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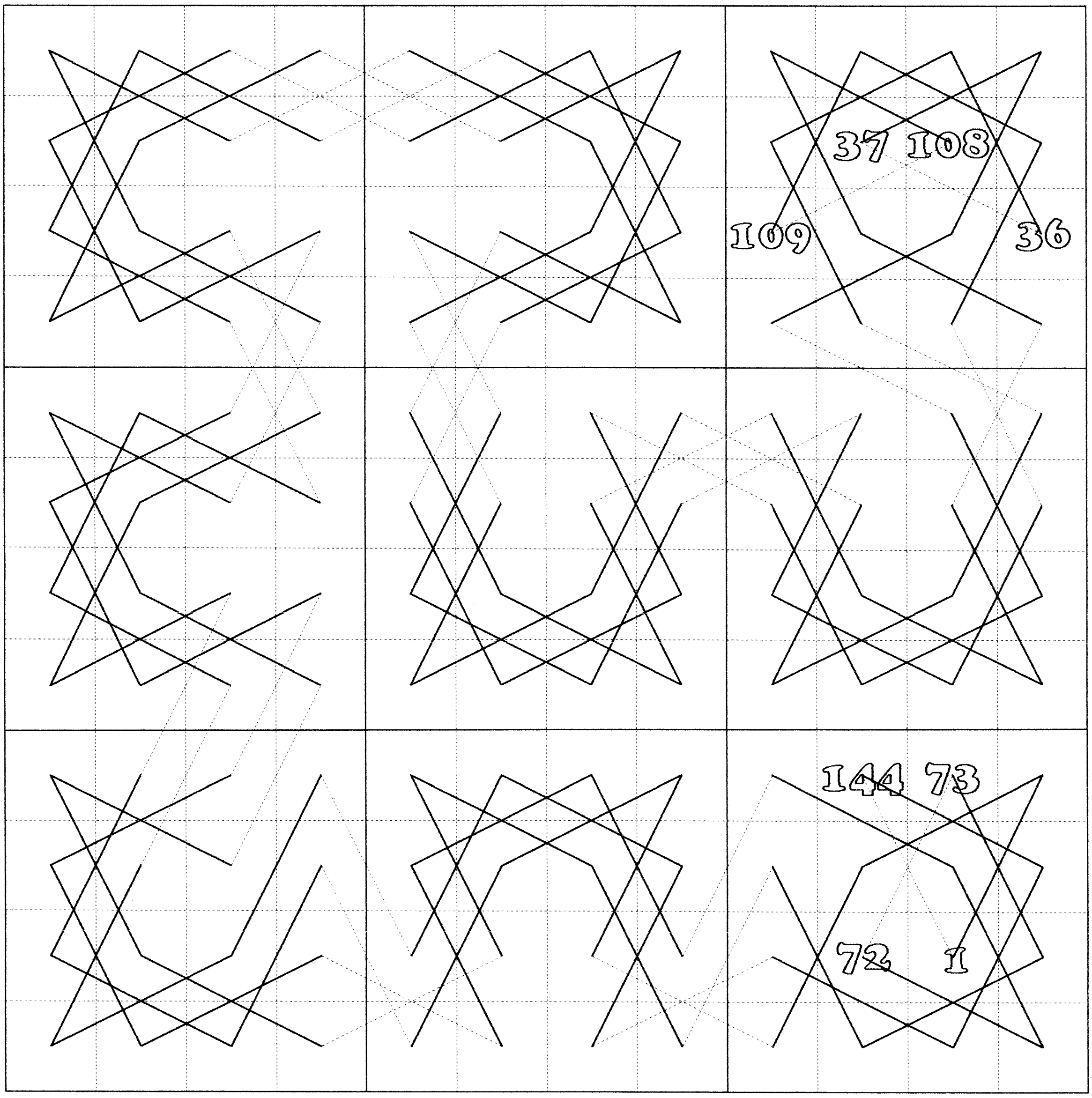
Our cover illustration shows the first ever magic knight's tour on a board larger than the 8×8. This was constructed sometime between 1852 and 1868 by Krishnaraj Wadiar (1790–1868), who was the Rajah of Mysore. This tour only became known in the West in S. R. Iyer's *Indian Chess* published in 1982. It is constructed on the 'squares and diamonds' principle. The ranks and files all add to 870 (290×3), and each 4×4 area adds to 1160 (290×4). For further history see pages 243–4. (The diagonals add to 980 and 608. A diagonally magic 12×12 tour has not yet been found, nor proved impossible, though several with one diagonal adding to 870 are known.)

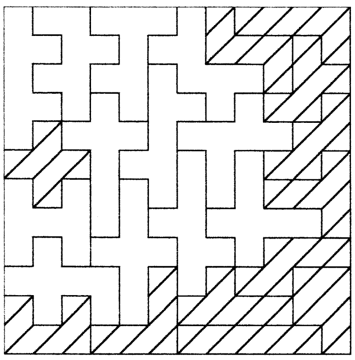


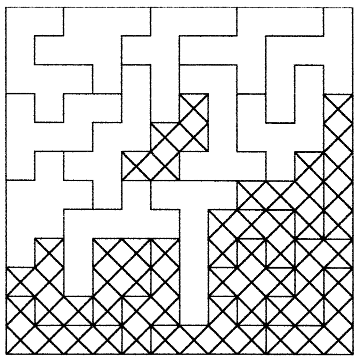
Editorial Meanderings

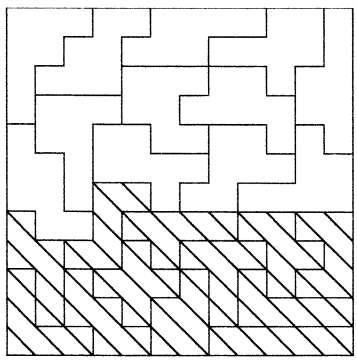
Errata. A few to report already alas: p.202 The title of the supplement to *Games Games Games* was *Ludography*, not *Ludology*. p.203 Joseph Horn's middle initial should be K not P. p.208 Bincknes should be Binckes (see p.230 in this issue). p.217 Tour III was used for the centre of the 10^2 tour (see p.212), not tour V, this was an error in the Bergholt text.

Method of of Binding. When you have the 6 issues of this volume of *The Games and Puzzles Journal* you should take them to your nearest 'instant print' shop where the plastic comb binders can be removed (using a special device which will cause no damage) and replaced by a single comb of larger size to give a bound volume. The cost of this should be minimal, especially as the print shop will be acquiring six small combs which can be reused. These combs usually have 21 rings, but to avoid weak links in the paper at top and bottom margins of A4 sheets one ring has to be cut off at the bottom.

Heptominoes. After much trial and error, here is a solution (A) I have found to the heptomino/pentomino problem, stated on p.202, which I proposed for *Games & Puzzles*: To arrange the 12 four-ended heptominoes and the 12 pentominoes to form a 12×12 square. When the board is chequered, seven of the heptominoes cover 5 cells of one colour and 2 of the other. These can be arranged so that they use 26 cells of one colour and 23 of the other. The remaining five heptominoes cover 4 cells of one colour and 3 of the other. These can be arranged so that they use 19 cells of one colour and 16 of the other. The difference of 3 in each case then balances out.

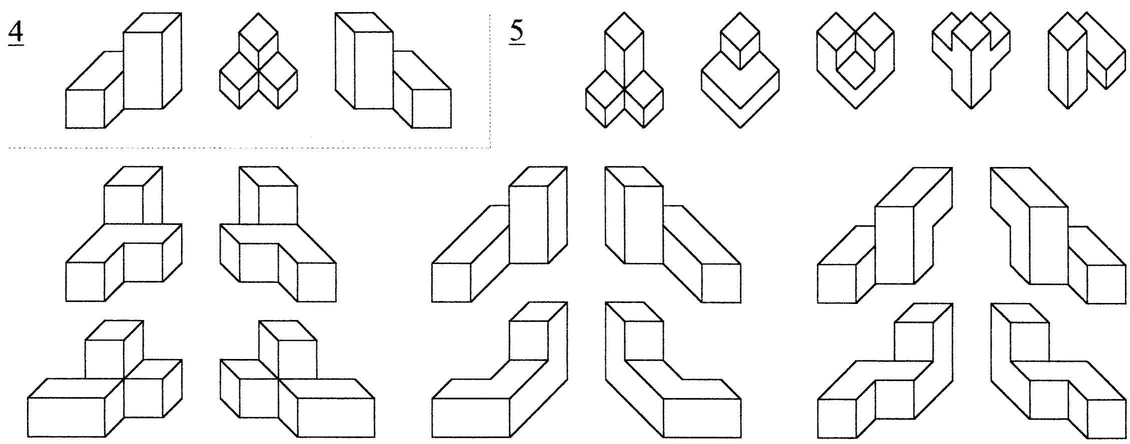

(A)


(B)


(C)

Also shown are solutions (B) using the 12 three-ended heptominoes with a 3-unit side and (C) using the 12 two-ended heptomino clumps (a 'clump', one of Walter Stead's terms, is a piece containing a 2×2 square). Can this be solved in two rectangles?

Polycubes. There are 3 non-planar polycubes formed of 4 cubes and 17 of 5 cubes as shown below:



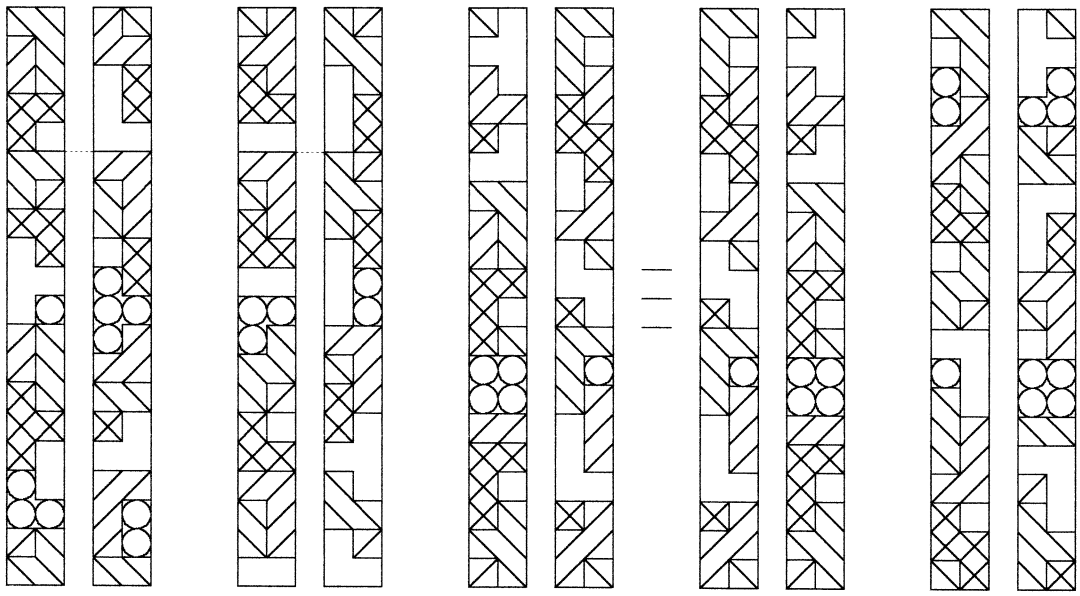
Polycube Constructions

continued, from notes by Walter Stead

The non-planar polycubes of 4 and 5 pieces are given in Mr Stead's notebooks in two different numberings and mixed up with the planar cases, which I think are best kept separate. They are illustrated opposite. The asymmetric cases occur in 'enantiomorphic' pairs.

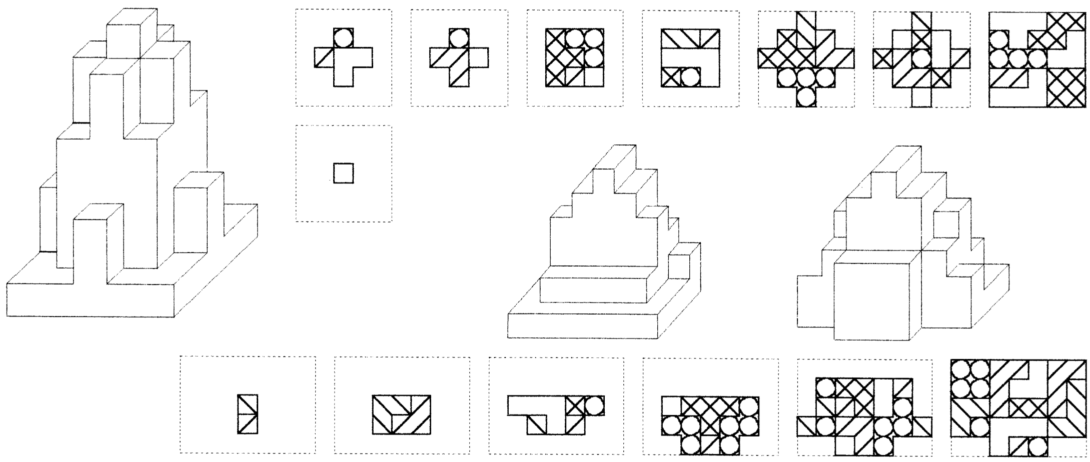
For problem (8) see p.247. The next six constructions in the notebooks are made by using 16 of the 5-cube non-planar pieces. In each case the piece omitted is specified, but presumably solutions omitting any other piece would be acceptable. (The descriptions, 'column', 'monument', 'well-head', etc used below have been added by the editor.)

(9) Square column 2×2×20. Five different solutions of this are given, omitting a different piece each time, the first by W. S. dated 20 March 1954, the third by R. Smith, this and the last two having 'no straight joints' (i.e. no fault planes). The first two solutions have fault planes 5 units from one end. The fourth solution is identical to the third apart from transposition of the layers, which means that the piece omitted is now the reflection of the asymmetric piece omitted in the third solution.



9(a) 9(b) 9(c) 9(d) 9(e)

(10) Monument with eight layers, as shown below. (11) Wellhead, is given for solution among the Problem Questions on p.247. (12) Monument with six layers, shown in front and rear views below.



to be continued

Thoughtwave

by Derick Green

Thoughtwave first appeared in the mid-1970s, and is a two-player abstract path-making game. Unfortunately it has been out of print for many years. [The inventor was Dr Eric Solomon., the games company was Intellect (UK) Ltd, and there is a note on the rules that says 'Theme, design and rules by Dinckes, Jarvis, Walsh and Gluck Ltd.']

The game is easy to learn but has considerable depth in its tactical play, a draw is almost impossible and the following simple handicap system allows players of different strengths to compete on equal terms. The stronger player simply gives the weaker player 1, 2 or 3 of his tiles.

A thoughtwave set is easy to make, a 10×10 board is needed and two sets of 24 tiles. The tiles and the squares on the board need to be the same size (1 inch or 2½ centimetres will do). The object of the game is to be the first player to connect opposite sides of the board with a 'thoughtwave'. Say player **A** North to South, and player **B** West to East. The squares connected do not need to be directly opposite each other.

The 'thoughtwave' with which the players attempt to connect opposite sides of the board is made from up to 24 tiles of five different types (Figure 1). In the original design the 'waves' were more rounded. A tile, once placed on the board, belongs to neither player and may be used by either player as part of their respective 'thoughtwaves'.

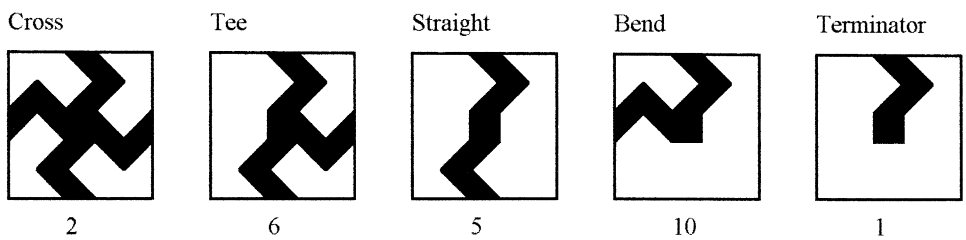


Figure 1. Full set of thoughtwave pieces for one player, showing the five types.

The most numerous tile is the Bend (10 tiles each) which connects two adjacent sides of a square. Bends are specially useful in the first few moves of a game to mark out potential thoughtwaves.

The T-piece (6 tiles each) connects three sides and the Cross (2 tiles each) connects four sides. Both become increasingly important as the game progresses.

The Straights (5 tiles each) connect two opposite sides and are useful as blockers. In Figure 2(a), **A** has a North to South Straight on e4 which is blocked by **B**'s West to East Straight on e2, so **A** is forced to divert and **B** has made a counter-threat.

Finally, the Terminator piece (1 tile each) connects one side only and should be used carefully. A player must keep his options open by laying down alternative routes. In Figure 2(b) **A**'s route at b3 and c3 is blocked by **B** playing his Terminator at c2. However, **A** has another route via a3 and a T-piece placed at a2, as shown in white, would guarantee **A** connecting to the southern edge of the board.

Thoughtwave is played in turns, with each player placing one tile in any empty square on the board. A tile once placed cannot be moved unless found to be illegally placed. In this event the guilty player removes his tile and misses one turn. A tile placed next to another must have connecting thoughtwaves, similarly blank edges must touch blank edges.

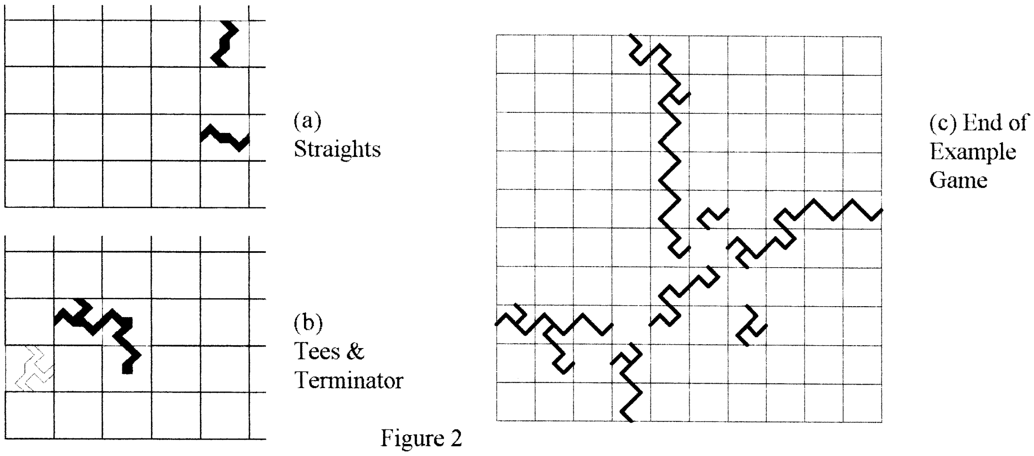


Figure 2

Two important facts to remember are that blank edges are just as important as connecting edges and too many T-pieces should not be used too early in the game.

For recording the game I have used a notation system using the four compass points and the square coordinates. For example in Figure 2(a) the Straight at e4 would be recorded as NSe4. The following short game is between two new players. Note that at the end of the game, Figure 2(c), squares d3 and f5 are useless as if either player places a tile into one square their opponent plays into the other and wins.

Illustrative Game: Player **A** (N-S), Player **B** (W-E)

1. NSe6	NEe5	2. NWf4	SWEg5	3. NEd10	SEh6
4. NSEe9	WEj6	5. NWe3	WEc3	6. NSWd2	NWEa3
7. NSe7	NEb2	8. NSd1	SEf6	9. SEe4	NWh5
10.SWe10	SWE b3	11. NSe8	WEi6	12. NSEg3	resigns.

(The tile patterns in Figure 2(c) have been simplified to make the drawing easier to do.)

Word and Letter Puzzles

Solution to Cryptic Crossword #10:

Across: 1. LIMPET 5. CASTANET 9. NEPOTISM 10. RHESUS 11. ORTHOGONALLY 13. VERY 14. EPISTLES 17. IMMERSER 18. MICE 20. ENCROACHMENT 23. CHIMES 24. OPULENCE 25. SPHAGNUM 26. LUSTRE

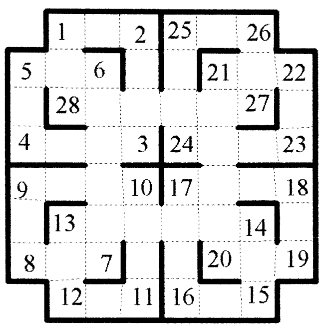
Down: 2. ICED 3. PROTOTYPE 4. TWISTS 5. COMMONEARTHWORM 6. SARDONIC 7. ARENA 8. EBULLIENCE 12. SEAMANSHIP 15. TEMPTRESS 16. ASSASSIN 19. SEQUEL 21. RUMBA 22. SCAR

Solution to A Question in Verse:

The hurt of a sad REVERSE
To her whom one REVERES
Is SEVERER far when she
In RESERVE conceals her tears.

Solution to Polygram: GUSTY, OREAD, BORED, BREAD, FINED, LODGE, TRIES, DEPOT, UNITE, INTER, LAYER, the middle letters spelling out SERENDIPITY.

A TANGLE-WORD ENIGMA



Clue to the Labyrinth:
1-4 To fix holiday trip round castle is first move
5-8 Nevertheless 13s the Earth with fish droplets
9-12 Cried by faithful listener to great bird
13-16 Roundabout tree anthropologists study well
17-20 Oriental Indian beverage, peak performer
21-24 Uncooked used to be bean currency in India
25-28 Part of 13 to limit poverty by affirmation

Knight's Tour News

Thanks to researches by Mr T. H. Willcocks considerably more biographical details of Ernest Bergholt have been brought to light. Mr Willcocks wrote as follows (10 June 1996):

Your remark about the paucity of information on Ernest Bergholt led me to do a little detective work. As prior to 1920 he had been a regular contributor of articles and author of books and as nothing appeared after that date, I thought he might have died not long after that date and in the *Calendar of Wills and Administrations* I found 'Bergholt see Binckes'. This led me to an entry: 'Ernest George Binckes or Ernest Bergholt of Letchworth, Hertfordshire, died 18 Nov 1925. Probate to widow Florence Jane Binckes otherwise Florence Bergholt.' Knowing the date of death, I discovered from the *Register of Deaths* that he was aged 69. This enabled me to discover that his birth was registered in the Jan/Mar quarter of 1856 under the reference Worcester b.c.300.

Having done all this spadework, it occurred to me that he might well have been a product of Oxbridge and it took only a few moments to find him in *Alumni Cantabrigienses*: 'Binckes, Ernest George (Alias Ernest Bergholt) Adm[itted] King's Oct 9 1875 Son of —, b. at Worcester, School: Christ's Hospital. Exhibitioner Matric[ulated] Mich[aelmas] 1875. Non-collegiate student 1876, BA 1879, MA 1894. Asst Master Bradfield College 1880–1, Private Tutor 1882–6, Journalist on the staff of The Dramatic Review 1886, Court and Society Review 1888, Editor of The Musical Standard 1888, Bridge Editor of the Onlooker 1901–7, Pastimes Editor of the Queen 1908, Cards Editor of the Field 1915. Fellow and Member of the Council of the Chartered Institute of Secretaries. Known professionally as Ernest Bergholt.' He seems to have been quite a polymath! I have no clue as to why he chose the nom-de-plume of Bergholt. Some family connection?

The fact that 'Bergholt' was adopted as a pen name is a surprise!

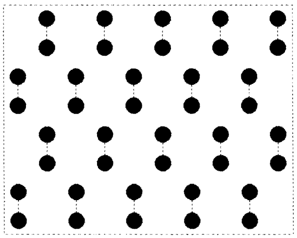
It looks as though Ernest Bergholt's 1915 dictum concerning knight's tours of the chessboard: "The total number of possible tours that can be made is so vast that it is safe to predict that no mathematician will ever suceed in counting up the total", is near to being falsified. Two workers at the University of Dortmund, Martin Löbbing and Ingo Wegener, claimed (*Electronic Journal of Combinatorics*, **3**, 1996, paper R5, 4pp) that the number of knight's tours on the 8×8 chessboard is exactly 33,439,123,484,294. Unfortunately their figure is wrong. It is supposed to count the number of distinct diagrams that represent closed tours; in other words, a tour with no symmetries is counted 8 times, while a tour with 180° rotational symmetry is counted 4 times, which would mean that this total should be divisible by 4, and it isn't! They used otherwise-idle time on 20 Sun workstations during a period of four months last year, so it may take months to rerun. Prof D. E. Knuth tells me that their method involved breaking the problem into approximately 383 million subproblems on partial boards! So an error in the adding up is not inconceivable.

In view of my interest in Figured and Magic tours readers would not expect me to agree with Bergholt's complaints, in his 6th Memoir (p.237), against the study of arithmetical properties of tours that do not derive from geometrical properties. I would say that the 'constant difference' properties do derive from geometrical properties, since they depend on what I call 'Piece-wise Symmetry'. Bergholt's concern however was with the overall symmetry of the diagram rather than of its component circuits.

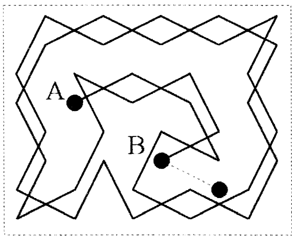
Memoranda 5–6 on the Knight's Tour

by Ernest Bergholt

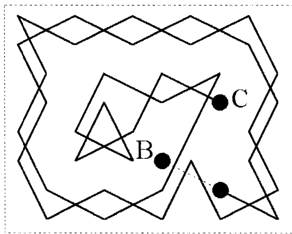
***Fifth Memorandum* (April 1, 1916).** *Explanation of a simple and general method applicable to all squares and rectangles having an even number of cells in both their sides.* For the sake of example, take the rectangle 10×8. It will be seen that the method will apply generally under the restriction enunciated. Mark the cells with dots in alternate pairs, as shown in the diagram below [16]. Fill up the whole of the dotted cells with a knight's path, beginning and ending on any convenient cells. This will be found easy if we skirt round the board with occasional excursions (when compelled) into the middle. For instance [17] starts at A and ends at B. On another similar rectangle, mark with dots all the cells left blank in [16]. On this second rectangle, commence a path on a cell which is one move from B and trace any other path, such as in [18] is shown ending on C. Superpose the two diagrams, and we have a complete tour commencing at A and ending at C. [See footnote for diagram.]



(16)

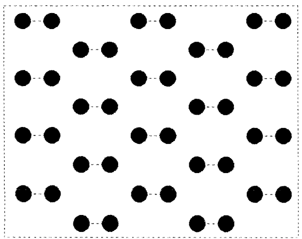


(17)

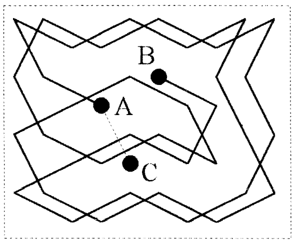


(18)

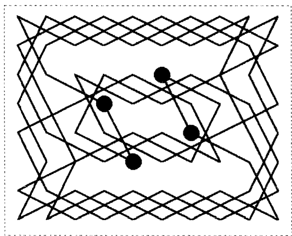
This tour will not usually be either reentrant or symmetrical. It can, however, be made both by a very simple modification to be presently explained. It will be noticed that neither of the above 40-cell paths can be directly duplicated on the same board with the half-turn that is necessary to attain binary symmetry (Euler type). If, however, we dot the cells along a side consisting of $4n + 2$ cells [19], instead of along a side of $4n + 4$ cells as is done above, the case is altered. In [20] the 40-cell path obtained as prescribed happens to be reentrant. But the move AB may be taken in the direction AC (shown by a dotted line) and if this be done (since C is conjugate to B) we can trace a duplicate of the whole path from C onward, and so obtain the symmetrical reentrant tour shown in [21].



(19)

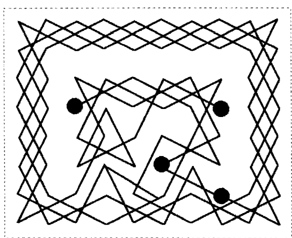


(20)

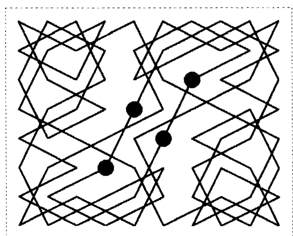


(21)

Here the editor inserts the tour formed by combining the half-tours (17) and (18), and shows. another tour constructed by the same method as (21), but less border-hugging.



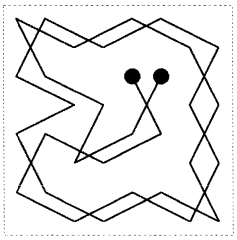
(A) = (17)+(18)

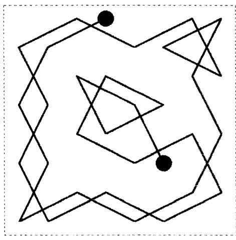


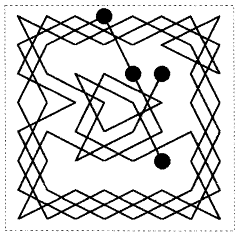
(B) example by G.P.J.

Note the 3rd memo was dated 30 March and the 5th memo 1 April, hence the missing 4th was 31 March.

Supposing, however, that both sides of our rectangle or square contain $4n + 4$ cells (as is the case with the ordinary chessboard), it is clear that this method does not directly give us binary symmetry. In such a case, we begin with two 32-cell paths, exactly as in the case of [17] and [18]. The two separate paths are shown in [22] and [23]; and the superposition of the two to form a non-reentrant tour is shown in [24].


(22)

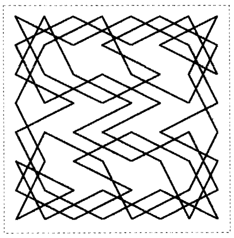

(23)

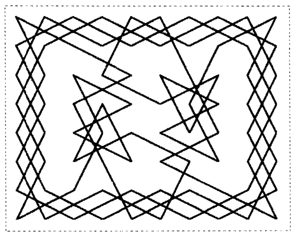

(24)

To show how [24] may be made reentrant and symmetrical. Number a board for binary symmetry (various boards for various purposes should be kept for ready reference, drawn on cardboard). Suppose our numbering to be as shown below [25].With 32 Lotto counters, trace out the tour [24] on this numbered board. It will be found, of course, that repetitions occur — when this happens we omit the repeating numbers. In the above case, these omissions will be found never to break the continuity of the path, which, commencing at A (22), will be as follows: 22:29.18.1.14.5.10.25.30. 17.2^{v}.13.6.9.26.21.4.15.32.23.8.11.24.7.12.3.16^{v}.31.20:27.28:19 where there are three optional moves [denoted here by a colon (:), the manuscript uses a single dot and an arc connecting the two numbers] and the final (19) does not join to (22), the initial.

1	2	3	4	5	6	7	8
16	15	14	13	12	11	10	9
17	18	19	20	21	22	23	24
32	31	30	29	28	27	26	25
25	26	27	28	29	30	31	32
24	23	22	21	20	19	18	17
9	10	11	12	13	14	15	16
8	7	6	5	4	3	2	1

(25)


(26)

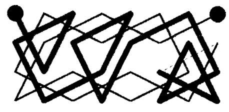

(C) by G.P.J.

A simple application of Euler's method will put this right. We find that (19) may be joined up at either of the two places marked ^{v} [the manuscript uses a slanting arrow]. Supose then we follow (2) with (19) and then run from (19) backward to (13) which joins to (22). Our tour will be 22:29.18.1.14.5.10.25.30.17.2.19:28.27:20.31.16.3.12.7.24.11.8.23.32.15.4.21.26.9.6.13.(22...&c). It will be convenient to begin with (29), coming to (22) last of all, because the last move (29.22) will then be an optional one, which will ensure that the tour does not 'short circuit' after 32 moves. (In cases where such a short circuit occurs and there is no optional move to put it right, a further simple application of Euler's method would be required.) Tracing out the above numbers geometrically the tour is as follows [26], and any other similar case can be similarly treated.

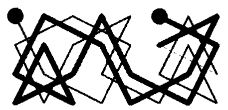
Further example: In the same way, transform the combination of [17] and [18] into a symmetrical reentrant tour on the 8 by 10 rectangle.

Here the 5th memo ends and Bergholt does not give a solution, but (C) is one possible answer the editor has constructed using Bergholt's methods. 'Euler's method' referred to above, simply consists in successive applications of what Murray (1942) calls 'The Bertrand Transformation' in which a chain of cells a...e.f...x where a terminal, say a, links with an internal cell, say f, is replaced by e...a.f...x, giving a new terminal, e. Bergholt's use of the phrase 'short circuit' must be the earliest use of this electrical engineering term in the context of tours.

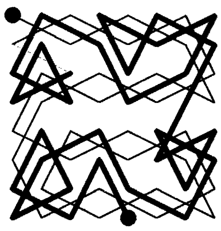
The above method, which I like to call the Domino Method, since it depends on dividing the board into two-square blocks, was probably derived by Bergholt from the method of constructing tours on the 4×8 half-chessboard, which are necessarily of this type. This method has been known since the time of Guarini (1512) and possibly earlier, though not fully expounded until Sainte-Marie (1877). Guarini showed his tour [27] by lettering the squares in two series printed in black and red, according to van der Linde's Quellenstudien... (1881), which also cites a tour [28] from a Florentine MS, date unclear, that covers the board with the 32 chessmen, the black and white pieces showing the domino pattern. This method of presentation was also used in earlier (Bonus Socius) MSs, but not showing the domino pattern.



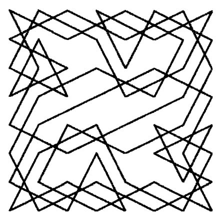
(27) Guarini



(29) Florentine



(29) De Montmort



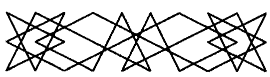
(30) G.P.J. after De M.

The tour [29] contributed by R. De Montmort to Ozanam's Récréations Mathématiques (1725) was also of Domino type. In [30] I give a symmetrization of this tour (found by a graphical method however rather than Bergholt's numbering approach). This retains all but one of the symmetric pairs in [29]. This tour has similarities with Bergholt's 'Arabesque' (p.209 of the last issue), suggesting that may also have been derived from the De Montmort tour.

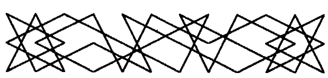
Sixth Memoir **(June 22, 1917**. Communicated to W. W. Rouse Ball, Esq. on Aug. 27, 1917.) *Some scattered notes on Euler, Ahrens, and Jaenisch, with a few other elucidatory historical comments.* [Capitalised headings inserted by the editor.]

EULER'S ERROR. In his memoir of 1759 (Berlin), Euler writes as follows: "Si la largeur (du rectangle) contient trois cases & la longeur 5 ou 6, il est impossible de les parcourir: mais, donnant à la longeur 7 ou plusieurs cases, on pourra réussir, pourtant sans rentrer." [Note: The meaning is "7 or more cells", I have no knowledge of *plusieurs* bearing this meaning in modern French. E.B.] And again: "Jusqu'ici les routes rentrantes en elles-mêmes ne peuvent avoir lieu; mais ... on pourra aussi remplir cette condition ... pourvu qu'il n'y ait pas moins de 5 cases dans un côté." [Bergholt's underlining.]

All later writers have blindly followed Euler in his assertion that no rectangle with 3 cells in one side can be made re-entrant (See Ahrens, 1910, p.350). It has escaped the observation of every previous writer that rectangles $p \times q$ can be re-entrant even when $p = 3$, provided q be not less than 10. Examples: (by E.B.) Complete binary direct symmetry, [as in [1]] never obtainable unless $p \times q$ is of the form $4m + 2$. [The tours [1] and [2] are identical, apart from inversion, to those published in Bergholt's series of problems in *The British Chess Magazine* 1918 (p.74). The introductory text there is: "In their assertions of 'impossibility' in the matter of knight's tours, mathematicians have been singularly unfortunate. The illustrious Euler, who was the first to investigate the subject methodically, states that a closed tour on an even-celled rectangle is only possible when there are not fewer than five cells in either of the sides. This statement has been blindly reproduced and endorsed by every subsequent writer on the subject, the latest being Professor Ahrens of Magdeburg, who has fitted it with an algebraical formula. Nevertheless, the 'authorities' are all wrong. It is true — and the proof given by Flye Sainte-Marie for the 4×8 rectangle is valid as a demonstration — that a closed tour can never be traced on a rectangle of which one side has *four* cells. Whence everybody has jumped to the conclusion that it is equally impossible when one side has only three. The following diagrams prove the error."]



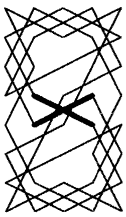
[1] 3×10



[2] 3×12

BERGHOLTIAN SYMMETRY. There is a further species of diametral symmetry which no previous writer has noticed. It can only occur when there is a central strip of cells operating as an axis. Example [3]. Arithmetically it is distinguished from the (Eulerian) diametral symmetry, in that the numerals in conjugate cells do not differ by (½)*pq* but sum to *pq* + 1. Thus: [4] 16 + 25 = 41, 3 + 38 = 41, 35 + 6 = 41 &c. The origin of the numeration must be suitably chosen [the note on this point bears the initials of W. W. Rouse Ball]. The shape of the board need not be rectangular: [5].

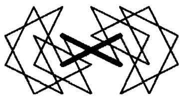
[Bergholt gave another tour of this type in *The British Chess Magazine* 1918 p.104, with the comment: "There are many ways of tracing a tour in complete binary symmetry on the 7×8 rectangle, although this is another of the problems which mathematicians have deemed to be impossible. One very neat (and easily memorised) tour is the following: [6]." Strangely, Bergholt does not mention the significant point that in this type of symmetry two of the knight's moves must cross at the centre point of the board. In the diagrams I have emphasised these moves by heavier lines.]



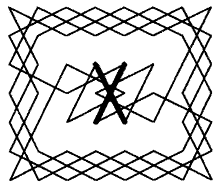
[3] 5×8

34	3	18	13	32
9	14	33	4	17
2	35	16	31	12
15	20	11	40	5
31	1	30	21	26
29	10	25	6	39
24	37	8	27	22
9	28	23	38	7

[4]

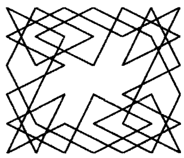


[5]

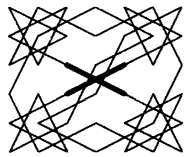


[6]

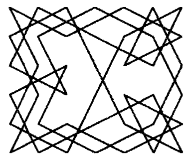
All three binary symmetries may occur on the same shape of board, e.g. rectangle of 7×6: [Diagrams 7, 8 and 9] Eulerian diametral, Bergholtian diametral and Binary direct symmetries.



[7] Eulerian



[8] Bergholtian



[9] Direct

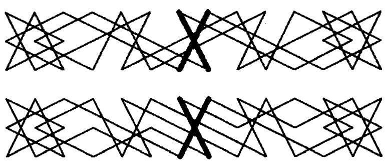
In a footnote Bergholt remarks "I have now adopted the nomenclature of Jaenisch, previously unknown to me" on symmetry, but apart from 'diametral symmetry' he makes no further use of these terms, which conflict with his own as defined in Memo 1.

G .L. Moore (1920) noted that there are four geometrically distinct symmetric closed tours on the 3×10 board of which two are in Bergholtian symmetry and two in direct symmetry. The captions show that Bergholt found one of these (it is dated 24 August 1917) and G. L. Moore the others. Two asymmetric tours are also possible.]

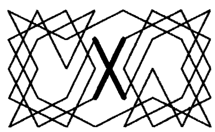
[1b] Moore

[1c] Bergholt

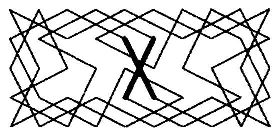
[1d] Moore



3×14 Murray



5×8 Papa



5×10 Murray

Col. U. Papa (1920) gave a 5×8 example of Bergholtian symmetry (which he called 'central diametral symmetry'), and Murray (1942) did some further work on small boards, giving 10 on the 3×14 board, 11 on the 5×8. and a 5×10 example.

NUMBERED COUNTER METHOD. Moving on to Jaenisch, Livre II (a book I have only recently succeded in procuring). I consider his whole treatment of the subject unnecessarily complicated and obscure. This is partly due to his mania for algebraic symbols (which are perfectly useless), and partly to his fondness for dwelling on arithmetical accidents which have no geometrical importance or meaning. When I began to work at the problem, I gave a fair trial to Vandermonde's notation of arithmetical coordinates, but soon abandoned it as cumbrous and unpractical. Vandermonde's purpose was merely to use it as an aid to symmetry, and that object can be better attained by other systems of notation. The numerations that I have finally found the most useful and convenient on the 8^2 board (after many trials) are the following:

Table A

Direct quaternary

9	15	5	27	27	5	15	9
3	21	11	17	17	11	21	3
13	7	25	29	29	25	7	13
23	1	19	31	31	19	1	23
23	1	19	31	31	19	1	23
13	7	25	29	29	2	7	13
3	21	11	17	17	11	21	3
9	15	5	27	27	5	15	9

Table B

Diametral to correspond

9	15	5	27	28	6	16	10
3	21	11	17	18	12	22	4
13	7	25	30	29	26	8	14
23	1	19	32	31	20	2	24
24	2	20	31	32	19	1	23
14	8	26	29	30	25	7	13
4	22	12	18	17	11	21	3
10	16	6	28	27	5	15	9

Table C

Oblique quaternary

9	15	5	23	27	13	3	9
3	21	11	17	1	7	21	15
13	7	25	29	19	25	11	5
27	1	19	31	31	29	17	23
23	17	29	31	31	19	1	27
5	11	25	19	29	25	7	13
15	21	7	1	17	11	21	3
9	3	13	27	23	5	15	9

Table D

Diametral to correspond

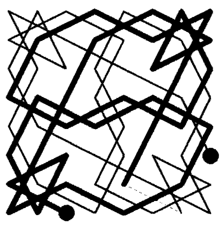
9	15	5	23	28	14	4	10
3	21	11	17	2	8	22	16
13	7	25	30	20	26	12	6
27	1	19	32	31	29	18	24
24	18	29	31	32	19	1	27
6	12	26	20	30	25	7	13
16	22	8	2	17	11	21	3
10	4	14	28	23	5	15	9

These four tables, written out on cards, and held in the hand to be used in conjunction with numbered counters, such as are found in any game of Lotto, will afford an extreme facility in the invention and manipulation of knight's tours of every kind. The following properties will be noticed in the Tables. — I. Every number joins by a knight's move to the next odd or the next even number, according as it is itself odd or even. (This applies equally to 31 and 1, 32 and 2.) — II. In Tables B and D, the odd numbers and the even numbers are congregated, so far as is consistent with the preceding property, in separate quarters of the board. — III. Every odd number is either directly (Table B) or obliquely (Table D) symmetrical with its corresponding even number, 1 with 2, 17 with 18, &c. — IV. In Tables A and C, the odd numbers 1, 3, ..., 31 are used in preference to the natural series 1, 2, ..., 16, in order to work more closely in correspondence with Tables B and D respectively. A chain may thus be transferred in number from one table to the other as rapidly as we can write, without risk of error.

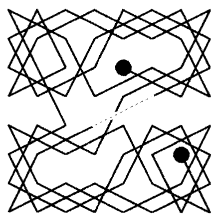
Every cycle of the first 16 odd numbers connecting on Tables A, C, by a series of knight's moves will yield either (a) four circuits of 16 cells, or (b) two circuits of 32 cells, on the Vandermonde plan, according as the circuit closes (a) in the same quarter of the board in which it began; or (b) on the same number in the diametrally opposite quarter, in which case the circuit of 32 will consist of two diametrally symmetrical halves.

QUATERNARY NEAR-SYMMETRY. For a complete tour of 64 cells, with the greatest possible quaternary symmetry, the problem is how to join these separate circuits, in diametral symmetry, with the least amount of disturbance of the perfectly symmetrical circuits that we start from. This problem was not solved by Vandermonde, nor did he use the oblique quaternary symmetry of Table C. So far as I can trace; the latter system was first used by Victor Käfer in 1842 (*Vollständige Anweisung zum Schachspiel*, Grätz, 1842) who finished the book with a chapter on the subject ("Vom Rösselsprunge") accompanied by a lithographed selection of tours arrived at by methods of trial.

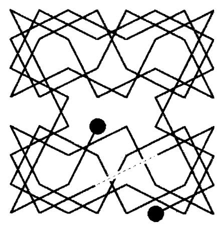
Using my own method on Table C, we can form the following cycle of 16, where no numeral is taken a second time: 1.5.3.25.15.13.17.19.11.9.7.23.21.27.29.31.(1...). The second (1) being in the quarter opposite to the first, we have here a re-entrant chain of 32, which we can immediately duplicate symmetrically by starting the same chain of numerals from one of the two remaining quarters [Diagram 10]. By joining any red cell [i.e. on the route shown here by lighter lines] to a black cell [on the darker route] which is a knight's move distant, and cancelling two other moves, we obtain a non-reentrant tour of 64. For instance, in the lower right hand quarter, join 17.15 (Table C), and cancel 17.13, 15.13. The dotted line shows the move supplied, and the two circles show the terminals of the tour, which is one given by Käfer.



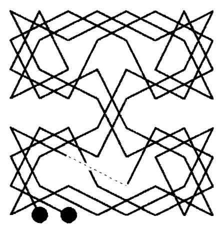
[10] Käfer 1842



[11] F.P.H. 1825



[12] E.B. after F.P.H.



[13] E.B. after F.P.H.

It will necessarily follow that if we number the tour, the dotted line will join 32 to 33, and the constant difference of the numbers in conjugate cells will be 16 [numbered diagram omitted]. The same numerical result would follow from a similar use of Table A to form a quaternary direct tour.

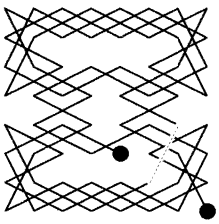
On p.24 (of his Vol.II) Jaenisch limits the joining of two such series, each of 32 cells, to a couple of cells obliquely symmetrical to each other. He would therefore join 29 to 29 (Table C) in Käfer's tour [10] above, but not 17 to 15. No extra symmetry of any kind is thus gained: in fact, the nearer to the centre we go, the more displeasing to the eye is the irregularity. It is one instance of how Jaenisch is often led astray by purely arithmetical considerations.

In an English edition of Philidor (*Studies of Chess*, 6th ed. 1825), the anonymous editor, "F.P.H.", constructs a tour [11] to have a constant numerical difference of 16 between conjugate cells, and also a constant difference of 2 between the conjugate cells of each quarter. That is, the additional restriction is imposed of always proceeding by 'squares and diamonds'. Here the numerical condition does not represent any geometrical symmetry, and appears to me to be quite irrelevant to the problem under consideration. If, however, the writer had proceeded to duplicate his half-tours in the manner described above, the result would have been far more satisfactory to the eye, while the numerals would have followed exactly the same laws, [12], or again, by duplication of the other half [13]. The dotted move is the one which breaks the symmetry by joining the two half-tours.

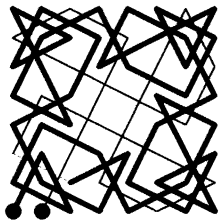
ARITHMO-GEOMETRIC TOURS. In the *Saturday Magazine* of Dec.11, 1841, p.229, Chas. Tomlinson gives a tour [14], in direct quaternary [near-]symmetry, formed by joining two half-tours in exactly the same way, to illustrate the same constant difference of 16. Tomlinson's article was subsequently reprinted as part of his *Amusements in Chess*, 1845.

The tour of F.P.H. reproduced above shows that the numerical property may exist without any corresponding geometrical symmetry, and that it is merely a subordinate and incidental consequence, when the latter is present. In fact, if the quaternary symmetry be obtained (as may be done) otherwise than by joining two 32-cell circuits, the numerical accident in question disappears.

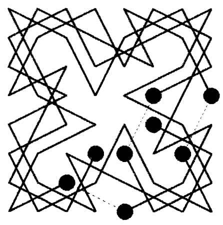
For example [15]. This illustrates the important fundamental principle (never before, I think, formulated, though it may have been dimly appreciated) that in any quaternary area of $8m$ cells (itself incapable of quaternary symmetry) we may always form two independent quaternary circuits of $8n + 4$ and $8s + 4$ cells respectively (where $n + s = m - 1$). Thus if $m = 8$, n may be 5, and $s = 2$, and we may form two circuits of 44 and 20, each of which shall itself be in complete quaternary symmetry. This symmetry will be oblique, according to the scheme of Table C. Referring to that Table, we see that the circuit in black shown [in 15] is 19.17.15.13.11.9.7.29.27.21.23.... taken four times ($4 \times 11 = 44$), while that in red [light line] is 1.5.3.25.31... taken 4 times ($4 \times 5 = 20$). Do not overlook the ambiguity in move 25.31; whichever progression is chosen, the same must be preserved in all four parts of the circuit. Here 3.25.31 are always drawn in a straight line. Now if we join cells 5 and 7 (lower left hand quarter) and cancel the moves 3.5 and 9.7 we form a non-reentrant tour of 64 which, morphologically, is wholly analogous [to those given above], but the numerical property of the latter tours will not be found. Because the irregular link joins points 44 and 45 of the chain instead of points 32, 33.



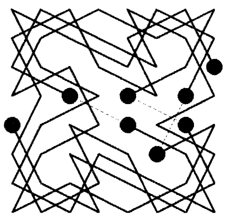
[14] Tomlinson 1841



[15] E.B.



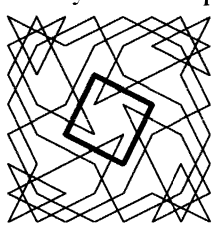
[16] Jaenisch 1862



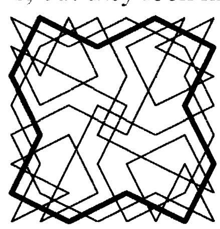
[17] F.P.H. 1825

If I am, therefore, unable to see any special elegance in F.P.H.'s tour merely on account of its arithmetic, I am still less able to admit Jaenisch's claim of 'elegance' for a tour [16] showing a constant diametral difference of 8 (such as his table on p.27, traced on his Plate vi, Fig.11). For such a difference can never have any symmetrical interpretation in the geometry of the tour. [Bergholt's manuscript does not include diagrams [14] and [16] but the editor inserts them to save readers having to consult the sources quoted. F.P.H. also gave a tour with difference 8; shown as diagram [17].]

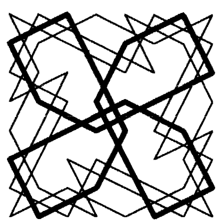
While researching magic two-knight tours I came across the following four 'pseudo-tours' which anticipate Bergholt's results in [15] above, and in [6] on p.214 of the last issue, by 44 years! They are due to "E.H." in the *Glasgow Weekly Herald* of 11 October 1873 (shown there in numerical, not diagram form). The paths can be joined together, if desired, to make near-symmetric open tours, but they look more attractive when left as separate circuits.



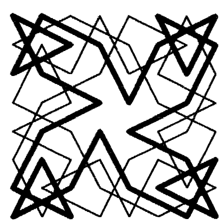
4 + 60



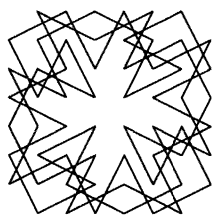
12 + 52



20 + 44



28 + 36



E.H. 1873

Given in the same source is this tour, omitting the corner cells. It is given in the form of a 'cryptotour' (23 Aug 1873) with the motto "The Sphinx Rediviva". The syllables on the squares, read in the sequence of the tour, give the verse: "The Sphinx of old, If truth was told, Was nicely sold, And put in a corner by Œdipus; But for our fox, There's no such box – Pursuit she mocks, And forages round like a greedy puss." (6 Sept 1873). If numbered from g6 to e7 the 8-cell ranks and files add to 240 or 242.

concluded on p.244

H. J. R. Murray's History of Magic Knight's Tours

with commentary by G. P. Jelliss

This and the following sections follow H. J. R. Murray's chapter on history in his 1951 manuscript *The Magic Knight's Tours, a Mathematical Recreation*, supplemented by notes from researches of my own and those of many helpful correspondents. Murray's text is shown between border lines. The manuscript was not completed, and Murray (1868-1955) was in his eighties when he wrote it. It was my original intention to quote this manuscript more or less verbatim as a chapter in my book, but so much new information has come to light that it will be necessary to rewrite the whole subject. Some paragraphs are taken out of sequence.

The Squares and Diamonds Method

The construction of magic knight's tours is one of the later developments of the Knight's Problem and was suggested by the discovery that it was possible to construct a knight's tour entirely composed of quartes. The first composer to give the figure which shows that the cells of the quadrant could be filled by four closed quartes was v. Warnsdorff (1823) but Ciccolini (1836) was the first to make a deliberate use of quartes by alternating squares and diamonds, although F. P. H. in the sixth edition of *Studies of Chess* (1825) had published a tour entirely composed of quartes, and v. Schinnern (1826) in Vienna had also composed tours in this way.

Attention to the numerical properties of tours was first directed by Euler (1759) when he showed that corresponding cells of a tour in diametral symmetry differed by 32. This was followed by attempts to construct tours in which corresponding cells differed by 16 and 8. F. P. H. (1825) solved both problems in [open] tours. Franz (*Schachzeitung*, 341) in 1847 solved the same problems with closed tours, which had considerable influence in Germany. Franz said that his tours were the nearest approach he had been able to make to a tour with equal-summed rows and columns, and he expressed doubts whether it was possible to do better. It was not then known in Germany that Beverley had already constructed a tour with both rows and columns equal-summed.

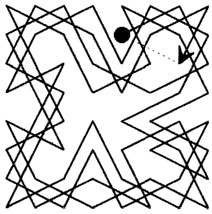
By 'quartes' Murray here means squares and diamonds. In the light of new evidence the work of von Schinnern (which seems to have gone unnoticed even in Germany) is seriously undervalued. Murray had seen Ciccolini's work, so his statement is more measured than other accounts which cite Ciccolini as originator of the method. Perhaps it should also be emphasised that I apply the description 'squares and diamonds' strictly to tours formed from the pattern of 16 circuits of 4 moves by deleting one move in each circuit and connecting the loose ends.

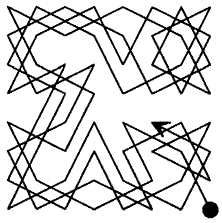
The earliest published examples of knight's tours constructed on the 'squares and diamonds' principle on the 8×8 chessboard were those given by 'F. P. H.' in 1825 (diagrammed on pages 236–7). Besides the 'constant difference of 8' property, F. P. H.'s second solution also adds to 260 in all the files, though the rank sums vary widely. The idea was apparently outlined in Warnsdorf's book of 1823 (which I've not yet seen), though according to Bergholt he did not give any actual tours of the type. Collini (1773) had used the squares and diamonds pattern on the central 4×4 area of the board, and Rudrata's 4×8 tour of 900AD can in retrospect be seen to use squares and diamonds at one end, so the elements of the idea had been around for a long time before resolving into a 'method'.

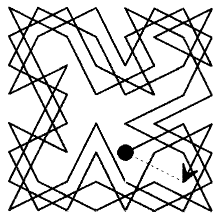
The first thorough account of the method was by Clemens Rudolph Ritter von Schinnern in his curious 36-page booklet *Ein Dutzend mathematischer Betrachtungen {A dozen mathematical contemplations}*, Vienna 1826 (with preface dated 1825). David Singmaster found a copy of this for sale actually in Vienna in February this year (with some pages still uncut!), and I am grateful to him for a photocopy of this (received in September), together with notes on the other 11 mathematical questions, which range from geometrical constructions to the relation of luck to skill in card games.

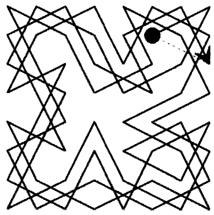
The article on tours is on pages 16–29 and has the title: 'Die Formeln für den geometrisch-aritmetischen Rösselsprung'. The 'formulas' referred to indicate the sequence in which the quarters are visited and whether the moves within them follow a square (*quadrat*) or diamond (*rhombus*) path.

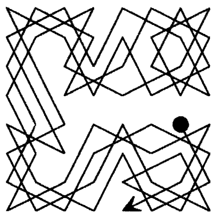
Although there are 8 tour diagrams (presented in numerical form) there are only 5 distinct tours, the others being reflections or reversals of these five. All five tours are *semi-magic* in Murray`s sense, i.e. their files all add to 260 but not the ranks (or vice versa). I show the five tours here in geometrical form, oriented so that the files are magic, and the last also in numerical form.


(7) rotated +90°


(8)


(9)


(10) inverted


(11)

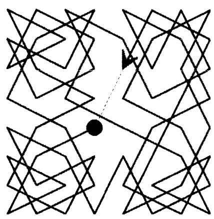
25	42	11	54	13	46	17	50
10	55	22	43	20	49	14	47
41	24	53	12	45	16	51	18
56	9	44	21	52	19	48	15
7	40	25	60	29	36	1	62
26	57	8	37	4	61	32	35
39	6	59	28	33	30	63	2
58	27	38	5	64	3	34	31

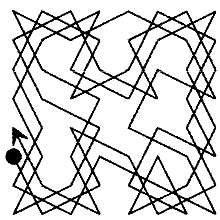
The five tours by von Schinnern.
By inverting (10) it bcomes clear that it is the same as (9) apart from choice of initial cell, and direction of description.

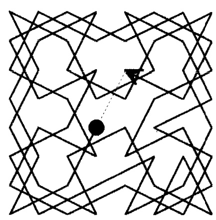
This last tour is the closest to a magic tour, since it has only two non-magic ranks, the first and fourth, which add to 260±4. It is impossible to get any closer than this to constructing a magic tour short of the real thing, which took another 21 years!

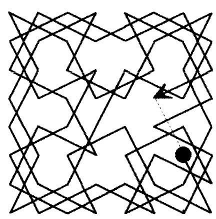
<u>The attribution of the squares and diamonds method to Ciccolini is an apocryphal tale.</u> This myth seems to have originated in Q. Poirson-Prugneaux, *Introduction pratique du jeu des echecs* (1849). From thence it was quoted by Lucas (1895) and Ahrens (1901), neither of whom had seen the Ciccolini book. In fact Ciccolini gave only <u>one</u> tour of squares and diamonds type, and then applied Euler`s method to derive other tours from it, which destroys the squares and diamonds structure. For example, the first tour derived from (A) is presented as 1–25, 64–43, 26–42, indicating that one follows the given tour as far as cell 25 (f4) then instead of going to 26 we go to 64 (e6), then back to 43 (c1) and from there to 26 (e2), ending at 42 (a2). Ciccolini`s purpose was to solve the Collini problem of finding a route between any two given squares of opposite colour.

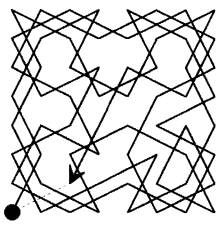
A similar project was carried out by J. E. Thomas de Lavernède (1839) who lists 256 tours, all of squares and diamonds type, using a complicated notation. I show only his first tour.

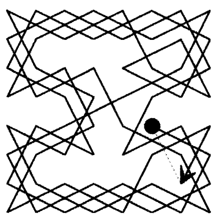

(A) Ciccolini

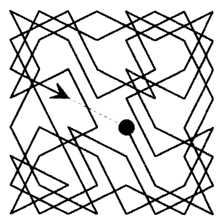

(B) Lavernède

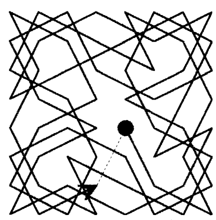

(C) Troupenas


(D) Troupenas (+90°)


(E) Anderson


(F) Tomlinson


(G) Franz


(H) Franz

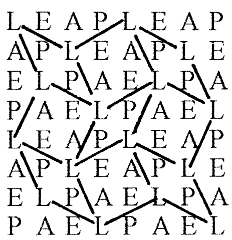
The squares and diamonds method is also expounded briefly by E. Troupenas in the early chess magazine *Le Palamède* (1842). His two examples were given wider circulation in Aaron Alexander's *Beauties of Chess* published simultaneously in French, German and English editions in 1846. [The tour (C) is quoted in Rouse Ball as 'Roget's solution', which it aint.] Troupenas derived this tour from one sent to him from England by a certain Mr Anderson. Tomlinson (1845) gave a semi-magic example. Franz's tour with difference of 8 is semi-magic but its ranks all add to 260±4, while his tour with difference 16 has all rows and columns adding to 260±2.

Roget's Method of Nets; or Straights and Slants

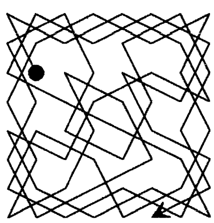
> The first writer to give a clear statement of the importance of quartes was Dr P. M. Roget, the Secretary of the Royal Society (better known as the author of his *Thesaurus of English Words and Phrases*, 1852) in an article in the *Philosophical Magazine*, 1840, but this article obtained no currency on the continent.

This note perpetuates another story which has become almost ineradicable from the literature. <u>The fact is that Roget's method of 1840 is quite distinct from the method of squares and diamonds!</u> Some tours of squares and diamonds type are Rogetian, but there are many that are not, and moreover most tours constructed by Roget's method are not of squares and diamonds type. This misunderstanding of Roget was pointed out by 'E.H.' in the *Glasgow Weekley Herald* 1873, where he criticises an account given in the *Westminster Papers* (which he satirises as the *Wastebasket Papers*). It seems that Murray may not have read Roget's original article but accepted the account in such otherwise reliable authorities as Charles Tomlinson's *Amusements in Chess* 1845 (which may have originated the confusion), and W. W. Rouse Ball's *Mathematical Recreations* (still in print).

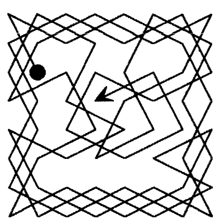
Roget considered the problem of constructing tours with specified first and last squares of opposite colour, which he believed to be a new condition, but was proposed by Collini in 1773. His method divides the board into "four separate systems, of 16 squares each" which he letters L, E, A, P (actually he used lower case for the vowels) as in the LEAP diagram shown below, where the L-net is drawn in. The 168 undirected knight moves on the board are then either of the types LL, EE, AA, PP (these moves form four 4×4 nets, each of 24 moves) or EL, AL, EP, AP (these each consist of six strands of three moves). Note that moves LP or AE (connecting different consonants or different vowels) are not possible. Roget's method is to traverse each of the four nets, L, E, A, P, separately as far as possible, so that the tour is in four parts, except that when the start and finish points are in the same net it is necessary to traverse this in two parts, and when the ends of the tour are L and P or A and E it is also necessary to traverse one of these nets in two parts. Roget gave three example tours, showing these three cases. Note that <u>none of these is of squares and diamonds type</u>. Roget's method satisfactorily solves the problem of presenting a tour as a conjuring trick, since, with practice, a tour can readily be drawn between any two cells of opposite colour quite quickly, even blindfolded!



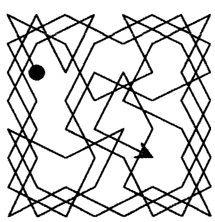
(A) LEAP diagram



(B) L–A tour



(C) L–L tour



(D) L–P tour

Murray (*BCM* 1949) introduced the useful terminology of 'straights' for moves of the types LL, EE, AA, PP and 'slants' for moves of types EL, AL, EP, AP, which is applicable to the description of any 8×8 knight's tour and can be extended to other even-sided boards. <u>In these terms, Roget's method is to use 'straights' as far as possible, keeping the 'slants' to a minimum.</u> (Murray uses the rather dull lettering ABDC in place of LEAP and attributes this 'indexing' to Collini, though in fact Collini did not use it, he merely used letters occasionally to mark all cells in a circuit. Lucas (1895) used an 'abcd' scheme to analyse Collini's method. Murray used capitals and transposed C and D.)

> Roget was the first to use the term 'magic' in connection with tours, which occurs in a card which he circulated among his friends in 1840. This card bore a chessboard with a semi-magic tour and in each quadrant [the 4×4 figure of the two squares and two diamonds] and the title 'Key to the Knight's Move as a Magic Square'.

It would be nice to discover a copy of this card, but I begin to suspect it may be apocryphal! The story of the card appears in Edward Falkener's *Games Ancient and Oriental*, 1892. The tour supposedly used on the card (F above) is given in Tomlinson's 1845 book, but without mention of Roget, the card or the motto. I have not seen the story in any earlier source than Falkener, and in view of other questionable statements in that book, I am reluctant to accept it as a reliable authority.

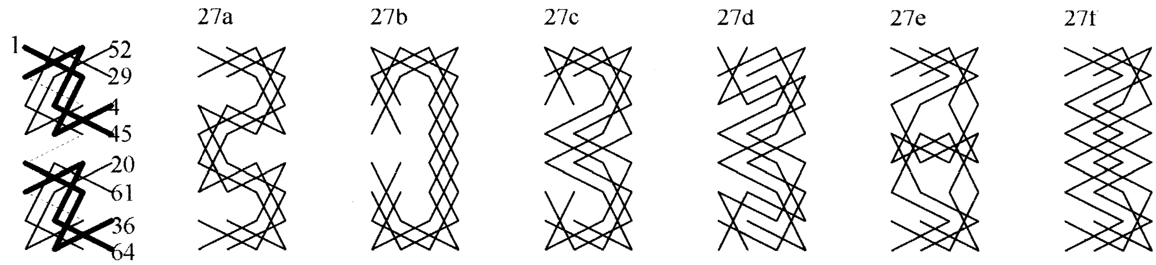
The First Composer of a Magic Knight's Tour: William Beverley

> The first composer to produce a magic tour was the Englishman, William Beverley of 9 Upper Terrace, Islington, and it was published in the *Philosophical Magazine* for August 1848, with a list of its numerical properties and the correspondence which accompanied the tour. From this it transpired that Beverley had composed the tour in the beginning of June 1847 and had sent it to his friend, H. Perigal Jr. of 5 Smith Street, Chelsea, on 5 June 1847. Perigal sent the letter to Roget on 29 March 1848. What was most remarkable about this tour was that it was not composed by the use of single-indexed quartes, and that Beverley seems to have known nothing about Roget's recipe for magic tours, or surely he would have sent it direct to Roget. Staunton published the tour in the November number of his *Chess Player's Chronicle* 1848 and Hanstein published it in the January number of the *Schachzeitung* 1849.

By 'single-indexed quartes' Murray means squares and diamonds, as opposed to the new type of quarte introduced by Beverley, which Murray calls 'two-indexed'.

Perigal's letter was addressed to the Editors of the magazine, not to Roget (though Roget may have been on the editorial board): «*Gentlemen, I inclose for insertion in the Philosophical Magazine a very interesting Magic Square, formed by numbering consecutively the moves of the knight in the grand tour of the chess-board. The knight's march has engaged the ingenuity of many eminent philosophers and mathematicians; but I believe that Mr W. Beverley is the first who has solved the difficult problem of converting it into a magic square. The principle upon which he has effected it, seems to be somewhat akin to that invented by Dr Roget, S.R.S., as explained in his paper on the Knight's Move in vol. xvi of the Philosophical Magazine. Yours very faithfuly, H. Perigal Jun.*» This implies that Beverley was not a mathematician; Perigal was (his name is associated with a visual proof of the theorem of Pythagoras). I submit this is why Beverley wrote via Perigal and not direct.

It may be noted that the symmetrical right-hand half in Beverley's tour (27a) appears also in von Schinnern's tours (7), (9) and (10). Subsequent composers have shown that this symmetrical path can be replaced by other symmetrical formations, as shown below: by Wenzelides 1849 (27b), Jaenisch 1862 (27cd), Reuss 1883 (27e) and Wihnyk 1885 (27f).



The left-hand Beverley half-tour can be joined to any of the six right-hand half-tours to give a magic tour.

The *Dictionary of National Biography* (Supplement 1901) mentions William Roxby Beverley (born at Richmond, Surrey 1814?, died at Hampstead, London, 17 May 1889) who is probably the knight's tour Beverley, though no direct proof of this has yet been established. He was a scene painter and designer of theatrical effects, and travelled round the country quite a lot in the course of this work. He is recorded as being in London from 1846 onwards, working at the Princess's Theatre, the Lyceum, Covent Garden and Drury Lane, and exhibited water colours at the Royal Academy.

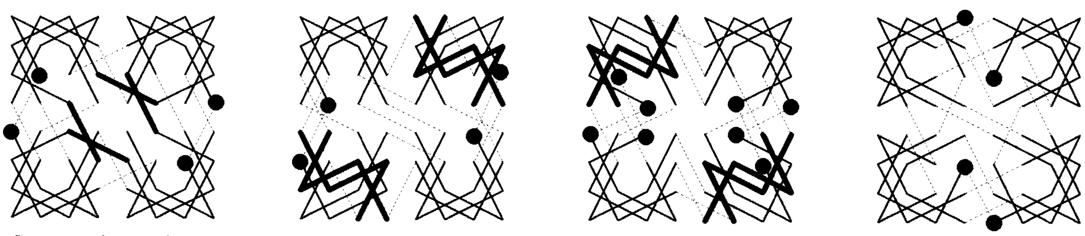
William Roxby Beverley had three older brothers, Samuel, Henry and Robert. Henry Roxby Beverley (1796–1863) controlled the Victoria Theatre, London for a short time, and "died at 26 Russell Square, the house of his brother Mr William Beverley the eminent scene painter". (This address is currently an Annex of Birkbeck College, University of London.) Their father William Roxby (1765–1842) was an actor-manager and adopted Beverley as a stage name, after his home town, the old capital of the East Riding of Yorkshire. Upper Terrace no longer exists; it was that part of Upper Street where Islington Town Hall now stands. Beverley's name is not there in the census records for 1851.

In view of Beverley's connection with the theatre it would be nice if the Anderson who submitted a tour to *Le Palamède* were John Henry Anderson (1815-1874) known as 'The Wizard of the North', an active stage magician and impresario of the time – who may perhaps have exhibited the knight's tour, using Roget's method? – but this is mere fanciful conjecture on my part, alas!

In his 1942 MS Murray claims: «Beverley's tour ... was not discovered by chance but was the fruit of a method based on mathematical analysis which was thought out in advance ... ». However the analysis he describes, in terms of 'contraparallel chains', is really Murray's own work. I suspect that Beverley found his tour by a more laborious method. Evidence for this is provided by the fact that, when all the tours of 'quartes' type (including the new quartes introduced in Beverley's tour) are oriented and arranged in sequence according to Frénicle's method for magic squares, Beverley's tour comes first in the list, and would therefore be the first discovered in a systematic search.

The Second Composer of Magic Knight's Tours: Carl Wenzelides

The second magic tour to be discovered was published in the February-March number of the *Schachzeitung*, 1849. It was the creation of Carl Wenzelides, a pensioned archivist of the Princes of Diedrichstein, who lived in Nicolsburg, Hungary. He was an invalid, confined for many years to his couch, who found a welcome relief to the tedium of his life in the composition of chess problems and knight's tours. He has told the story of his research on the knight's problem in two most human and interesting articles in the *Schachzeitung*, the first completed on 1 October 1848 (Sztg, IV, 41 and 242) the second completed on 25 March 1850 (Sztg, V, 212, 230). Extracts from his correspondence with Hanstein are [also] given in the *Schachzeitung* XIII, 174.» In his second article he records his discovery that the 'quartes' (the name is his own invention) form a 'system' and describes his attempts to obtain magic tours in diametral symmetry. His first success was obtained on 19-20 February 1849. By the time this tour was published he had obtained three other solutions. His first tour was hailed in the *Schachzeitung* as "the knight's tour in its highest perfection" and quite put Beverley's tour in the shade. Later German references to magic tours omit any mention of Beverley's priority, though Wenzelides himself yielded the crown to Beverley and said that to construct an open magic tour cost 'incomparably greater trouble' than a closed and symmetrical magic tour. In all he obtained seven symmetrical magic tours of which only five were published (12a, 12b, 12e, 12m 00m) and one varant of Beverley's tour (27b) which, strangely enough, he thought an improvement on Beverley's setting.



Symmetric magic tours by Carl Wenzelides. 12b is formed by rotating the four darker moves in12a.

Thus 12a was the first magic tour of squares and diamonds type, and also the first symmetric example. Among the 16 symmetrical magic tours now known, 00m is still the only multimagic example, that is it gives two distinct arithmetical forms, when numbered from b6 or c5. The dark lines in the diagrams for 12e and 00m are 'Beverley quartes'.

In a separate note attached to his 'Early History' MS Murray notes that in «*Schachzeitng*, May 1858, pp.174-5, Von Oppen, at the insistence of v. Jaenisch, gave the three magic tours which Carl Wenzelides had sent to the *Schachzeitung* in 1849, but which were not published then, probably owing to the death of Hanstein, the then editor.» The following passage, confirming this, appears in an article by Jaenisch in *The Chess Monthly* 1859 pp.149–150: «Mr C. Wenzelides, an Austrian amateur, was the first to give, in the Schachzeitung for 1849, a tour of this kind [rows and columns adding to 260 and the two diagonals adding to 520] and one which besides is symmetrical. He has since announced (Schachzeitung 1850, p.238) the discovery of six more symmetrical tours, the sum of whose figures in each rank and file amount to precisely the same. Death having unfortunately prevented him from giving them to the public, they would have been entirely lost if the Privy Councillor O. von Oppen had not succeeded in finding three of them, among some old letters addressed to the Editors of the *Schachzeitung*. And connoisseurs certainly owe a debt of gratitude to Mr Von Oppen for the publication of these beautiful solutions in the number of that magazine for May 1858. These three posthumous solutions of Mr Wenzelides however, do not make the total sum of the numbers upon the great diagonals amount to 520.»

Donald Knuth wrote to me 24 May 1994 as follows: «I see that Ahrens and Rouse Ball give the spelling Wenzelides, as in the German *Schachzeitung*. But de Jaenisch says Vencélidès was Hungarian; de Jaenisch was in correspondence with Vencélidès [shortly] before the latter's death. I found Karl Wenzelides, 'polyhistorian', listed in *Biographisches Lexikon des Kaiserthums Österreich* by Wurzbach, 1856-1891; this is almost certainly our man! Born September 1770 in Troppau (now Opava in the Czech Republic), died 6 May 1852 in Nikolsburg (now Mikulov). He wrote poetry and music, besides works on the Bronze Age, etc; many of his books and letters were in the Troppauer Museum.»

Jaenisch seems to have been confused about Wenzelides' nationality. Perhaps he should be called 'Austro-Hungarian', though the Empire of that name was not formed until 1867, after his death. Probably the Czech republic now claims him! Returning to Murray's text:

> In the same year (1849) the *Schachzeitung* announced that A. F. Svanberg, Professor of Mathematics in the Stockholm University, had also discovered four magic tours 'as a result of mathematical reasoning', but these were never published and are not to be found among Svanberg's papers now preserved in Stockholm. All that we know of them is that they were found later than Wenzelides's first tour and that one was 'concordant' with Wenzelides's first tour, whatever that may mean.

The Third Composer of Magic Knight's Tours: The Rajah of Mysore

> It was not known in Europe until 1938 that Indian players had also busied themselves with magic tours and that a closed unsymmetrical magic tour had been discovered in Mysore on 31 July 1852. A contemporary silk handkerchief bearing this tour, which it ascribed to Maharajah Kristna Rajah Wodayer Bahaudah, the Rajah of Mysore, was exhibited at the Margate Easter Chess Congress, 1938. The tour is [also] given in N. Rangiah Naidu's *Feats of Chess* 1922; it had been independently discovered by Francony in 1881.

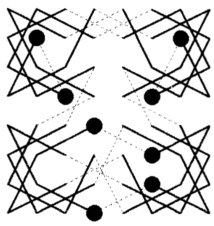
Another sighting of this silk was reported by Major J. Akenhead in a letter dated 12 March 1947 to *Fairy Chess Review*, vol. 6, April 1947, p.84: «I was in Mr A. Hammond's (Emil, Burlington Gardens) yesterday and found that he had a piece of silk framed on which was a magic knight-tour invented, as the wording stated by Maha Rajah Kristna Rajah Wodaye, Behauder Rajah of Mysore, on 31st July 1852.» The present whereabouts of the silk (or silks?) is unknown.

In search of some biographical detail about the Rajah I got involved in the history of India. The *New Standard Encyclopaedia* (1933) has: «Mysore. Native state of S. India ... situated on the Deccan Plateau, surrounded by British territory. It is ruled by a Hindu maharajah, under British protection.» *Chambers Biographical Dictionary*: «Haidar Ali (1728–82) Indian soldier and ruler. By his bravery he attracted the notice of the Maharaja of Mysore's minister, and soon rose to power, ousting both prime minister and raja. ... He waged two wars against the British. ... Taking advantage of the war between Britain and the French (1778), he and his son Tippoo Sahib ... routed the British, and ravaged the country to within 40 miles of Madras, but were ultimately defeated.»

Tippoo became sultan, but after further conflict was killed at Seringapatam 1799, which was followed by British annexation of Mysore.

My next information is from *The Golden Book of India* by Sir Roper Lethbridge, 1893, pp.362-8 (in the India Office Collection, British Library, Blackfriars Road). In summary: «The British resolved that Mysore revert to control of the family of its ancient rulers. An infant son of Chamraj by name Krishnaraj was placed on the *gadi*. During the minority of the Maharaja Krishnaraj 1799–1810 the state was administered by a *Diwan* or Prime Minister, the famous Purnaiya. The affairs of the state however fell into disorder after the retirement of Purnaiya; and the misgovernment of the Maharaja was terminated by the British Government assuming the direct administration in 1831, retaining the Maharaja as the titular sovereign. On the 18th June 1865 the Maharaja adopted as his son and successor the young prince Maharaja Chama Rajendra Wadiar, whose adoption was sanctioned in 1867. Maharaja Krishnaraj died in 1868. The rulers of Mysore were called *Wadiars* or *Wodeyars*, *Wodeyar* being a plural or honorific form of *Odeya*, Kanarese for 'Lord'.» Assuming that the age of majority in India was 21 this would give the date of birth of Krishnaraj as 1790.

My next source is S. R. Iyer's *Indian Chess* (NAG Publishers, Delhi 1982) — brought to my notice by David Pritchard. This is part of a work by Pandit Harikrishna Sharma Jyotishacharya, who was «son of Venkataraman, who lived in Aurangabad in the Maharastra State. He wrote it in Saka 1793 (AD 1871) ... Harikrishna compiled a voluminous encyclopaedic work ... of 6 parts ... the 20th chapter in the 6th part of that work ... treats in 12 main sections of many indoor and outdoor games prevalent in his part of the country. The extract printed here forms the 8th section.... This book was printed in Devanagari script by Venkateshwar Steam Press, Bombay in Saka 1822 (AD 1900).» The English commentary in Section IX reads: «Now some ways of horse-movement are being explained ... some of those ways mentioned by the King of Karnataka, H. H. Shri Krishna Udayar, are quoted below.» Karnataka is the modern name (since 1973) for Mysore, so the King of Karnataka may be identified with the Rajah of Mysore.



11	46	21	52	13	44	19	54
22	49	12	45	20	53	16	43
47	10	51	24	41	14	55	18
50	23	48	9	56	17	42	15
35	8	25	64	29	40	57	2
26	63	36	5	60	1	30	39
7	34	61	28	37	32	3	58
62	27	6	33	4	59	38	31

59	30	5	36	61	28	3	38
6	33	60	29	4	37	64	27
31	58	35	8	25	62	39	2
34	7	32	57	40	1	26	63
19	56	9	48	13	24	41	50
10	47	20	53	44	49	14	23
55	18	45	12	21	16	51	42
46	11	54	17	52	43	22	15

43	14	53	20	45	12	51	22
54	17	44	13	52	21	48	11
15	42	19	56	9	46	23	50
18	55	16	41	24	49	10	47
3	40	57	32	61	8	25	34
58	31	4	37	28	33	62	7
39	2	29	60	5	64	35	26
30	59	38	1	36	27	6	63

27	62	37	4	29	60	35	6
38	1	28	61	36	5	32	59
63	26	3	40	57	30	7	34
2	39	64	25	8	33	58	31
51	24	41	16	45	56	9	18
42	15	52	21	12	17	46	55
23	50	13	44	53	48	19	10
14	43	22	49	20	11	54	47

This is further confirmed by the inclusion of the above-mentioned 8×8 magic tour, which is #37among the 82 tour diagrams given at the end of the book. This tour is an asymmetric reentrant tour of squares and diamonds type, and is fourfold magic, in that it can be numbered magically from four different origins by cyclic shift of the numbering (i..e from f5, d1, b7 as well as f3).

Of even greater interest is the inclusion of a 12×12 magic knight's tour, diagram #3, also formed on the squares and diamonds principle, which we show as our front cover illustration. This is the earliest known magic knight's tour on a board larger than the 8×8. The existence of this tour was not known to Murray. No other magic tour on a larger board is known until 1885, when Wihnyk gave a 16×16 example, and no other on the 12×12 is known before 1932 when E. Lange added a border to an 8×8 tour. The snake-like pattern of this tour is similar to that of tour (11) by von Schinnern.

History of magic tours to be continued.

Continued from page 237. Bergholt's 6th Memoir concludes with a 'Supplementary Note on Colini', not reproduced here, which queries the spelling of Collini's name, and says: «We will now consider the crux of the problem, which arises out of the Vandermonde enunciation of quaternary symmetry, viz. How to preserve complete diametral symmetry in joining up quaternary circuits into one reentrant tour of 8*m* cells.» Bergholt's study of this problem led to three further memoranda (now termed *Memoirs* and addressed to H. J. R Murray instead of to W. W. Rouse Ball) which expound his methods of 'mixed quaternary symmetry', and will form a chapter in my book, *Knight's Tour Notes.*

Puzzle Answers

1. A question of proportion.

If a and b are the short and long sides of an A4 sheet of paper then, by definition, $b/a = \sqrt{2}$ and (since an A0 sheet has an area of one square metre) $a \times b = 10^6\ \text{mm}^2/2^4 = 62500\ \text{mm}^2$, from which we can deduce more accurate values than cited: $a = 210.2241$ mm and $b = 297.30178$ mm.

If a and b are the short and long sides of a 'golden rectangle' then $(b-a)/a = a/b$. If we multiply by b/a we get $(b/a)^2 - (b/a) - 1 = 0$, whence $b/a = (\sqrt{5} + 1)/2 = 1.6180339 = \Phi$, while multiplying by a/b we get $(a/b)^2 + (a/b) - 1 = 0$, whence $a/b = (\sqrt{5} - 1)/2 = 0.6180339 = \phi$. Both Φ and ϕ may be termed 'golden ratios'. They are related by $\Phi \times \phi = 1 = \Phi - \phi$.

(a) If the equal margin that must be left all round an A4 sheet to leave a print area that is a golden rectangle is m mm then we require $(b - 2m)/(a - 2m) = \Phi$ so $m = (a\Phi - b)/2(\Phi - 1)$. Substituting the above figures $m = 3.4665$ cm, or 1.365 inches, a surprisingly large amount.

(b) If the margin that must be cut from one side of an A4 sheet to leave a golden rectangle is m mm then $b/(a - m) = \Phi$ so $m = a - b/\Phi$ whence $m = 2.648$ cm or 1.0425 inch.

(c) If a sheet of size a by b has a domino cut from one end it is reduced to $b - (1/2)a$ by a. And if the reduced sheet sheet is of the same proportions as the whole then $a/b = (b - (1/2)a)/a$; multiplying by b/a we get $1 = (b/a)^2 - (1/2)(b/a)$, whence $b/a = [\sqrt{17} + 1]/4 = 1.2807764$ (whose inverse is 0.7807764: another pair of inverses, like Φ and ϕ, with a constant difference.).

(d) If a sheet of size a by b has a rectangle whose sides are in the ratio m/n cut from one end, and the remaining rectangle is the same shape as the whole, we get the same equations as in (c) with m/n substituted for 1/2 whence

$$b/a = [\sqrt{(m^2 + 4n^2)} + m]/2n,$$

the inverse of which is

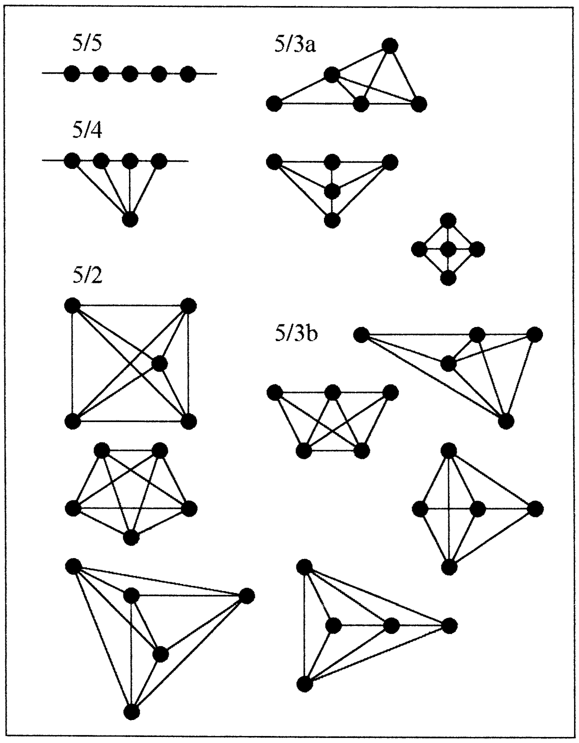
$$a/b = [\sqrt{(m^2 + 4n^2)} - m]/2n,$$

denoting these 'generalised golden ratios' by Ψ and ψ we have $\Psi \times \psi = 1$ and $\Psi - \psi = m/n$.

For readers whose algebra is rusty, it may be worth noting that in the above much use is made of the relation $(x - y)(x + y) = x^2 - y^2$ and of the solution to the quadratic equation $ax^2 + bx + c = 0$ which is $x = [\sqrt{(b^2 - 4ac)} - b]/2a$.

2. Plantations.

The question was: How many different types of plantation are possible with five trees? There are 5 types in terms of lines and numbers of trees per line, but 12 types topologically different.



Those who wish may pursue this count further to 6, 7 or more trees, but it becomes rapidly more difficult and less recreational, so we next begin to confine our attention to plantations with more regular properties. See **9** in the Puzzle Questions section.

3. Cryptarithm

The puzzle, by T. H. Willcocks, was to find 8 integers which will solve both sums:

```
BRYANT     NORMAN
BRYANT     NORMAN
NORMAN     BRYANT
```

The amusing little catch in this question is that two simultaneous equations of the form a + a = b and b + b = a only have the solution a = b = 0. So the two equations must be solved separately, in which case we find that the first part has four solutions: 2×215947 = 431894, 2×231947 = 463894, 2×410789 = 821578 and 2×431789 = 863578, while the second part has three solutions: 2×450624 = 901248, 2×475124 = 950248 and 2×475624 = 951248. The two underlined solutions use the same set of digits (01245789). Puzzle Question **10** continues this series.

Some hints on methods of solution of cryptograms may be of interest to some readers. It is a matter of reducing the range of values possile for each letter by deducing conditions applying to it. For example, in the first of these simple two-line additions one notes that by convention the leading digits B and N are not 0.

In the units column T + T = N or 1N, thus T is not 0 or 5 and N must be even. Since N is even we have in the first column B + B = N, with no carry over from the second column. Since T and B are different we must have T + T = 1N. Thus we know that T is 6, 7, 8 or 9, while B is 1, 2, 3 or 4, and N is 2, 4, 6 or 8 respectively. We can now split the problem into these four cases.

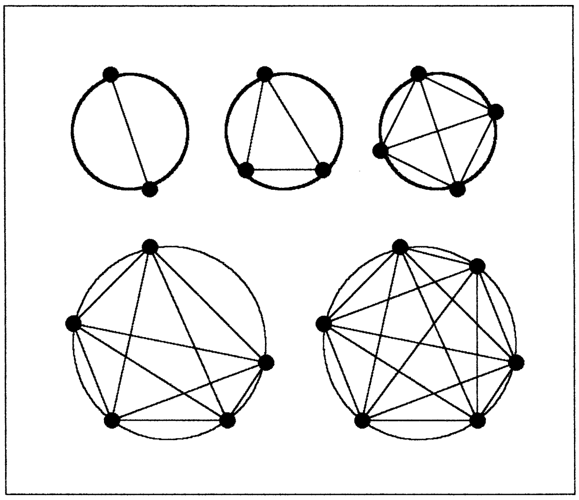
If N = 2 then we have from the tens column that A = 5, and from the hundreds column M = 0. Since there is no carry over into the first column R < 5, and since 1 + Y + Y = R or 1R the R is odd. These conditions, together with B = 1, mean R = 3. But this means R + R = O = 6, and we already have T = 6. So N=2 is impossible.

We then go on to the other three cases with similar reasoning.

4. Regions in a Circle

This question by R. J. Cook, was to connect *n* points on a circle to each other by straight lines with no multiple crossovers and to count the resulting number of regions. His answer is as follows [the notation *n*C*r* means the number of 'combinations' of *n* things taken *r* at a time, that is $nCr = n!/(n-r)!r!$]:

For 2, 3, 4 and 5 points on the circumference the numbers of regions are 2, 4, 8 and 16, respectively, suggesting that for *n* points there are 2^{n-1} regions, but this conjecture fails when there are more than 5 points.



The correct solution depends on a variant of Euler's theorem: If there are *p* lines intersecting at *m* crossings, there will be $p + m + 1$ regions. Since a line joins every pair of points, there will be *n*C2 lines, so $p = n$C2. Every 4 points produce a pair of intersecting lines and hence one crossing, so there will be *n*C4 crossings and so $m = n$C4.

Thus the number of regions for 12 points is $p + m + 1$, i.e. 12C2 + 12C4 + 1. This equals 66 + 495 +1 or 562. (Note that *n*C*r* = 0 when $r > n$, whatever your calculator says.)

The general solution is *n*C2 + *n*C4 + 1, the sequence running: 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, ... (Note the answer to case 10 is once more a power of 2, but 2^8 not 2^9).

5. A Spherical Argument

Professor Cranium claims that <u>any</u> argument involving infinity is invalid (even if it leads to the right conclusion), especially one involving adding up an infinity of 'infinitesimal' pieces. He is a 'finitist'.

A less controversial explanation of the fallacy in the argument presented is that circular triangles cannot be flattened into plane triangles without distortion, no matter what their size, so long as it is a definite size. When the base reduces to zero, the triangle becomes a circular arc, which <u>can</u> be flattened without distortion, but it is no longer a triangle and has no area.

To apply the argument correctly it is necessary to estimate the degree of distortion of the triangles, which evidently tends to $4/\pi$, but no sound argument has yet been put forward.

6. The Fate of the Dirigible

Dawson's solution is as follows: "If there are any odd nodes, i.e. points where an odd number of matches meet, we must start at one and finish at another,and if there are more than two we cannot solve the puzzle at all. As a matter of fact our balloon has two. Possible routes betwen these nodes are 120, and as we may start at either end, there are in all 240 ways of doing as required." He then tabulates the routes.

If anyone knows of algorithms or general formulae for enumeration of unicursal tours like this I would be interested in the details. Tarry's treatment, as given in Rouse Ball's *Mathematical Recreations* must surely have been surpassed?

7. Arithmetrication

In positional numeration with base A, a number of four digits like dcba (where each digit takes any of the values 0, 1, 2, ..., A–1) represents $dA^3 + cA^2 + bA + a = ((dA + c)A + b)A + a$.

However, instead of using only one base of numeration, like ten in the customary denary system, we could use several. Thus an expression such as dcba would mean $((dC + c)B + b)A + a = dCBA + cBA + bA + a$. In these expressions d, c, b and a are integers, $a \leq A-1$, $b \leq B-1$ $c \leq C-1$.

This sort of scheme is often used in measurement systems, where the bases A, B, C are known as 'conversion factors'.

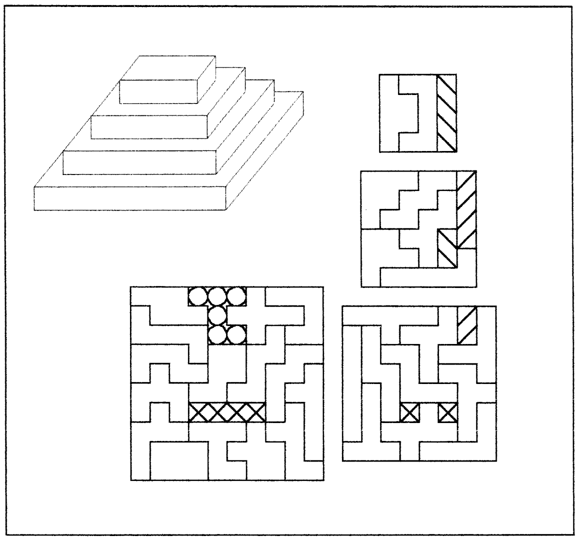
Possibly the oldest scheme of this type is our division of time: year of 12 months, month of 4 weeks, week of 7 days, day of 24 hours, hour of 60 minutes, minute of 60 seconds.

The 'Imperial' system of length is another example: mile = 8 furlongs, furlong = 10 chains, chain = 22 yards, yard = 3 feet, foot = 12 inches.

When the conversion factors are integral, as is usual, the relatonship $a \leq A-1$ is equivalent to $a < A$. But when A is non-integral (as in the case of the little used measure 'pole' which is 5½ yards) these two statements are not equivalent. In the example given the the $a \leq A-1$ rule is violated, since 5 exceeds the allowed limit of 4 for yards when used in conjunction with poles.

8. Polycube Construction

The problem was to construct a dais of the design shown below, using the 35 six-cube flat pieces (i.e. hexominoes) and one duplicate piece. The solution given by Walter Stead (21 June 1954) is shown in plan form.



Puzzle Questions

9. Plantations

How many different plantation patterns are possible (a) of 6 trees with every tree in a line of at least three? (b) of 7 trees with every tree in at least two lines of three or more?

10. Cryptarithms

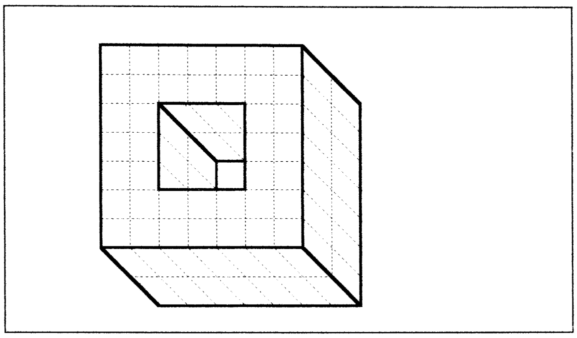
Continuing his investigations into cryptosums of the form a + a = b, Mr T. H. Willcocks wrote in September with the first problem below, to which I have added the second. These both have unique solutions (the first using all ten digits, the second only nine of the ten — you have to deduce which digit is omitted).

```
SPRING    WINTER
SPRING    WINTER
AUTUMN    AUTUMN
```

Also possible are 2×AUTUMN = SPRING and 2×SPRING = WINTER, but these have three and four solutions respectively. SUMMER, oddly is not a 'summer'.

11. Polycube Construction

This is another problem from Walter Stead's notebooks. You are asked to construct a square wall of height 2 units and width 2 units with a three-unit square hole in the centre, as illustrated (a 'well-head'?), using 16 of the 17 non-planar five-cube pieces. The solution given has a fault plane – perhaps this could be avoided?



12. On the Horizon

A holidaymaker on the promenade, at height *h* above sea-level, can see ships on the horizon at a distance *d*. A coast-guard on the nearby cliff-top can see ships twice as far. How much higher up is he? (A rough approximation will be sufficient. More ambitious solvers may wish to treat the problem with greater generality.)

13. The Naming of Numbers

Our present number names, apart from those of the first few digits, are derived from the denary, base ten, numbering system. An objection to this scheme is that the choice of ten as the base is an arbitrary one, apparently resulting from the biological accident that we generally have ten bony appendages at the ends of our arms.

Of course, other bases have been proposed, such as two or twelve, but any base (except perhaps two which is the minimal base) is open to the objection of being an arbitrary choice.

Can we then devise a system that is independent of an arbitrary choice of base?

Professor Cranium knows of a group of 'Primitives' who have something very close to such a system. They revere the 'Fundamental Theorem of Arithmetic' that every number greater than one is either a prime or can be uniquely factorised into primes. These prime factors, arranged in order of magnitude, give a unique expression for the number. For example $4 = 2\times2$, $6 = 2\times3$, $8 = 2\times2\times2$, $9 = 3\times3$, $10 = 2\times5$, ...

Accordingly, by assigning consonantal letters to the prime numbers (such as t = 2, k = 3, n = 5, d = 7, ...) and denoting multiplication by a vowel (say e) the Primitives then have pronounceable names for a large range of composite numbers. For example 4 = tet, 6 = tek, 8 = tetet, 9 = kek, 10 = ten (coincidentally!) 12 = tetek, 14 = ted, 15 = ken, 16 = tetetet, ...

The main problem with this scheme is that we only have a few consonants, so we soon run out of names for prime numbers. One scheme the Primitives have devised for getting round this difficulty is that they can add one to any number by following it with the vowel i (with the sound as in yeti or spaghetti). Adding this feature to the above system we get for example 11 = teni, 13 = teteki, 17 = tetetetí, 19 = tekeki, and so on.

However, there is a problem with this rule since it allows some higher numbers to have the same name. This gives us our first question: What are the first two numbers with the same name?

A way round this problem is to put the i also at the start of a number to which one is added, so that the two i's act like a bracket. For example 29 is tekeni and 58 is teitekeni (where the e and i combine and are pronounce like A). We may go into further elaborations of this scheme in due course. meanwhile: What is the name of the 'Number of the Beast' in Prime-Speak?

14. Contrarian Currency

In Contraria the principal units of currency are the clink and the clonk (the names derive of course from the sound of the coins knocking together in ones pocket.

Upon entering Contraria a traveller changed a note of his own currency, a Ruritanian Dollar, into Contrarian currency, but the clerk, being confused by the fact that in Contraria prices are always quoted with the smaller values first, paid the traveller clinks instead of clonks and clonks instead of clinks.

Upon returning from his journey the Ruritanian was amazed to receive, in exchange for what Contrarian money he had left, the correct sum of exactly one Ruritanian Dollar! Upon doing his accounts he found that he had also spent the equivalent of exactly one dollar.

The next year he took another holiday in Contraria, and exactly the same events occurred, except that now, due to inflation resulting in new exchange rates, he found he had spent the equivalent of two dollars.

How many clinks in a clonk?

15. Mystic Rectangles

Magic rectangles 3×4, formed of the numbers 0, 1, 2, ..., 11, are impossible, but 'mystic rectangles' in which the ranks and files all add to a multiple of the same 'mystic constant' are possible. What is the constant, and what is the chance of finding a mystic array at random?

16. The Old Egyptian Camel Tax

In Ancient Egypt, I am told, a law was passed that a man's camel herd upon his death must be divided between his dependents; no two receiving the same number, and each receiving a unit fraction of the herd (i.e. 1/2, 1/3, 1/4 etc), any camels left over going to the state as tax. How few camels should a man with six dependents keep to avoid paying the tax?

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