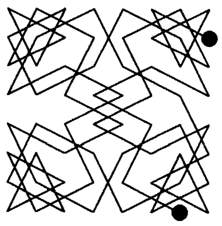
It will be noticed, in the two patterns given today, that I have introduced, in addition, a certain amount of quaternary symmetry. This is always possible to a greater or less extent, according to the character of the design; but, as I said last week, it can never be extended over the complete tour.

***Queen* 8/i/1916.** It was pointed out by Euler in 1759 (*Memoirs of Berlin*) that the general solution of the knight's tour is so vast and indefinite that it becomes a necessity to limit the problem in some way, the obvious course being to confine the attention to symmetrical designs. On the ordinary board of 64 squares, Euler's success did not go beyond symmetry of the binary type; but in the records of the Paris *Académie des Sciences* for 1771 is a memoir by Vandemonde in which the author makes a systematic attempt to achieve complete quaternary symmetry. In this attempt he necessarily failed. He constructed two separate closed paths of 32 squares each, fulfilling all conditions, but, in joining them together, found himself driven to abandoning even the binary form which had been previously illustrated by Euler.

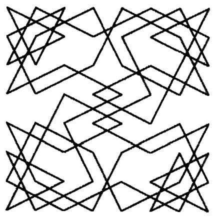
In tour [5] I have given Vandermonde's attempt the most complete quaternary expression that is possible by changing his closed tour into an open ('non-reentrant') one. Here, out of the total 63 moves, there is only one that has no counterpart; the remainder fall into fourteen groups of 4, and two groups of 3. This is a form that, in an open tour, can always be attained without difficulty. [A handwritten note in the margin says: "All possible tours of this kind have been catalogued." This probably refers to the work of Paul de Hijo (1882).]

In tour [6] I have expressed Vandermonde's design in the most perfect closed form that is possible. Binary symmetry is preserved throughout—as should always be done. There are fourteen symmetrical groups of 4, and four centrosymmetrical pairs. This is the utmost that can be done with any closed tour; the form must therefore be regarded as the knight's tour in its highest perfection. [A hand-written marginal note says: "The design of this, however, I have since improved."] Tours [7] and [8] show similarly perfect designs, for which I think I may claim considerable elegance.

5.



6.



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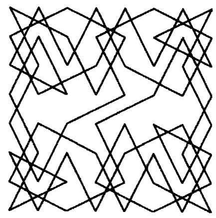

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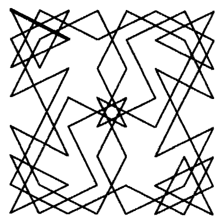
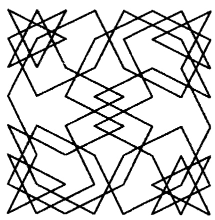
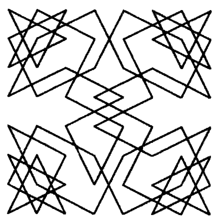


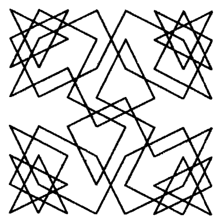
(A) Bergholt



(B) Vandermonde



(C) Jelliss



(D) Jelliss

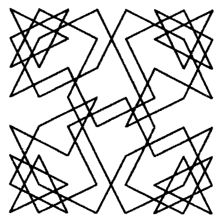


Diagram (A), which occurs amongst the 'Miscellaneous Articles and Notes', must be the version referred to, the 'improvement' being in direct quaternary symmetry of the central area, but it differs from Vandermonde's original tour (B) by 8 changes, as against 6 for [6]. In fact it is possible to symmetrize (B) by only four changes, as I have done in (C).

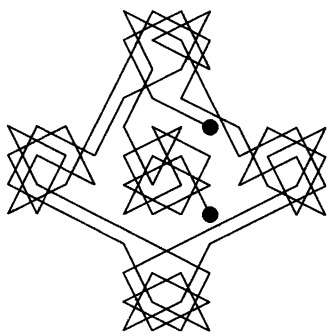
Vandermonde actually began with four 16-move circuits in direct quaternary symmetry, first joining them in pairs and then joining the resulting two 32-move circuits. At each step his deleted and inserted moves form a rhombus. The tours (C) and (D) show how he could have used his method to produce symmetric tours from the four circuits, with deletion of only one move from each circuit. Although six moves are deleted and six inserted in his process, the minimum of four is reached since two of the moves inserted at the first stage are those deleted at the second stage.

***Queen* 22/i/1916.** In the *Queen* of Nov. 12, 1910, Mr H. E. Dudeney invited its readers to complete a knight's tour (he called it a 'path', because he objects to the former term when the course is non-reentrant) over the 80 squares of a Greek cross. "There are, of course," he wrote, "many ways of doing it; but the reader should try to find as symmetrical a path as possible. Symmetry in these knight's-path puzzles is always a pleasing feature when the path is indicated by lines drawn from square to square." For purposes of comparison, I reprint Mr Dudeney's solution as tour [9]. When he published it, he wrote: "We select, ..., one of the more elegant ways of solving this puzzle. Perfect symmetry is not possible, but our path approaches it." *Sed longo intervallo*. It will be seen that, from the point of view of symmetry, the solution given was very unsatisfactory. It is interesting to recall it, however, as it appears to mark the high-water level achieved by previous investigators. In tour [10] I submit one of my own solutions of the same problem. This is non-reentrant, and is an example of the highest degree of quaternary symmetry that it is possible, on this form of board, to attain.

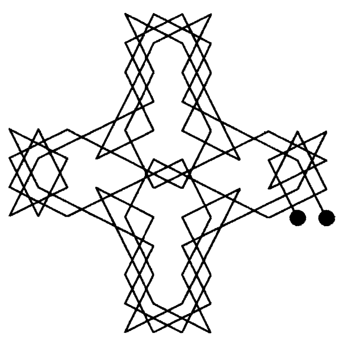
Reentrant tours over the same board may be constructed on Vandermonde's plan, by joining up unsymmetrically two symmetrical half tours. But if the condition be that binary symmetry be in no place violated then tour [11] shows absolutely the best type that can be achieved. The degree of symmetry in tours [10] and [11] is exactly the same as in the previous cases of tours [5] and [6], respectively, over the ordinary chess-board.

As we are now choosing our own shape for the board, however, there is no reason why we should not choose it to the best advantage. We have only to take in four more cells to obtain a shape that will permit of our exhibiting complete quaternary symmetry. This is shown in tour [12].

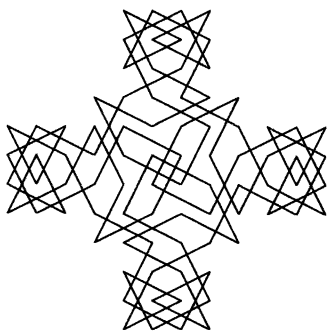
9. Dudeney



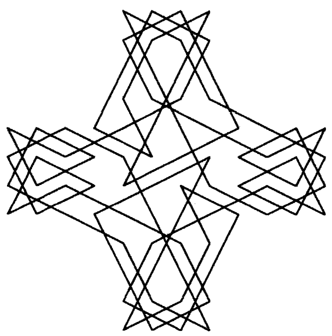
10.



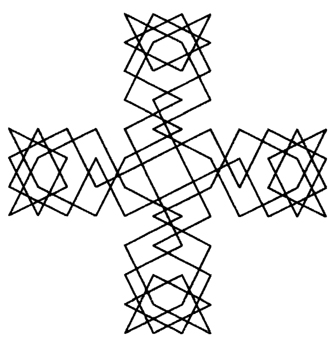
12.



11.



(E) Murray

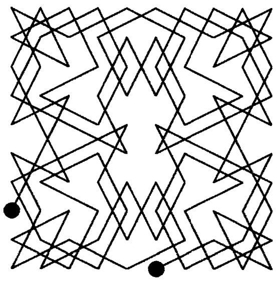


[A handwritten note alongside [11] asserts: "I have since constructed several still more remarkable for their elegance"; but no others are given among the notes. The editor inserts another example (E) from H. J. R. Murray's 1942 manuscript, "selected from some 60 tours on this board in mixed quaternary symmetry composed by the author".]

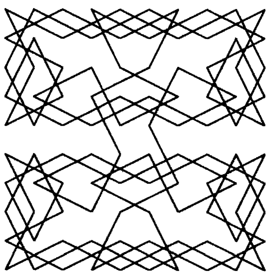
The same complete symmetry [as in 12] can be shown on square boards of 6, 10, 14, ... &c, cells to the side. Also on the ordinary chess-board, if we (1) omit or (2) add four cells symmetrically to the centre. Also on square boards of 7, 11, 15, ... &c, cells to the side, if we omit the central cells.

***Queen* 29/i/1916.** In illustration of my statement last week as to tours on the board of 10^2 cells (the French draught-board), I give this week four assorted specimens—as a gardener might say. If we try for what I may call direct symmetry, i.e. symmetry with respect to the horizontal and vertical axes, it will be impossible to achieve complete success. We shall be driven (1) to an open tour, containing a single unrelated move, as in tour [13], or (2) to a closed tour with at least two unrelated pairs of moves, as in tour [14]. The last-named design shows the joining up of two completely symmetrical paths over separated halves of the board. It is very easy to join up four paths over separated quarters of the board; but as this has been previously shown by Euler, I need not give another example here.

13



14



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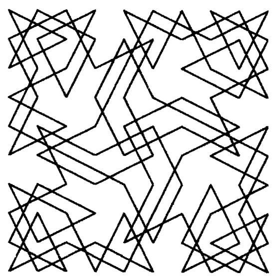
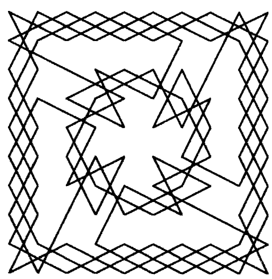


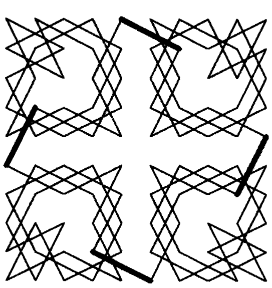
16



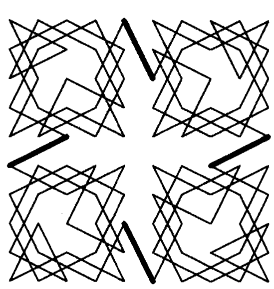
Tours [15] and [16] show how complete quaternary symmetry may be obtained by giving the pattern four successive quarter-turns, as we pass round the board. The nature of the pattern in [15] makes these quarter-turns obvious to the eye; scrutiny a little more attentive will show that they exist equally in tour [16]. This is the oblique symmetry (or central symmetry), which I have discussed in a previous article. Tour [16] we may call the 'picture-frame' pattern; it consists of a central symmetrical 6^2 surrounded by a regular border. On the ordinary chess-board, the same plan (with a central 4^2) was applied in a crude unsymmetrical way first by De Moivre (1725) and afterwards by Moon (1843). [The same idea was shown by Ali ibn Mani c.1350.] Tour [16] is easy to memorise, the plan of it being clear and simple. A systematic progress round the margin of the board is interrupted four times, at regular intervals, 9 cells of the central 36 being taken on each occasion. [A handwritten note declares: "Here the articles, due to war exigencies, abruptly terminated."]

There is a convenient space here in which to reproduce a "Note on the combinations of tours on the 5^2 board to form symmetrical tours on the 10^2 board" by H. J. R. Murray, dated 2/i/1917.

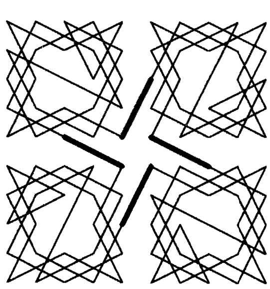
F



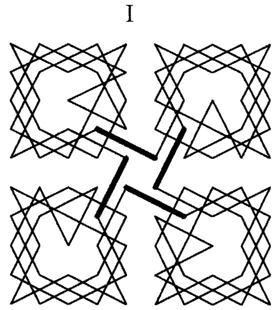
G



H



I

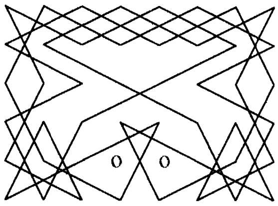


"Since a reentrant tour over the 5^2 board is impossible, and of the two terminals of any open tour one at least is an angular point, while the other may be any square of the same colour, it is very easy to combine four of these tours in oblique quaternary symmetry. There are four different methods of linking, illustrated in the tours F, G, H, I. Since there are 6 tours on the 5^2 which begin and end on the pair of cells used in F and H, and 8 using the same terminals as for G and I, there are 28 tours of this type on the 10^2 board."

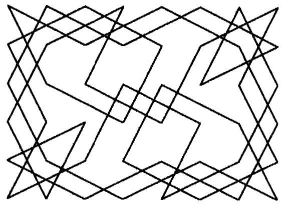
Memoranda 1–4 on the Knight's Tour

by Ernest Bergholt

First Memorandum **(Feb 24 1916).** I define 'symmetry' as either (1) direct with respect to a central axis, or (2) oblique (or skew) with respect to the central point. These are binary forms. On some square boards, tours may be constructed in quaternary skew symmetry. On no board, however, is complete quaternary direct symmetry (with respect to a pair of rectangular axes) possible.



(1) Direct Symmetry

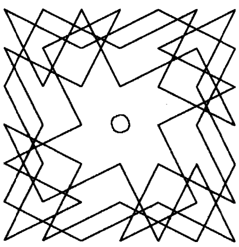


(2) Oblique Symmetry

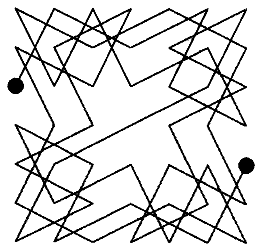
By a simple application of the 'Law of Parity', it can be ascertained by inspection of a given figure whether the symmetry required is (a) *a priori* possible, or (b) definitely impossible. If we number the cells of a board alternately odd and even it will always take an even number of knight's moves to go from odd to odd or from even to even; and an odd number of moves to go from odd to even or vice versa. (On chequered boards, the difference of parity between cell and cell is marked by difference of colour.) On any board which is not numbered alternately we have only to count the cells from terminal to terminal in vertical and horizontal directions. [In other words count the number of wazir moves instead of knight moves. In his 1942 MS Murray notes alternatively that "one may make a succession of bishop moves from one cell to the other; if this is possible the number is even, if not, it is odd." I have omitted two numbered diagrams here, and substituted coordinates instead of numbers in the text.]

The *a priori* condition for binary direct symmetry is that from one cell to its directly symmetrical counterpart cell must be the same parity of knight's moves as the parity of half the total number of cells. It is convenient to count from corner to corner. Thus in the 8 by 6 rectangle above, half the board contains 24 cells (even); from a6 to h6 is an odd number of moves; therefore binary direct symmetry is impossible. But from a6 to h1, or (more quickly counted) from d4 to e3, is an even number of moves; therefore oblique binary symmetry is *a priori* possible. We thus see how the omission of two cells in (1) makes direct binary symmetry possible, because half 46 is an odd number.

The possibility of quaternary skew symmetry is similarly foreseen by comparing the number of cells in a quarter of the board with the number of moves from cell to counterpart cell. For example, if we omit the central cell in the 7^2 there are 12 cells in each quarter of what remains, and from a7 to its skew counterpart g7 is an even number of moves. Therefore quaternary skew symmetry is *a priori* possible; and I show it below (3). Join the centre cell, opening two terminals in binary symmetry, and we have a non-reentrant tour of the 7^2 with maximum quaternary symmetry (4).



(3) Quaternary Oblique Symmetry



(4) Open tour with Maximum Quaternary Symmetry

[It is unfortunate that some authors (e.g. H. S. M. Coxeter, *Introduction to Geometry*, 1969, p.39) apply the description 'direct' to transformations that do not reverse cyclic order, which is the opposite to Bergholt's use of the term.]

First Method of Quaternary Skew Symmetry. On the foregoing considerations I have based an easy method for quaternary skew symmetry (so far as is possible) over any kind of board. For example: take the ordinary 8^2 chess-board and number the squares at random over a quarter of the board, numbering each of the other three quarters to correspond [5]. If we now omit any four cells bearing the same number, complete quaternary symmetry will be possible over the remaining 60 cells.

We must choose for (temporary) omission such cells as can readily be joined up later on, either the four '12' cells or the four '14' cells, say '14'. To enable these cells to be taken in later, we must see that the 60-cell tour comprises one of the moves 6.9, 5.9, 4.7 [5.13, 6.13]. With these two precautions, any reentrant path of 16 cells which does not repeat the same number twice solves the problem. Practically it is most convenient to use 16 numbered counters (Lotto counters answer every purpose). If the tour of 16 cells does not come out at first trial, the counters are very easily manipulated (on Euler's plan) until success is obtained; a minute or two is the longest time I have found necessary.

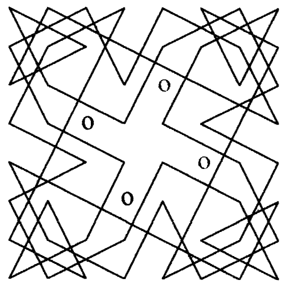
(5)

A B

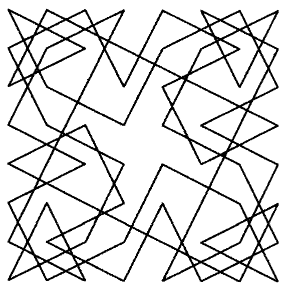
1	2	3	4	16	9	8	1
8	7	6	5	15	10	7	2
9	10	11	12	14	11	6	3
16	15	14	13	13	12	5	4
4	5	12	13	13	14	15	16
3	6	11	14	12	11	10	9
2	7	10	15	5	6	7	8
1	8	9	16	4	3	2	1

D C

(6)

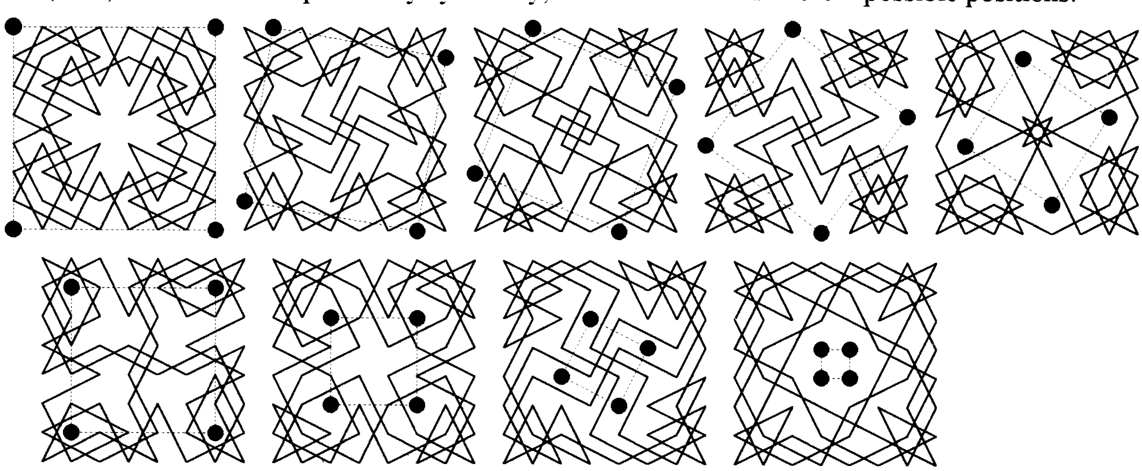


(7)



For an illustration: proceed alternately (4 cells at a time) in diamond and square formation. 6.1.10.13.15.8.3.12.16.7.4.11.5.2.9... The cycle reenters from '9' to another '6', and it contains the move 7.4. As there are 15 moves we can never return to the same '6', nor to the one in vertically opposed quarter, but always travel to an adjacent quarter, that is, one chain, if it starts in quarter A aways leads to B or C, never to D, nor back into A, and so we go on successively in four identical cycles all round the board [6]. If we remember on two occasions to pass directly from '7' to '4', and on the other two occasions to go 7.14.14.4, we take in the four missing cells, and complete the whole tour of 64 [7]. We thus introduce three pairs of moves without quaternary counterparts, and the omission of 7.4 twice leaves one other pair without counterparts. We thus have four pairs in all which are in binary symmetry only; an irreducible minimum in all quaternary solutions over $(4n)^2$ boards.

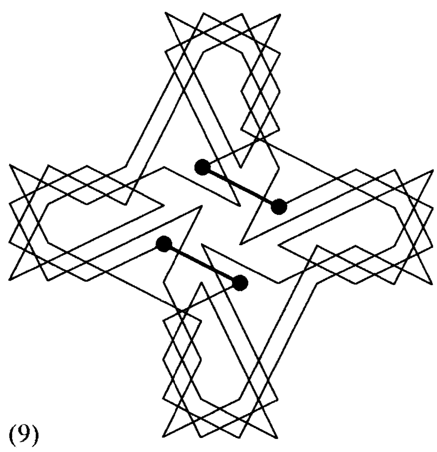
Among the Murray manuscripts there is also a study, The Knight's Tour, by G. L. Moore dated 1920, which follows up some of the work of Bergholt. In particular it includes this set of (8^2–4) board tours in quaternary symmetry, with cells omitted in the 9 possible positions.



As an additional example, let us now apply the same method to the solution of the Greek cross of 80 cells [8]. We will omit temporarily the four cells '10' and we have to see that our path includes one of the four moves 5.13, 5.14, 11.13, 11.14. I go, as before, in diamond and square formation alternately, as far as I can, wherefore; 6.1.17.14.5.2.18.13.8.3.19.16... Here I am driven into the central square '9'. Going on to '11', I cannot complete the diamond (for '9' cannot be repeated), so I go on to 7.4.20.15; and then '12' will join me to another '6'. *Da capo*. The cycle is complete. One point is noticeable. The move 9.11 may be taken in either of two ways. It does not matter which way we take, but having once decided we must not change during the tour. All the 9.11s must be taken at the same angle from the previous move 16.9. Below is the tracing [9]; I take the angle 16.9.11 as obtuse.

				1	2	20	19				
				4	3	17	18				
				5	6	16	15				
				8	7	13	14				
19	18	15	14	9	10	12	9	8	5	4	1
20	17	16	13	12	11	11	10	7	6	3	2
2	3	6	7	10	11	11	12	13	16	17	20
1	4	5	8	9	12	10	9	14	15	18	19
				14	13	7	8				
				15	16	6	5				
				18	17	3	4				
				19	20	2	1				

(8)



(9)

***Second Memorandum* (March 14 1916).** Being a report to Mr W. Rouse Ball Esq. on the Memoir of P. Volpicelli on the subject, published at Rome, 1872. [The first six Memoranda are all recorded as 'Communicated to W. W. Rouse Ball Esq.' on the dates cited. I was tempted to leave out much of this Memorandum, but it is such a splendidly scathing review, and some positive points are made by Bergholt on general methodology.]

This book contains 389 quarto pages (besides separately inserted large-sheet diagrams) and professes to be a "Complete and General Solution of the Problem, by means of the Geometry of Position, upon any board whatever." *"Quid dignum tanto feret hic promissor hiatu? Parturient montes, nascetur ridiculus mus."* It was a very necessary thing to examine carefully into this work, and I am glad to have had the opportunity of doing so. I have been through the whole of it, and regret that the many hours' work which the task entailed has turned out to be absolutely unproductive. There is not a *scintilla* of merit or interest (as regards the professedly original portion of the work: Part II, pp.70-389) in the book from beginning to end.

The author numbers the cells of his boards on the plan of the cartesian coordinates, the bottom left corner of a square board being taken as origin. Then he makes a table of all the cells which are a knight's move from each of the cells of the board. This he calls a 'directive table' (*Tavola Direttrice*). Starting from any given cell, he proposes to tabulate every possible sequence of knight's moves, following out every path until it comes to an obligatory stop. This, he says, when completely carried out, will be a complete and general solution of the problem, free from all trial, and he plumes himself on being the first person who has ever been able to give such a complete and general analysis. The alleged 'solution', as will be seen at once, far from being free from all trial, is trial pure and simple. Also, it is an absolute impossibility to carry it out. Also, there is nothing in it whatever that is original.

Euler pointed out long ago that it was hopeless to attempt to solve the problem generally by trial, as the number of resulting paths would be so enormous that they could never be exhausted. He further pointed out that, even if possible, the task would not be worth while, as by the method he described and used, one could obtain as many complete tours as one wanted (not, of course, necessarily reentrant, nor symmetrical). Furthermore, the notation of Volpicelli is due, as is well known, to Vandermonde. There is therefore neither novelty, nor utility in any part of Volpicelli's 'method', if it can be dignified by such a title.

The absurdity of the whole work becomes more than ever manifest when we examine into the results which our author actually sets out. On page 87 he starts out to analyse the rectangle of 4×3 cells, and on page 88 succeeds in tabulating all the paths (those which end prematurely, as well as those which visit all the 12 cells which start from cell 24. (Tavola 4). This work is wholly unnecessary. Instead of seeking to abridge his trials (the true aim of all methods of solution of this and kindred problems), he takes a foolish pride in setting down every possible false step that can be taken.

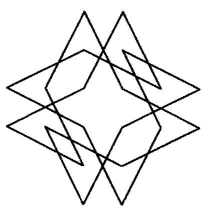
Any person of ordinary intelligence would proceed as follows. From 24 we can go to 12 or 32, but we need only consider one of these initial moves, as the tours resulting from the other will be merely reflections. Take 24.12. From 12 we must go to 31, by the usual 'corner' rule (obviously, when a cell connects only with two cells—31 with 12 and 23—the moment we reach, say 12, we must take in 31 and 23. By similar reasoning, the first 7 moves are forced: 24.12.31.23.11.32.13. Then we have a choice: 21 or 34. Whichever we take, the remaining moves are forced, and we have therefore only the two completions: 21.33.14.22.34 and 34.22.14.33.21, both of which may be reflected on the horizontal axis. These are Volpicelli's i, k, l, m. (The above solutions can be obtained in less time, of course, than it has take me to write out the process.)

The analysis of this little rectangle absorbs the energies of Volpicelli up to p.105. Then he says he will analyse the Greek cross of 20 cells, and to this he devotes no fewer than 59 pages, examining all the paths which can continue the three opening moves 13.32.24. My own analysis of the cross would be on the following lines. I should first try for a symmetrical closed tour. Quaternary is at once seen to be impossible, so I number the cells to try for a binary [10]. Notice that 4.9 and 3.6 are 'optional' moves (i.e. can be taken in two ways each). Of these two ways, one crosses into the other half of the cross (and I write these moves as 4×9 and 3×6) while the other way, in each case, does not cross (written 4–9, 3–6). Half a minute's examination shows that if we start with 6.1.8, there is one way only of going through the ten numbers and returning at end to 6, viz: 6.1.8.5.4-9.2.7.10.3×6... [11]. We shall also find that it makes no difference whether we move 4–9...3×6 or 4×9..3–6 since we only get another phase [12] of the same tour. [The tour was given by Euler, 1759.] This is therefore the unique solution. (If we take 4–9 and 3–6 or 4×9 and 3×6, we only get a closed path of 10 cells. This exemplifies a general principle to which I will return in another memo.)

		1	2		
		4	3		
5	6	7	8	9	10
10	9	8	7	6	5
		3	4		
		2	1		

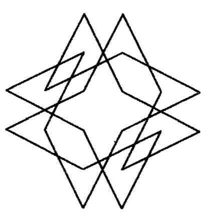
(10)

4-9 and 3x6



(11)

4x9 and 3-6



(12)

		1	2		
		4	3		
5	6	7	8	9	10
16	15	14	13	12	11
		17	18		
		20	19		

(13)

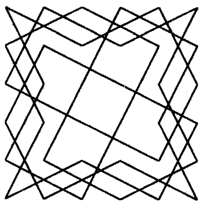
It has hardly been necessary to use counters for so simple an analysis, but to get all the possible open tours beginning 16.17.6 I should number my cross from 1 to 20 [13], the closed tour being 16.17.6.1.8.11.18.15.20.13.10.3.12.19.14.5.4.9.2.7, and should employ Euler's method of breaking and reconnecting the chain, with a card diagram in my left hand, and 20 Lotto counters on the table, which can be easily and rapidly re-adjusted with the right hand. Noticing that in a closed tour the following are fixed sequences: (1) 16.17.6.1.8.11.18, (2) 15.20.13.10.3, (3) 12.19.14.5.4, (4) 9.2.7 it follows that, for an open tour from 6 only one of these sequences can ever be broken, and that then one of the separated links must be the other terminal of the tour, e.g. 16.17.6.1.8(15...3)(12...4)(9.2.7)18.11 where 8 and 11 have been separated and 11 has become the second terminal. It is also seen that 9.2.7 can never be broken (to make 2 a terminal), for then 7 would have to come between 12 and 18 and 18.7(12...4)(15...3) would be a forced sequence, leaving only 9.2. But 9 does not join to 3. Observing all these limitations of trial (wholly ignored by Volpicelli) his ten open tours p.158 are readily formed.

When I first tackled the knight's tour problem, I tried Vandermonde's notation, but found it perfectly useless, and, in fact, inconvenient. I now use the natural numbers from 1 onward, arranged in various ways, according to the object aimed at. To use a fixed diagram and moveable counters is a great practical improvement on Euler's plan of continually changing the numbers on a diagram, which soon becomes wearisome, as we are forced either to perform confusing mental calculations, or to keep on writing out the diagrams afresh.

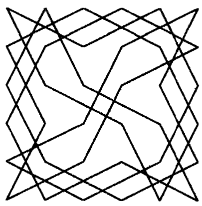
I intended originally to make some remarks on Volpicelli's historical introduction (pp.1–69) but am afraid this memo is already long enough. The introduction shows clearly that V. had never seen most of the works he mentions. Of the knight's tours on the ordinary board of 8^2 cells, V. only succeeds in arriving at 48, all of a very commonplace and uninteresting type.

Third Memorandum **(March 30, 1916).** The analysis of the 6^2 for quaternary oblique symmetry will be found both instructive and easy, according to the method I have previously described. A quarter of the board is numbered in any way, and the other three quarters are then numbered to correspond, a quarter-turn being given to each quarter, as we go round the whole board, e.g. [14].

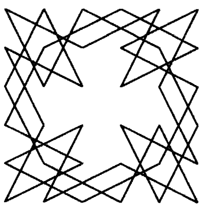
With this diagram before us we take 9 Lotto counters, numbered 1, 2, 3, ..., 9; and starting always with the four numbers 2, 9, 4, 6 in that order, we investigate the number of possible arrangements of the remaining five counters, so that the last counter shall be a knight's move from our first counter (the 2). Eliminating the duplicates which are reflections of each other on the diagonal (\), we find that there are five, and five only: I. 2.9.4.6.7.3.1.5.8, II. 2.9.4.6.7.3.5.8.1, III. 2.9.4.6.1.8.5.3.7, IV. 2.9.4.6.7.3.5.1.8, V. 2.9.4.6.7.1.3.5.8. Each of these cycles, four times repeated, gives a completely symmetrical tour of the board.



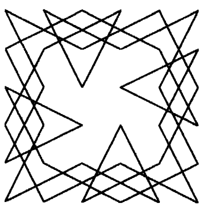
I



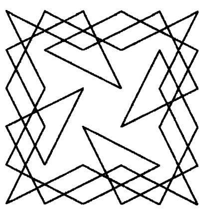
II



III



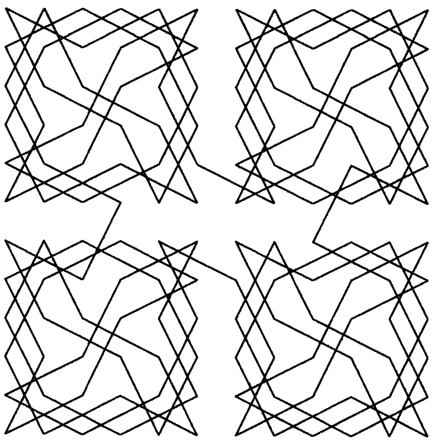
IV



V

9	8	7	5	6	9
6	3	4	2	3	8
5	2	1	1	4	7
7	4	1	1	2	5
8	3	2	4	3	6
9	6	5	7	8	9

(14)



(15)

Tour III was used for the centre of the 10^2 in tour no.16 published in *Queen* of Jan 29, 1916. Tour II I have used for a 12^2 [15]. [These five tours were found, using a similar method, by Paul de Hijo (alias the Abbé Jolivald) in 1882 (and possibly by Carl Adam in 1867) and have been independently rediscovered many times since.]

Fourth Memorandum **(March 31 1916).** [No fourth memorandum is included in Murray's papers. I wrote to Trinity College, of which W. W. Rouse Ball was a Fellow, and received the reply (14/viii/1991) that their Library catalogue contains no entries for Bergholt or H. J. R. Murray and that the W. W. R. Ball entry lists only papers of his own composition. So this memo is either lost or was omitted due to misnumbering. A fragment in the 'Miscellaneous Notes' suggests that the memo may have duplicated de Hijo's work on the "Complete enumeration of the four knight's problem (4 reentrant tours of 16 cells each) in oblique quaternary symmetry." The correct total of 140 is given, together with the numbers having each type of centre, using de Hijo's letter designations for these centres. (See *Chessics* #24 p.92 for more details on this topic.)

To be continued.

PI FOR POLYGONS

by Professor Z. I. Cranium

The editor has offered me an occasional page to expound some of the ideas in my unpublished work on *Rational Mathematics*, in which I show how almost all mathematics, certainly all that needed in scientific applications, requires only rational numbers (i.e. ratios of integers).

The system of so-called 'real' numbers, which consists of the rational numbers augmented by others, appropriately termed 'irrational' (because they cannot be expressed as ratios of integers), has been designed to enable mathematicians to apply numbers to 'continuum geometry' which, while it has a venerable history, bears little relation to reality. In this geometry points and lines have no width, and between any two points it is always possible to find another quite distinct point, in fact a 'continuum' (which is a 'super-infinity') of points. None of these ideas, in the age of atoms, quantum mechanics and the uncertainty principle, bear any relation to reality; they are as out-of-date as phlogiston.

It is true that we have one or two symbols, like π, e and ✓2 (sorry I can't find a proper square root sign on this machine) that are usually said to represent irrational numbers, but in fact whenever we do any calculations with them we substitute rational numbers, the value chosen depending on the particular context. The Greek geometers called the diagonal of a square 'incommensurable' with its sides, since, if the side is of length s, the diagonal is of length (✓2)s, and ✓2 cannot be 'exactly' expressed as a ratio of integers. This overlooks the fact that s cannot be 'exactly' determined either. If s is 1.0000 to four places of decimals, then the diagonal is ✓2 = 1.4142 with equal accuracy.

Let us consider the meaning of π. The 'circular constant', denoted , since about 1706, by the Greek letter pi (π) is defined geometrically to be the ratio of the circumference of a circle to its diameter. Since in rational geometry a circle cannot be distinguished from a regular polygon with a large number of sides it would therefore be convenient if we could define π for any polygon with n sides. We denote this ratio by $\pi(n)$.

Various definitions can be considered. The definition which gives the simplest formulae in the case of a regular polygon is the ratio of the semi-perimeter of the polygon to its circumradius (the distance from centre to vertex). If the length of each side is $2s$ then the semiperimeter is ns and the circumradius is $R = s \sin(c/2n)$, where c = cycle (360°). We thus find that $\pi(n) = n \sin(c/2n)$.

As n increases, $\sin(c/2n)$ decreases, but the product $n \sin(c/2n)$ varies less and less. Within a given context, in which calculations are made to a given number of decimal places, it takes a constant value for all values of n above a certain bound. It is quite natural to say that the values of $\pi(n)$ approach a limiting value, which we can denote by π, without the affix n, but we do not imply by this form of wording that the 'limit' is that of an 'infinite sequence'. In a different context, where a higher degree of precision is sought, the value of π will be defined to more decimal places, but, we maintain, there is no real context in which π is determined to an 'infinite number of decimal places'.

The area of the polygon works out to be $ns^2/[\tan(c/2n)]$, and this is equal to $\pi(n)rR$, where r is the inradius (the distance from centre to mid-point of side), which is $s/[\tan(c/2n)]$. As n increases, r and R become more nearly equal, and the area of the polygon can then be expressed accurately enough by the familiar 'circle' formula πR^2.

We find the following successive values of $\pi(n)$ to four places of decimals:

π(3) = 2.5980(7)	π(10) = 3.0901(7)	π(32) = 3.1365(4)
π(4) = 2.8284(2)	π(11) = 3.0990(5)	π(64) = 3.1403(3)
π(5) = 2.9389(2)	π(12) = 3.1058(2)	π(128) = 3.1412(7)
π(6) = 3 exactly	π(13) = 3.1111(0)	π(256) = 3.1415(1)
π(7) = 3.0371(8)	π(14) = 3.1152(9)	π(512) = 3.1415(7)
π(8) = 3.0614(6)	π(15) = 3.1186(7)	π(1024)= 3.1415(8)
π(9) = 3.0781(8)	π(16) = 3.1214(5)	π(2048)= 3.1415(9)

From these values we conclude that around 2^{11} = 2048 sides are necessary to form a circle with π accurate to the fifth decimal place (i.e. 1 part in 100,000).

SPELL SPORT

by G. P. Jelliss [© 1996]

'Spell Sport', published here for the first time, is a word and letter game on a board of 26 by 26 squares, using tiles of two-square (domino) shape, each tile bearing two letters (possibly the same), one in each square. On one side of the tile the two letters are printed in 'across' fashion, and on the other side the same two letters are printed, in the same sequence, in 'down' fashion. This two-sided property of the tiles is the main feature of the game for which I believe some originality may be claimed.

Q	U

Q
U

The total number of tiles possible, using all combinations of letters, would be 26×26 = 676, but many of these combinations do not occur in actual words (e.g. QZ) or occur only rarely (e.g. BT). The number of tiles actually used is 338, half this total, just sufficient to cover the board completely, since each tile covers two cells. This also makes the equipment easy to manufacture and pack, by printing the tiles on a single sheet of card of the same size as the board, leaving the purchaser to cut out the tiles—or, with more sophisticated manufacture, to push them out. The pairs of letters chosen to appear on the tiles are those combinations of letters that most commonly occur in everyday English words. After several trials I have selected the pairs of letters as shown in the following chart. The first ten rows of the chart are formed on a systematic basis, showing all possible combinations of the vowels A, E, I, O, U preceded and followed by the consonants B, C, D, F, G, L, M, N, P, R, S, T, V. These 130 tiles are duplicated in the complete set. The remaining 78 tiles, occupying the last six rows of the chart, constitute an unsystematic selection of other common letter combinations, i.e. those consisting of two vowels or two consonants, or combining a vowel with the less-used consonants H, J, K, Q, W, X, Y, Z. Some of these pairs have been chosen not for their frequency of occurrence but simply to ensure that each letter appears at least twice.

It may be possible to improve the selection here, I do not claim to have found the best solution yet, possibly there are too many Us and Vs. In an earlier version all 338 tiles were differently lettered, and I found that 301 of the pairs occurred in the list of 850 words of 'Basic English'published by C. K. Ogden in 1930 (and reproduced on p.356 of D. Crystal's *Cambridge Encyclopedia of Language*). However, this proved to have too many double-consonant combinations, which restricted word formation.

Using the two printed charts I have been able to make a set of tiles by copying them four times using the enlargement facility on my photocopier (which magnifies by a factor of 1.22, roughly A4 to B4), ending up with tiles with roughly half-inch squares. I pasted the sheets from the first chart onto card, and when dry pasted the sheets from the second chart onto the other sides, and, when the glue was dry again, cut the cards up into the individual tiles. It needs special care of course to get the fronts and backs of the tiles to align correctly.

In play it is permitted to place the tiles using combinations of the letters H, I, N, O, S, X, Z either way up, so that for instance the I O tile also serves for O I. Using a different type face it might also be allowed to interpret the inverted M as a W and vice versa.

Some authority of course needs to be agreed beforehand to decide the validity of words. This can be any dictionary that happens to be available to the players, and not necessarily the most comprehensive. Hyphenated words are allowed, spelt without the hyphen.

With this equipment it is obviously possible to play a range of 'Spell Sports'. The rules offered here are not the final word on the subject. Readers may like to experiment with alternatives. Play between the two players will normally be for a series of games (a 'rubber'), the object being to achieve the highest score over the series. The method of scoring is explained below. A single game ends when one of the players completes a connected path of tiles from one side of the board to the other. By so doing this player doubles her score for that game, while her opponent's score is undoubled. These scores are added to the running total for the rubber. The first player doubles if the path is across the board, and the second player doubles if the path is down the board. A path that goes across the board may meander, but must not touch the upper or lower sides of the board, though it may have offshoots that do so. Similarly a path down the board must not touch the right or left sides. The cells of the board can obviously be named by lettering the columns a,b,c.... and the rows A,B,C,... so that aA is the top left cell, zA the top right, aZ the bottom left and zZ the bottom right.

Chart of the 338 tiles (across side): repeat the first ten rows.

A B	A C	A D	A F	A G	A L	A M	A N	A P	A R	A S	A T	A V
E B	E C	E D	E F	E G	E L	E M	E N	E P	E R	E S	E T	E V
I B	I C	I D	I F	I G	I L	I M	I N	I P	I R	I S	I T	I V
O B	O C	O D	O F	O G	O L	O M	O N	O P	O R	O S	O T	O V
U B	U C	U D	U F	U G	U L	U M	U N	U P	U R	U S	U T	U V
B A	C A	D A	F A	G A	L A	M A	N A	P A	R A	S A	T A	V A
B E	C E	D E	F E	G E	L E	M E	N E	P E	R E	S E	T E	V E
B I	C I	D I	F I	G I	L I	M I	N I	P I	R I	S I	T I	V I
B O	C O	D O	F O	G O	L O	M O	N O	P O	R O	S O	T O	V O
B U	C U	D U	F U	G U	L U	M U	N U	P U	R U	S U	T U	V U
A I	H A	J A	W A	E X	M B	C H	B L	S M	C R	C S	C T	S W
E A	H E	J U	W E	O X	S C	G H	C L	K N	D R	N S	F T	T W
E E	H I	K E	W I	A Y	N D	P H	F L	R N	F R	P S	N T	G Y
I O	H O	K I	A W	Y E	L F	S H	L L	M P	P R	R S	R T	L Y
O O	H U	E Q	E W	I Z	N G	T H	P L	S P	T R	S S	S T	R Y
O U	H Y	Q U	O W	Z O	C K	W H	S L	S Q	W R	T S	X T	T Y

To begin the game the tiles are shaken in a bag and each player takes a random selection of twelve. These tiles are not hidden but are displayed in front of the player, where the opponent can see them. The opponent is permitted to use any number of tiles from the opponent's set, but at a penalty of 4 points for each stolen tile. When a player is left with six or fewer tiles at the end of a play he may take a random selection of another six.

The players alternately place a series of tiles on the board, to form words. Any tile may be placed by either player to read across or down. Each tile placed must have at least one side either against an edge of the board or adjacent to a tile already placed, either in the same turn of play or earlier in the game. When placed adjacent to another tile, the two adjacent letters must form part of an across or down word of at least three letters. Not more than two words may be formed in one go, one across and one down. These words may make use of letters of existing words, either by extending them, at one or both ends, or by forming a junction or cross formation. A player who completes a circuit of words doubles his score for that play.

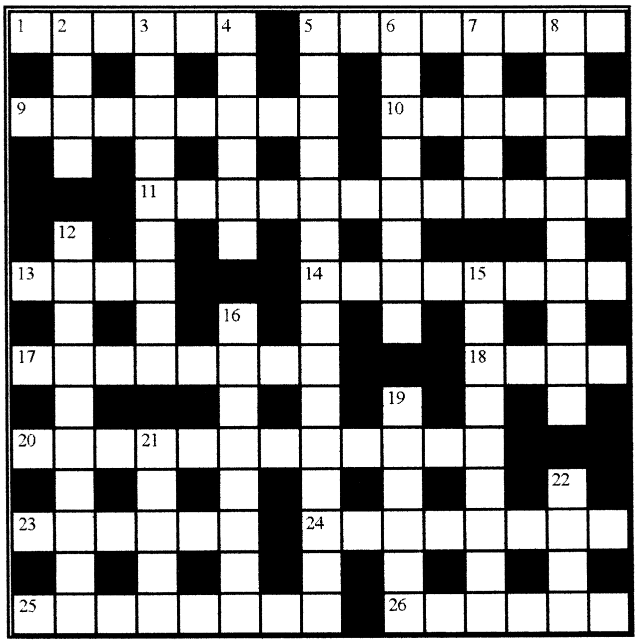
If a single word is formed, across or down, it scores the number of letters in it. If two separate words are formed, one across and the other down, these scores are added. If two joined words are formed however these scores are multiplied. Thus in a normal go the player will seek to form two joined words, one 'across' and the other 'down'. The join need not occur at a newly placed tile.

Chart of the 338 tiles (down side): repeat the first ten columns.

A B	E B	I B	O B	U B	B A	B E	B I	B O	B U	A I	E A	E E	I O	O O	O U
A C	E C	I C	O C	U C	C A	C E	C I	C O	C U	H A	H E	H I	H O	H U	H Y
A D	E D	I D	O D	U D	D A	D E	D I	D O	D U	J A	J U	K E	K I	E Q	Q U
A F	E F	I F	O F	U F	F A	F E	F I	F O	F U	W A	W E	W I	A W	E W	O W
A G	E G	I G	O G	U G	G A	G E	G I	G O	G U	E X	O X	A Y	Y E	I Z	Z O
A L	E L	I L	O L	U L	L A	L E	L I	L O	L U	M B	S C	N D	L F	N G	C K
A M	E M	I M	O M	U M	M A	M E	M I	M O	M U	C H	G H	P H	S H	T H	W H
A N	E N	I N	O N	U N	N A	N E	N I	N O	N U	B L	C L	F L	L L	P L	S L
A P	E P	I P	O P	U P	P A	P E	P I	P O	P U	S M	K N	R N	M P	S P	S Q
A R	E R	I R	O R	U R	R A	R E	R I	R O	R U	C R	D R	F R	P R	T R	W R
A S	E S	I S	O S	U S	S A	S E	S I	S O	S U	C S	N S	P S	R S	S S	T S
A T	E T	I T	O T	U T	T A	T E	T I	T O	T U	C T	F T	N T	R T	S T	X T
A V	E V	I V	O V	U V	V A	V E	V I	V O	V U	S W	T W	G Y	L Y	R Y	T Y

Here is how a sample game might begin. 'Across' takes the tiles AD, BA, DU, ID, IS, IT, OX, SH, TO, UF, UV, WO and 'Down' takes AL, BU, CE, DR, ER, ET, GA, GO, HA, HI, PI, VO. (1) 'A' claims 'D''s ET and spells out WO/AD across, from square aM to dM, and D/UV/ET down from dM to dQ, thus getting rid of the awkward UV, and scoring 4×5 = 20 less the 4-point penalty, giving 16. 'D' now plays PI/ER/CE down from, say, nA to nF, and VO/C/AL across from lE to pE, scoring 6×5 = 30, and having only 6 tiles left he can take 6 new ones, which prove to be AT, FA, OC, OU, PH, RY. (2) 'A' extends VOCAL/IS/T across with TO/SH down scoring 8×4 = 32, and enabling him to take 6 new tiles: AR, ER, NU, TI, UL, YE. 'D' immediately steals ER from 'A' to play HI/G/H across (to the H of TOSH) and GO/PH/ER down, scoring 4×6 – 4 = 20.

Cryptic Crossword – *10* – by Querculus



A Question in Verse

A Sigh of Sorrow

The hurt of a sad - - - - - - -
To her whom one - - - - - - -
Is - - - - - - - far when she
In - - - - - - - conceals her tears.

Each of the four missing words is made up of the same seven letters. This is problem 57, dated 13/v/1921, in T.R.Dawson's manuscript of 'Original Puzzles' (for more details see p.224). More questions of this type are invited.

ACROSS

1. Mine is attractive. (6)
5. Throw a veil over Spanish rattler. (8)
9. Relative bias makes mine stop. (8)
10. Rushes about like a monkey. (6)
11. Leaving swine on friend, go straight. (12)
13. Extremely light pistol. (4)
14. Sends messages in its sleep. (8)
17. In what grimmer service than that of a Baptist? (8)
18. I'm opposed to the Establishment, like the poorest churchgoers. (4)
20. Slow invasion turns on cement arch. (12)
23. I'm inches, literally, from bells. (6)
24. Encouple riches with disorder. (8)
25. Maps hung this way gather moss. (8)
26. Brilliance of ormolu streaked with grey. (6)

DOWN

2. Returns on the first of the last, frozen. (4)
3. The professional, just as expected, is experimental. (9)
4. Oliver's laurels? (6)
5. Creature to make no exotic neutral turn nor mammoth cower! (15, two words)
6. Scornful academic in third Indian costume. (8)
7. Where battle commenced at the end of an era. (5)
8. Licences hundreds lost after blue letters display exuberance. (10)
12. Knowing the ropes, join, give brief reply, and start cheering. (10)
15. Impermanent hairpiece for a trying female. (9)
16. As serpentine as wrongdoing can produce? (8)
19. Erase Quelch from the next series. (6)
21. Strange graduation dance. (5)
22. A mark that mascara will hide. (4)

POLYGRAM

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GUTSY
OARED
ORBED
DEBAR
FIEND
OGLED
RESIT
TOPED
UNTIE
NITRE
EARLY
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Anagram each across word to give a different down word or phrase of similar meaning, but not necessarily in the first column.

Puzzle Questions

If there is sufficient response to these questions we will award points and publish a solver's ladder in each issue. You have two months to respond, though any later contributions will of course be given proper consideration. Several of the questions have two or more parts.

1. A question of proportion.

In the BBC radio programme 'The Square on the Pythagoras', transmitted in October 1995, and intended as a humorous rough guide to mathematics, the presenter stated that an A4 sheet of paper provides an example of a rectangle with sides in the golden ratio. This is a comment I have come across in a number of publications. The ratio of the sides of all sheets of paper in the international standard system is in fact $\sqrt{2}$.

A true golden rectangle has the property that it produces a smaller sheet of the same proportions when a square is cut from one end, whereas the A4 sheet does so when cut in half.

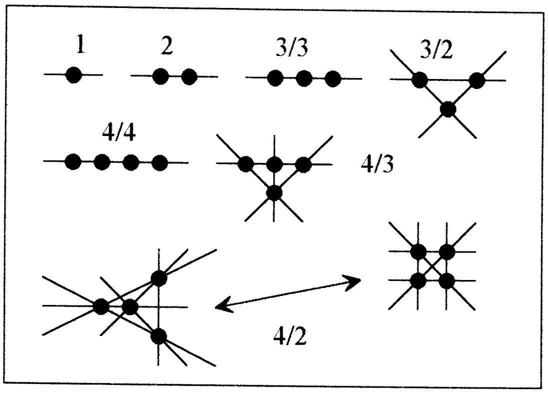
On another occasion I recall a TV presenter stating that a television screen is an example of a golden rectangle because its proportions are in the ratio 5:3, two successive numbers in the Fibonacci sequence. In fact the ratio is nearer 5:4, in my limited experience.

My questions are: (a) What margin,all round, must be left on an A4 sheet (210×297 mm or $8\frac{1}{4}\times11\frac{3}{4}$ in) to leave a print area that is a golden rectangle? (b) What width of margin must be cut from one side of an A4 sheet to leave a golden rectangle? (c) What shape gives a smaller sheet of the same proportions when a domino (i.e. a two-square shape) is cut crosswise from one end? (d) More generally, when a rectangle m squares $\times$ n squares is removed?

2. Plantations.

Problems of the planting of orchards have long been popular (John Jackson's *Rational Amusements for Winter Evenings* London 1821 had ten examples). Here we begin a progressive series of questions on this topic.

With one or two trees there is only one pattern of planting, but with three trees we can put them in a row (a line of 3, which we denote by 3/3) or in a triangle (lines of 2, which we denote by 3/2). With four trees there are three patterns when considered numerically (4/4, 4/3 and 4/2), but the last of these can exist in two topologically distinct forms: one in which one point is within the triangle formed by the other three, and one in which each tree is outside the triangle formed by the other three. If we try to deform the triangle form gradually into the quadrilateral form the central point has to cross one of the sides of the triangle—thus forming the 4/3 pattern—or else passes through a vertex, forming a simple triangle.



This leads to our first question in this series: How many different types of plantation are possible with five trees? There are two answers of course, (a) according to the number of lines and the numbers of trees in the lines, and (b) taking account of the topological considerations.

3. Cryptarithm.

This will be another regular feature. This example was sent by our long-term correspondent Mr T. H. Willcocks on his 1995 Christmas Card. He describes it as "A little alphametic I composed for the amusement of a friend."

```
 BRYANT     NORMAN
 BRYANT     NORMAN
 ------     ------
 NORMAN     BRYANT
```

Find 8 integers which will solve both sums.
I understand Mr Bryant has given permission for this play on his name to be published. We tried to find alternative words but none were as effective.

4. Regions in a Circle.

This question is specially contributed by Mr R. J. Cook of Frome. A number of points on the circumference of a circle are connected by straight lines which link each point to every other point, and no more than two lines pass through an intersection where lines cross. This divides the circle into a number of regions: 2 for 2 points, 4 for 3 points, 8 for 4 points, and so on. How many regions are formed when there are 12 points on the circumference? More ambitious solvers may be able to give an answer for n points.

5. A Spherical Argument

For our Logic Department Professor Cranium offers the following argument to find the area of a sphere of radius R. The circumference is $2\pi R$. Divide the sphere in half along an 'equator' and divide each half into N parts by equally spaced 'meridians'. As N increases, these three-sided parts can more and more easily be flattened out to form triangles of height $2\pi R/4$ and base $2\pi R/N$. So their combined area is ½ (base) (height) × $2N$ = ½ $(2\pi R/N)(2\pi R/4)\,2N = \pi^2R^2$. But as is well known, the correct formula is $4\pi R^2$. Where is the fallacy in the argument? Can you provide a correct proof on similar lines?

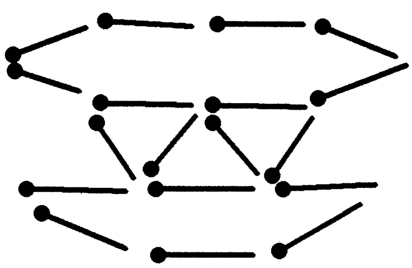
6. The Fate of the Dirigible.

There is a manuscript booklet of 'Original Puzzles' by T. R. Dawson in the BCPS Archive, many obviously inspired by the work of Loyd and Dudeney, from which I propose to quote some examples from time to time.

There are 104 puzzles complete with solutions, and at the end there is a list of further problems, going up to number 157, where only the title is given. These were evidently published somewhere. I believe Dawson had a column in the London Evening Standard *during the 1930s.*

Against most of the earlier puzzles there are no publication details, though initials indicate that they were shown to various friends such as, H. A. Adamson, F. Douglas and H. D. Benjamin.

Here is Puzzle 14, in Dawson's 'Original Puzzles'; when composed in 1914, it was topical. My drawing is regrettably crude compared with the original.



"Arrange 18 matches as shown in the figure, giving a more or less satisfactory representation of a dirigible in full flight. We know the flight is full, because the propeller is revolving so fast we cannot see it. Unfortunately for the aeronauts they miscalculated the height of a lamp-post and were reduced to matchwood.

The problem we propose here is to discover in how many ways our balloon can be reduced to matchwood: removing matches one at a time in a continuous chain."

7. Arithmetrication

In *Teach Yourself Arithmetic* (by L. C. Pascoe 1958) the following curiosity is quoted (from the *Second Penguin Problems Book*): the length of 3 miles, 7 furlongs, 9 chains, 3 poles, 5 yards, 1 foot and 7 inches is multiplied by 2 and then divided by 2, showing that it is equal to the apparently greater length of 4 miles and 1 inch! It is stated that "The fault lies in our curious system of measurement". This is a bit of pro-metrication propaganda. In fact the fault lies in not following the correct convention—which is, what?

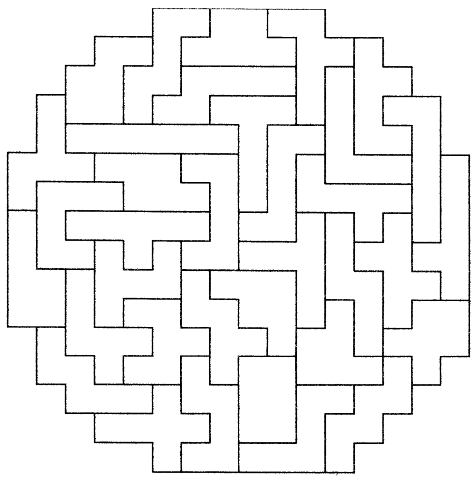
8. Polycube Construction

This problem is from the W. Stead manuscript, where it is dated 21/vi/1954.

Using the 35 six-cube flat pieces (hexominoes) and one duplicate construct a truncated step pyramid (dais), formed of four square layers: $10^2 + 8^2 + 6^2 + 4^2 = 216 = 6^3$.

The condition may be added that as many pieces as possible be laid flat.

For readers who may not have a set of the 35 hexominoes to hand, here they are, in as near as I can get to a circular area.



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Volume 2, Issue 13. Published 28 May 1996.