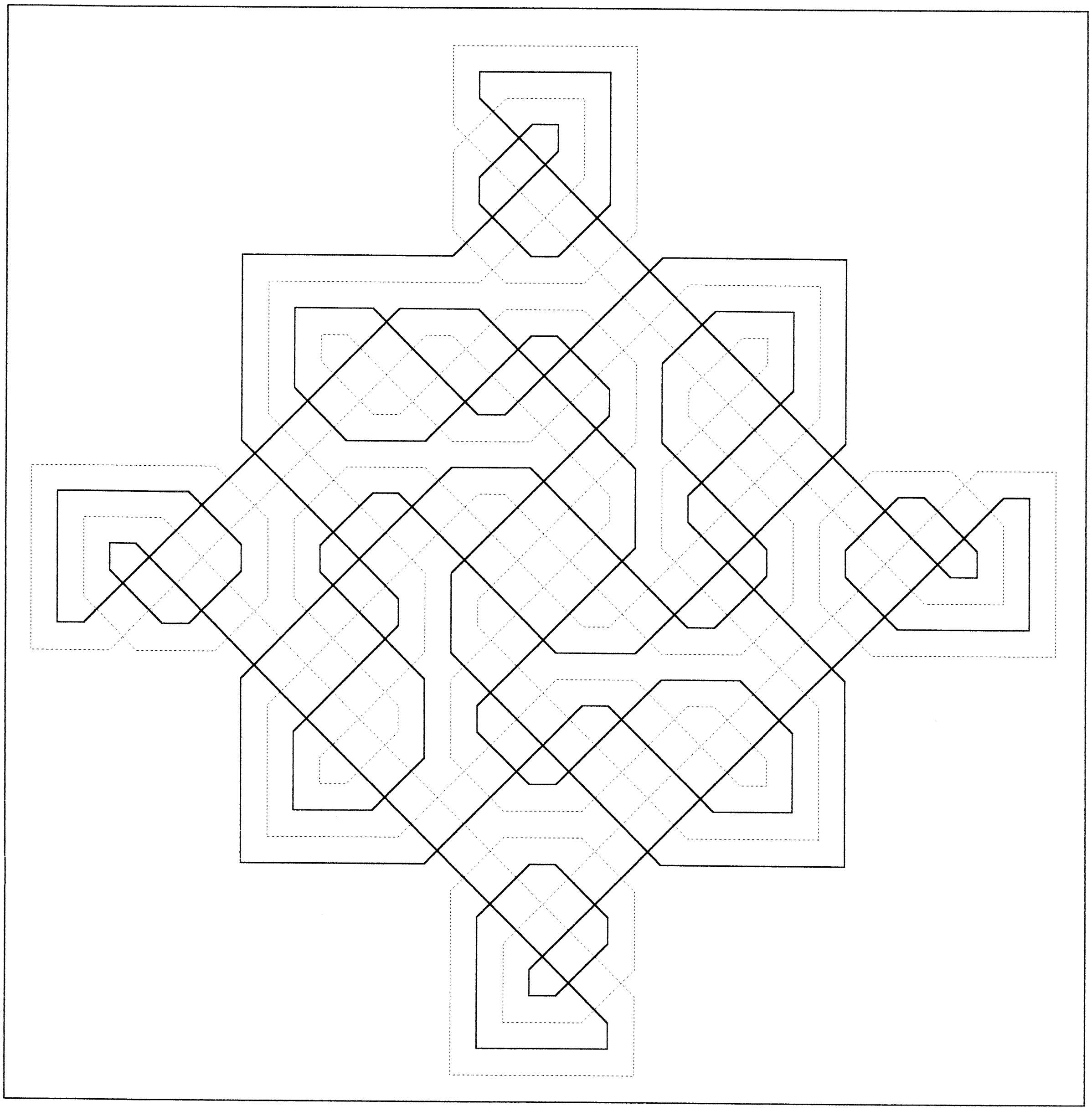
THE GAMES AND PUZZLES JOURNAL

December 1997
Published by

G.P.Jelliss
Top Floor
63 Eversfield Place
St Leonards on Sea
East Sussex
TN37 6DB

ISSN 0267 369X

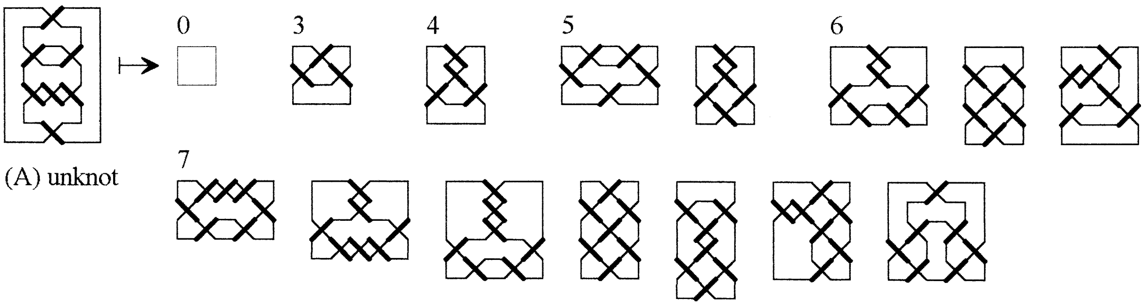
The main theme of this issue is knots and king-tours. Our cover illustration shows how the *Book of Kells* interlacing-ribbon design can be represented as a pair of simultaneous king tours (a pseudotour) of a board in the form of a Celtic Cross, made up of 13 'chessboards' (each 8×8). It may be that the pattern was designed in this way (probably beginning with a single king tour on a cross formed of 13 square boards 4×4 as shown on page 263).



Editorial Meanderings

Errata. In issue 14: On p.240 '*Weekly*' is misspelt in line 17, and 'blindfolded' in line 35 is a solecism, since according to *The Chambers Dictionary* (1993) 'blindfold' is not from the 'fold' of cloth used to cover the eyes but from 'blind-felled', i.e. 'struck blind', so is already in the past tense. In the diagram of the left half of Beverley's tour on p.241 the number 64 should be on c1, and the number at d1 should be 13. On p.246 column 1, line 2, the first word should be 'cryptarithms' not 'cryptograms'.

The Knot Book by Colin C. Adams (W. H. Freeman and Co, New York, 1994, £23.95) subtitled *An Elementary Introduction to the Mathematical Theory of Knots*. A 'knot' in mathematical terms is a closed curve in three-dimensional space. If one knot can be deformed so that it is congruent to another without passing through itself anywhere, the two knots are of the same type. In particular a knot that can be simplified into a circle is a 'trivial knot' (or an 'unknot'). It is by no means always obvious that a given knot is an unknot, as example (A) shows. (In the book the knots are shown as curved ribbons, but in accordance with the main theme of this issue I show them as king paths, with the intersections arranged diagonally; the heavy line passing over the lighter line of course. Presenting the knots in this formalised way may perhaps give clues to classifying them or analysing their structure.)

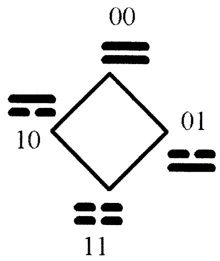


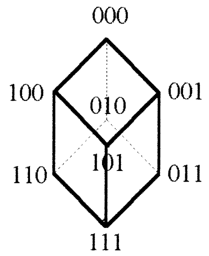
(A) unknot

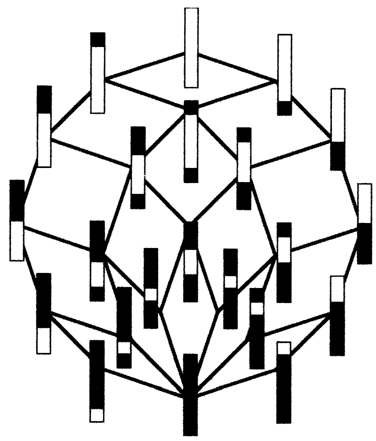
A 'compound' knot is one formed from two knots by deleting a small part of each knot and joining the ends in such a way that the joining lines do not cross each other or any parts of the existing knots. A 'prime' knot is one that is not formed by joining two knots in this way. A knot can be classified by the minimum number of crossovers in a representation of it. The numbers of prime knots P(n) of 0 to 13 crossings are: P(0) = 1, P(1) = 0, P(2) = 0, P(3) = 1, P(4) = 1, P(5) = 2, P(6) = 3, P(7) = 7, P(8) = 21, P(9) = 49, P(10) = 165, P(11) = 552, P(12) = 2176, P(13) = 9988. Alain Caudron calculated P(11) in 1978. The last two totals were computed by Morwen Thistlethwaite in 1981 and 1982. The totals do not count reflected knots as different. The cases up to 7 crossings are illustrated above.

Language of the Lines: The I Ching Oracle by Nigel Richmond (Wildwood House London 1977). I came across this book, and the next, in the Hastings public library where I go occasionally in the vain hope of finding something on Mathematics or Puzzles, but since Hastings library stocks only about five books on Mathematics as compared with whole bookcases on Obscurantism of various kinds I settled on the latter in default. Such sources can sometimes suggest ideas worth researching. This book reminded me of my articles on 'Chessics and the I Ching' in *Chessics* 13 and 14 in 1982, so I thought it might be a subject worth revisiting. As before I found the mystic sayings incomprehensible, though in this book they are apparently not translations from the Chinese but derivations by the author based on his own analysis of the lines.

Recapping from *Chessics*: The I Ching symbols are formed of horizontal lines, one above another, some of which have a break in the middle. Thus in the case of symbols with n lines there are 2^n in all. If we represent a continuous line (yang) by 0 and a broken line (yin) by 1 and write these digits horizontally instead of vertically (the I Ching symbols are 'read' from the bottom line upwards) we get a series of binary numbers. A single 'change' to a symbol, replaces a yang by a yin or vice versa. It follows that these changes connect the n-grams into the form of an 'n-dimensional hypercube'. In the case of 'digrams' a '2-dimensional hypercube' is simply a square around which the digrams occur in the sequence: 00, 01, 11, 10, which is equivalent to 0, 1, 3, 2 (note: not numerical order). In the case of the trigrams the '3-dimensional hypercube' is just a cube. Mr Richmond arranges the trigrams in the cyclic change sequence: 001, 011, 111, 110 100, 000 with the 101 and 010 giving short circuits, except that for some reason he twists the cycle into the form of a figure of eight. In our diagram below this cycle is the sequence round the outer edges of the cube diagram.





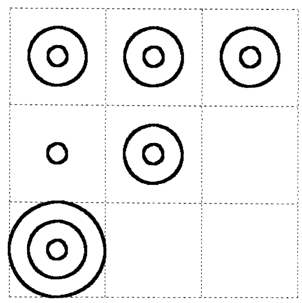


I noticed from these diagrams that the symbols with only one or two sequences of 0s or 1s are those round the edges. The n-digit symbols of this type number $2n$ and form a uniquely defined cyclic sequence of changes, from all 0s to all 1s and back again. The third diagram, in which the hexagrams are shown as six-segment wands, the segments being white for 0, black for 1, has the 12 hexagrams of this type round the edges. In the cube diagram the 101 point connects to three vertices, dividing the hexagonal outline into three rhombs, and the 010 point does likewise. This is a general property: the n-grams which contain only one sequence of 0s divide the $2n$-gon into $n(n-1)/2$ rhombs (shown for n = 6 where there are 15 rhombs). The same is true for the n-grams that contain only one sequence of 1s.

Numerology by Austin Coates (published by Frederick Muller 1974). This is another book from the Obscurantist shelves of Hastings Library. The author associates the first nine numbers with the 26 letters as in these 3×3 diagrams. (The middle diagram, not in the book, is implied by another chart.)

3	6	9
2	5	8
1	4	7

CLU	FOX	IR
BKT	ENW	HQZ
AJS	DMV	GPY



Mr Coates uses this chart to plot the letters of, mainly famous, people's names and suggests that certain patterns correspond to character traits. This seems very hard to believe, especially considering the arbitrary nature of the alphabetical order. For example FRANCIS BACON has the chart shown and a chapter is devoted to interpreting the chart for the former Lord Chancellor of England. It is not clear how the chart would be interpreted for the modern artist of the same name.

Something similar might be done using the phonetic scheme of 4×7 array shown later in this issue, based on actual similarities in the sounds of the letters used in the name rather than their alphabetical position. I offer this idea for any who may be interested in developing it, either for this purpose or in other recreations. It may be that there is a tendency for people either to grow to fit the name given to them, or in the case of film stars and authors, to adopt names that suit them better. Personally I would caution against regarding this type of 'numerology', which is more appropriately called 'numeromancy', as anything other than an amusement.

Irregular News by Ricardo Calvo, Av. Paralela 2, 1C, E-28220 Majadahonda, (Madrid) Spain. Two correspondents (Mike Pennell and Ken Whyld) independently sent me copies of issue 6 of this publication, since that particular issue was full of items on Knight's Tours. Ricardo Calvo subsequently sent me copies of earlier issues. The topics dealt with concern the early history of Chess and Magic Squares and other mystical symbolism.

Issue 6 contains diagrams of two 8×8 magic squares named the 'Safadi' and 'Mercury' squares, shown below (reoriented with the 1 at the top left corner). These are what I call 'natural' magic squares, since they have half their numbers in natural order and half in the reverse of natural order. After studying these I found that there is a third magic square of this type, the 'Intermediate' shown below. (Can anyone tell me if this magic square also has a name and a history? Since magic squares have been worked on for centuries I don't expect it is an original result.)

'Safadi'

1	2	62	61	60	59	7	8
9	10	54	53	52	51	15	16
48	49	19	20	21	22	42	41
40	39	27	28	29	30	34	33
32	31	35	36	37	38	26	25
24	23	43	44	45	46	18	17
49	50	14	13	12	11	55	56
57	58	6	5	4	3	63	64

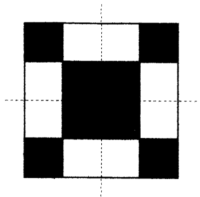
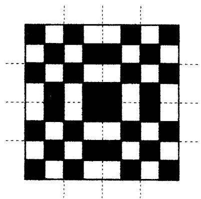
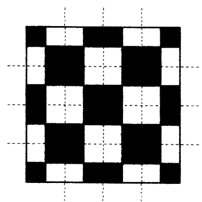
Intermediate

1	63	3	61	60	6	58	8
56	10	54	12	13	51	15	49
17	47	19	45	44	22	42	24
40	26	38	28	29	35	31	33
32	34	30	36	37	27	39	25
41	23	43	21	20	46	18	48
16	50	14	52	53	11	55	9
57	7	59	5	4	62	2	64

'Mercury'

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

The following are the patterns of the squares occupied by the moved and unmoved numbers:

A magic square similar to the Intermediate square, in that the ranks and files contain the same sets of numbers, though arranged in a different sequence, is given in *The Oxford Companion to Chess* (1984, p.199) by D. Hooper and K. Whyld. (It seems to have been taken from the works of Pavle Bidev, who in turn ascribes it to H. J. Kesson, 1881.) Mr Calvo notes that the numbers in the Safadi magic square that occupy the cells of the 'squares and diamonds' used in knight's tours add to half the magic constant 130. This remains true in the Intermediate square but not in the Mercury. It may be noted that in the Mercury these cells are all of one colour in the black and white diagram, whereas in the first two they are half white and half black. In the Safadi the quarters are chequered boards, in the Mercury the sixteenths (i.e. the 2×2 blocks) are chequered; the Intermediate has both these properties.

Mr Calvo is also involved in producing a glossy magazine called ***Artedrez*** (*Revista de Arte en Ajedrez*). Issue 4 contains three articles in English. First one by Ken Whyld on 'Robin Hood: Chess Player' (King John was a player, and one legend identifies Robin with a certain Fulk Fitz-Warine with whom Prince John quarrelled when a boy, smashing the chess board over his head). Second one by Mike Pennell on 'Chess with a Magnifying Glass' about chess illustrations on stamps. Third the essay by H. J. R. Murray on 'The Early History of the Knight's Tour' with notes by myself. Unfortunately the printers have made errors in this, since diagrams 14–17 are all the same, the correct artwork having been misplaced, and some captions are in error, but I hope this can be corrected in the next issue.

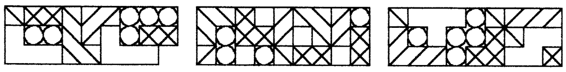
Figured Tours: *A Mathematical Recreation* by G. P. Jelliss. This offshoot to my knight's tour studies is a 22-page A4-size comb-bound booklet containing more than 225 tours showing specified numbered cells in geometrical formations. It includes all 100 of Dawson's tours showing the square numbers in a knight chain and has my Fibonacci tour (*GPJ* #13) on the cover. Price is £3.50 in UK (£4.00 Europe or surface mail, £4.50 airmail overseas).

Polycube Constructions

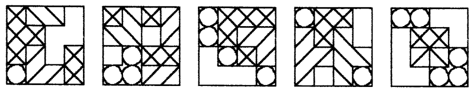
continued, from notes by Walter Stead

For problem (11) see the Puzzle Solutions p.269.
Here are two more problems using 16 of the 5-cube non-planar pieces, covering 80 cells.
(13) Block $3\times3\times9$ with one corner missing.
(14) Block $4\times4\times5$. Problem by Dennison Nixon, solution by F. Hansson.

(13)



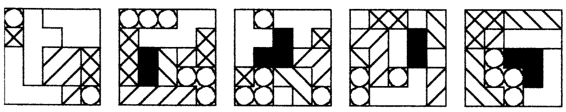
(14)



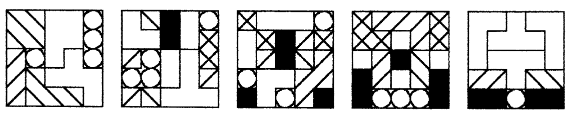
The next problems use the non-planar 5-pieces augmented by some or all of the flat 5-pieces:

(15) Cube of side 5, using the 17 non-planar 5-pieces and 8 of the planar 5-pieces. Four solutions are given: (15a) using the flat pieces that are 'orthogonally unsymmetrical'; this is described as "fairly difficult but a very interesting task". (15b) using "the 8 flat pieces which may be cut from 3×3"; this is "difficult". These two are marked 'W.S. April 1954'. (15c) and (15d) use the straight piece; in (15d) it passes through the centre of the cube. (15d) can be modified to give $5\times5\times6$ by using all the flat pieces plus a duplicate of the flat P-piece. Also other combinations of flat pieces can be substituted in the first layer.

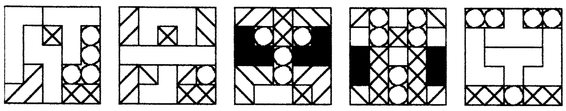
15a



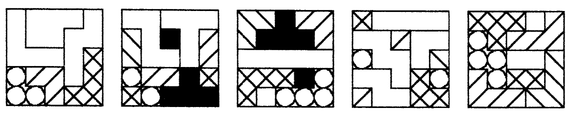
15b



15c

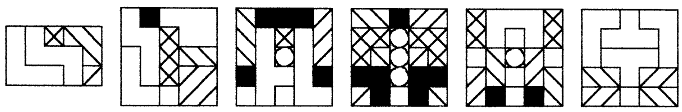


15d

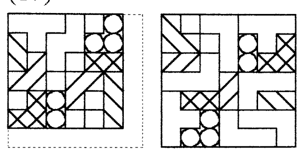


(16) W-Monument. This uses all the 5-pieces to make a plinth in the shape of a cube of side 5, with a slab 3 by 5 on top, and the W-shaped piece standing on its two points on the slab. In this it is noticed that all the enantiomorphic pairs are symmetrically placed and all orthogonally symmetrical single deckers are on the centre plane.

(16)



(17)



(17) Dais consisting of 7^2 base with 6^2 on top, having a common corner. This uses all the "two-decker 5s" that is all the 17 non-planar 5-pieces (total 85 cubes).

I have received a circular advertising a magazine ***Puzzle Fun*** which is apparently solely devoted to polyominoes. Seventeen issues are listed, totalling 302 pages and 566 problems. Each issue covers one particular topic. It is produced by Rudolfo Marcelo Kurchan, Parana 960 5to 'A', (1017) Buenos Aires, Argentina. Subscription for 5 issues is $25 (US currency).

Mr Kurchan also recommends ***Cubism for Fun*** which is the English language Newsletter of the Dutch Cubists Club and is devoted to mechanical puzzles, but I have no other details. Subscription is 30 guilders to Lucien Matthijsse, Leonapad 12, NL-3402 EP, Ijsselstein, The Netherlands.

The Saga of Fermat's Last Theorem

This article was going to be a short review of the book ***Fermat's Last Theorem*** by Simon Singh (Fourth Estate, London 1997 – xxii + 362 pages – recommended retail price £12.99, though the book-shop chain Ottakar's have it at £6.49) but it has expanded until it is an attempt to summarise, to myself as much as to anyone else, the salient events in the history of this grit in the clam of mathematics, that has led to the accretion of a remarkable, many-layered pearl of interconnected ideas.

The book is a very readable, in fact almost un-put-downable, account of the history of this famous problem, and the more easily understood aspects of the mathematics, leading up to the proof that was published by Andrew Wiles in *Annals of Mathematics* in 1995. The book is based in part on the author's 1996 BBC TV *Horizon* documentary.

Wiles first reported his results in lectures at Cambridge on 23 June 1993, but a flaw was found in part of his argument which took another 16 months to fix, with some help from Richard Taylor. The proof has now been adjudged correct and on 27 June 1997 Wiles collected the Prize of 100,000 Marks offered by the German industrialist Paul Wolfskehl in 1908, now worth $50,000.

In this account I have also consulted the article '**Fermat's Last Theorem — A Theorem at Last**' by Roger Cook (University of Sheffield) *Mathematical Spectrum*, vol.26, no.3. pages 65–73, 1994, for a litle more technical detail. To understand the proof fully it would be necessary to read Wiles's 109 page paper, and to be familiar with the mathematics in all the sources which it cites — which cover practically all branches of modern number theory!

The earlier parts of the book cover topics that will be familiar to anyone who has read standard accounts of the history of mathematics or of elementary number theory. They owe a lot to Eric Temple Bell's books *The Last Problem* (which Wiles saw in his local library in 1963 when he was ten and inspired his life-long interest in the subject) and *Men of Mathematics*, which contains lively biographies of many of the other figures who contributed to the subject.

Fermat's Last Theorem is, of course, that there are no whole numbers x, y, z in {1, 2, 3, ...} such that $x^n + y^n = z^n$ for $n > 2$. This is in contrast to the case when $n = 2$, where the equation expresses the theorem of Pythagoras and the numbers can be taken as representing the sides of a right-angled triangle. In this case the equation has an infinity of whole-number solutions. (Solutions are given by any two numbers b, c such that hcf $(b, c) = 1$ and $b > c$ and $x = b^2 - c^2$, $y = 2bc$, $z = b^2 + c^2$.)

Correspondence between Marin Mersenne and Pierre de Fermat indicates that Fermat discovered the result around 1637. It did not appear in print however until 1670, five years after his death, when his eldest son Clément-Samuel de Fermat brought out an edition of C. G. Bachet's commentary on the *Arithmetic* of Diophantus (who is thought to have lived c. AD 250). This edition included his father's observations, many of which had been written in the margins of his own copy (alas no longer extant) of Bachet's original 1621 edition. These notes included the notorious one (in Latin) that **"I have a truly marvellous demonstration of this proposition which this margin is too narrow to contain"**.

Fermat did in fact indicate a proof for the case of $n = 4$ using his 'method of descent' in connection with another problem. This method shows that if there is a solution, say $u^4 + v^4 = w^4$ then there must be a smaller solution (u', v', w'). However, since the solutions are positive whole numbers there must be a smallest solution, and for this no smaller solution can exist. This is a contradiction, so the assumption that a solution exists must be false. This result also accounts for all multiples of 4, since $x^{4k} = (x^k)^4$.

In the next century Leonhard Euler clarified Fermat's proof of this case and in 1753 adapted it to prove the case $n = 3$ (and all multiples of 3). A further breakthrough was made by Sophie Germain around 1800 showing that when n is a prime for which $2n + 1$ is also prime, then if there is a solution x, y or z must be a multiple of n, which puts a very tight restriction on any solutions. These methods led to the proof of the case for $n = 5$ in 1825 by J. P. G. L. Dirichlet and Adrien-Marie Legendre, and for $n = 7$ by Gabriel Lamé in 1839. About this time prizes including 3000 Francs and a gold medal were offered by the French Academy of Sciences for a resolution of the problem.

In 1847 both Lamé and Augustin Cauchy announced that they were close to a complete proof. Their methods, surprisingly, require the use of complex numbers (involving i, the root of minus 1). These numbers appear in the argument because they permit the factorisation of expressions involving

nth powers into n factors; for example $x^2 + 1 = (x + i)(x - i)$. Ernst Kummer showed that their arguments were flawed because of the non-applicability of unique factorisation in the realm of complex numbers, and eventually in 1857 the medal was awarded to Kummer for his work.

"Kummer showed that by employing extra techniques ... the problem of unique factorisation could be circumvented for all prime numbers up to and including $n = 31$. However, the prime number $n = 37$ could not be dealt with so easily. Among the other primes less than 100, two others, $n = 59$ and 67, were also awkward cases. These so-called irregular primes, which are sprinkled throughout the remaining prime numbers, were now the stumbling block to a complete proof." This difficulty however was one of calculation time rather than principle and with the development of computers the calculations were gradually carried to ever greater numbers: up to 125,000 by Samuel Wagstaff in 1977, and up to 4 million more recently.

However, showing that a result is true for all numbers up to a certain limit does not prove that it is true for all cases — the next case may be the exception. An important step in Wiles's proof was provided in 1984 by Gerhard Frey who showed that if Fermat's conjecture was wrong and a solution to Fermat's equation existed, say $u^n + v^n = w^n$, then the '**elliptic equation**' $y^2 = x^3 + (u^n - v^n)x^2 - u^n v^n$ has some very strange properties — so strange that it probably cannot exist! (Elliptic equations are of the general form $y^2 = x^3 + ax^2 + bx + c$.)

This step connected the Fermat conjecture with the theory of elliptic equations, and the theory of elliptic equations had, some years earlier, in 1955, been connected to another theory, that of '**modular forms**' by the Japanese mathematician Yukata Taniyama who conjectured that the two theories, though very different in form, were in fact isomorphic. In 1971 the conjecture was proved for certain classes of elliptic curves by Goro Shimura. Another step made in 1986, by Ken Ribet, was to prove that the Frey elliptic equation, if it existed, was not modular, i.e. it would be a counter-example to the Taniyama conjecture. So if the Taniyama conjecture could be proved the Frey equation would not exist and Fermat's equation would have no solution. This is what Wiles set out to prove.

A diversion occurred in 1988 when Yoichi Miyaoka put forward a proof of Fermat's conjecture based on work by Gerd Faltings published in 1983 using a completely different approach involving the apparently unconnected subject of differential geometry. The four-dimensional surfaces represented by equations of the type $x^n + y^n = k$, where the variables are complex, are all punctured with holes. "Faltings was able to prove that, because these shapes always have more than one hole, the associated Fermat equation could only have a finite number of whole number solutions." Miyaoka thought he had adapted this result to show that the number was not only finite but zero, however, after a few weeks a flaw in Miyaoka's argument was found that could not be repaired (much to Wiles's relief).

What Wiles has done is to prove the Taniyama conjecture for a further class of 'semistable' elliptic equations. Since Frey's equation is semistable this proves that it is modular, contradicting Ribet's result, thus showing that the equation does not exist, so Fermat's equation has no solution.

The most inadequate part of Singh's book, no doubt because the subject is virtually impossible to explain without actually doing some difficult mathematics, is his treatment of the subject of modular forms and their connection with the Fermat problem. He indicates merely that they are analogous to plane tessellations but in a space of four dimensions, represented by two complex coordinates. These were studied by Henri Poincaré towards the end of the 19th century. Singh illustrates the concept with one of M. C. Escher's designs (Circle Limit IV) showing a circular tiling formed of bats and angels that proliferate and become smaller and smaller towards the outer periphery.

The above is by no means the whole story. Many other contributors to the saga have not been mentioned, and the mathematics has not been adequately expounded, but perhaps it will encourage younger minds than mine to look into the subject in more detail in the original sources.

It is evident from the sheer complexity of the proof, involving so many new and different branches of mathematics that did not exist in Fermat's time that Wiles's proof is not Fermat's. The question still remains open as to whether Fermat really did have a '**demonstrationem mirabilem**'. After all it does seem very strange that it is necessary to go so far outside the basic concepts of number theory into so many diverse fields of study. Why has it been necessary to develop such a high-tech atom cruncher to crack so seemingly small a nut? Perhaps this is not yet the end of the saga after all.

Professor Cranium's Simplified Spelling

The Cambridge Encyclopedia of Language by David Crystal (Cambridge University Press, 1987) and ***The Oxford Companion to the English Language*** (abridged edition) by Tom McArthur (Oxford University Press 1996) describe, somewhat incompletely, various schemes that have been proposed for phonetic spelling of English. These either use many special symbols, more than the usual 26 letters, that are not readily obtainable even on word processors, or they make use of digraphs such as ch, th, oe, oo, that represent sounds that are clearly not produced by voicing the two letters of which they are formed. The Shaw Alphabet, designed by Kingsley Read in the 1950s, uses 40 letters of special design, resembling short-hand or some Indian scripts.

I propose here a simplified spelling scheme using only two additional letters, though some of our existing letters are assigned to different sounds. The two additional letters proposed are s and z with a line through them, thus ~~s~~ and ~~z~~, representing the sounds of ss in 'fissure', and si in 'vision', that are often represented by the digraphs sh and zh. The line through can be done on the Lotus AmiPro word processor by using the Text/SpecialEffects/Strikethrough (or Overstrike) facility and is probably available in other word processing systems. On a typewriter it can be done by overtyping with a hyphen. The letters x and q are redefined to represent the two sounds of th as in 'thin' and 'then' respectively (as a mnemonic note that x has a cross-piece and is unvoiced like t, while q is shaped like and is voiced like d). Lastly, c is redefined to represent the sound usually shown as ch or tch.

p 'pop'	t 'tot'	k 'kick'	s 'cease'	h 'who'	r 'rare'	u 'hut'
b 'bib'	d 'dad'	g 'gag'	z 'zoos'	l 'lull'	w 'wood'	o 'top'
f 'fluff'	x 'thin'	c 'church'	s 'shush'	m 'mum'	y 'yield'	i 'sit'
v 'verve'	q 'then'	j 'judge'	z 'vision'	n 'nun'	a 'cat'	e 'pet'

We recognise 20 consonants, in 5 groups of 4 (the first five columns above). The Shaw alphabet uses a separate letter for the 'ng' sound, but I consider that this modified n sound occurs naturally when n is followed by g or k and the degree by which the g is sounded is a matter of dialect. If it is wished to emphasise that the g is hard it can be doubled, thus 'finger' (finggu) but 'singer' (singu).

The 'semi-consonants' r, w, y, which are also called 'semi-vowels', we classify among the vowels. These letters assume their tensed consonantal sounds only when they precede another vowel. In addition to these consonantal sounds we also use r, w, y as vowels. The indefinite vowel sound is the commonest sound in English yet has no letter of its own. We represent it as u normally, but by r when it follows another vowel. The letter r already has this effect when it occurs at the end of a word following a vowel (except in some accents where it retains its consonantal roll in this position). The w represents the vowel in 'good' and y the vowel in 'cheap'. The vowels a, e, i, o, u, have their usual short sounds. This scheme is not entirely unambiguous, but seems about the best that can be done without introducing extra letters or accent marks which are traditionally not favoured in English.

Other vowel sounds are represented by combinations of vowels with the semi-vowels r, w, y thus: ey for the sound in 'rain/rein/reign', iy can be used as an alternative to the vowel sound of y in 'tee/tea', oy in 'boy/buoy', uy in 'buy/bye/by', aw in 'cow', ow in 'low/lo', uw in 'too/to/two', ar in 'baa/bah/bar', er in 'pair/pare/pear/pere', ir or yr as in 'beer/bier', or in 'fort/faught', ur as in 'fur/fir', uwr in 'poor', uyr in 'fire', awr in 'sour', yuw in 'new/nu'. Note that if another vowel follows these combinations, the consonantal form of r, w, y appears, e.g. 'daring', 'viewing', 'being'.

Of course we leave out silent letters like b in 'debt' or p in 'pneumatic' and we put in letters that are pronounced but not shown like y in 'pew' (pyuw) or u in 'chasm' (kasum). Note that dy becomes j as in 'duke' (juwk), and ty becomes c as in 'tune' (cuwn), and sy becomes ~~s~~ as in 'sure' (~~s~~uwr).

Here are some well-known verse and prose extracts presented in this phonetic form. I have found it possible to get used to this system quite quickly. Comments or suggested improvementsare invited.

Wintu
Wen uysikulz hang buy qu worl
And Dik qu sepud blowz hiz neyl
And Tom berz logz intuw qu horl
And milk kumz frowzen howm in peyl
Wen blud iz nipt and weyz by fawl
Qen nuytly singz qu steyring awl
Tuwit! Tuwuw! u mery nowt
Wuyl grysy Jown dux kyl qu pot.

Wen orl ulawd qu wind dux blow
And kofing drawnz qu parsunz sor
And burdz sit bruwding in qu snow
And Maryunz nowz lwks red and ror
Wen rowstid krabz his in qu bowl
Qen nuytly singz qu steyring awl
Tuwit! Tuwuw! u mery nowt
Wuyl grysy Jown dux kyl qu pot.

Wilyum Seykspyr

U Sy Durj
Fwl faqum fuyv quy farqu luyz
Ov hiz bownz ar korul meyd
Qowz ar purlz qat wur hiz uyz
Nuxing ov him qat dux feyd
But dux sufur u sy-ceynj
Intuw sumxing ric and streynj
Sy-nimfs owrly ring hiz nel
Hark! Naw uy hyr qem —
Ding-dong bel!

Wilyum Seykspyr

Jaburwoky
Twoz brilig and qu sluyqy towvz
Did guyr and gimbul in qu weyb
Orl mimzy wur qu borogowvz
And qu mowm rarxs awtgreyb

Luwis Karul

Elejy Riten in u Kuntry Curcard
Qu kurfyuw towlz qu nel ov parting dey,
Qu lowing hurd wuynd slowly owr qu ly,
Qu plowman howmwud plodz hiz wyry wey,
And lyvz qu wurld tuw darknes and tuw my.

Tomus Grey

Qu Getizburg Adres
Forskor and seven yirz ugow owr farquz brort forx on qis kontinent u nyuw neysun, konsyvd in libuty, and dedikeyted tuw qu propozisun qat orl men ar kryeyted ykwul. Naw wy ar engeyjd in u greyt sivil wor, testing wequr qat neysun, sow konsyvd and sow dedikeyted, kan long enjuwr. Wy ar met on u greyt batulfyld ov qat wor ...

Eybruham Linkun

(The Cambridge Encyclopedia includes several versions of this in different reformed spellings.)

Qu Ruwbuyat ov Owmar Kuyam

Uweyk! for Morn in qu Bowl ov Nuyt
Haz flung qu Stown qat pwts qu Starz tuw fluyt
And low! qu Huntur ov qy Yst haz kort
Qu Sultun`z Turet in u Nuws ov Luyt.

Edwud Fitzjeruld

Weyk! for qu Rudy Borl haz teyken fluyt
Qat skaturz qu slow Wiket ov qu Nuyt;
And qu swift Batsmun ov qu Dorn haz driven
Ugeynst qu Star-spuykt Reylz u fuyry Smuyt.

Framsis Tomsun

Barkleyunizum

Qer wuns woz u man huw sed: "God
Must xink it eksydingly od
　　If hy fuyndz qat qis triy
　　Kontinyuwz tuw biy
Wen qerz nowun ubawt in qu Kwod."

Ronuld Noks

"Dyr Sur, Yor astonisments od;
Uy am orlweyz ubawt in qu Kwod;
　　And qats wuy qu triy
　　Wil kontinyuw tuw biy
Sins obzurvd buy Yorz feyxfwly, God."

Anonimus

(The editor is fond of these parodies or sequels and is thinking of making an anthology of them.)

DICE RINK

by Derick Green

For this game four dice, in two pairs, one pair white and the other coloured, and a 4×4 board are all that is needed. I have no idea who the designer was, but have been told the game was originally produced commercially by Whitman Australia Pty Ltd in the early 1970s, under the uninspired name 'Relate', and may have been based on a mathematical puzzle from the 1950s.

Call the numbers 1 and 2 'low' and regard them as equivalent, the numbers 4 and 5 as 'high' and also equivalent to each other, while the numbers 3 and 6 (easily remembered as being divisible by three) are considered distinct. We call low, high, 3 and 6 the four 'values' of the dice. (The game can alternatively be played with specially prepared dice with coloured instead of numbered faces, say purple, green, yellow and red as the four values. The two purple faces are adjacent, as are the two green faces, and red is adjacent to yellow.)

To start, the first player places one die on one of the four squares on the first rank. The second player responds with a die on one of the four squares on the fourth rank, taking care that its uppermost face does not duplicate the high, low, 3 or 6 shown on the opponent's die. The first player puts down the third die on the first rank, obeying the same restrictions, and the second player places the last die on the fourth rank, ensuring that the four dice now show the four different values, high, low, 3 and 6 on their upper faces. The game is then ready for play. (A simpler alternative opening procedure is for player **A** to start in b1 and c1 and player **B** in b4 and c4.)

The game is won when one player has moved both dice to the opponent's starting line. At each turn a player rolls one die through ninety degrees to an adjacent square of the board. The value now shown must not duplicate that on the player's other die but may duplicate that on one of the opponent's dice. If this happens, the opponent must roll that die or, if no legal roll to another square is possible, rotate it on the same square, to show any face different from that on the other die. A player must roll, rather than rotate, if at all possible. This applies even if one of the dice has already reached the opponent's side of the board; the player must move it if the opponent matches the combination on it.

Illustrative Game: Notation: A move is shown by the starting number, direction of movement, and ending number. Set-up: **A** places 1 at b1, **B** places 6 at b4, **A** places 3 at c1, **B** places 5 at d4, to give the position shown (Figure 1). Play: **1**. 1N4 (This move forces **B** to move his die on d4 and allows **A** to advance b2 on move 2) 5S4 **2**. 4N6 6W3 **3**. 3W5 4W6 **4**. 6N3 3S2 **5**. 5N6 6W3 **6**. 3W2 2S4 **7**. 2E3 3E6 **8**. 6N2 4S5 (**B**'s mistake. A better move would have been 4N2 forcing **A**'s 2S6 delaying the win by forcing **A** to move back.) **9**. 2W4 5N4 **10**. 4N1 and wins. Game end (Figure 2)

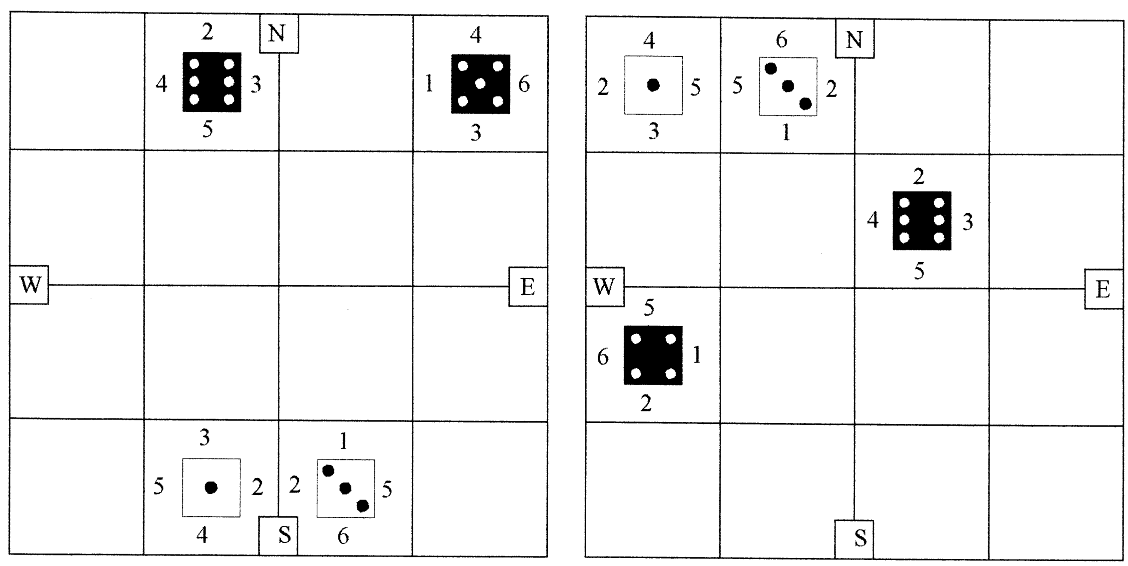


Figure 1 — Game start

Figure 2 — Game end

The game is not easy to analyse but with some thought it is possible to see a couple of moves ahead. It has been calculated that there are 276,480 possible starting positions, though I would not like to argue the point over this number. Players should always be aware of the number of moves possible by their own and the opponent`s dice. For example in Figure 3, where both dice are player **A**`s the die on b2 has four possible moves, but the die on a1 only one.

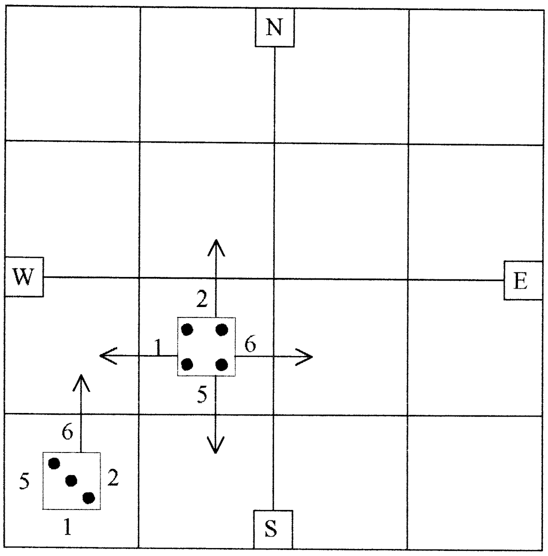


Figure 3 — Moves available

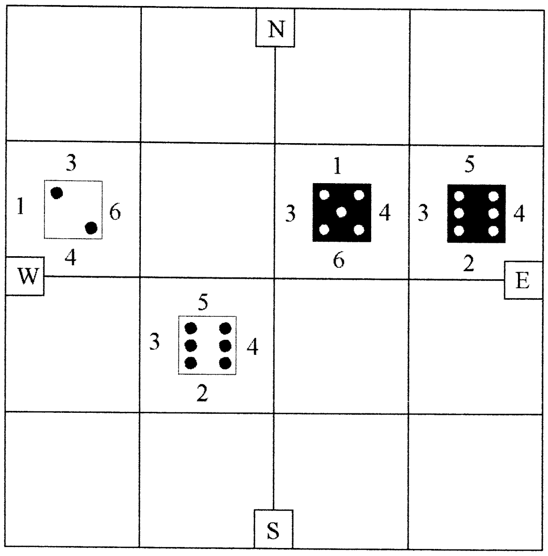


Figure 4 — Endgame

Illustrative Endgame: Figure 4 is taken from a postal endgame. **A** has just moved to b2 with **1**. 5N6, **B** can only make one move **1**... 6N2. After this **A** moves **2**. 2N4 then **B** can only play **2**... 5N6. Now **A** plays **3**. 6N2 and **B** cannot roll the die on d4 so rotates to 5 (with 4 at north). Now **A** is forced to rotate a4 which he does to 6 with 3 at north. **B** resigns, since **A** cannot be prevented from moving 2N1 next turn. Despite this example, it is often good practice not to allow the opponent to rotate.

Moving into corners restricts movement and it is often appropriate to move backwards in the middle game to free up the board, although the player who can get one die on the opponent`s back rank first can often use this die to sandwich opposing dice, to cut down the opponent`s possible moves. Dice Rink is not easy to analyse and tactics and strategy vary with each game because of the variety of opening positions possible.

As regards variants of the game, a friend suggested that the first player has the advantage and put forward the idea that to even things up **A** should set-up in b1 and c1 only, while **B** can still use any of his four back-row squares for initial placement. The theory being that **A** would only have a choice of four possible opening moves but **B** could have five. I disagree, because with careful placement **A** can restrict **B**`s opening moves anyway and **B** gets to place his dice after **A** and so knows what possible moves **A** can make. The idea of three dice per side (using all six of the normal dice values) and a bigger board (say 6×6 or 4×6) has also been discussed, but not tried in practice.

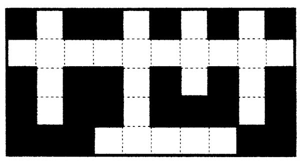
Of the 30 games I have on record from the postal group 15 have been wins for the first player, 2 draws agreed on by both players (although I believe both are wins by the first player with good play) and 13 are wins by the second player.

Illustrative Game 1. (Postal play). The opening position for **A** in this game is well thought out but the game was perhaps lengthened by his to and fro moves from a1 to b1 and back again. Start: **A**: b1(top 5, front 4) d1(top 3, front 6) **B**: b4(top 2, front 4) d4(top 6, front 4) Play: **1**. 5W6 6S3 **2**. 3W2 2E1 **3**. 2N1 (**B** has moved in such a way as to trap the die on c4 on the back row until d3 can be moved) 1W2 **4**. 1E3 (mistake, better would have been 1N5 keeping b4 on the back rank) 3N6 **5**. 6E5 2S3 **6**. 3W1 3S5 **7**. 5W6 (delaying b2`s advance) 6S3 (forced move) **8**. 1E3 3N6 **9**. 6E5 5E1 **10**. 5W6 6S3 **11**. 3S2 1S4 **12**. 6W5 (better would have been 6N3) 4N1 **13**. 2N3 3W5 **14**. 5W6 5N6 **15**. 6E5 (this die is trapped and should have been advanced earlier) 6W4 **16**. 5E1 1N3 **17**. 3N5 4W1 (4S5 would have been better) **18**. 1N3 3N6 **19**. 3N6 6W5 (if 17...4S5 had been played 6E2 could now have been moved) **20**. 5N4 5E6 (better would have been 5S3) **21**. 6E2 1S5 **22**. rotate 6(2N) 6S3 **23**. 6W3 3S1 **24**. 2N4 wins. Final position: **A**: c4(top 3, front 2) d4(top 4, front 2) **B**: a3(top 5, front 1) c2(top 1, front 3).

Illustrative Game 2. (Over-the-board play). Start: **A**: b1(top 4, front 2) d1(top 3, front 1). **B**: b4(top 6, front 3) d4(top 1, front 4) Play: **1**.3N6 (this allows an initial advance; **B** can either move 6E5 or 6S4, both of which allow 4N5) 6S4 **2**. 4N5 4E5 **3**. 5E1 1S3 (this may seem to restrict **B**'s movement, but in fact forces an advance; **A** must move 1S4, 1W5 or 6S3 all of which allow **B** to advance and keep up the pressure) **4**. 6S3 3S6 **5**. 3W5 (mistake, 1W5 with 1W6 to follow, forcing **B** to move back) 5W4 **6**. 5W4 (mistake, 5E3 etc) 4S1 (keeping up the pressure, **B** is only two moves away from victory, **A** is five away. **B** can therefore allow **A**'s single die through.) **7**. 1N3 6S4 **8**. 4E5 4N6 **9**. 3N6 6N3 (This move sems strange but if 6S4 **A** moves 5W4 forcing 4W6 and blocking **B**'s second die.) **10**. 5E3 3S6 **11**. 6E2 (mistake, better would have been 6W5) 1S3 **12**. 3W5 6S4 wins. Final position: **A**: c1top5 front 1) d4(top 2, front 3) **B**: b1(top 3, front 1) d1(top 4, front 6).

Word and Letter Puzzles

by G. P. Jelliss



Clueless Crosswords. In issue 7 of *WordsWorth* (see review in issue 13 p.202) Ted Clarke mentioned the problem of creating a 'pangram' crossword containing just the 26 letters of the alphabet, once each. He gave an example within a 7×11 area. I was able to find an arrangement within a 5×10 area, as shown here — can you complete it without clues to the words? Can you do better?

In the July issue (vol.2 no.3) Ted gave a Clueless Crossword 13×13, without blocks, of the type in which the letters are represented by numbers and you have to determine which number represents which letter, each letter appearing at least once. He made it difficult by showing only 62 of the numbers, the words being chosen for their unusual letter-patterns.

Round Britain Quiz. This returned to BBC Radio 4 in August. Suggestions for questions were invited, and I sent in a couple of ideas, though only a somewhat mangled half of one was apparently used, unless there was something in a programme I missed. If you have any ideas (for the next series) write to The Producer, Round Britain Quiz, POBox 27, Manchester M60 1SJ.

As we mentioned on p.166 of *GPJ* vol.1 the types of questions asked are what I call 'Enigmas', that is questions requiring slightly obscure knowledge, usually from a variety of unrelated subject areas. We only gave one Enigma in vol.1, so let's try to continue the series: **Enigma 2**: What holds together: (a) a poet's anti-muse, (b) an inappropriately named archbishop, (c) a philosopher's potion, (d) a futurist's underworld, (e) an ex-assistant surgeon's room-mate, and (f) the relief of Lucknow.

Tanglewords. Due to constraints on space no explanation was given of the Tangleword puzzle in the last issue. I hope solvers made sense of it. As far as I know this is an original idea. The words are entered into the diagram in a continuous sequence, the last letter of each word being the first letter of the next word. Clues are given in the form of enigmatic sayings that consist of definitions of a sequence of (in these examples four) successive words. Our second Tangleword follows the same pattern as the first, but the words are longer. No letters are entered in the four 'holes' in the diagram.

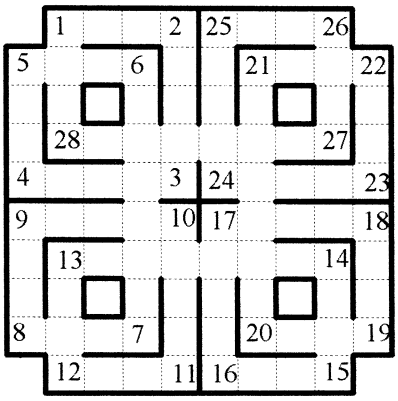
Jigsaw Crosswords. Crosswords of this type, by my favourite crossword puzzle composer 'Araucaria' (John Graham) who may have originated the idea, appear occasionally in *The Guardian* newspaper (there was one in the 5th July issue). In my version I have reduced the grid to 13×13 instead of 15×15 and follow the rule that there is only one word, and clue, for each letter (on the larger grid two of the letters are clued twice, though the two words start on the same square, one across and the other down). Solve the clues and fit the 26 words into the grid wherever they will go. Since the pattern is symmetric some clue must be sought to decide which words are to be entered across and which down.

TANGLEWORD PUZZLE

TANGLE-WORD 1 SOLUTION

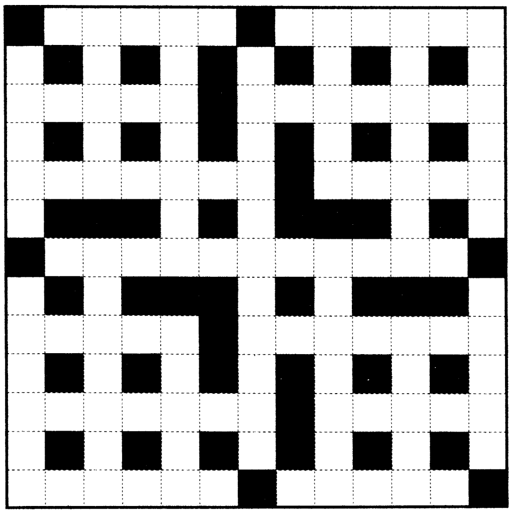
	S	E	T	A	R	C	
Y	E	T	O	N	R	A	W
E	Y	R	U	N	E	P	A
K	O	O	R	A	Y	O	S
W	E	P	T	E	A	S	T
E	C	I	R	C	L	E	E
D	O	C	U	I	P	L	A
	R	A	E	N	A	M	

TANGLE-WORD PUZZLE 2



Clue to the Labyrinth:
1-4: Rumour slow speech enthusiast is torn.
5-8: Get round resistance controller by automatic switch summons.
9-12: Be real, read intensively papal edict, cloth-bound.
13-16: Very poor deep sleep partly open burnt down.
17-20: Mislead the abominable snowman, muslim chief flashing by in the sky
21-24: Prepare to eat sort of piece of turf by the chef's tall hat
25-28: Very bad area under roof projectors: frozen water

JIGSAW CROSSWORD



A. 400 are seen to be growing together. (11)
B. Wash clubman. (5)
C. Officer isolated in pass (7).
D. Tardy as the sun. (7)
E. Vetoers necessary right. (7)
F. Zealot spread a twitch. (7)
G. Surly one of seven. (6)
H. All in 'Henry V' is lawful. (6)
I. Hiram in Pilgrim's dress? (5)
J. Colloquially explosive (5)
K. Wound if in cut knee. (5)
L. Real tin wash place. (7)
M. Understatedly I make it come. (7)
N. Simpleton: Newton in a state. (5)
O. Compound from dehydrogenated cattle skin? (5)
P. First charge across. (5)
Q. Planetary motion? (11)
R. Steal her majesty's diamond. (5)
S. Convert wise woman of the south. (6)
T. X can be held. (7)
U. More ideal person. (7)
V. Rival of invisible agent. (5)
W. The cartoonist at last made to dance. (5)
X. None return from unknown element. (5)
Y. Rode in state over there. (6)
Z. Marked down, don't charge across. (5)

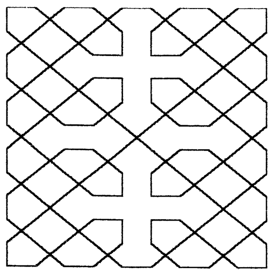
TIE-BREAKER: Compose a short verse on the name of Sergei Rachmaninov. For example, a clerihew; one by E. C. Bentley, the inventor of the genre, is: John Stuart Mill / By a mighty effort of will / Overcame his natural bonhomie / And wrote 'Principles of Practical Economy'.

PRIZE: Eight 1980s issues of the Language Quarterly *Verbatim* will go to the best set of solutions of the above five puzzles received before 1 February 1998. Please use photocopies (or copies of other kinds) to avoid damage to the journal. The Editor's decision is final.

Knight's Tour Notes

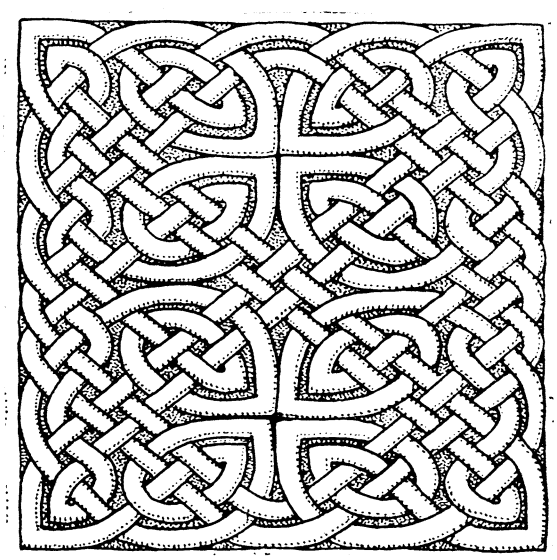
King Tours

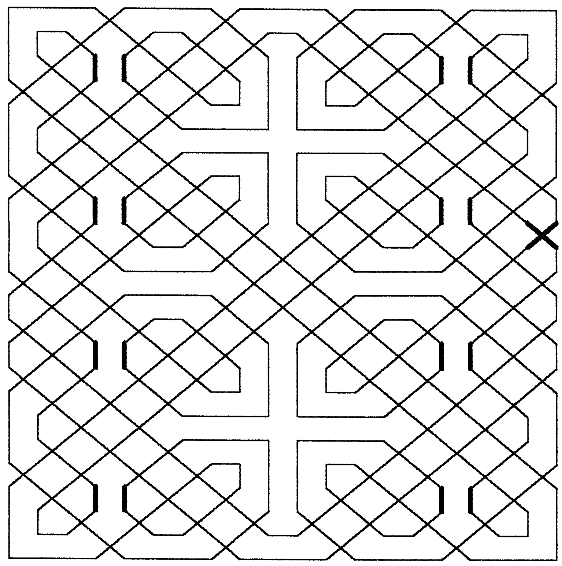
One of the main characteristics of 'Celtic Art' during its later period is the use of interlace patterns depicting a ribbon passing alternately over and under itself. No doubt such patterns derived originally from depictions of real interlacings, as in the plaiting of hair, rope-making (where several strands are woven together to give greater strength), lace fastenings for boots, leggings or coats, or in the passes and stitches used in weaving, tapestry and embroidery.



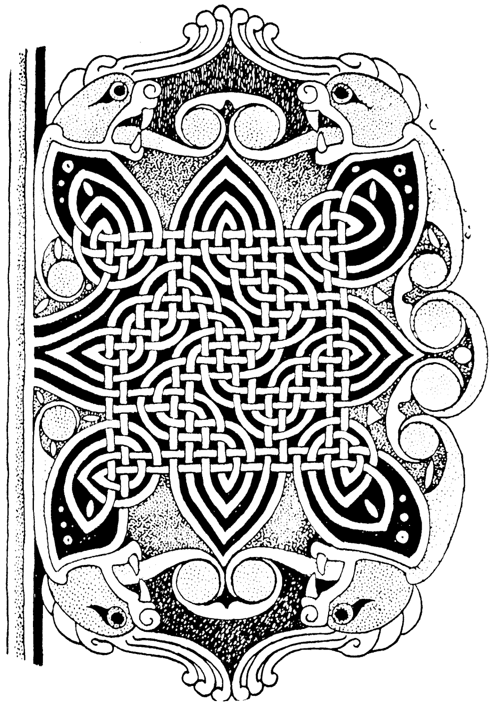
Many of the interlacings can be represented graphically by a path of king moves on a lattice of squares, the cross-overs being made by diagonal moves of the king, though this does not necessarily imply that they were designed this way. I give here however two diagrams which it seems to me almost certainly derive from underlying king-tour patterns. It may be noted that these king tours avoid sharp turns (of 45°) since this would result in small interstitial areas, not allowing room for the definite width of the ribbon.

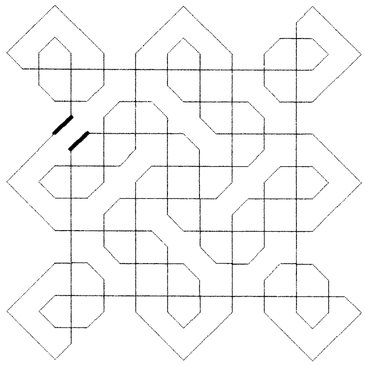
A nearly square king tour underlies the design of a panel from the Saint Madoes stone, Perthshire. The illustration below left is reduced from George Bain, *Celtic Art: The Methods of Construction*, 1951, p.46. This is a 20×24 king tour, which can be formed by 'doubling' a simpler king tour 10×12 (inset) and altering some links to turn the pseudotour into a true tour.





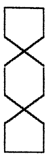
The 'binding knot' from the *Book of Kells*, late 8th or early 9th century is possibly the most elaborate of these designs. The illustration shown here is also reduced from Bain (p.54). A similar figure occurs in Nigel Pennick, *Mazes and Labyrinths*, 1990. By redrawing the knot design on squared paper, I found that this knot becomes two circuits of king moves, on a board in the shape of a quadrate cross (formed of thirteen 8×8 squares), as shown in our cover illustration. The following quotation from Alcuin (735–804), cited in Lloyd and Jennifer Laing, *Art of the Celts*, 1992, is perhaps what the design illustrates: "the four rivers of the virtues flowing out of one bright and health-giving paradise, irrigating the whole breadth of the christian church".

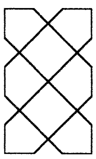


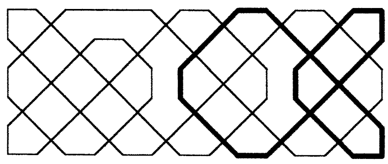


Examination of the knot shows that the two threads of which it is constructed follow each other in parallel throughout, thus the pair of threads define a single path. Replacing the two king paths by one line midway between them results in a single king tour of a quadrate cross formed of thirteen 4×4 squares, as shown inset here. This cross tour explains why the threads come to a point at the four cardinal points instead of just curving round: they correspond to the corners of the square part of the quadrate cross. It seems evident to me that in this case at least the king tour was consciously the basis of the design of the Kells knot.

The Laings describe the cover of "A strange object known as the *soiscél Molaise* ... made around AD 1000 as a *cumdach* or book-shrine for the Gospel of St Molaise, using pieces of a house-shaped shrine of the early eighth century" which uses various interlacing patterns to fill gaps in the design. Among these are patterns that may be represented by king paths on 2×6, 4×6 and 6×14 boards. The width of the ribbon and the under–over lacing of the lines is not shown here. Obviously the first is a simple plait. The second is a 'prime knot' in the modern topological theory of knots. The third is a pseudotour, consisting of two linked circuits.





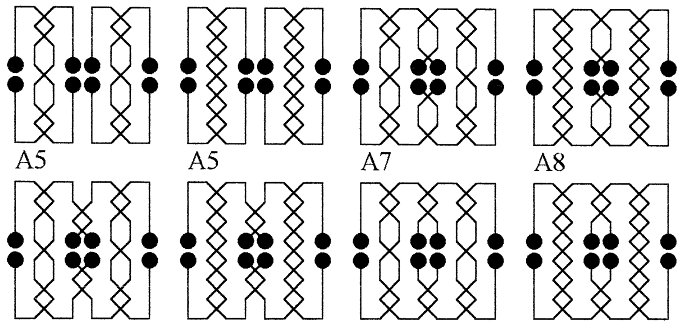


Similar but more elaborate designs are used in the upper and lower borders on a page illustrating an Eagle symbol in the *Book of Dimma*, late 8th century, and in the left and right borders on the Lion symbol page of the *Book of Durrow* of the mid 7th century. These can be simplified to pseudotours 4×28 and 4×34, which may have been the basis for the designs.

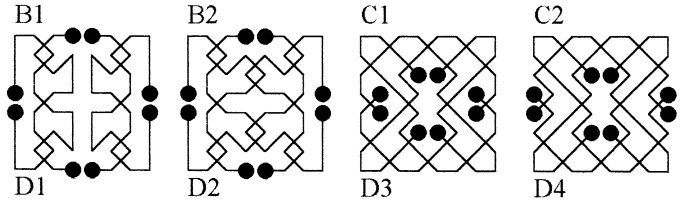
Diagonally Magic 8×8 King Tours with Double Axial Symmetry

In a letter dated 28 September 1995, Tom Marlow wrote: "I have lately been looking at magic tours again and enclose diagrams of some King tours that may be of interest. They follow those in *Chessics* Summer 1986 and, I think, complete the set in that category." The sheet of diagrams itself is dated 12 July 1995. I repeat here the diagrams A, B, C, D from *Chessics* (vol.2, issue 26, p.120) and follow these with diagrams E, F, G from the letter; with apologies to Mr Marlow for the delay in publication of this excellent and fascinating work.

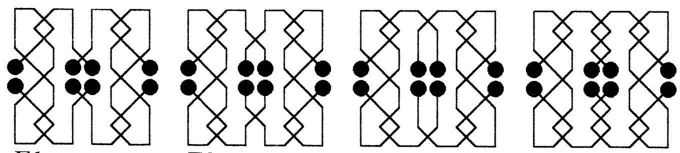
A1 A2 A3 A4 A5 A5 A7 A8



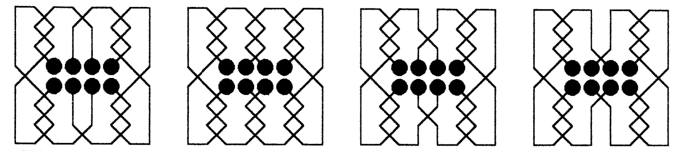
B1 B2 C1 C2



D1 D2 D3 D4

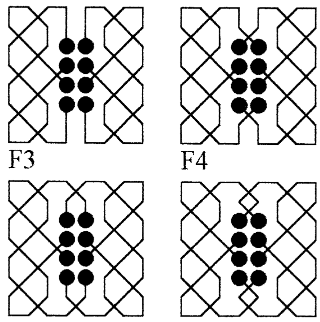


E1 E2 E3 E4

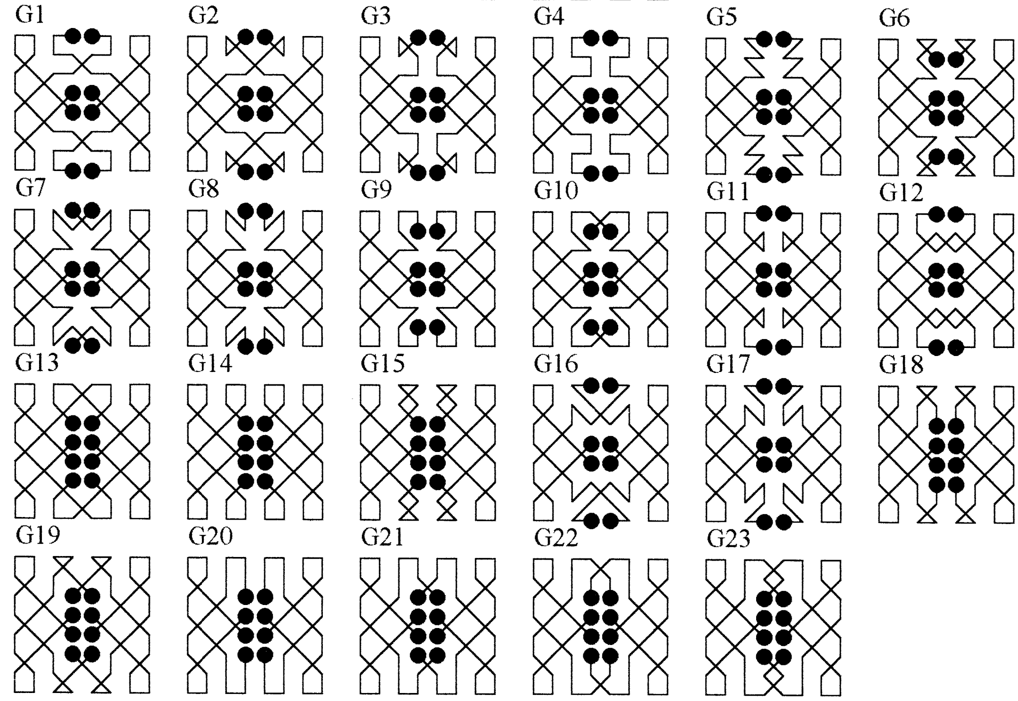


Diagonals:
(A: 4, 54, 55, 49, 17, 23, 22, 36)
(B: 4, 51, 52, 55, 23, 20, 19, 36)
(C: 45, 4, 25, 24, 56, 57, 36, 13)
(D: 5, 54, 55, 48, 16, 23, 22, 37)
(E: 5, 54, 55, 48, 16, 23, 22, 37)
(F: 60, 12, 57, 1, 33, 25, 44, 28)
(G: 60, 58, 11, 1, 33, 43, 26, 28)
Since D and E have the same diagonal perhaps they should count as one class. The diagonal shown is the one with the smallest number in it. To find the entries in the other diagonal subtract from 65.

F1 F2 F3 F4



G1 G2 G3 G4 G5 G6 G7 G8 G9 G10 G11 G12 G13 G14 G15 G16 G17 G18 G19 G20 G21 G22 G23



The Enumeration of Closed Knight`s Tours

Mario Velucchi has very kindly sent me a copy of a Technical Report, TR-CS-97-03 dated February 1997, by Brendan D. McKay of the Computer Science Department, Australian National University, Canberra, ACT 0200, Australia, which follows up the enumeration work of M. Löbbing and I. Wegener reported on p.230 in the last issue. Dr McKay reports the total number of geometrically distinct closed tours of the 8×8 board to be 1,658,420,855,433. This figure awaits confirmation.

I give here an incomplete table of enumerations of closed knight's tours on rectangular boards of up to 64 cells. The total 9862 for the 6×6 board was an early computer result by J. J. Duby (1964). Most of the larger totals were communicated to me in 1994 by Prof. D. E. Knuth. For the 3-rank boards he has a complete recursion relationship and listed T up to 3×100. It should now be possible for the computer workers to complete this table.

By G we denote the number of geometrically distinct knight`s tours, by S the number of geometrically distinct tours with binary symmetry, by Q the number with quaternary symmetry and by T the total number of tour diagrams. On oblong boards we have $T = 4G - 2S$, and on square boards, which can additionally be self-superimposed by 90° rotation, the formula is $T = 8G - 4S - 6Q$.

I find it difficult to understand why the number of diagrams total, T, is the one most writers quote. To my way of thinking the number of geometrically distinct tours, G, is by far the most significant total. Counting all the orientations of a tour as different is a bit like counting a mixed herd of zebras and ostriches by the number of legs!

The tours with binary symmetry are in three classes $S = S_A + S_E + S_B$ according as the symmetry is Axial (axis of symmetry), Eulerian (centre of symmetry, not passed through) or Bergholtian (centre of symmetry, passed through twice).

Some general results: Closed tours require a board with an even number of cells (result known to Euler 1759). Closed tours are impossible on any boards 4×n (proved by Sainte-Marie 1877). Quaternary symmetry requires a square board of singly-even side, i.e. 2(2n+1) = 4n+2, i.e. 6, 10, 14 .... Bergholtian symmetry requires a board odd×even. Axial symmetry requires a singly-even×odd board, i.e. (4n+2)×(2m+1). Eulerian symmetry is impossible on a board (4n)×(2m+1). These results were proved in my 'Notes on the Knight`s Tour' in *Chessics* #22, Summer 1985, p.64.

Table: Closed knight`s tours of rectangular boards

cells	board	Q	S_A	S_E	S_B	G	T
30	3×10	0	2	0	2	6	16
	5×6	0	2	0	0	3	8
36	3×12	0	0	0	0	44	176
	6×6	5	0	17	0	1245	9862
40	5×8	0	0	0	11	11056	44202
42	3×14	0	12	8	4	396	1536
	6×7	0	265	263	19	267183	1067638
48	3×16	0	0	0	24	3868	15424
	6×8	0	0	2817	0	13873444	55,488,142
50	5×10	0	1133	606	247	?	?
54	3×18	0	146	62	84	37078?	147728
	6×9	0	10264	11105	1862	?	?
56	7×8	0	0	0	10984	?	?
60	3×20	0	0	0	176	362192?	1448416
	5×12	0	0	0	4429	?	?
	6×10	0	0	?	0	?	?
64	8×8	0	0	608,233	0	1,658,420,855,433	13,267,364,410,532

The number of 10×10 tours with quaternary symmetry was estimated by W. H. Cozens (1960) to be 200,000. The accurate value should clearly now be within the power of the computers.

H. J. R. Murray`s History of Magic Knight`s Tours

with commentary by G. P. Jelliss

In the preceding issue we gave notes on the first three composers of magic knight`s tours and their precursors. Here we continue to follow H. J. R. Murray`s chapter on history in his 1951 manuscript *The Magic Knight`s Tours, a Mathematical Recreation*. As before, Murray`s text is shown between border lines. The commentary is less extensive than before (more next time).

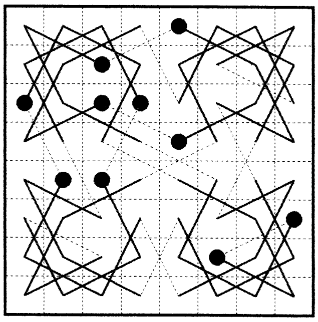
The Fourth Composer of Magic Knight`s Tours: C. F. de Jaenisch 1859/1862

C. F. v. Jaenisch devoted the second volume of his *Traité des Applications de l`Analyse mathématiques au Jeu des Echecs*, 1862, to the Knight`s problem. In this he summarised and developed the position reached by Wenzelides and added six new magic tours. Two of these (27c, 27d) are modifications of Beverley`s tour in which a different symmetrical arrangement of the right-hand half of the tour replaces those used by Beverley and Wenzelides. Four are composed of quartes, of these two (12n, 12o) are in diametral symmetry, one (00a) has corresponding cells differing by 8, and the other (00e) is a modification of one by Wenzelides (00m). E. Falkener, *Games Ancient and Oriental*, 1892, devoted a chapter to magic tours which includes three tours which he claimed to have discovered 'some 30 years ago', i.e. about 1862, but all of these are identical with Jaenisch`s tours, and I can only conclude that Falkener`s memory was at fault.

Carl Friedrich Andreyevich Jaenisch (1813–1872) was from St Petersburg and like that city his name is given in different forms: German 'von J', French 'de J', or with neither prefix, or sometimes in a different transliteration from the Cyrillic as 'Yanich'. His tours 12n and 12o were first published in *Chess Monthly* 1859 in an article previewing his Treatise. The numbers 12n, etc, refer to the catalogue of magic tours given in *Chessics* 26 1986, and are not the numbers assigned to them by Murray.

The fivefold magic knight`s tour (00a), by Jaenisch

2	15	50	29	**64**	27	38	35
51	30	**1**	14	39	36	63	26
16	3	**32**	**49**	28	61	34	37
31	52	13	4	**33**	40	25	62
54	**17**	**48**	41	12	5	60	23
45	42	53	20	57	24	11	**8**
18	55	44	47	6	**9**	22	59
43	46	19	56	21	58	7	10



58	7	42	21	**56**	19	30	27
43	22	**57**	6	31	28	55	18
8	59	**24**	**41**	20	53	26	29
23	44	5	60	**25**	32	17	54
46	**9**	**40**	33	4	61	52	15
37	34	45	12	49	16	3	**64**
10	47	36	39	62	**1**	14	51
35	38	11	48	13	50	63	2

50	63	34	13	**48**	11	22	19
35	14	**49**	62	23	20	47	10
64	51	**16**	**33**	12	45	18	21
15	36	61	52	**17**	24	9	46
38	**1**	**32**	25	60	53	44	7
29	26	37	4	41	8	59	**56**
2	39	28	31	54	**57**	6	43
27	30	3	40	5	42	55	58

34	47	18	61	**32**	59	6	3
19	62	**33**	46	7	4	31	58
48	35	**64**	**17**	60	29	2	5
63	20	45	36	**1**	8	57	30
22	**49**	**16**	9	44	37	28	55
13	10	21	52	25	56	43	**40**
50	23	12	15	38	**41**	54	27
11	14	51	24	53	26	39	42

18	31	2	45	**16**	43	54	51
3	46	**17**	30	55	52	15	42
32	19	**48**	**1**	44	13	50	53
47	4	29	20	**49**	56	41	14
6	**33**	**64**	57	28	21	12	39
61	58	5	36	9	40	27	**24**
34	7	60	63	22	**25**	38	11
59	62	35	8	37	10	23	26

The tour 00a is especially interesting since it is magic when numbered from five different origins, the most ever achieved. The geometrical form and all five numberings are shown above. (There are of course a further five numberings by reversing all these five, numbering in the opposite direction. Also, of course, each square can be presented in eight different orientations by rotation and reflection.)

The Age of Magic Knight's Tours 1876–1885

Jaenisch's contemporaries regarded his treatment of magic tours as definitive, and no new magic tours were discovered during the next fourteen years. Then in 1876 Charles Bouvier, a Frenchman, published a magic tour (05a) which struck new ground in containing two pairs of irregular quartes, and Dr Exner published a memoir, *Der Rösselsprung als Zauber-quadrat*, {The knight's tour as magic square} which contained fifteen magic tours, of which three (27i, 34a, 34e) were new and another gave a fifth arithmetical version of Jaenisch's (00a). I have not seen Dr Exner's memoir, but Parmentier (*Complément* 5, note**) says that it gives an ingenious but very laborious method of construction with consecutive quartes in the same quadrant (which Wenzelides called double-quartes). E. C. Caldwell contributed a new magic tour (05b) to the *English Mechanic* in 1879.

A period of great activity in the composition of magic tours opened in France in 1880, largely stimulated by M. A. Feisthamel in his chess column in *Le Siècle* {The Age} 1876–1885, in which he published all the known magic tours and new ones as they were produced. Two new tours appeared in 1880, six in 1881, fourteen in 1882, thirty-nine in 1883 and two in 1884. The contributors of these tours all adopted pseudonyms: A = Béligne (2 tours); Adsum = C. Bouvier (4 tours); Célina = E. Francony (11 tours); F = Feisthamel (1 tour); Paul de Hijo = Abbé Jolivald (4 tours); Palamède = Ct Ligondès of Orleans (31 tours); X à Belfort = Prof C. E. Reuss (7 tours). Altogether this group added 63 geometrical and 80 arithmetical magic tours. The majority are composed of quartes, but they also include tours in which no use is made of quartes. The first of these was contributed by Charles Bouvier in 1882. None of these composers gave any indication of the methods used to obtain his tours.

M. Wihnyk of Frauenburg, Kurland, contributed two articles on magic tours to the *Schachzeitung* in 1885 (pages 98 and 289). The first gave a modification of Beverley's tour in which irregular quartes were employed, in order to correct Wenzelides's statement that irregular quartes could not be used in magic tours (but which Charles Bouvier had done in 1876). The second article showed how Beverley's tour could be extended to give magic tours on all boards of $4n\times4n$ (n>2) cells, and thus opened a new field for explorers.

As we noted in the last issue Murray was not aware of the 12×12 magic knight tour constructed by the Rajah of Mysore between 1852 and 1868.

The Cataloguers of Magic Knight's Tours

The period of activity in France ended with General Parmentier's paper on the Knight's Problem which was read at the Marseilles meeting (1891) of the French Association pour l'Avancement des Sciences, which he supplemented at the Pau meeting in 1892. These papers were subsequently published with diagrams of 83 geometrical and 110 arithmetical solutions of magic tours. This remained the standard collection of magic tours for forty years.

Although four new solutions were published in French chess columns between 1892 and 1912 (three by Ct Ligondès and one by Grossetaite) the main interest of French composers turned to the construction of magic two-knight's tours in quaternary symmetry. Ct Ligondès (who had previously printed privately his magic tours as he discovered them) collected all the magic two-knight's tours in quaternary symmetry in a privately printed work in 1911. This work contained all the 292 solutions to this facet of the knight's problem.

M. B. Lehmann, in the fourth edition of his *Neue mathematische Spiele*, 1932, included geometrical diagrams of all the known magic tours on the chessboard, and a selection of tours on larger boards which had been composed by E. Lange of Hamburg by adding a frame to the chessboard and using suitable parent tours. Subsequently Lehmann found a new magic tour which he published in *Le Sphinx*, August 1933. This brought the number of geometrical magic tours on the chessboard to 88. This was the position when I began study of these tours in 1935.

We will continue with Murray's own work and later developments in the next issue.

RATIONAL MATHEMATICS

by Professor Z. I. Cranium (Ivory Tower Institute of Technology)

As noted in issue 13 the editor has kindly allotted me an occasional page wherein I may ride some of my hobby horses; the principal of these is known as *Finitism*, which is essentially the belief that the concept of 'infinity' is unnecessary. In all practical applications in which infinity is used, such as in the methods of differential and integral calculus, as customarily presented, it will be found that the concept is no longer present in the final results, as applied, and in fact it can be eliminated from the argument.

Continuity. My rival, Doctor Owell of the Problematic Institute, has challenged me to define 'continuity' within a finitist context. This I proceed to do.

In many problems we encounter an 'ordered set' or **range** of values X, from a 'first' value x_0 to a 'last' value x_1 and between them a sequence of $n - 1$ values, which we can write as $x_{1/n}, x_{2/n}, \ldots, x_{(n-1)/n}$, so that we can put $x_0 = x_{0/n}$ and $x_1 = x_{n/n}$. Thus there are $n + 1$ values in all in the range X (and n 'intervals' between them).

A range Y that contains all the values of a range X and others besides that are interposed between the values in X is said to be **finer** that X; or alternatively X is said to be **coarser** than Y.

A variable q that takes a succession of values in a range X , passing through all values between x_r and x_s in X, is said to vary **continuously** in X. It will also vary continuously in any range coarser than X, but will not necessarily vary continuously in finer ranges.

A variable that does not take certain values in a range, while taking values to either side of those not taken, is thus **discontinuous** in that range. A variable whose values are all separated by intermediate values that it does not take is termed a **discrete** variable. A discontinuous variable may be discrete for part of its range and continuous for the rest.

If we are able to increase the accuracy of our measurements of a physical variable, by a more 'microscopic' study, it may be that the variable that we have previously been able to treat as continuous now becomes discrete. This, for example, is the case with the density of a liquid, which is a continuous variable provided it is determined for volume units considerably greater than that of the individual molecules of the material. (A cubic centimetre of water contains more than 10^{22} molecules.) Conversely, if we consider a particular system from a more 'macroscopic' point of view, that is disregarding finer details, then a variable that we have treated as discrete may become continuous.

Accuracy. One of the practices that annoys me considerably is to see quantities cited without any indication of the degree of accuracy claimed for them. One of the figures currently bandied about is that of the Age of the Universe, which is said to have begun, at the 'big bang' 15, 000 million years ago. (Patrick Moore, *Atlas of the Universe* 1994, p.190 – is only one of many references that could be cited.) How accurate is this figure? To the nearest million? The nearest thousand million? The nearest 5000 million? If it is that inaccurate, does it really have any meaning at all?

***Are Calendars Full of Holes*?** On an entirely different subject: I've been thinking about calendars recently and began looking up a few terms in *The Chambers Dictionary* (1993 edition) to try to clarify my ideas. There I found 'summer' defined as "the warmest season of the year; the period between the summer solstice and the autumn equinox". However, 'mid-summer' is defined as 'the middle of the summer; the summer solstice, occurring about 21 June in the Northern Hemisphere." Am I confused or is Chambers? Can something start in the middle?

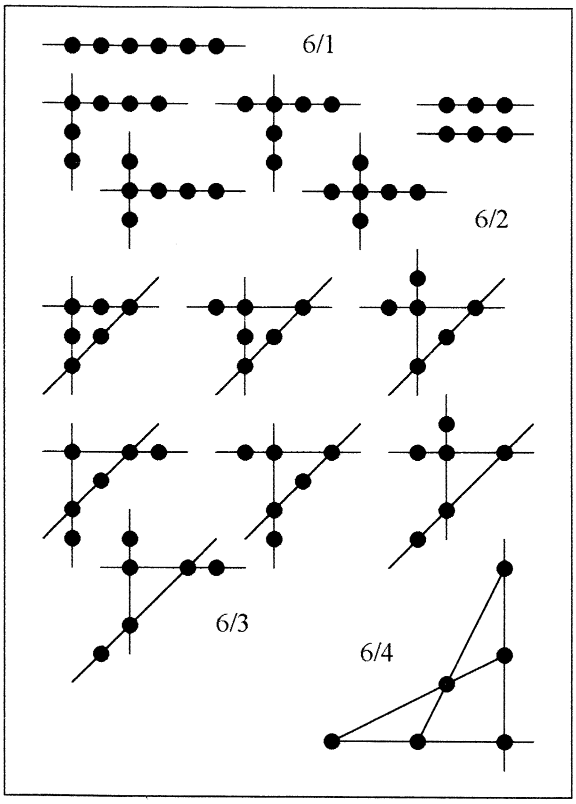
In case I was seeing things I also checked up on 'winter': "the cold season of the year, in northern temperate regions, from November or December to January or February; astronomically, from the winter solstice to the vernal equinox" and 'midwinter': "the middle of winter; the winter solstice, occurring about 22 December in the northern hemisphere."

The same self-contradictory nonsense appears in the *Concise Oxford Dictionary* (1978 edition, admittedly out of date). Can anyone resolve this quandary for me?

Puzzle Answers

9. Plantations

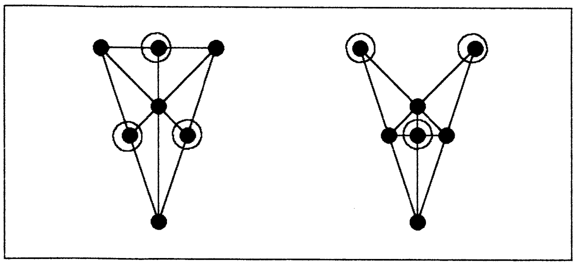
(a) How many plantations of 6 trees with every tree in a line of at least 3?Answer 14.



In the 3-line case, three trees occur at the vertices of a triangle and the third tree on each line can occur between the vertices or externally.

The 4-line case is the first in which each tree belongs to two lines of three.

(b) The impossibility of forming a plantation of 7 trees in 7 lines of three is known in projective geometry as Fano's Axiom. The maximum for 7 trees is 6 lines of 3:



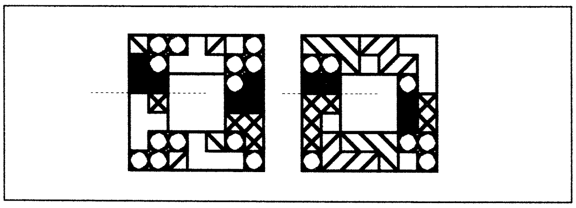
If we label the seven points in the first diagram ABC (outer triangle) DEF (inner triangle) G (centre) and define seven lines to be ADB, BEC, CFA, AGE, BGF, CGD and DEF we get a 'finite projective geometry' in which Fano's axiom is contradicted, but many other projective properties remain true.

10. Cryptarithms

The solutions to the seasonal cryptarithms can be presented in the following tabular form:

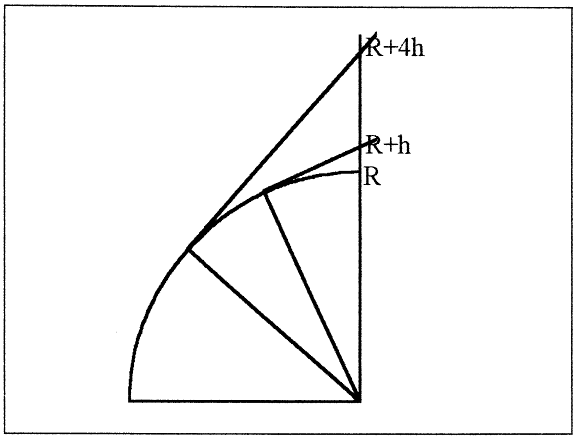
```
                      0123456789
2×SPRING=AUTUMN:      PUSTAIRMNG
2×WINTER=AUTUMN:      EMNWT-RAUI
2×AUTUMN=SPRING:      RASUMTIPGN
                      GTRAUNSMPI
                      GASTUNRMPI
2×SPRING=WINTER:      EGRSPNWTI-
                      EGRTSNI-PW
                      ITGSRPEWN-
                      RNTESGI-PW
```

11. Polycube Construction



The above construction by Walter Stead uses 16 of the 17 non-planar five-cube pieces. There is a fault-plane in the left-hand wall, indicated by the dotted lines. Can this be avoided?

12. On the Horizon



Assuming the Earth a sphere of radius R, an observer at height h above sea-level will see a distance $\sqrt{[(R + h)^2 - R^2]} = \sqrt{[2Rh + h^2]}$. Since h is small compared to R we can ignore the term in h^2 giving the approximation $\sqrt{(2Rh)}$. Thus to see twice as far you need to be four times higher up: $2\sqrt{(2Rh)} = \sqrt{[2R(4h)]}$. This question was suggested when the editor moved to his present top-floor address overlooking the sea and found he could see more ships on the horizon.

13. The Naming of Numbers

The first two numbers with the same name are 37 = 36 + 1 = (tetekek)i and 38 = 2×19 = 2×(18+1) = te(tekeki). To avoid this ambiguity we can introduce another letter, say l, to mark the beginning of the bracket to which the i adds a unit, except when the i affects the whole. (The use of 'l' is an improvement on using two 'i's as was suggested in the problem statement.) Thus 37 = tetekeki and 38 = teltekeki.

The first number whose name is altered by this notation is 22 = telteni 2×(2×5 + 1), then 23 = teltenii is the first ending with two 'i's.

If we take the 'number of the beast' to be 666 it is equal to 2×3×3×37 = tekekeltetekeki, which is a name to rival Rumpelstiltskin!

Further on this topic in problem 19.

14. Contrarian Currency

Professor Cranium corrects my fantasy geography by pointing out that the country bordering Contraria should be Raritania, and not Ruritania which is a quite different place, but this does not affect the numerical problem.

If a Raritanian dollar is worth a clonks plus b clinks and there are x clinks in a clonk then 1RD = $b+ax$ clinks. However, the traveller was given $a+bx$ clinks, and it is implied that this equals $2(b+ax)$, that is $(x-2)b = (2x-1)a$. We also have the conditions $0 \leq a < b < x$. The smallest solutions for (a,b,x) are: (0,1,2), (1,3,5), (2,5,8), (3,7,11), (4,9,14), (5,11,17), (6,13,20), (7,15,23), ... These are found by taking successive values for x, e.g. for $x = 11$ we find $9b = 21a$, which divides through by 3 giving $3b = 7a$, which is satisfied by $a = 3$, $b = 7$, less than 11, as required.

On the second occasion 1RD = $d+cx$ and the traveller receives $c+dx$ which equals $3(d+cx)$ so $(x-3)d = (3x-1)c$ with $0 \leq c < d < x$. The smallest solutions for (c,d,x) are: (0,1,3), (0,2,3), (1,5,7), (1,4,11), (2,8,11), (3,11,15), (2,7,19), (4,14,19), (5,17,23), ...

So the fewest number of clinks in a clonk is **11**. The next higher solution is 23.

15. Mystic Rectangles

It is a general rule that if the entries in an array r×s are any r×s numbers in arithmetical progression a, a+b, a+2b, ..., a+(rs−1)b, then constant row and column sums (i.e. magic rectangles) are possible only when r and s are both even or both odd. The reason for this is that the total T of all the entries is rsa + rs(rs−1)b/2 and if rs−1 is odd then for the row and column sums T/r and T/s to be whole numbers both r and s must be divisible by 2 (i.e. even). On the other hand if rs−1 is even then rs is odd, which implies that both r and s are odd.

Although magic odd×even rectangles are not possible, we can consider 'mystic' rectangles in which the row and column sums are all multiples of the same constant (preferably the largest possible constant).

This question, for the 3×4 case, formed part of an unpublished article that I wrote for *The Dozenal Journal*, but following the death of its editor, Don Hammond, no issue has appeared since 1992. The back page of some issues carried a 3×4 array of the 12 dozenal digits, as they might appear on a telephone button pad or calculator.

The symbols proposed for digits ten and eleven by the Dozenal Society of Great Britain are an inverted 2 and 3, while the US Society advocates a crossed X and the hatch symbol #. Three of these symbols not being readily available even on modern word processors does not help their cause. I find it convenient to use A for 10 and B for 11. (This conforms to the now established notation used in computing for base 16, digits 0, ..., 9, A, ..., F; a notation that provides for all cases up to base Z = 35.)

For a 3×4 array of dozenal digits the mystic constant is eleven. The following are all the 'basic' mystic arrays (in which 0 is at the top left corner and the other elements in the first row and column are arranged in order of magnitude).

In all the arrays two columns sum to 11 and two sum to 22. There are 14 arrays in which the three rows sum to 22:

```
01AB 01AB 029B 038B 038B 039A
2785 5782 435A 4A26 5917 425B
9346 6349 7681 7915 6A24 7681

039A 048A 056B 056B 057A 057A
5728 5719 1498 327A 2B18 326B
61B4 6B23 A273 8491 9634 8491

0589 0589
14B6 462A
A237 7B13
```

and there are 11 arrrays in which the rows sum to 11, 22 and 33.

0128	0128	0137	0146	0146	0236
4A35	57BA	589B	2785	3A27	1489
7B69	6394	62A4	93AB	8B59	A5B7

0236	047B	047B	048A	0679
48AB	2153	3152	2135	3152
7195	96A8	86A9	96B7	84AB

Magic or mystic properties are not affected by permutation of the rows or columns (their sums remain the same) so each of the above 25 basic arrays is one of a class of $4!\times 3! = 144$ mystic arrays. This total $25\times 144 = 3600$ includes the rotations and reflections.

The number of ways of entering the 12 digits on the array is $12! = 479{,}001{,}600$. So the probability of finding a mystic array by chance is $3600/12! =$ **1/133,056**.

In the dozenal system (base C) these totals have simpler expressions, with more zeros: the number of arrays is $21_C\times 100_C$ and C! = $114{,}500{,}000_C$, so the chance is $1/65{,}000_C$.

16. The Old Egyptian Camel Tax

We require a set of six distinct unit fractions adding to unity. The two smallest cases are: 1/3 + 1/4 + 1/6 + 1/8 + 1/12 + 1/24 where the least common denominator is 24, and 1/2 + 1/5 + 1/8 + 1/10 + 1/20 + 1/40 lcd 40. The fewest number of camels to keep to avoid the tax is **24**.

For three fractions the solution is 1/2 + 1/3 + 1/6 lcd 6. For four fractions: 1/2 + 1/4 + 1/6 + 1/12 lcd 12 or 1/2 + 1/3 + 1/9 + 1/18 lcd 18 or 1/2 + 1/4 + 1/5 + 1/20 lcd 20 or 1/2 + 1/3 + 1/8 + 1/24 lcd 24 or 1/2 + 1/3 + 1/7 + 1/42 lcd 42. For five fractions: 1/2 + 1/4 + 1/8 + 1/12 + 1/24 lcd 24 or 1/2 + 1/5 + 1/6 + 1/10 + 1/30 lcd 30 or 1/2 + 1/3 + 1/12 + 1/18 + 1/36 lcd 36 or 1/2 + 1/4 + 1/6 + 1/18 + 1/36 lcd 36 or 1/2 + 1/4 + 1/5 + 1/20 + 1/40 lcd 40.

This type of puzzle is usually presented with the total one less, and the last unit fraction missing. The problem is then to share out, say, 17 camels in the proportions 1/2 and 1/3 and 1/9 without dividing any camel. This is of course strictly speaking impossible, since 1/2 + 1/3 + 1/9 = 17/18 not 1. The dilemma is resolved by borrowing a camel from someone, making the division, and then giving the odd camel back. This device gives the nearest whole-number solution.

An algorithm for expressing a fraction $m/n < 1$ as a sum of unit fractions was given by Fibonacci (AD 1202). The first fraction in the expression is $1/q$ where q is the next whole number greater than or equal to n/m (note the inversion). To find the next unit fraction apply the same rule to $m/n - 1/q = (mq - 1)/nq$, and so on.

In the case of fractions of the type $(n-1)/n$, if n is large, the unit fractions determined by the Fibonacci method follow the sequence 1/2 + 1/3 + 1/7 + 1/43 + ... where each denominator is one more than the product of all the preceding denominators; the next denominator is $2\times 3\times 7\times 43 + 1 = 1807$.

Puzzle Questions

17. Plantations

(a) How many ways are there of planting 7 trees in 5 lines of 3, every tree being in at least two lines of three? (b) The same for 8 trees. The results are not unconnected.

18. Cryptarithm

Solve: **(fast)^2 = unfasten**.
(each letter represents a different digit)

This result just 'came into my mind' while I was thinking of something else, more about this with the solution. Perhaps there are in fact thousands of answers to the problem, but having tested some other groups of numbers at random this doesn't seem to be the case. I've tried to reason out all possible solutions but so far without success — so I'm relying on readers to help me out with this one. Is my dream solution the only answer — surely it can't be, can it?

19. The Naming of Numbers

As explained in Problem 13 the Primitives name numbers by expressing them as a product of prime factors in order of magnitude, denoting 1 = i, 2 = t, 3 = k, 5 = n, 7 = d, e = ×; the higher primes being formed by following the name of the preceding even number by i, signifying addition of one, and to avoid ambiguity l is used to indicate which part of the name the i acts on.

Professor Cranium notes two points of sociological or anthropological interest not mentioned before: that it is from the Primitives that we get our sayings "tea for two" and "beyond our ken" (meaning greater than fifteen).

Another of their sayings that has joined the vernacular is "tet for tat" (meaning exchange of one thing for something equal) this arises from a further refinement to the Primitive system of numerary nomenclature in which the letter 'a' indicates the operation of "raising to a power". Thus tat = $2^2 = 4 = 2\times2$ = tet.

This innovation allows some names to be expressed more concisely or by using lower numbers. The rule is followed that 'a' takes priority over 'e', thus tat is used for 4 in preference to tet, and most number-names are affected: 8 = tak, 9 = kat, 12 = tatek (i.e. $2^2\times3$), 13 = tateki, 16 = tatat (which is unambiguous since $4^2 = 2^4$), 17 = tatati, 18 = tekat, 19 = tekati, 20 = taten, 24 = takek, 25 = nat, 26 = teltateki, 27 = kak, 28 = tated, 29 = tatedi, and so on.

Is this system free of ambiguities? If not where do the first ambiguities occur? What is the name of the number of the beast in this system?

Answer the same questions for another tribe of primitives that use only t for 2 and k for 3, so that 5 = tati and 7 = teki.

20. Digitology

What is the smallest number which may be divided into three different parts such that each part multiplied by three gives the same digits in the answer? With the right approach this is quite easy — but with the wrong approach ....

This is Problem 76, dated 20 April 1923, from T. R. Dawson's manuscript book of *Original Puzzles* (mentioned in issue 13, p.224).

21. A Snooker Question

In snooker, what is the maximum number of breaks, all equal, that can occur in a game if no penalties are awarded?

For the uninitiated: there are 15 red balls each scoring 1 point and six 'coloured' balls, yellow, green, brown, blue, pink, black, valued at 2, 3, 4, 5, 6, 7 points respectively. A white ball is cued to knock the other balls into the pockets.

If a red is pocketed (potted) this permits a colour to be potted on the next shot. (In snooker red is not a colour, but black is!). The colour is then replaced on the board (respotted) and another red may be potted, followed by a colour, and so on, until the last red has gone, and any colour taken after the red has been respotted. Then the six colours must be potted in order of increasing value. A break ends as soon as a pot is missed.

Thus, as is well known even beyond snooker circles, the maximum score achievable in a single break, by potting the 15 reds, each followed by the black, and then the six colours, is $(1 + 7)\times15 + (2 + 3 + 4 + 5 + 6 + 7) = \mathbf{147}$.

If the first player misses a pot the other player takes over. Thus if the first player scores 72 by potting 9 reds and blacks, then the second player can win by scoring the remaining 75 points, or he can level the game by reaching 72, say by potting three pinks instead of blacks after the reds. This is the case of two equal breaks.

22. Knotty

Draw the 21 different prime knots (p.250) with 8 intersections, in the form of King Tours, using diagonal lines only for the intersections.

23. Parsing the Port

This question is from Mr R. A. Watson. Fifteen barrels of wine were offered for sale, containing the following numbers of gallons: 15, 16, 18, 19, 21, 22, 23, 25, 28, 31, 33, 34, 37, 39, 40. One barrel contained port, the others sherry. Two merchants between them bought all the sherry. No-one bought the port. One merchant bought twice as many gallons of sherry as the other. No barrel was divided. Which barrel contained the port?

24. I.O.U.s

This is another question from the *Original Puzzles* manuscript by T. R. Dawson. Problem number 4, dated 1914.

Four card players, playing for low and variable stakes, have made a practice of scribbling I.O.U.s as the game proceeds, using them also as coin tokens from deal to deal, settling up at the close of play. After some play one evening, they held respectively the following papers: A, 2s, 5d, ½d; B, 9d, 1½d; C, 1s, 3d; and D, 2s 6d, 6d. At the close of the next hand the only payment was 2d from B to A. How can it be arranged without writing out a fresh I.O.U.? [s = 12d]