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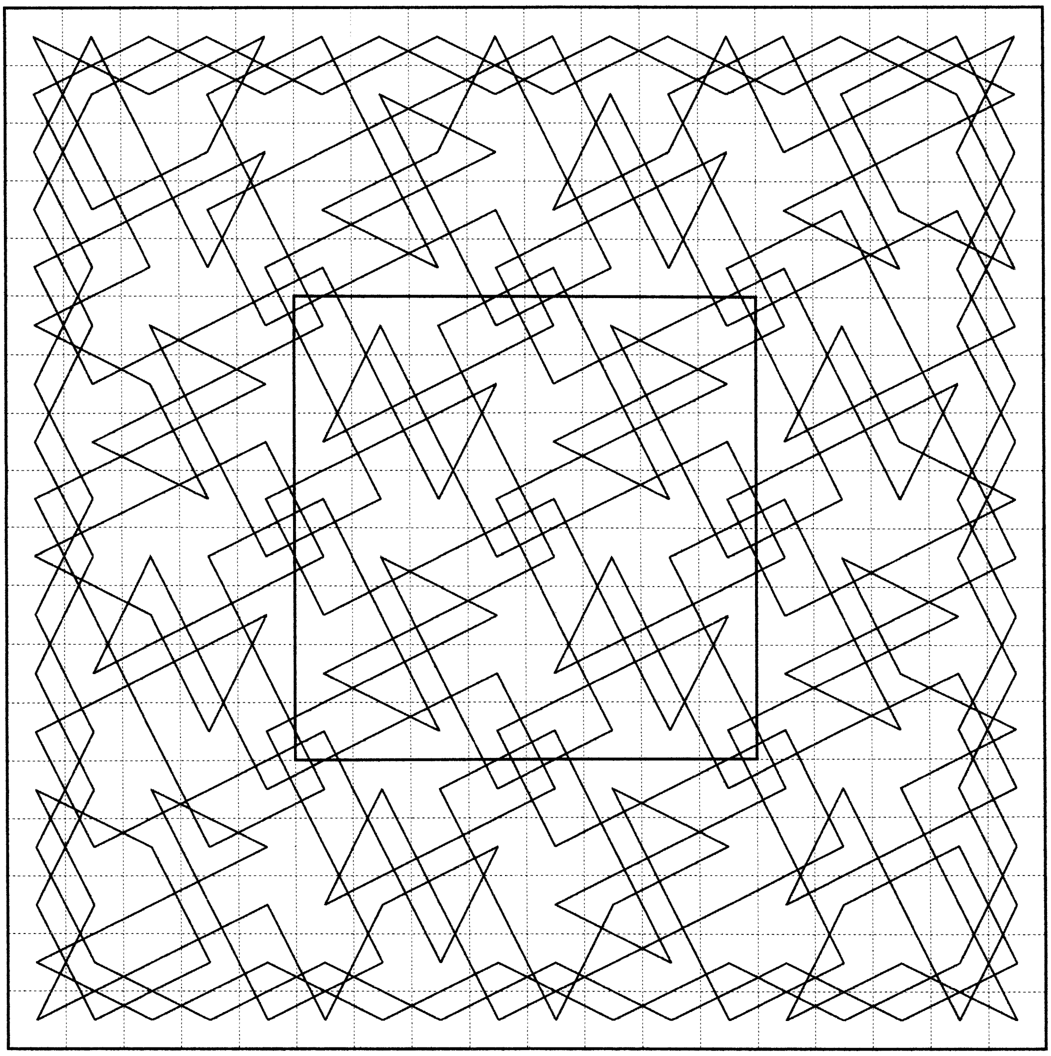
The main theme of this issue is *Symmetry in Knight's Tours*. Our cover illustration shows a knight's tour of the 18×18 board showing quaternary symmetry, with the central area using a repeating (wall-paper type) pattern. This was to be the cover illustration of a larger booklet on the subject, but financial circumstances have prevented me from publishing this at present, so some of the main results are summarised in this issue of the journal.



18×18 tour with 90° rotary symmetry. G. P. Jelliss 1998 using a repeating pattern by A. Sharp 1925

Editorial Meanderings

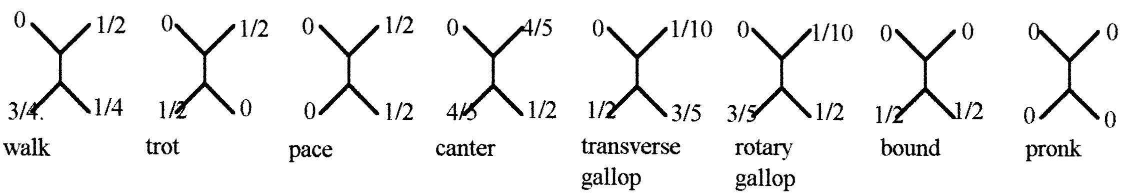
Apologies for the delay of over a year since the last issue of the journal. The obstacles preventing publication now being resolved, the remaining two issues should appear quickly. Note my new address.

Life's Other Secret, *The New Mathematics of the Living World.* Ian Stewart, (1998 Allen Lane The Penguin Press, £20). Much of this book is inspired by the pioneering book on the subject by D'Arcy Thompson, *On Growth and Form*, 1917, which is quoted in most of the chapter headings.

This is another of those books seeking to explain mathematics to the general public without actually doing any mathematics. For example in chapter 7 we read: "[Alan] Turing discovered an unexpected unity in animal markings: They can all be produced by the same type of equation. Such equations describe what happens to chemicals when they react together and diffuse over a surface or through a solid medium, so we call them 'reaction-diffusion equations'. Turing's equations do not match biology precisely: they are best viewed as a particularly simple example of the *kind* of mathematical scheme that must govern pattern formation in animals." Recent developments in this field (based on 'symmetry-breaking') are discussed and illustrated with colourful plates of patterns on big cats, fish and shells and computer simulations thereof, but nowhere do the actual equations appear.

Likewise in chapter 6 the well known relationship of Fibonacci numbers to plant and flower growth is reprised: The same subject is covered more fully in the same author's *Nature's Numbers* (1995 Weidenfeld & Nicolson). But in neither book is there any 'real' mathematics (i.e. algebraic formulae).

Chapter 9 contains some interesting notes on mathematical modelling of quadrupedal animal (and hexapedal insect) gaits. The motion of the legs forms a cyclic pattern and taking 1 to represent the time of one cycle and 0 to be the time at which the left foreleg is moved the phase relationships for the eight most common quadruped gaits are as follows (simplified from figure 75 and using fractions instead of decimals). Apparently camels normally 'pace', hence their swaying gait:



The bipedal gait of humans would be (0, 1/2). The most common insect gaits are said to be the 'tripod' in which two sets of three legs (providing a triangular base) move successively, and the 'metachronal wave' in which the six legs move successively, from back to front on the right, then back to front on the left (though the text does not correspond to the illustration in Figure 76).

The Eight by Katherine Neville, Headline Book Publishing 1990 (first published by Ballantine Books, Random House, 1988). This is a nearly 700-page thriller about a quest stretching over centuries to reassemble a chess-set supposedly presented by the Moors to Charlemagne in 782 and which, through symbols on the squares and pieces, reveals the secrets of alchemy. Because of this dangerous knowledge it is concealed for 1000 years in 'Montglane Abbey' but, with the coming of the French Revolution the board and pieces are dispersed and its reassembly becomes the object of rival forces. The story is told in alternating episodes set in the 1790s and the 1970s which are connected at the end. Almost every notable historical figure of the 18th century pops up at some point in the tale.

Of particular interest to us here is a short episode "The Chess Master's Tale" giving an account of a supposed meeting at the court of Frederick the Great of Prussia around 1748 involving the mathematician **Leonhard Euler**, the composer **Johann Sebastian Bach** and the chess-master and musician **André Danican Philidor**, who tells the tale. I give an extract:

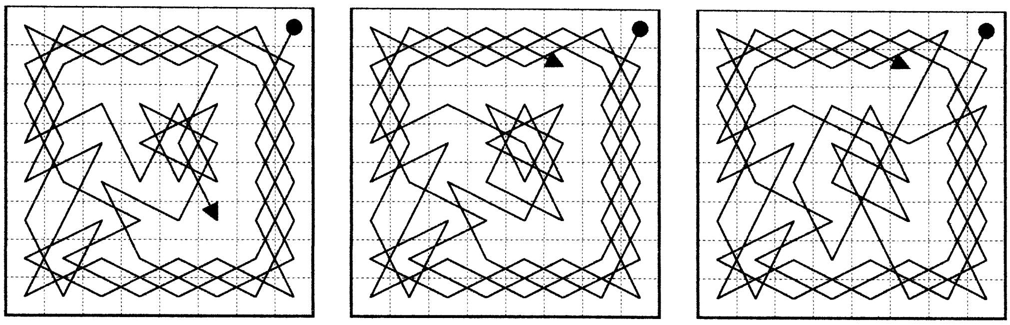
"... *Euler took me aside. 'I'd prepared a gift for you,' he told me, 'I've invented a new Knight's Tour, a mathematical puzzle. I believe it to be the finest formula yet discovered for the tour of a Knight across the chessboard. But I should like to give this copy to the old composer tonight, if you don't mind. As he likes mathematical games, it will amuse him.' Bach received the gift with a strange smile* *and thanked us genuinely. 'I suggest you meet me at my son's cottage tomorrow morning before Herr Philidor departs,' said Bach. 'I may then have time to prepare a little surprise for both of you.' Our curiosity was roused, and we agreed to arrive at the appointed time and place. The next morning, Bach opened the door of Carl Philipp's cottage and showed us inside. He seated us in the small parlour and offered us tea. Then he took a seat at the small clavier and began to play a most unusual melody. When he'd finished, both Euler and I were completely confused. 'That is the surprise!' said Bach with a cackle of glee that dispelled the habitual gloom from his face. He saw that Euler and I were both totally at sea.' 'But have a look at the sheet music,' said Bach. We both stood and moved to the clavier. There on the music stand was nothing other than the Knight's Tour that Euler had prepared and given him the prior evening. It was the map of a large chess board with a number written in each square. Bach had cleverly connected the numbers with a web of fine lines that meant something to him, though not to me. But Euler was a mathematician, and his mind moved faster than mine. 'You've turned these numbers into octaves and chords!' he cried. 'But you must show me how you've done it. To turn mathematics into music – it is sheer magic!' 'But mathematics are music,' Bach replied. 'And the reverse is also true.' ..."*

The trouble with such quasi-historical fiction is that one can never be quite sure whether a character or place or event is fact, fiction or something in between. The meeting of Philidor, Euler and Bach is just possible (Philidor wrote his book on chess in 1749, Bach died in 1750, Euler presented his paper on knight's tours in 1759 but had clearly worked on the subject for some time). The idea of converting a knight's tour to music is intriguing (more on this below), and some modern composer may like to take up the idea, but Bach's composition is probably a fiction.

As might be expected from the title, there is much talk of symbolisms involving the figure of eight, including the entwined snakes on the caduceus of Hermes, the octaves of John Newlands which were a precursor of the periodic table of elements of Mendeleev, the course of a Venetian ceremonial procession, and use of the same symbol turned on its side to represent infinity, (it is noted in the book by Gullberg reviewed below that this symbolism originated with John Wallis in 1655).

The end of the story I find somewhat disappointing. The chases and fights are typical of many a standard thriller and could have been cut back to give more room for events exploring the ethical dilemmas implicit in the possession of the esoteric knowledge imparted by deciphering the formula concealed in the chess set, which would have made the book more of a philosophical novel.

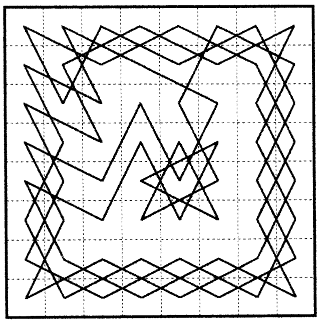
Fillet and Field Tours. Back in August 1997 Ken Whyld sent me a copy of an article in German by Frank Bernhauer on a knight's tour in the *Manasollasa* (translated 'Freude des Geistes' which I take to be not 'Joy of Ghosts' but 'Delight of the Spirit' a traditional sobriquet for chess). This is described as a 'Fürstenspiegel' ('Princely Mirror') written for King Somesvara III of the Kalyani area in central India (c.1150). The tour is described in the form of a, slightly corrupt, list of two-letter coordinates (more on this in the next section). Bernhauer gives the open tour shown in the first diagram below.



However, the sequence of syllables as listed on the third page of the article, ends with the cell a knight's move from the corner; this suggested to me that the tour was intended to be reentrant, and led me to the second diagram above which seems the most likely interpretation in my view. Another point I noticed was that the move sa-/-de in the third line, where there is some doubt about the text, can be made in a straight line, so perhaps the intermediate move might have been intentionally omitted. This could be incorporated as in the tour shown in the third diagram, but this seems less probable.

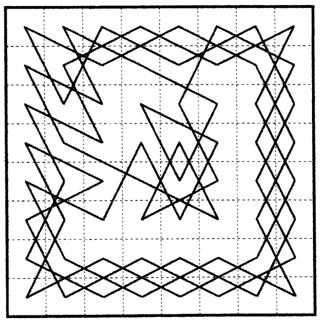
These tours are of the type termed 'fillet and field' by Haldeman (1864). The middle diagram (when closed) has a complete 'braid' along two sides of the board. The following three are later examples on the same plan. The tour in the sixth diagram was sent to me by Franco Pratesi of Firenze, Italy, in December 1996, it is from a book published in Firenze in 1836, *Raccolta di Venticinque Nuovi Problemi di Scacchi* ..., which is attributed to Gasbarri. His tour has the added feature of a 3×4 tour embedded in the centre of the 8×8 tour.

Euler (1759)



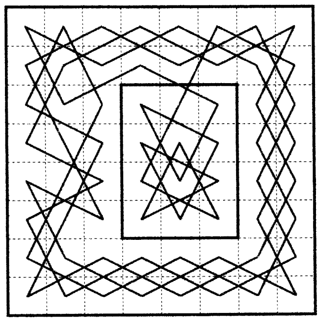
numbered a1–b3

Dollinger (1806)



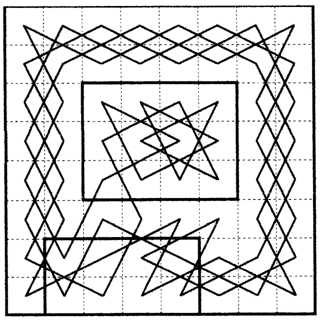
numbered c1–a2

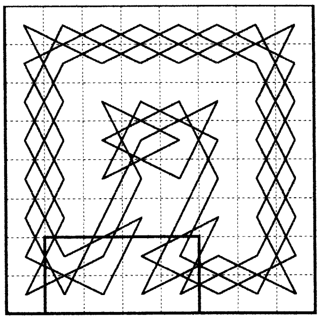
Gasbarri (1836)

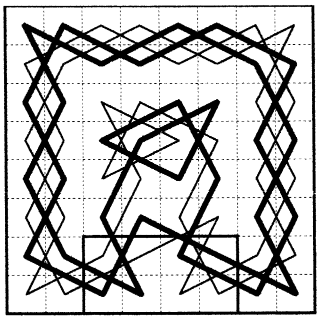


numbered d7–f6

In my knight's tour notes dated 18 July 1990 I enumerated all tours (19) that have a maximum length border braid (i.e. along three sides of the board, plus a bit of the fourth). Two of these include a central 3×4 subtour. I show one of these below (see the Puzzle Questions for the other) and another which is the only one showing a gap from edge to centre (it may minimise the number of crossovers). I classified these tours according to the way the 20 moves are partitioned that join up the eight braid-end cells bcde 1 and 2. The partitions are 2,2,2,14 (4); 2,2,8,8 (7); 2,2,5,11 (6); 2,5,5,8 (2).







If the ends of the braid are symmetrically placed, at cdef 1 and 2, a true tour is not possible. The best that can be done is a pseudotour formed of two superimposed paths, either asymmetric as shown or symmetric formed by using one of these paths together with its reflection left to right.

Mathematics from the Birth of Numbers by Jan Gullberg, W. W. Norton & Co 1997. This book of over 1000 pages, produced in camera-ready form by the author himself, comprises a complete course in history and foundations of mathematics in one volume, and appears to be generally sound. However the account of knight's tours, p.209, contains a couple of errors: Beverley's magic tour (reflected in the principal diagonal) is attributed to Euler, and the magic tour ascribed to Wenzelides is not one of his but one by Jaenisch. On p.469 Stonehenge is moved to Cornwall instead of Wiltshire.

Board Games Studies 1: an 'academic journal for historical and systematic research on board games' from Research School CNWS, Leiden University, P.O.Box 9515, 2300 RA, Leiden, Holland. To appear annually (subscription to three issues Dfl.125 + 15 for international cheques). The 1998 issue has articles on Mancala, Inca Dice and Board games, Edward Falkener, book reviews, etc.

The Eagle Bookshop, 103 Castle Road, Bedford MK40 3QP produces a catalogue of second hand books on Mathematics and Physics, having taken over the stock of F. E. Whitehart who has retired.

Incantatory and Musical Knight's Tours. The use of coordinates to record positions and moves in chess goes back at least to the time of al Adli (c.840). In the *Manasollasa*, mentioned above, the files are lettered by consonants c, g, n, d, t, r, s, p and the ranks (top to bottom) by vowels similar to a, â, i, î, u, û, e, ê. (Bernhauer has overbars instead of caps, and ai instead of ê, presumably these indicate corresponding longer vowel sounds). This system makes it possible to present a tour as a sequence of syllables that is pronounceable and looks like Sanskrit, but (as far as I am aware) makes no sense. The tour given (middle interpretation), when split up into groups of four syllables for clarity, runs:

pa si pu se / tê ne cê gû / nî cu gi ca / nâ ta sâ pî
sû pê re dê / ge dû gu ci / ga dâ ra pâ / sî pû sê te
nê ce nû gê / cû gî câ na / tâ sa pi su / pe rê de nu
tû rî di tu / ri dî ru ti / du rû tî ni / cî gâ da râ.

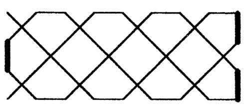
A similar consonant-vowel system of coordinates is used in the telegraphic code for transmission of chess games, known as the Gringmuth notation in which the files a–h are lettered BCDFGHKL on White's half of the board and MNPRSTWZ on Black's side, while the ranks 1–4 and 8–5 are lettered AEIO, so that each cell has a two-letter designation. For example castling king-side with the White king is shown as GAKA and with the Black king as SAWA. Beverley's tour in this notation, divided into four-cell sections again (read row by row as usual), is almost an incantation:

MAPENORI	SAWEZOTI	GOKILAHE	FACEDOBI
DEBACIFO	GEKALIHO	SIWOZETA	RENAMIPO
BODICAFE	HALEKOGI	TOZIWASE	RONIPAME
PIMONERA	TEZAWISO	HILOKEGA	FICOBEDA

Converting a knight's tour into music as Bach is supposed to have done in the novel reviewed above could perhaps be done by a similar method. In the simplest method the consonants could indicate the pitch, i.e. the eight notes of a scale, and the vowels could indicate the duration of the notes, say 1, 2, 3 or 4 beats, so that in regular tours like Beverley's where each vowel occurs once in each group of four, there would be 10 beats to each bar. Or alternatively the vowel could indicate a lower note sounded as a chord with the 'consonantal' note; say with frequency ratios of 2/3, 3/4, 3/5, 4/5 (1/1 is the ratio of a note to itself and 1/2 is the ratio of a note to its double frequency, which is the 'octave' above, so the ratios must be between 1 and 1/2, and are supposed to be more harmonious if in simple numerical proportions). The incantation can then be sung to the music. A reader with musical ability may like to try out these ideas. Would the results be musical or cacophony? I suspect it might be similar to change-ringing of bells. Coincidentally, the following was squeezed out of the last issue on king tours:

King-Tours and Change-Ringing Another context in which interlacings like king tours can be seen is that of change-ringing. This is a method of ringing four or more bells, each tuned to a different pitch, so that a different sequence of notes is followed each time the full set of bells is rung. The simplest pattern in which four bells can be played, known as 'plain bob minimus', is shown in the diagram. The highest pitched bell 1 'hunts' through the other bells to the back, stays there one more change, then hunts forward again to the front. Eight different sequences are thereby rung. (Joining 1–2 and 3–4 in the final column and 2–3 in the first column gives a king tour.)

1	2	2	4	4	3	3	1	1
2	1	4	2	3	4	1	3	2
3	4	1	3	2	1	4	2	3
4	3	3	1	1	2	2	4	4



More elaborate change-ringing schemes were designed by Fabian Stedman (b. 1631) who wrote two books on the subject, *Tintinnologia* and *Campanologia*.

Polycube Constructions

continued, from notes by Walter Stead

Frans Hansson of Sweden who was a long-term correspondent of Walter Stead, and contributed numerous articles to *Fairy Chess Review* on dissections and tours and other topics, proposed a whole series of polycube problems with 5-cube pieces in a letter to Stead dated 24 June 1954. His first group of problems use only one piece from each enantiomorphous pair (i.e. interchange of the 3-dimensional pieces with their reflections is considered possible). In the note-books these constructions are termed 'Type I' and those using both pieces of the 'E-pair' are 'Type II'.

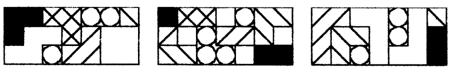
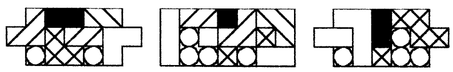
An obituary in the former Scandinavian chess problem magazine *Stella Polaris*, March 1969, gives Frans Hansson's dates of birth and death as 25 September 1887 to 28 August 1968.

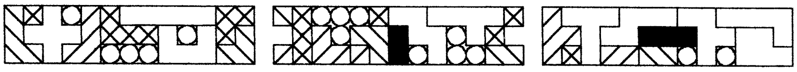
The following are the Type I problems proposed by Hansson.

(1) Arrange the 2-decker pieces of 4 and 5 cubes in $7\times3\times3$.

(2) The 2-decker 5-cube pieces in $7\times3\times3$ with the 8 corner cubes omitted.

(3) All (23) 5-cube pieces in $13\times3\times3$ with centre cube of each end face omitted.

(1)  (2) 

(3) 

(4) All 5-pieces in $5\times5\times5$ with all 8 corners and the centre cubes of two opposite faces omitted.

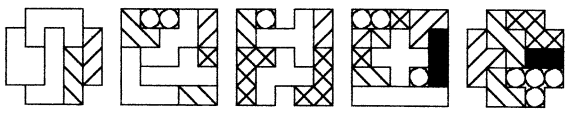
(5) All 5-pieces in $5\times5\times5$ with cross-shaped 5-cube hole in centre of two opposite faces. The real cross-shaped 5-piece being perpendicular to the holes and equidistant.

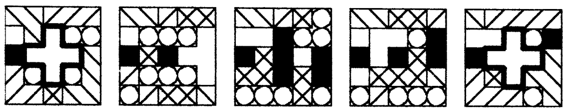
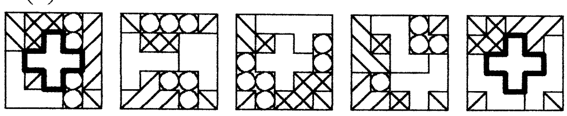
(6) As in (5) but with the cross-shaped parallel to the cross-shaped holes, and equidistant.

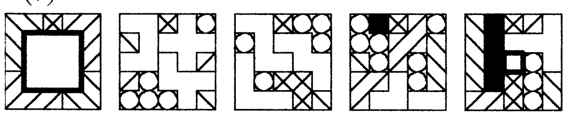
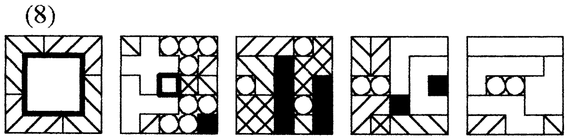
(7) All 5-pieces in $5\times5\times5$ with single cube removed from centre of one face and 3×3 from centre of opposite face.

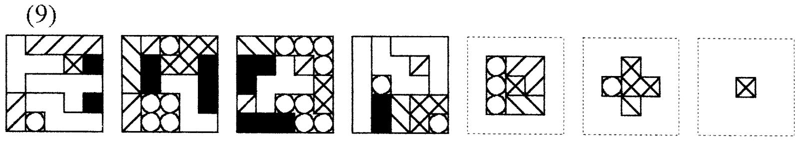
(8) As in (7) but with single cube removed from second layer centrally.

(9) All 5-pieces in $5\times5\times4$ crowned with 3×3, plus 5-cube cross, plus single cube.

(4) 

(5)  (6) 

(7)  (8) 

(9) 

to be continued

The Game of Dodgem

by T. W. Marlow

This game, invented by Colin Vout [1] is simple in concept but difficult to play well. It is played on a 3×3 board as shown in Figure 1. Each side has two counters which start on the squares shown. The players alternate in making single moves and a move consists of a step by one piece to a laterally adjacent square — there are no diagonal moves. White can move up or left or right but never down. Similarly Black moves up and down or to the right but not to the left. Finally, when a White counter is on the top row then a move can take it off the board. The same applies to Black when a counter is in the right-hand column. The aim is to get both counters off the board and the first to do so is the winner. It is possible for one side to be unable to move at its turn — the situation known as stalemate in chess. The side in this position is then immediately declared the winner, i.e. the giver of stalemate is punished.

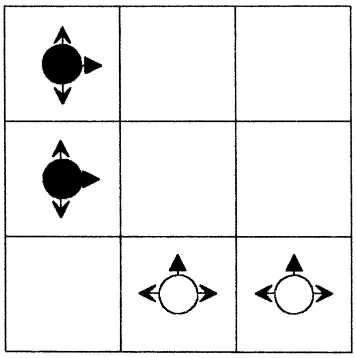


Fig. 1

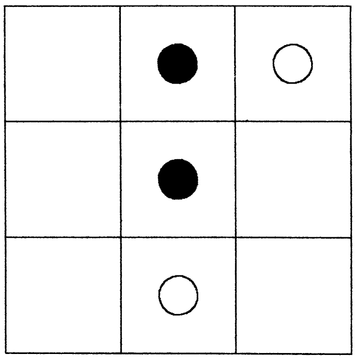


Fig. 2

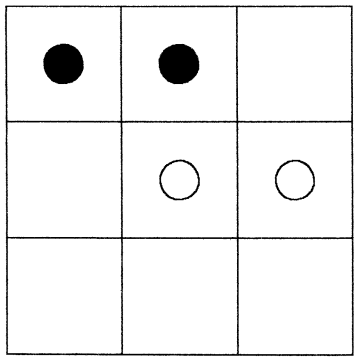


Fig. 3

A full analysis of the game has been made by Berlekamp, Conway and Guy [2] which shows that the first to play, customarily White, can always win. However, despite the simple concept, the strategy is tricky and it is easy to go wrong. An example is the position in Figure 2, where White is to play. The obvious move of taking c3 off is fatal as the play runs: **1. c3 off c3 2. a1 c2 etc.** and Black gets both his counters off first. Correct is **1. a1 c2 2. a2 c2 off 3. a3 b2** and this wasted move by Black gives the game to White. Figure 3 shows a curious situation in which White can win whichever turn it is to move. If White started the game then it must be his turn to play and win by **1. c3 a2 2. c3 off c3 3. b3 etc.** or **2. ... a3 3. a2 etc.** But suppose the players had agreed on Black playing first and so now having the move. Play continues **1. ... c3 2. b3 c3 off 3. c3 etc.** or **2. ... a2 3. b3 off c3 off** and again White wins. Note that in the last sequence **3. ... b2** would have lost by giving stalemate.

The analysis can be used to set up a computerised version of the game, allowing the computer to know all the strategy. Taking White against the computer, and so playing first, is good practice as any inaccuracy will be punished by the machine. I have such a programme but only for the Acorn platform at present. Anyone sending a disc is welcome to a copy.

(The address to write to is: T.W.Marlow, 24 Saxon Way, Saffron Walden, Essex CB11 4EG)

There is an obvious extension of the game to an $n \times n$ board with each side having $n - 1$ counters. It seems that no analysis has yet been made for any n greater than three.

References:

[1]. C. Vout and G. Gray, *Challenging Puzzles*, Cambridge University Press, Cambridge, 1993.

[2]. E. R. Berlekamp, J. H. Conway and R. K. Guy, *Winning Ways*, Vol.2, Academic Press, London, 1982.

Pagoda

by Derick Green

Pagoda is one of the more recent versions of the ancient game of Go-Moku, which is believed to be as old as the game of Go or Wei Ch'i. Go Moku is normally played on a Go board of 18×18 squares. Pagoda, Pente, Go Moku, Renju, Peggity, Peg Five and Spoil Five amongst many others have all been marketed in the West since about 1880 and all belong to the five-in-a-row family of games. All of these games have some slight variation in their rules, for this article however I will concentrate on the rules to Pagoda.

Pagoda was marketed by MB Games in the mid-1980s and was sold in an eye-catching large red, yellow and gold cardboard tube. The board was made of vinyl and backed to make it more durable. Each player was supplied with a set of playing stones, one in red the other pale yellow, and each set was in its own small bag, sealable with a piece of cord. No designer is credited but information on the history of the game states that Pagoda is based on the traditional Japanese game of Ninuki Renju.

Pagoda is played on a 17×17 squared board and each player has 40 playing stones that I will refer to as black and white within this article (the first player using the black pieces). The object of the game is for each player to be the first to make a continuous line of five stones. The winning five-in-a-row must be in a straight line exactly five stones long and may be horizontal, vertical or diagonal. Six or more stones in a row are <u>not</u> winning positions. In Figure 1, a, b and c are winning positions for black, d however does not win.

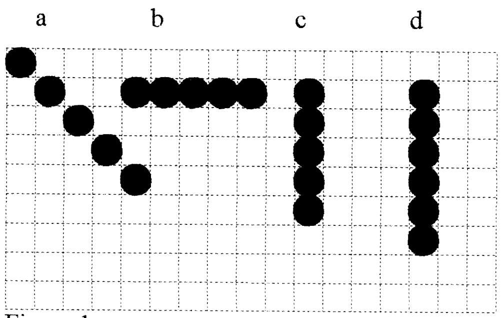


Figure 1

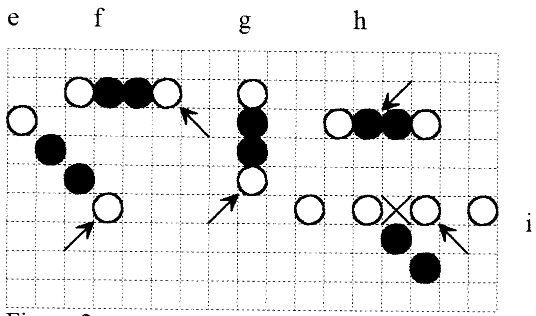


Figure 2

A player may also win by capturing five pairs of the opponent's stones. A player captures a pair of the opponent's stones by trapping them between two of his own. In Figure 2, e, f and g, White captures two black stones in all three examples, with the move marked by an arrow. Note however that a player may place a stone between two opponent's stones without being captured: in example h Black's stones are safe.

The board starts empty and in turn each player places a stone on any vacant square on the board. Note that in Pagoda there is no restriction on the first stone being placed in the centre square. Once placed, no stone may be moved unless removed by capture.

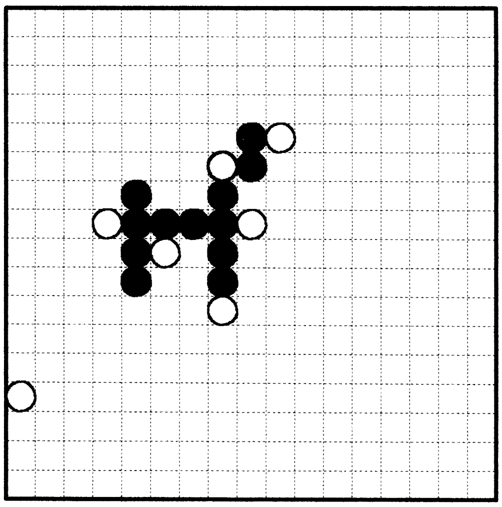
For handicap play the weaker player gives the stronger one, two or three stones. These are then placed on the board before the play begins.

Each game is quite different. However, it is important to attempt to create a chain of four-in-a-row with empty squares at each end. Whichever end your opponent plays to you can play at the other end and win. It is also good sense to avoid placing two stones side by side as it is better to place stones with a space between them. In Figure 2, i, White has played at the position shown by the arrow. If Black plays at X, White captures Black's stones and if Black plays elsewhere, White plays at X and wins.

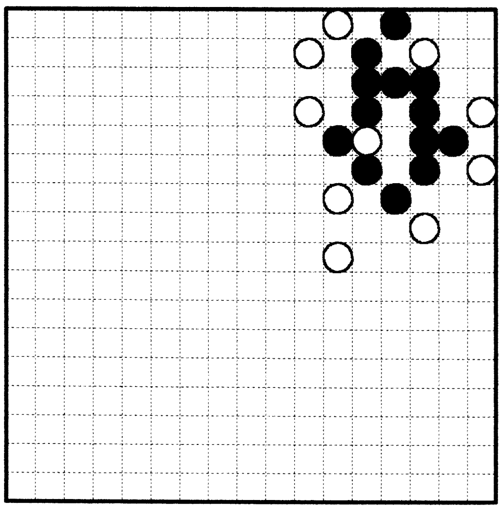
The following are two games from postal play. The notation is simple: square coordinates and a cross for capture of two stones. Both these games are quite short compared to most I have played. They do however show how different games can be, and show how the game is played quite clearly. Draws are possible, but I have never come across one in any of my games or those played within my postal group.

Example Game 1. 1. h8 a4 2. f10 (the one-space jump) g9 3. h10 g11 4. g10 i10 5. e10 d10 6. h9 h7 (blocking here avoids too many pairs, e.g. g11/h11 and h11/i10) 7. h11 h12 8. i13X h12 9. i12 j13 (blocking Black's row of three at f9 would have led to another capture) 10. e8 (multiple threats, row of five to i12, row of three to e10, or a row of four to h8) f9 11. e9X f9 (forced, and the game is lost) 12. e11 and White resigns.

Example Game 2. 1. n11 l9 2. n15 l11 3. l13 n13 4. m12 o10 5. p13 (Black has created an interesting diamond pattern, threatening four-in-a-row in several directions) m14 6. o12 q14 7. l15X o16 8. o14 q12 9. m16 l17 10. n17 k14 (if White allows Black to play here, Black would have two rows of four and a win) 11. m14 n13 12. m15 k16X (any attempt to block Black's potential row of five at m16-m12 by playing at m13 would fail after a capture at o13) 13. m14 m13 14. 013X m13 15. o15 and White resigns.



End of game 1



End of game 2

Solutions to Word Puzzles

Solution to Jigsaw Crossword 1

	O	**X**	I	**D**	E		**G**	**R**	U	**M**	P	**Y**
K		E		A		**Q**		H		E		O
N	I	N	N	Y		**U**	T	O	P	I	A	N
I		O		S		I		M		O		D
F	A	N	A	T	I	C		**B**	A	T	H	E
E				A		K				I		R
	A	**C**	C	R	E	S	C	**E**	N	C	E	
S		O				I		S				**J**
W	A	L	T	**Z**		**L**	A	T	R	**I**	N	E
I		O		E		V		O		H		L
T	E	N	A	B	L	E		**V**	I	R	A	L
C		E		R		R		E		A		Y
H	A	L	L	A	L		**P**	R	I	M	E	

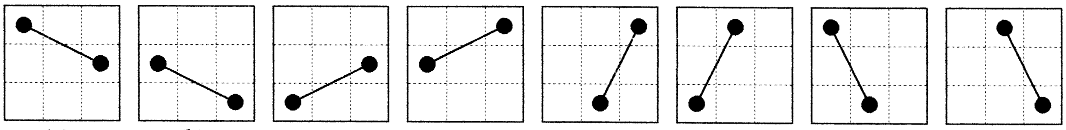
Solutions continue on page 292

Solution to Enigma 2: What connects: (a) a poet's anti-muse, (b) an inappropriately named archbishop, (c) a philosopher's potion, (d) a futurist's underworld, (e) an ex-assistant surgeon's room-mate, and (f) the relief of Lucknow. **Answer LOCK.** (a) 'The Person from **Porlock**' who interrupted Coleridge when he was writing down his dream of Kubla Khan. (b) The late Catholic Archbishop **Warlock** of Liverpool. (c) The **Hemlock** administered to Socrates. (d) The **Morlo(c)ks** were the underground dwellers in *The Time Machine* by H. G. Wells. (e) According to *A Study in Scarlet* Dr Watson was an Assistant Surgeon in the Fifth Northumberland Fusiliers, and later shared rooms in Baker Street with a certain **Sherlock** Holmes. (f) Sir Henry **Havelock** led an attempt to relieve Lucknow during the Indian Mutiny, but was besieged there himself until reinforcements arrived.

Knight's Tour News

Symmetry in Knight's Tours

Amount or Degree of Symmetry. In counting patterns a clear distinction must be made between the pattern considered as a geometrical object, and a DIAGRAM of the pattern. Assuming that a rectangular diagram is printed with its sides in a given orientation, one pattern may nevertheless have eight different appearances:



(a) (b) (c) (d) (e) (f) (g) (h)

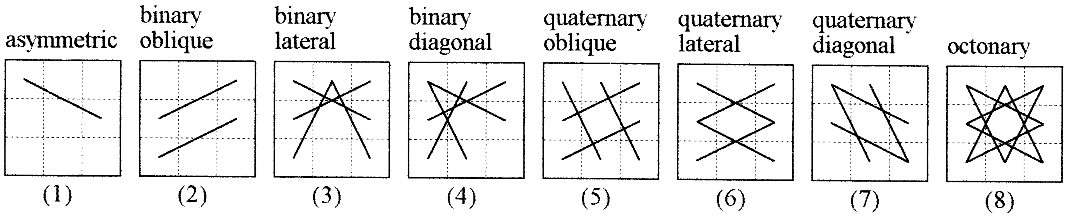
From diagram (a), (b) is formed by a half-turn, or 180° rotation, (c) and (d) by reflection in the horizontal and vertical medians, (e) and (f) by a quarter turn, or 90° rotation, clockwise and anticlockwise respectively, (g) and (h) by reflection in the principal and secondary diagonals.

In the case of an ASYMMETRIC pattern all eight diagrams will be different, as shown above. However, in the case of a SYMMETRIC pattern some diagrams will look the same. If all eight diagrams are the same we call the symmetry OCTONARY; lines through the centre will divide such a pattern into eight congruent COMPONENTS. If the diagrams occur in two sets of four alike the symmetry is QUATERNARY and lines can be drawn through the centre to divide the pattern into four congruent components. If the diagrams occur in pairs alike the symmetry is BINARY and the pattern can be divided into two congruent components. (On the same scheme, asymmetry could be termed *unary* symmetry, since it is formed of one component.)

Instead of measuring the amount of symmetry in a pattern by counting the number n of congruent components we could use $d = \log_2 n$ (i.e. $n = 2^d$) which is the *degree* of symmetry. Patterns with degrees of symmetry 0, 1, 2 and 3 may then be referred to as *nully*, *singly* (or *simply*), *doubly* and *triply* symmetric, corresponding to unary, binary, quaternary and octonary. (Terminology like this was used by Archibald Sharp *Linaludo* 1925.) In cases where n is not a power of 2, d becomes fractional. For example when $n = 3$, as in an equilateral triangle, then $d = 1.585$ approximately.

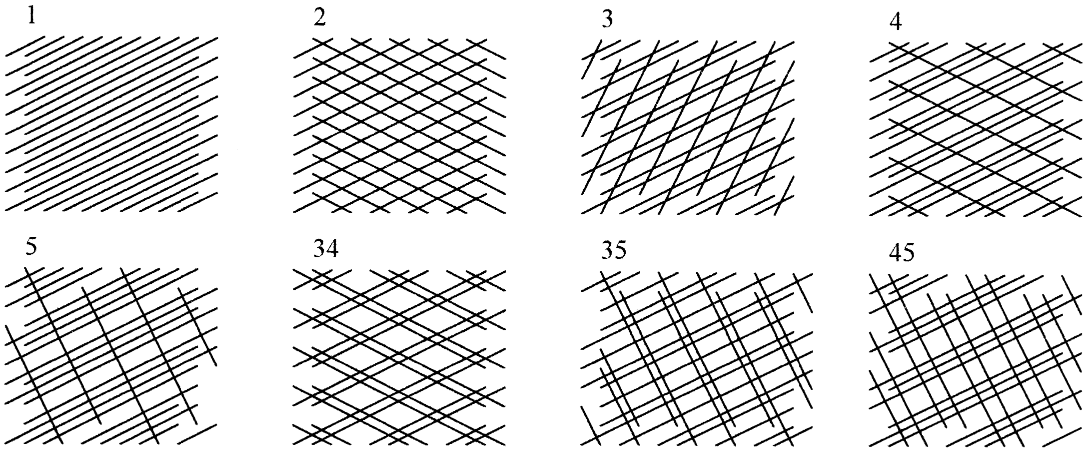
Reflective and Rotational Symmetry. A line in which a pattern may be reflected without alteration is called an AXIS of symmetry (plural AXES). Axes on rectangular boards must either be lateral, through cell centres (odd side) or along the sides of cells (even side), or when the rectangle is a square the axis can be diagonal. A point about which a pattern may be rotated (other than a multiple of 360°) without alteration is called a CENTRE of symmetry; if one exists it is unique. Centres of symmetry of rectangular boards are either at the centre of a cell (odd×odd boards), at the corner of a cell (even×even boards) or at the mid-point of a side of a cell (odd×even boards). A pattern can have both an axis and a centre of symmetry; in fact any pattern that has two axes is necessarily symmetric by rotation about the point of intersection of the axes, the minimum angle of rotation being twice that between the two axes.

Following Bergholt (1917) we call a pattern DIRECT if it has at least one axis of symmetry and OBLIQUE if it has a centre of symmetry <u>but no axis</u> of symmetry. These two classes of symmetry are thus mutually exclusive.



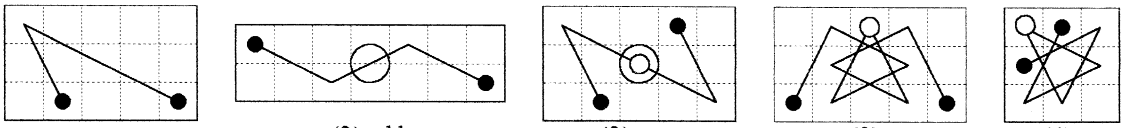
In the case of patterns of knight moves on square boards eight types of symmetry can be distinguished as illustrated above. On rectangular boards we call lines parallel to the edges LATERAL (in preference to *orthogonal* used by Murray and others), lines at 45 degrees to the edges DIAGONAL, and other lines SKEW. In the direct symmetries the axes can be lateral or diagonal. The octonary symmetry is necessarily of direct type, since octonary oblique symmetry requires a 45° rotation and the acute angles between knight moves are fixed at 37° and 53° to the nearest degree (they are the acute angles in a right-angled triangle with sides of 3, 4 and 5 units) and in any case the lattice of squares is not invariant to 45° rotation. Patterns of knight moves, considered apart from the board on which they are drawn, can have skew axes. For example a single knight move has INTRINSIC quaternary direct symmetry, but the cells it links form a pattern that does not reflect in the skew axes, so the knight move and board together do not have this symmetry.

Translational Symmetry. Knight-move patterns can also be constructed that can be repeated at regular intervals so as to cover an area of any size. This type of design is said to exhibit TRANSLATIONAL SYMMETRY of the type seen in wallpaper patterns. The simplest such patterns are those consisting solely of straight lines of knight moves, and according to a study I made in 1985, reported in a Christmas and New Year card I sent that year, there are eight possible patterns of this type, as shown below.



Pattern 1 is the basic pattern formed of one set of close-packed parallels all in the same direction. Patterns 2, 3, 4 and 5 are formed from pattern 1 by rotating every second, third, fourth or fifth line through a suitable angle. Patterns 34, 35 and 45 are similarly formed from pattern 1 by rotating every third and fourth, third and fifth or fourth and fifth pairs of lines. More complex patterns involving non-straight arrangements of knight moves can be used for the central area in large tours, such as our front cover example.

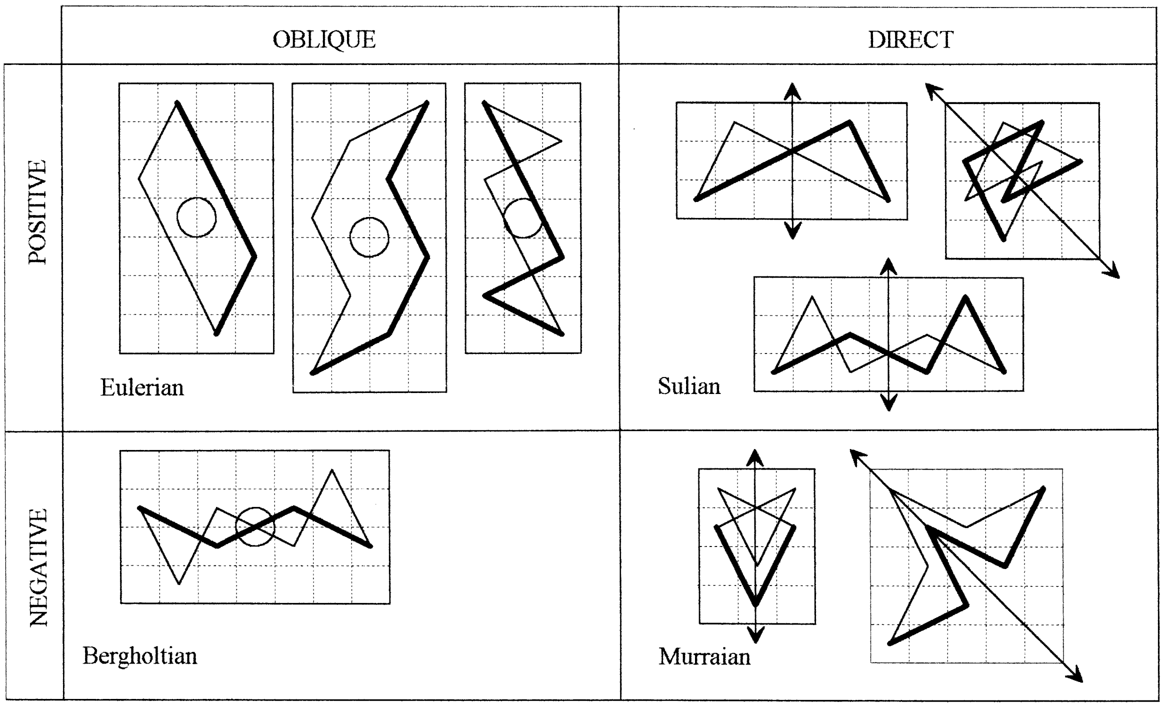
Symmetry in Open Paths. A pattern formed by an open knight path on a rectangular board can exhibit the first four of the above types of symmetry: asymmetric, binary oblique, binary lateral or binary diagonal. For <u>oblique</u> symmetry the mid-point of the path must be at the centre point of the board. Two cases can be distinguished: <u>(2)-odd</u>: The centre of the board is the mid-point of the edge of a cell, and the mid-point of the middle move of the path. The path therefore has an odd number of moves (even number of cells) and uses a board of odd×even dimensions. (2)-even: The centre of the board is the centre of a cell, and the two middle moves of the path meet there, making a two-move straight line. The path therefore has an even number of moves (odd number of cells) and uses a board odd×odd. For direct symmetry the mid-point of the path must lie on the axis, i.e. on a lateral or diagonal line through the board centre. Since a knight move is always skew it cannot cross a lateral or diagonal axis at right angles, so a knight path in direct symmetry must have an even number of moves (odd number of cells), the mid-point of the path being the centre of a cell on the axis. There are two cases; (3)-even: lateral axis, board odd×any, (4)-even: diagonal axis, board square. Only case (4) can occur on a board even×even.



(1) (2)-odd (2)-even (3) (4)

Numbering: If the cells in an open path are numbered 1 to E then the numbers in the end cells add to $E + 1$, and in the case of a symmetric open path the numbers in any pair of cells related by the rotation or reflection (i.e. cells x moves from one end, numbered $x + 1$, and x moves from the other end, numbered $E - x$) also add to the constant value $E + 1$. When the number of cells E is odd (or the number of moves $E - 1$ is even) there is a middle cell in the path, and the number on it is $(E + 1)/2$, which is the average of all the numbers 1 to E.

Symmetry in Closed Paths. All eight of the types of symmetry (1 – 8) listed above can occur in patterns formed by closed knight paths on rectangular boards. If A and A' are a pair of corresponding cells in a closed path with binary symmetry then the closed path consists of two joined open paths A—A'. For binary symmetry these two paths must either be congruent, one being the rotation or reflection of the other (we call this POSITIVE symmetry), or each must itself be symmetric, with the same type of symmetry (NEGATIVE). If B and B' are another pair of corresponding cells then in the positive case the points occur in the cyclic sequence ABAB... along the path, but in the negative case in the sequence ABBA... (or AABB... which is the same).

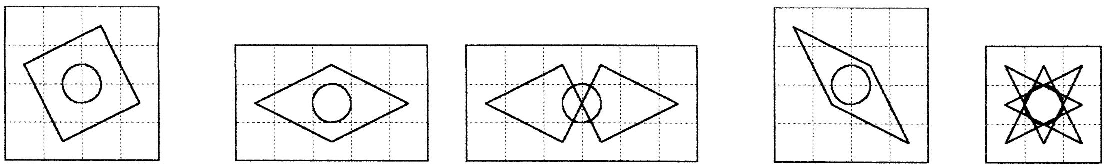


We thus have four types of binary symmetry in closed paths. Namely: positive oblique = **Eulerian**, negative oblique = **Bergholtian**, positive direct = **Sulian** and negative direct = **Murraian**, so-called since examples were shown by Euler, Bergholt, Suli and Murray.

These can be further subclassified according to whether the number of moves in the connecting paths A—A' are even or odd, or equivalently, whether the centre is on a cell corner or centre (even) or cell edge (odd) or whether the axis is through cell centres (even) or cell edges (odd). Eulerian and Sulian symmetries can be even or odd, but Bergholtian must be odd and Murraian must be even.

Numbering: If we number the cells of a closed path 1, 2, ..., E (beginning at any cell and proceeding in either of the two directions) and the cell corresponding to cell a under binary symmetry is the cell numbered a' then the cell corresponding to $a + 1$ will be either $a' + 1$ or $a' - 1$. These correspond to the cases of positive and negative symmetry defined geometrically above. In the positive case numbers in corresponding cells have constant difference $a' - a$, while in the negative case they have constant sum $a' + a$. In the positive case we always have $|a' - a| = E/2$ throughout. In the negative case, when the move 1 to E passes through the centre we have $a' + a = E + 1$ throughout, but if the origin of numbering is differently placed there will be two different values for the constant $a' + a$ along different sections of the tour.

In the case of quaternary symmetry the paths A—A' (where A' is the cell related to A by 180° rotation) will be congruent and formed of two congruent parts. Examples:



(5) quaternary oblique (6) quaternary lateral, positive and negative (7) quaternary diagonal (8) octonary

Symmetry in Rectangular Tours. The arguments in the following theorems were sketched in my article in *Chessics* 22 (1985) but are given here in more detail.

THEOREM 1. An open knight's tour of a rectangular board can only have oblique binary symmetry, and at least one side of the board must be odd. *Proof*: (a) For direct symmetry of an open path with an odd number of moves the middle move would have to cross the lateral or diagonal axis at right angles, which is impossible for a knight's move, which is skew. (b) For direct symmetry of an open path with an even number of moves the mid-point of the path must be the centre of a cell on the axis. The path cannot enter any other cell on the axis since the other half of the path would by symmetry also enter the same cell, thus making a closed path. For the path to be a tour all cells on the axis must be entered; thus there can only be one cell on the axis, so the board must be $1\times n$, but no knight moves are possible on such a board. (c) For oblique symmetry of an open path the mid-point of the tour must be either the end-point or the mid-point of a knight's move, i.e. a cell centre (even number of moves) or the mid-point of the side of a cell (odd number of moves). Thus the board cannot be even $\times$even, since at the centre of such a board the corners of four cells meet, so the board must have at least one side odd. (d) The oblique symmetry canot be quaternary since that would require four end-points.

THEOREM 2. A rectangular closed knight's tour requires a board with at least one side even. *Proof*: On a chequered board each move is to a different colour, so an even number of moves is needed to bring the knight back to the cell on which it started; and the same number of cells, one as the destination of each move. So at least one side of the board must be even, since on an odd$\times$odd board the number of cells is odd.

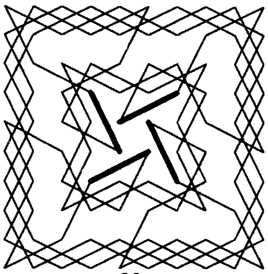
THEOREM 3. A rectangular closed knight's tour with direct binary symmetry requires a board with one side odd and the other singly-even, and cannot show direct quaternary symmetry. *Proof*: (a) If A and A' are corresponding cells, not on the axis of symmetry, in a tour with direct symmetry, then it consists of two paths A-A'. (b) The two paths cannot be themselves symmetric, with their mid-points on the axis, since there would then only be two cells on the axis and the board would be $2\times n$ on which no knight's tours are possible. In other words the symmetry must be Sulian and not Murraian. (c) The two paths A-A' must be reflections of each other in the axis and there must be no cells on the axis. The axes are therefore lateral, not diagonal, and the cells A and A', being equidistant from the axis and on opposite sides of it, must be of opposite colour when chequered. (d) The paths A-A', connecting cells of opposite colour, are therefore of an odd number of moves. The tour therefore occupies twice an odd number of cells. The board must therefore be odd×singly even, i.e. $(2m+1)\times2(2n+1)$, that is $(2m+1)\times(4n+2)$. The axis is the bisector of the even side. (e) It follows that direct quaternary symmetry is impossible in a rectangular knight's tour since the side parallel to an axis must be odd and the side perpendicular to it even, and if this is true for one axis it cannot be true for the perpendicular axis.

THEOREM 4. A symmetric rectangular closed tour on a board even×even can only be Eulerian, while on a board with one side a multiple of four and the other odd it can only be Bergholtian. On a board singly-even×odd all three types, Sulian, Eulerian and Bergholtian may be possible. *Proof*: (a) As proved above direct symmetry requires singly-even×odd and so is impossible on both types of board mentioned. (b) Bergholtian symmetry reqires a board odd×even since the centre point must be the mid-point of a knight's move and therefore the mid-point of the side of a cell , so on the even×even board only Eulerian symmetry remains (it can be binary or quaternary). In Eulerian symmetry on an odd×even board the corresponding cells A and A' (on a line bisected by the centre point of the board) are of opposite colour, so the number of moves in A-A' is odd and the number of moves in the tour is twice this, i.e. singly even. Therefore if the even side is not singly-even, i.e. if it is a multiple of 4, only Bergholtian symmetry remains feasible.

THEOREM 5. A rectangular closed knight's tour with oblique quaternary symmetry requires a square board with singly-even side. *Proof*: Such a tour consists of four equal paths, the board must be square, for the 90° rotations to leave it invariant, and even-sided, for closure. On an even-sided square the 90° rotation of a cell is of opposite colour to the original cell, and so the path joining them is of an odd number of moves. The whole tour is thus 4 times an odd number of moves, i.e. it contains 2 as a factor only twice. The square boards on which quaternary tours are possible therefore have side: 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50 and so on.

10×10 Knight's Tours with Quaternary Symmetry

The five solutions for the 6×6 board (known to deHijo 1882) were diagrammed in the Bergholt article in *GPJ*13 p.217. Examples 10×10 were given by Euler (1759), Bergholt (1916), Kraitchik (1927), Murray (1942) and Cozens (1960). Here are some of my own construction: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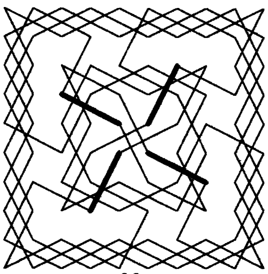
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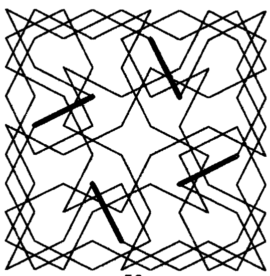
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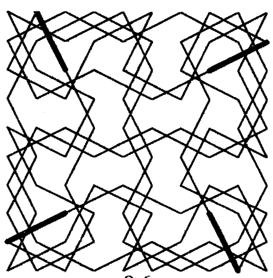
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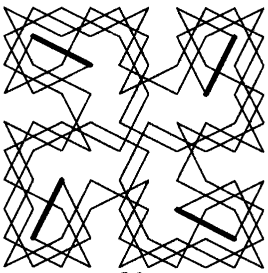
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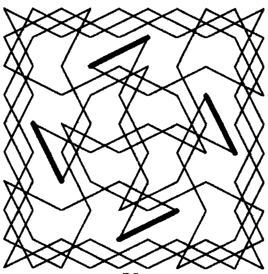
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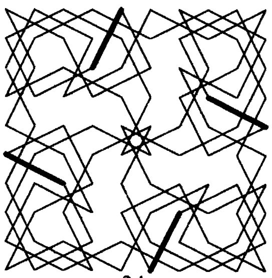
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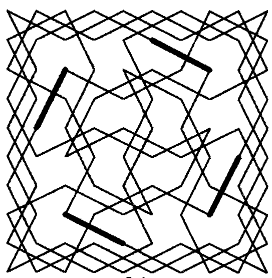
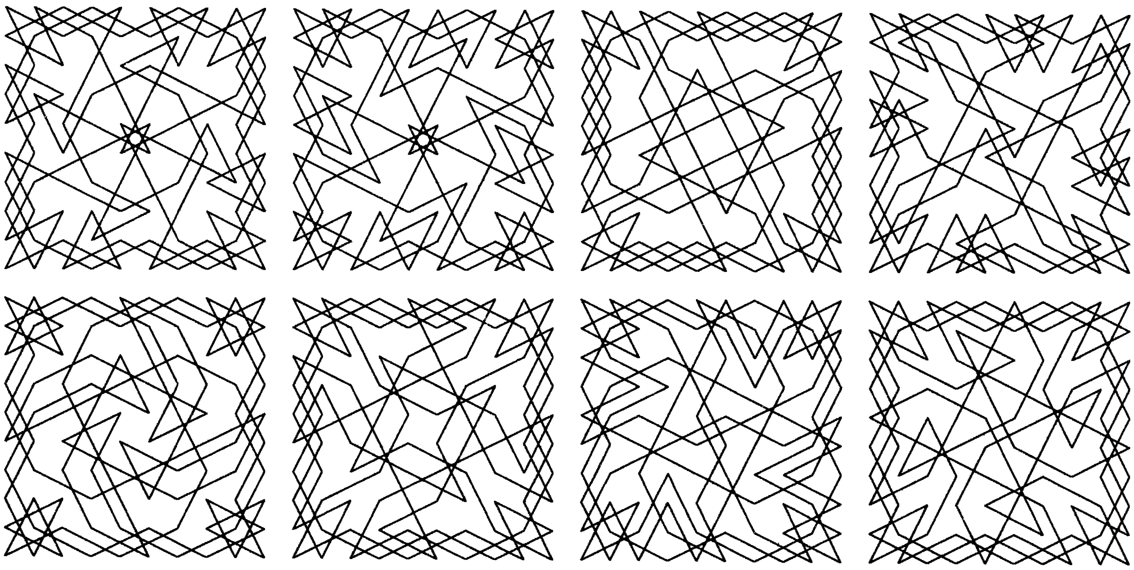
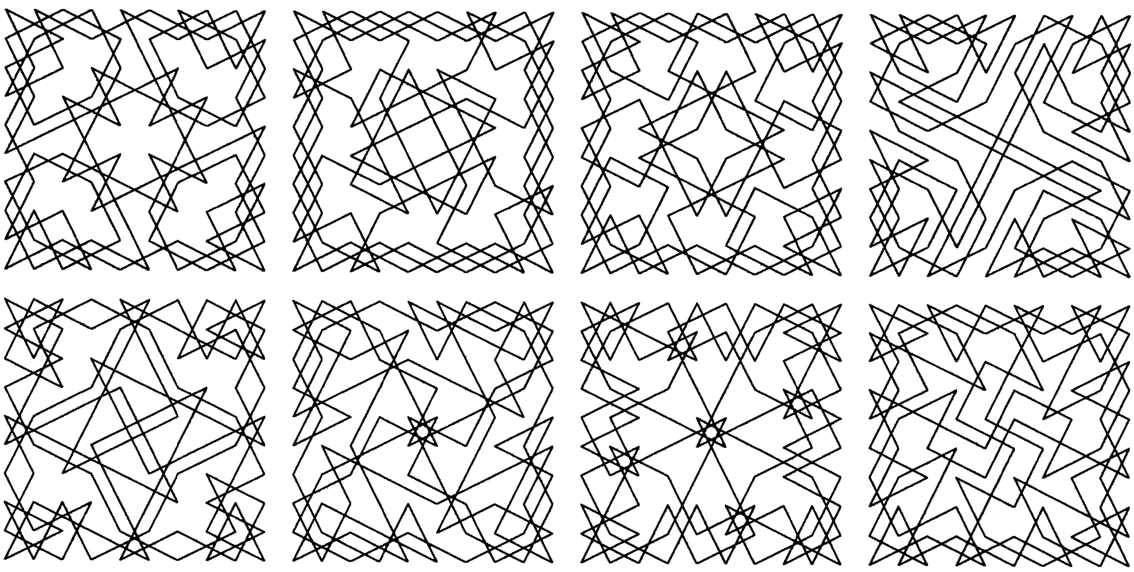

34

The family of 10×10 tours with four slants and in quaternary symmetry, has 408 geometrically distinct members (2×32 + 3×34 + 2×52 + 2×86) by my calculations (not checked). The above diagrams show one tour for each of the 8 workable positions of the four slants. The enumeration is made reasonably simple, since each 25-cell knight path joining the slants is equivalent to a wazir tour of a 5×5 board; its ends occuring on the pair of cells determined by the slants.

In the first issue of *Chessics* (1976) I proved that every 8×8 knight's tour contains a right angle, but in *Chessics* 22 (1985) I showed that this result does not hold true on the 10×10 board, by giving a tour, in binary symmetry, which had no right angles. Subsequently I have enumerated 40 examples in quaternary symmetry, of which I give a selection, one for each of the central angle formations; apart from the first two which are the only ones <u>not</u> including the d8-h6, h7-f3, g3-c5, c4-e8 square.



The following eight were constructed just for the interest of their patterns. The central St John or Maltese Cross is impossible to achieve on the 8×8 board. The first three date from 1986.



The last tour shown above is an example of a 'celtic tour', so-named by Prof. D. E. Knuth, that is a tour in which no 'size 1' triangles occur. Size 1 triangles are the smallest possible formed by knight moves, for example by a1-c2, b2-d1, b1-c3, and have an area 1/120 of a board-square. The triangle formed by a1-c2, b1-c3, c1-b3 is size 2, which has an area 1/30. (See Puzzle Questions p.296: Puzzle 32. Knightly Triangles for more on this.)

The Enumeration of Closed Knight's Tours

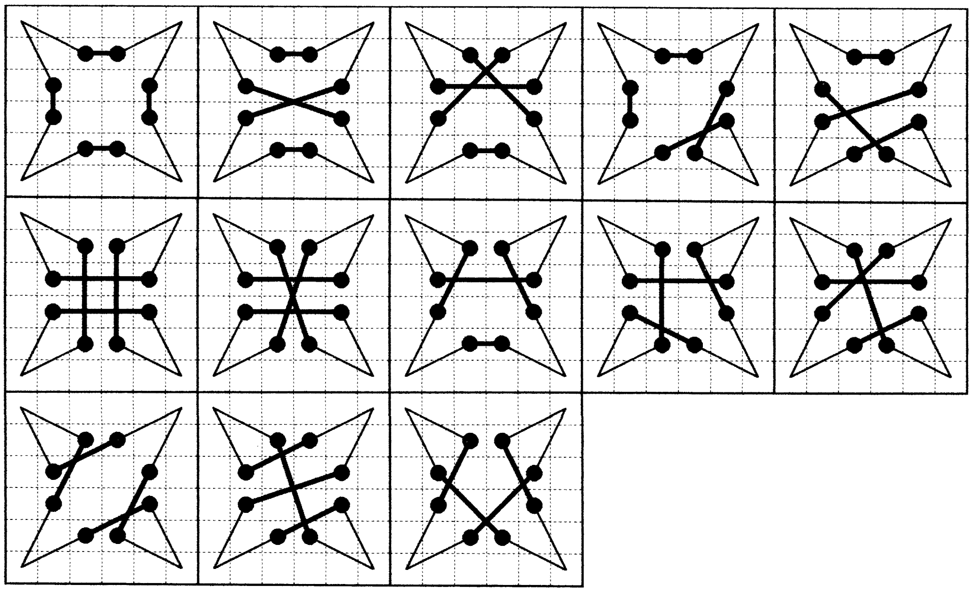
New Results by T. W. Marlow

Tom Marlow has applied his computer to filling two gaps in the table of numbers of closed knight's tours in the last issue, finding: (a) **197064** symmetric closed tours on the **6×10** board and (b) **415902** closed tours with quaternary symmetry on the **10×10** board (more than twice W. H. Cozen's estimate). These are the numbers of geometrically distinct tours. He describes the process as follows:

The method of search was, in each case, to start a tour at c2, proceed to a1 and then to b3, numbering these squares as 1, 2, 3. The chain was then extended to a length of 30 cells in the 6×10 case or 25 cells in the 10×10 case and at each step the corresponding cells, after 180° rotation in the first case or after repeated 90° rotation in the second case, were marked as unavailable. At this stage a check was made as to whether a link was possible, to b8 or i3 on the 6×10 and to h5 on the 10×10, and if so a tour recorded. After this, or at any previous dead end, the programme systematically back-tracked and tried a new step forward until all steps from b3 had been tried. The total was then recorded. The actual totals recorded were 394128 and 831804 but each geometrically distinct tour can be diagrammed in two forms in each case, one a reflection of the other in the vertical median. The routine described counts both versions, so the totals must be halved. (Because of the 180° rotational symmetry of the tours, reflection in the horizontal median has the same effect as reflection in the vertical median.) In the case of the 10×10 board the same count was found by a second (more time-consuming) check as follows. The start was made at b1 and extended by the same method until a link to j2 could be sought. A link to a9 would produce a tour that is a 90° rotation of a tour linking at j2 so is not geometrically different and should not be counted. This method produced the same count.

In the 10×10 case Mr Marlow also notes: "Tours fall into two types. The first links b3 to b8 and then, via a10, c9 to h9, i8 to i3 and h2 to c2 — call this a circular tour. The second type runs b3 to i3, h2 to h9, i8 t0 b8 and c7 to c2 — call this a lop tour since it loops the lop at each corner. The count of the two sorts is as follows: circular tours 206937, loop tours 209065, total 415902."

This interesting distinction between circular and loop tours was new to the editor, and led him to consider the general case of tours (not necessarily symmetric) on rectangular boards. There appear to be 13 types of connection, as shown below. Only the three types in the first column can occur in tours with Eulerian symmetry; the first two are the circular and loop types mentioned above.

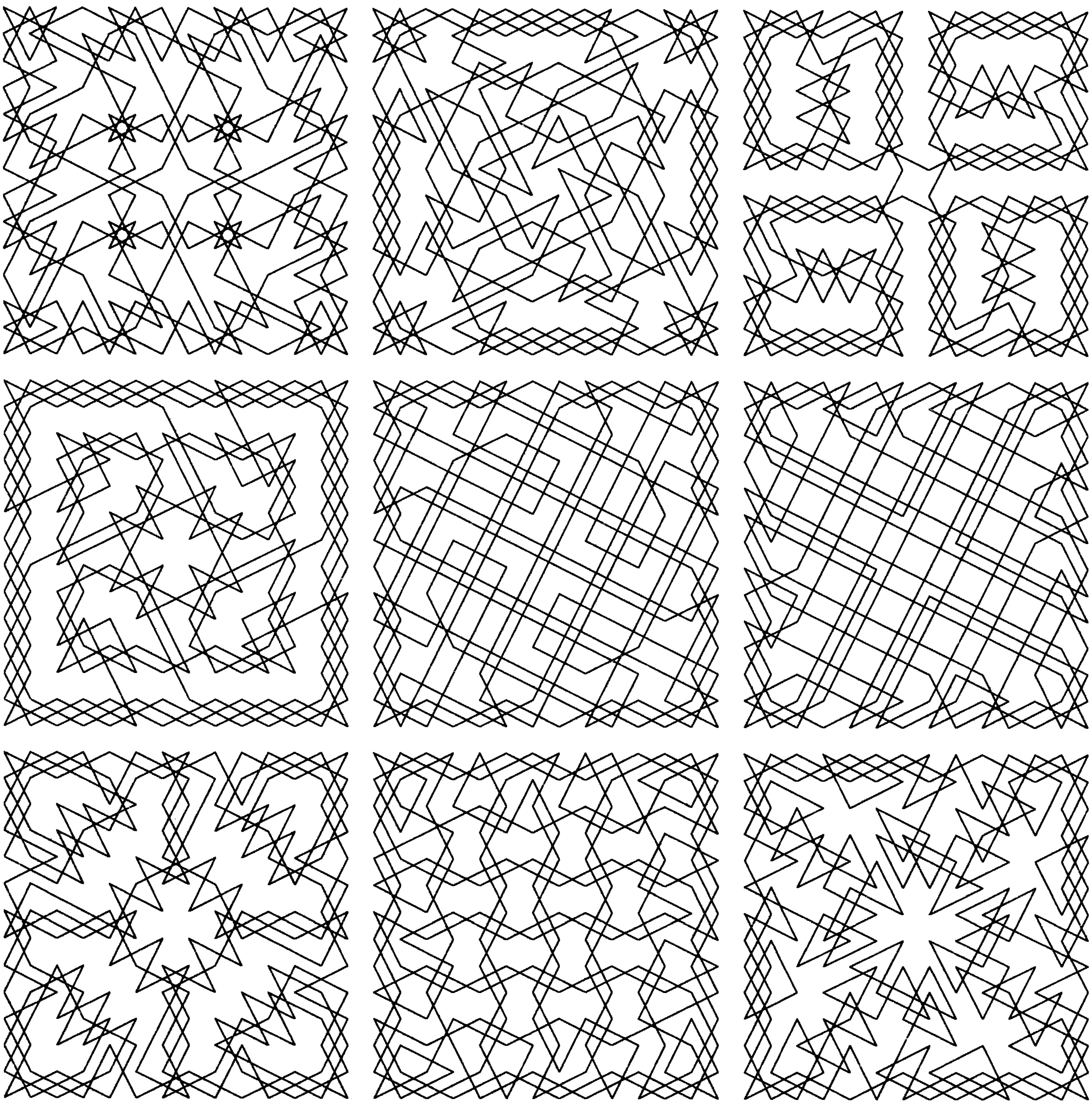


Only the types in the second column can occur in tours with Bergholtian symmetry, and only those in the third column in those with Sulian symmetry. The dark lines represent sequences of connecting moves that can be quite irregular. Two connections shown by dark lines of the same move-pattern do not necessarily represent congruent sequences of moves.

14×14 Knight's Tours with Quaternary Symmetry

By G.P.Jelliss

This board, like the 10×10 and 6×6, admits complete tours with quaternary symmetry, but the earliest printed example I have come across is the first diagram below by Archibald Sharp from his book *Linaludo* (1925). The next two are from Kraitchik (1927). Other examples like the third here, formed of four 7×7 open tours joined together, were given by Murray and by Kraitchik, but this board offers scope for much more interesting constructions. The other six are my own, the first from 1986.



My first example places the 10×10 tour incorporating a Maltese cross within a complete border braid as frame. The next two attempt tartan plaid effects, based on the '35' and '45' arrangements of nightrider lines (where every 3rd and 5th, or every 4th and 5th line in a set of parallels is turned at right angles). The fourth is another Maltese cross design. The fifth has a central mosaic pattern. The sixth includes sequences of seven successive three-knight-move triangles in the central region.

For an 18×18 tour with oblique quaternary symmetry see our front cover illustration.

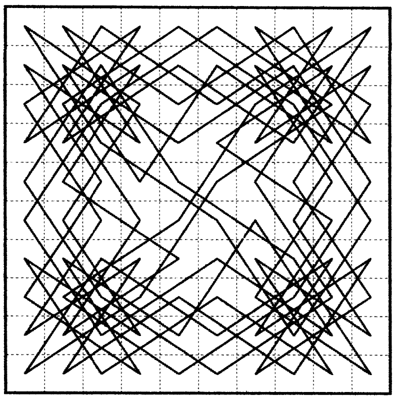
Longer Leaper Tours with Quaternary Symmetry

Including New Results by T.W.Marlow

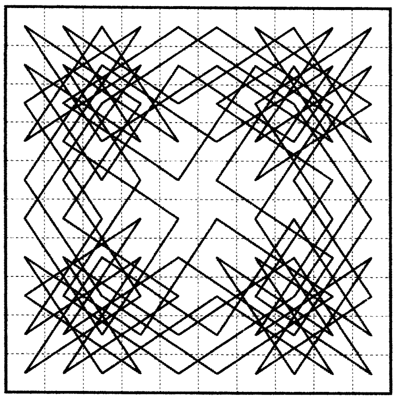
Closed tours by Zebra ({2,3}-mover) and Giraffe ({1,4}-mover) on the 10×10 board were found by A.H.Frost as long ago as 1886 (in M.Frolow, *Les Carrés Magiques*, Paris 1886, Plate VII). Examples with 90° rotary symmetry were first shown by myself* and W.H.Cozens in *Chessics* 1978.

Tom Marlow has (March 1998) applied the same computer program as he used for the count of 10×10 knight tours with quaternary symmetry, with minor adaptations, to count the Zebra and Giraffe tours, finding only 6 geometrically distinct Zebra Tours (2 circular, 4 looping), and 50 geometrically distinct Giraffe tours (24 circular, 26 looping). Here are the six Zebra tours.

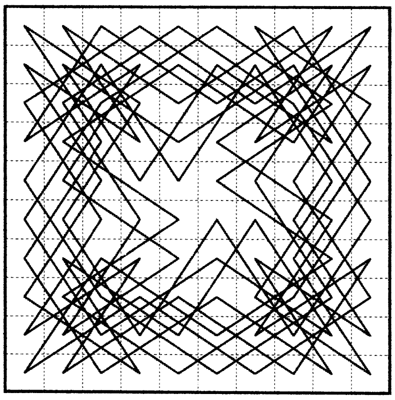
Marlow 1 (1998)



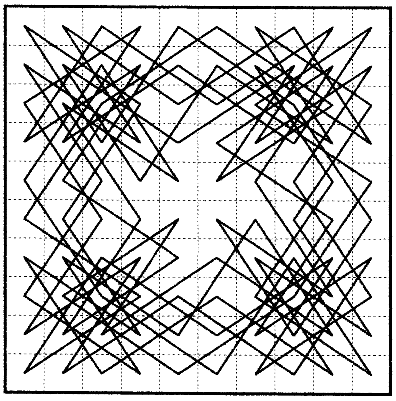
Marlow 2 (1998)



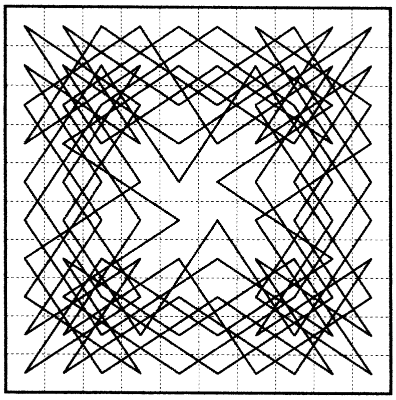
Marlow 3 (1998)



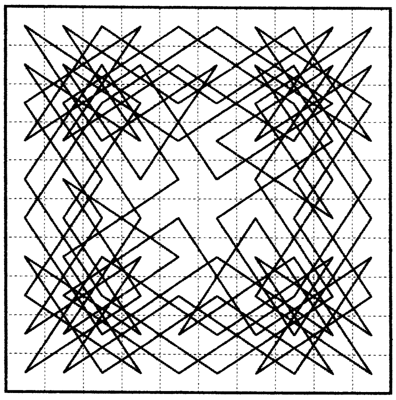
Marlow 4 = Cozens 1978 (reflected)



Marlow 5 (1998)



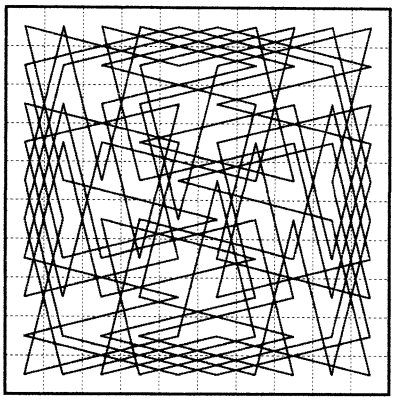
Marlow 6 = Jelliss 1978 (reflected)



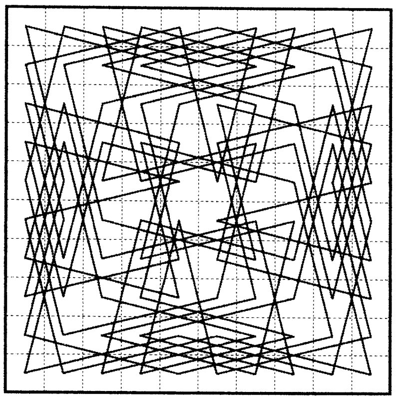
Tom Marlow presented the tours in the form of arrays of numbers (1 at c4, 2 at a1, 3 at d3, and so on). In converting these tours to diagram form using the Drawing facility in AmiPro it is only necessary to join up the dots for the first 25 moves, then this quarter-tour can be copied and rotated 90° repeatedly to give the other three quarters, which join together seamlessly. (*see following note)

*My recollection, and my notes, said that my above Zebra tour was published in *Chessics* 1978, but on referring back I find only the Cozens solution given there, so I suppose I suppressed my example to give the limited space to another contributor. Here are two 10×10 quaternary Giraffe tours.

Cozens (1978)

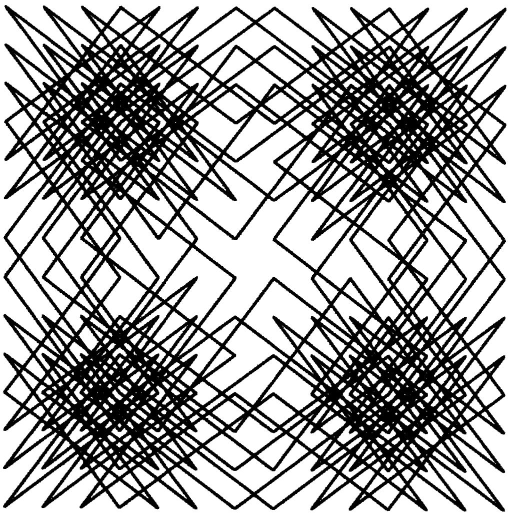


Marlow (1998)

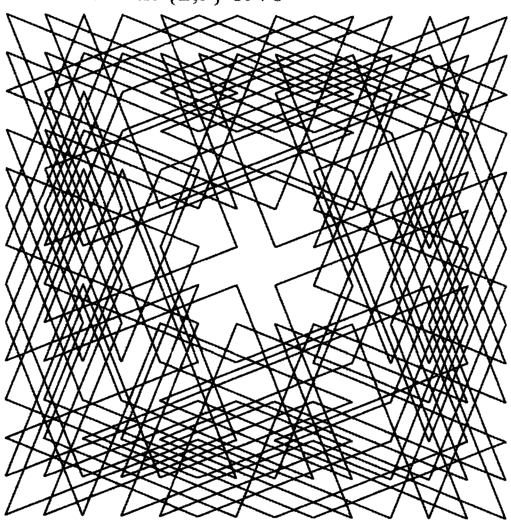


Here are similar quaternary tours by {3,4}-mover and {2,5}-mover on the 14×14 board given by T.H.Willcocks in *Chessics* 1978. For reasons of clarity, the hand-drawn diagrams used there only showed one quarter of the path. Now that drawings can be made with computer precision we are able to show the full tours with reasonable clarity, but I have left out the background squares in these cases.

T. H. Willcocks {3,4} 1978



T.H.Willcocks {2,5} 1978



H. J. R. Murray's History of Magic Knight's Tours

with commentary by G. P. Jelliss

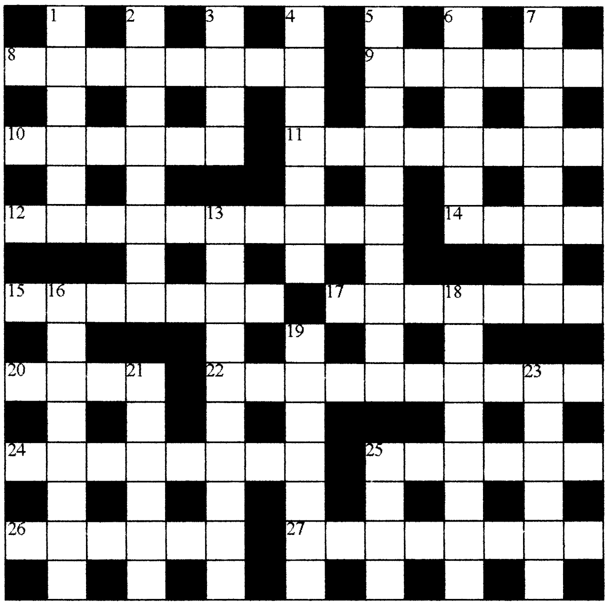
Murray's Own Work, and Subsequent Work on Magic Knight's Tours

Just a few notes to round off this series for the present. H. J. R. Murray added a further eight magic knight's tours to the collection of 8×8 tours, and also extended his methods to larger boards.

Shortly after Murray's death in 1955 H. E. de Vasa and T. H. Willcocks applied Murray's methods and developed other methods, to construct the first diagonally magic knight's tours (i.e. with the two main diagonals also adding to the magic constant) on boards of sizes $4n\times4n$ with $n > 3$.

Following the publication in *Chessics* in 1986 of the catalogue of 8×8 magic knight's tours, and details of their construction by the method of quartes, T. W. Marlow applied a computer to the task of enumerating all tours of regular quartes type and found five new tours which were published in *The Problemist* in January 1988. This work has recently been independently confirmed using more powerful computer methods by Michael Gilpin of Michigan Technological University.

Cryptic Crossword Number 11 by *Querculus*



ACROSS

8. Instrument of Zoroastrian glee. (8)
9. Word of mouth builds confidence in me. (6)
10. Appointment in the sin-bin. (6)
11. Instrument produces change in unfashionable psychic. (8)
12. Subterranean pendant to static tale. (10)
14. Do two down in operatic tale. (4)
15. Royal Naval float sustains damage in bow. (7)
17. Ayre left Wagnerian theatre for ism in heavy metal at 83. (7)
20. Sharp as a Don Rodrigo. (4)
22. Engels sent round: Senior bids sit you down in it as you like it. (10)
24. King found in tomb with one instrument. (8)
25. Mother Carey's chickens are, in jest or myth. (6)
26. Dr Carey's flock ascend in the pink. (6)
27. Drum lice ruined instrument. (8)

DOWN

1. In favour of healthy gain. (6)
2. Call round iron instrument maker. (8)
3. Giant sublet brogues? (4)
4. Giant, like O'Connor and Lynam together. (7)
5. Time alarm goes off, after one, is irrelevant. (10)
6. Third class depression. (6)
7. Article about wind instrument. (8)
13. Hundred hover round short angel, passing the baton. (10)
16. An angel's instrument? (8)
18. Make nothing from old Minoan instrument of Corelli. (8)
19. King's epithet confused due to yarn. (7)
21. Murder mother in a public place. (6)
23. One who goes to law about Frenchmen in season. (6)
25. QC kills, without law and order. (4)

Solutions to Word and Letter Puzzles in GPJ 15

See also p.281

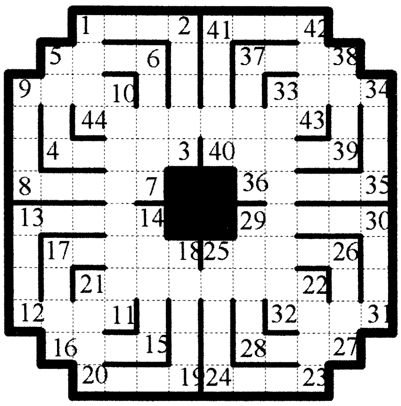
Solution to Clueless Crossword

	Q			M		F		V	
J	U	X	T	A	P	O	S	E	D
	I			W		G		R	
	Z			K				B	
			L	Y	N	C	H		

Solution to Tangleword Puzzle 2.

	W	O	R	D	E	V	I	L	
T	O	U	R	R	U	C	O	O	K
N	N		H	A	Q	I		F	I
E	S	R	E	W	O	R	H	T	N
R	E	V	O	L	T	O	V	I	D
E	X	I	S	T	D	E	C	O	Y
G	P	A	T	H	E	T	I	C	E
A	M		A	U	Z	E		O	T
P	I	R	T	M	A	M	A	M	I
	L	L	U	B	R	A	J	A	

Tangleword Puzzle 3 to solve

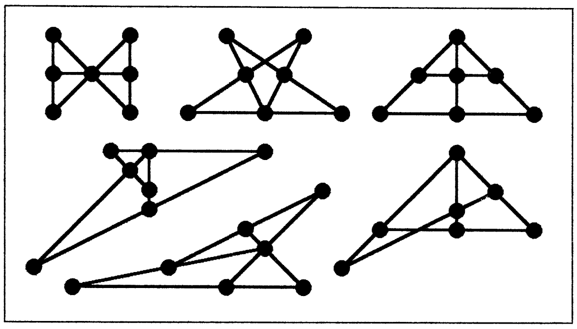


Clues
1-4. Three oxygenate soil instrument
5-8. Big cat songs remove foliage
9-12. Away strict one: walk thataway
13-16. Mollusc dig: leave, throw up
17-20. Muscle woman: plant flyer
21-24. Dainty jug: lasso, throw out
25-28. Crown unfortunately ego talent
29-32. Competitor, idle, deviates flight
33-36. Centre crown dazzle acquires
37-40. Errors discredit ash pole
41-44. Shipwreck boss noble ruffian

Puzzle Answers

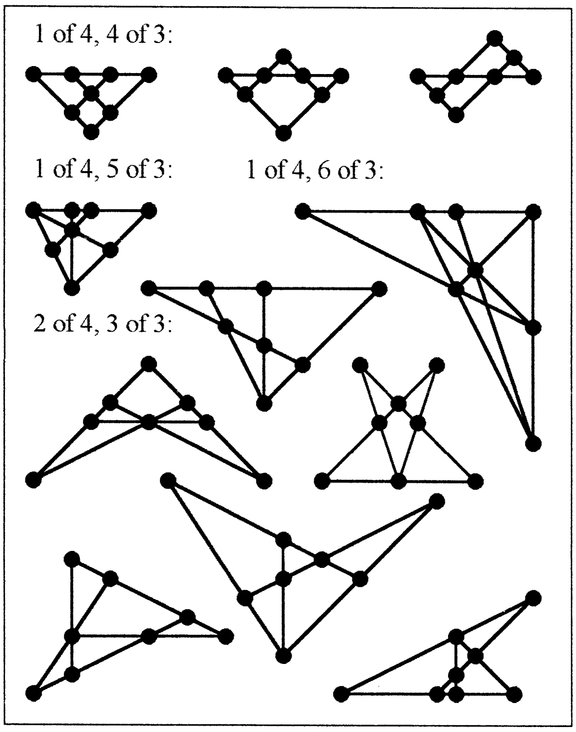
17. Plantations

There are 6 ways of planting 7 trees in 5 lines of 3, with every tree in at least two lines:



No arrangement incorporating a line of 4 is possible since the other 3 points form a triangle whose sides can only pass through 3 of the 4 points on the other line, so one of the 4 points on this line has no other line passing through it.

Arrangements with a line of 4 are however possible with 8 trees. The 6 plantations with 2 lines of 4 correspond to the 6 plantations of 7 trees in the first diagram, above.



18. Cryptarithm

The solution to (fast)² = unfasten that I found is **3792² = 14379264**.

This result came into my mind in a rather strange way that seemed inexplicable. The date when I thought of it was 19/9/1997 which is an unusually repetitive date, but doesn't seem to have any connection with the result. I was running over in my mind the argument of Professor Cranium in *GPJ*13 (p.218) about the side and diagonal of a square not being 'incommensurable' because both can only be determined to the same degree of exactness. If the side is 1.000 then the diagonal is 1.414 to the same number of decimal places. I wondered if I could recall what the next four digits of the square root of 2 were and 1.4143792 suddenly came into my head, from whence I know not. The four digits 3792 stuck in my mind, and I felt sure they were wrong, and so I got hold of a pocket calculator nearby and checked what the square root of 2 was, namely 1.4142136. So what was 3792, if anything? I checked the square root of 3, finding 1.7320508, and this also bore no relation to the number thought of. Then I thought, perhaps 3792 is the square root of something else? So I squared it on the calculator and to my surprise the same digits reappeared in the middle of the square! The words 'fast' and 'unfasten' were fitted to the result later.

I found this sudden unexplained appearance of a numerical result out of nowhere into my head rather disturbing. Usually finding such results takes a good bit of hefty reasoning and several pages of calculations. To have it presented to one out of the blue is a bit startling. It also goes against the Protestant Work Ethic — finding such results should not be so easy, they should be the reward of much mental labour.

Had I recalled this result after reading it somewhere? I looked up 3792 in *The Penguin Dictionary of Curious and Interesting Numbers* by David Wells, but it is not mentioned there. (See the Stop Press note added at the end of this article for the latest development.)

I thought that perhaps there might in fact be thousands of answers to the problem and that I'd just hit on one, but after testing some other groups of numbers chosen at random this didn't seem to be the case. Ted Clarke, of *WordsWorth*, phoned soon after publication to say that he had applied his computer to the problem and could confirm that the answer was unique.

Then Tom Marlow wrote at the end of January 1998 as follows: "I found the solution ... by hand. My computer tells me that there are 9 four-digit numbers as below that are embedded in their own squares. Most are trivial and the fourth is clearly the only solution to your delightful discovery."

1000^2 = **1000**000
2500^2 = 6**2500**00
3760^2 = 141**3760**0
3792^2 = 14**3792**64
5000^2 = 2**5000**000
6000^2 = **3600**0000
6250^2 = 390**6250**0
7600^2 = 57**7600**00
9376^2 = 8790**9376**

The 9376 is the only other case without terminal zero. Numbers of this type, which reappear as the last digits of their square are known as **automorphic** numbers. A. H. Beiler in *Recreations in the Theory of Numbers* (Dover Publications 1964, p.147) says that the last digits of automorphic numbers in base ten, apart from 0 and 1, must be in the following two sequences: ...6259918212890625 or ...3740081787109376.

In August 1998 I mentioned the puzzle to a chess correspondent, Fabio Dulcich in Italy, and he also solved it using a computer. Later he reported four new results. He found the first two "in an old file note" and the other two by a new computer search. These are all the results less than 10^9, excluding numbers ending in zero.

495475^2 = 245**495475**625
971582^2 = 943**971582**724
66952741^2 = 4482**66952741**3081
177656344^2 = 3156**177656344**6336

It seems that Tom Marlow also found the first two cases in the above list, sending his results to *Computer Weekly* in April 1998. This resulted in the editor of that magazine presenting the original cryptarithm (in the inferior form brok^2 = unbroken), alas without acknowledgement of *The Games and Puzzles Journal* as the source.

Stop-Press: Having received a catalogue from the Eagle Bookshop (see p.276) I visited them on 25/3/1999, and found a copy of Martin Gardner's *Mathematical Carnival* (Pelican 1988), which I do not recall having read before, and on p.169 in chapter 13 on Random Numbers is the 3792 result, quoted from Birger Jansson's book *Random Number Generators* (1966). Apparently John von Neumann (1903–1957) had proposed a method to generate random numbers by choosing a number, squaring it, taking the middle digits of the square, squaring that, and taking the middle digits of the result, and so on; but if you start with 3792 the result is anomalous! It is mentioned that "The same thing happens if you start with such six digit numbers as 495475 and 971582." (Hence Dulcich's 'old file note'.)

19. The Naming of Numbers

The first ambiguity is 12 = (2^2)×3 = (tat)ek and 64 = 2^(2×3) = ta(tek), where ^ means 'to the power'. The number of the Beast in the revised system is $666 = 2\times3^2\times37 = 2\times3^2\times(2^2\times3^2 + 1)$ = tekat-el-tatekati (the hyphens help pronunciation).

In the system using only t=2, k=3 the first ambiguity occurs at 17 = (2^4)+1 = (tatat)i and 32 = 2^5 = 2^(4+1) = ta(tati), instead of tan. The name of the Beast is unaffected.

20. Digitology

The three numbers 102 = 3×34, 120 = 3×40, 201 = 3×67 provide the three parts which when added give 34 + 40 + 67 = **141**, which is thus the required smallest number which may be divided into three parts each of which when multiplied by 3 gives the same digits. Dawson commented "An example of the very great simplicity sometimes obtained by working backwards!"

21. A Snooker Question

The last six balls potted must be the colours in the sequence 2, 3, 4, 5, 6, 7. These cannot be partitioned into scores adding to 7 or to 7+6 = 13, but we can have 7+6+5 = 18, preceded by 4+3+2 +4+1+3+1, then 7+1+6+1+2+1 then two similar breaks, then 5+1+3+1+3+1+3+1, that is 6 breaks of 18 (total 108, 39 less than the maximum).

The next possible total is 7+6+5+4 = 22 after 3+2+7+1+4+1+3+1, then 7+1+6+1+6+1, and three similar breaks, that is 6 breaks of 22 (total 132, 15 less than the maximum).

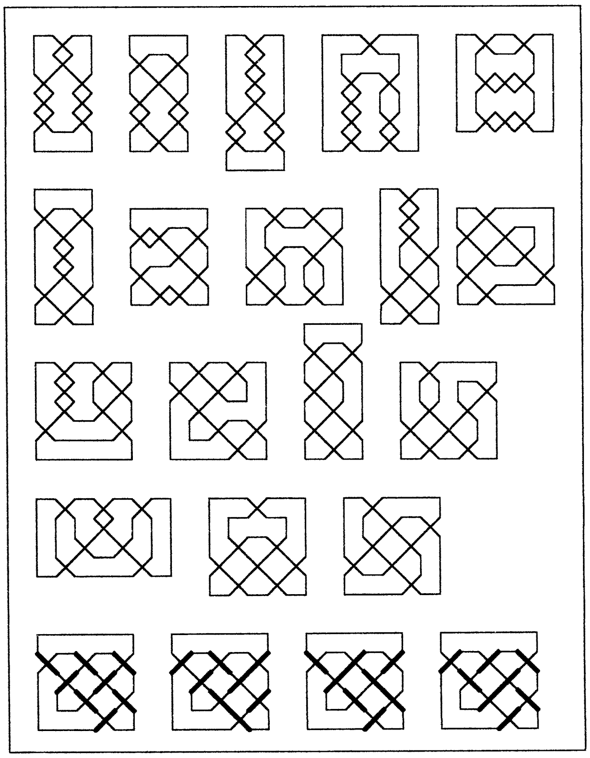
The next possible total is 7+6+5+4+3 = 25, but 6 breaks are not possible.

Answer: 6 breaks of 18 or 22.

22. Knotty

The 21 prime knots with 8 intersections, shown as king-paths with diagonal cross-points. In most of the knots the crossings are alternately over-under (under-over is merely equivalent to turning the knot over), and accordingly we can show these without darkening the upper part.

However, there are four knots, the last four in the diagram here, which can be shown by the same move-pattern but differ in the way the overs and unders are taken. (These are 17, 19, 20, 21 in the catalogue in *The Knot Book* by Colin Adams, and the equivalence is not pointed out there.)



23. Parsing the Port

One merchant purchases twice as much sherry as the other. Therefore total gallons of sherry must be divisible by 3. Casting out 3s from the number of gallons in each barrel gives remainders respectively as follows: 0, 1, 0, 1, 0, 1, 2, 1, 1, 1, 0, 1, 1, 0, 1. Casting out 3s from the sum of these figures gives a remainder of 2. This is the remainder when the total for all 15 barrels (14 sherry and 1 port) is divided by 3. Remainder 2 corresponds with the 23-gallon barrel. By omitting this barrel we have the required 14 barrels having total gallons divisible by 3. The 23-gallon barrel contained port. (The sherry barrels could have been shared out in various ways, say, 15 + 18 + 19 + 21 + 22 + 31 = 126; 16 + 25 + 28 + 33 + 34 + 37 + 39 + 40 = 252.)

24. IOUs

Solution by Dawson: Pool the whole of the papers: 2½s, 2s, 1s, 9d, 6d, 5d, 3d, 1½d, ½d. Now A had 2s 5½d and receives 2d, so he wants 2s 7½d from the pool. We may apportion him either 2½s + 1½d or 2s + 6d + 1½d. B had 10½d and lost 2d, so he wants only 8½d and this MUST be 5d + 3d + ½d. C wants his 1s 3d back, and as the 3d is used, it MUST be 9d + 6d; hence A's is fixed as 2½s + 1½d; leaving D to recover his 3s as 2s + 1s. No player has then any IOU that he put into the pool! (In the original of course 2½s is 2/6 and 2s is 2/- using the 'shilling stroke'.)

Puzzle Questions

25. Cryptarithms

There are two cryptarithms in T.R.Dawson's MS of Original Puzzles, they are numbers 2 and 18, both dated 1914.

(a) *Alphabetical Arithmetic*: Solve with the nine digits and zero.

(b) *The Multipl-add-subtractor*: These letters representing digits find the numerical values throughout. Each of the stages in the sum may be addition, subtraction or multiplication and we cannot say which a priori.

(a)

R E P N H
I R
= A I K R
I
= S F A H
R
= R I A

(b)

A
A
R
A
E D
A
D R

Dawson uses 'digits' in (a) to exclude zero, but in (b) to include zero, as is the modern usage.

In each problem when the values of the letters are deduced and they are arranged in typewriter order (0 after 9) they spell out a 'key word' or words.

26. Wire-Framed Boxes

(a) You are given four pieces of wire, all of length 3 units. You are required to bend the pieces and place them together, welding the ends, so as to form a cubical frame. How many ways are there of doing it?

(b) You are given pieces of wire, all of the same length, a whole number of units, to form boxes in the manner described above. The boxes must have right-angled corners with sides multiples of the given unit.

(1) What is the smallest length that will make two differently shaped boxes of the same volume?

(2) Four differently shaped boxes with two different volumes?

(3) Two differently shaped non-square boxes of the same volume?

(4) Four differently shaped non-square boxes of two different volumes?

(5) Six differently shaped boxes with three different volumes.

I worked out the above results myself, but the idea is not new. Algebraically it is a matter of finding triples (x,y,z) and (a,b,c) such that x+y+z = a+b+c and xyz = abc. Can anyone refer me to earlier sources?

27. No Mean Speed!

I went on a long-distance cycle ride recently and in trying to estimate the time I would take for the journey came up with the following questions, which are an old chestnut, but worth revisiting:

(a) If I cycle up a symmetric hill at 5 miles per hour and down the other side at 15 mph, what is my average speed? No it is not 10 mph!

(b) If I cycle up a symmetric hill at 5 mph how fast must I cycle downhill to reach an average speed of 10mph?

28. The Sixfold Way

A point noted by Don Hammond, the late editor of *The Dozenal Journal*, was that prime numbers, other than 2 and 3, are always of the form $6n \pm 1$. This result, incidentally, is surprisingly absent from books on number theory that I have consulted. I like to call numbers of the form $6n \pm 1$ **primals**.

In base 6, which uses just the six digits 0, 1, 2, 3, 4, 5, the primals are those numbers ending with 1 or 5. This is one of the reasons I advocate base 6 as a good choice for human, as opposed to computer, numeration.

Primals are either the larger primes (that is the primes other than 2 and 3) or products of the larger primes only (not having 2 or 3 as a factor).

The first few pairs of primals are pairs of primes (5, 7), (11, 13), (17, 19). Then comes the first pair with one non-prime member (23, **25**) where the second member is composite: $25 = 5 \times 5$. After another prime pair (29, 31) we meet the first pair with the lower member composite (**35**, 37), where $35 = 5 \times 7$.

What is the first pair of primals in which both members are composite?

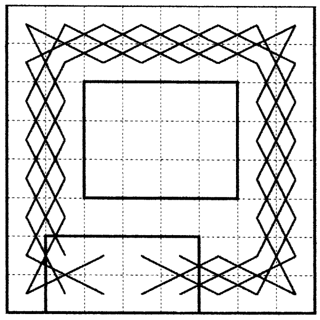
29. Birthdates

If a person is born on the first leap day of the next millennium, when will their first birthday occur and how old will they then be?

This is more of a logic problem than mathematical.

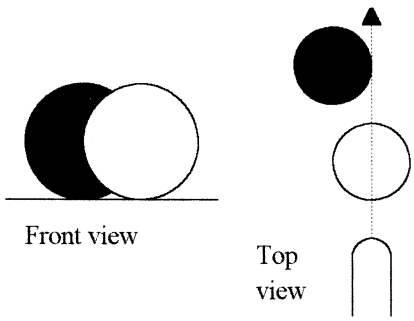
30. Knight's Tour

Complete the other knight's tour with maximum border braid and central 3×4 subtour mentioned on page 276.



31. Snooker Shots

In a 'half-ball shot' at snooker the white ball is cued so that the line along the cue through the centre of the cue ball is directed at the edge of the object ball. Ignoring complicating effects such as spin and friction, at what angles to the cue line should the balls be expected to move after a half-ball impact?



32. Knightly Triangles

Despite what may appear on some hand-drawn knight-tour diagrams it is impossible to have three knight-lines concurrent. They always form a triangle, and moreover the triangle is always a 3:4:5 right-angled triangle.

(a) Taking a square of the board to be unit area, and numbering the triangles from the smallest area upwards, show that the area of a size k triangle is $k^2/120$.

(b) Draw a triangle of knight-lines whose area is as near as possible to a unit area.

(c) What is the area of a triangle of three successive knight moves?

See the 10×10 and 14×14 knight's tours on earlier pages for examples.