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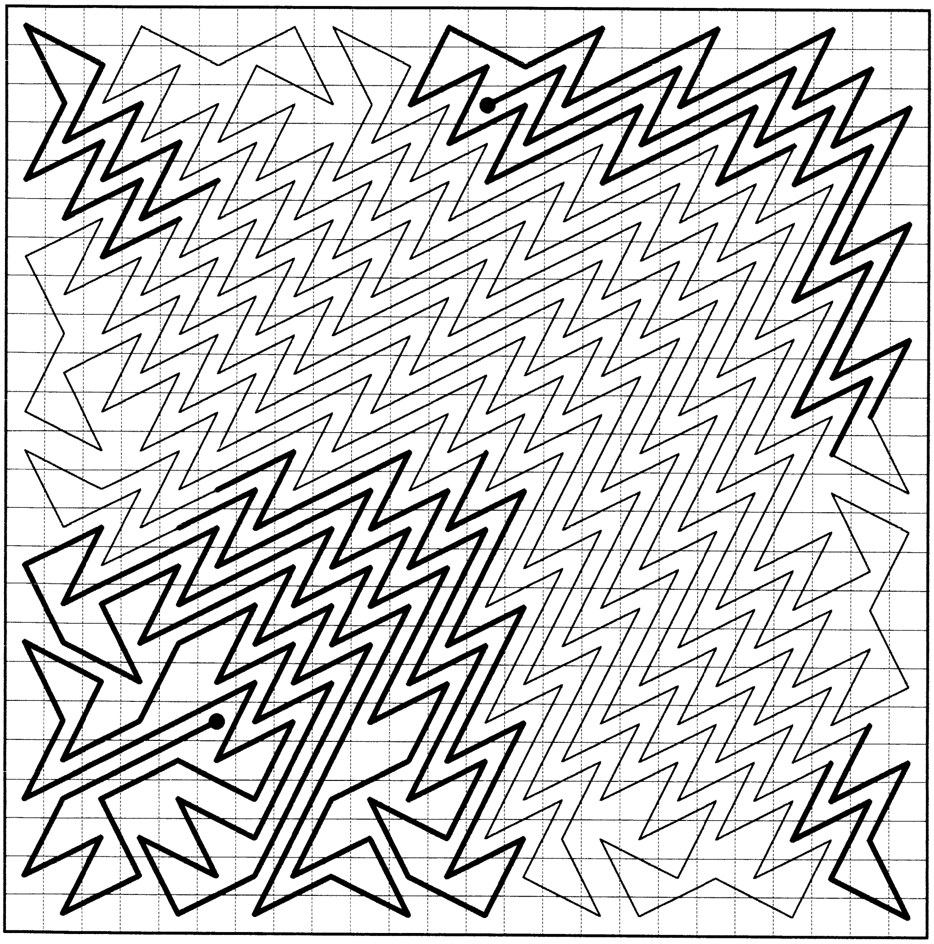
The main theme of this issue is non-self-intersecting paths. Our cover illustration shows a non-intersecting knight path by Robin Merson, which covers 454 cells on a 24×24 board, leaving 122 cells unused. The heavy lines solve the 16×16 case in 182 cells.



Further results by Robin Merson on square boards up to side 31 are tabulated, with some further examples and theory, on pages 305-310, and you are challenged to try to improve on his results, and (in the Puzzle Questions) to construct a maximal non-intersecting path, open or closed on the 32×32 board.

Editorial Meanderings

I've been reading a number of books about numbers and give a few notes on topics unearthed.

The Fascination of Numbers, by W.J.Reichmann, Methuen 1957 (reprint 1963). This old book is quite a nice introduction to number theory, very simply explained, readable and yet accurate (though, considering this is a reprint with corrections, a surprising number of minor misprints).

On page 52 he tells us: "Every prime greater than 3 differs from a multiple of both 4 and 6 by a difference of 1 (either added or subtracted). Unfortunately it does not follow that every number so related to 4 and 6 is necessarily a prime." This cancels my statement on p.296 about the $6n \pm 1$ result being absent from books on number theory, though he does not develop the point further.

On page 33 this result is new to me: "Fifth powers [in base ten] have exactly the same end-digits as have their roots." This is slightly unexpected since squares and fourth powers end in 0, 1, 4, 5, 6 or 9, and cubes in any digit. He argues: "This is another way of saying that the difference between any two successive fifth power numbers is always a number having 1 as its end digit." He proves it thus: $(n+1)^5 - n^5 = 5n^4 + 10n^3 + 10n^2 + 5n + 1 = 5(n^4 + 2n^3 + 2n^2 + n) + 1$. The expression in brackets is even when n is odd since $n^4 + n = n(n^3 + 1)$ and when n is odd then $(n^3 + 1)$ is even.

The following is a pretty little conjuring trick: The victim is asked to think of any number of any number of digits. He is then to add up the digits and to take the result from the original number. He is then to cross out any one digit (once). He is then to permute the remaining digits in any order and reveal the resulting number. The conjuror can then name the digit crossed out! This she does by calculating the '**digital root**' of the revealed numbers (by adding the digits, then adding the digits of the result, and so on) and subtracting it from 9. The trick works since subtracting the sum of the digits from any number leaves a number that is a multiple of 9, whose digital root is therefore 9.

The **magic subtraction square** shown below is also from this book. It has the property that if in each row, column and diagonal we subtract the first number from the second, and the result from the third, the magic constant 5 results. Thus from a, b, c we calculate $c-(b-a) = (a+c)-b$, which is the same whichever end of the line of three we start. Our Puzzle Question 35 asks for an extension of this idea. The four-page treatment of congruences is barely adequate, but includes an argument that I have modified to make into Puzzle Question 40 on our back page. See also Question 47.

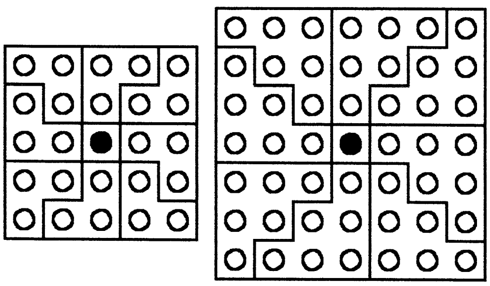
magic subtraction square

2	1	4
3	5	7
6	9	8

'magic sigma square'

3	1	5
8	6	4
7	2	9

odd squares are of the form 8t + 1



remainder of (m×n) ÷ 9

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	0
2	0	2	4	6	8	1	3	5	7	0
3	0	3	6	0	3	6	0	3	6	0
4	0	4	8	3	7	2	6	1	5	0
5	0	5	1	6	2	7	3	8	4	0
6	0	6	3	0	6	3	0	6	3	0
7	0	7	5	3	1	8	6	4	2	0
8	0	8	7	6	5	4	3	2	1	0
9	0	0	0	0	0	0	0	0	0	0

Number 9 | The Search for the Sigma Code, by Cecil Balmond, Prestel -Verlag 1998. This little book is provoking but ultimately rather disappointing. It consists largely of a compilation of references to 'nine' in proverbs and religious literature, with about 30 pages in italic devoted to a story in which nothing much happens, interspersed with much quasi-mystical musing illustrated with childish art.

The 'sigma code' referred to is simply the procedure of finding what Reichmann calls the 'digital root', but which Balmond denotes by $\Sigma(n)$. This of course is the same as the remainder when n is divided by 9, except that in place of the remainder 0 it takes the value 9. Balmond gives the impression that he has just thought of this idea himself, but it is antique.

On pages 33-4 when introducing the numbers 1 to 8 he notes that: "A curious property of 4 is that any odd number squared, when divided by four, leaves a remainder of one." and: "all prime numbers when squared and divided by eight leave a remainder of one!". This last is obviously untrue for the prime number 2. The correct statement should be that all odd numbers squared (not just odd primes) are of the form $8t+1$, since: $(2n+1)^2 = 8[n(n+1)/2] + 1$.

This result, which goes back at least to Diophantus, can be illustrated visually (as in our diagrams above) by a square array in which the central counter is isolated and the rest divide into eight equal triangles (such diagrams do not appear in the book reviewed).

On pages 202-3 is a similar result: "All primes to the sixth power, except for number 3, have digits that sum up to unity!" By this he means that they give remainder 1 on division by 9. This is easily proved using the fact (see our Problem 28) that all primes greater than 3 are of the form $6n \pm 1$. Thus, in the binomial expansion of $(6n \pm 1)^6$ every term other than the unit contains the factor 6×6 [either in the form of $(6n)^2$ or, in the term of degree 1, as $(6C5)\times6n$] and so is divisible by 9. We can thus state, more generally, that all primals to the sixth power are of the form $9n+1$. The fact that the property holds for 2 as well is incidental ($2^6 = 64 = 7\times9 + 1$).

On pages 174 and 205 appear the 'magic sigma square' (see preceding diagram). This is derived from the 3×3 magic square by circular permutation, moving n to $n+1$ and 9 to 1, and has the property that every line of three adds to a multiple of 9, (i.e. 9 or 18). In other words it is a '**mystic square**' as defined in our Problem 15 (see *G&PJ* 14 and 15).

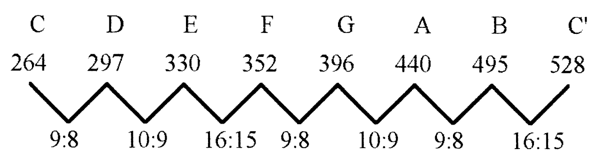
The author makes much play with the "sigma code values of the multiplication table" (p.90), which he transforms into a circular 'mandala' (p.122), and a figure of eight mandala (p.134) where the 9's are all condensed to one point, but does not seem aware that this table (if we substitute 0 for his 9) is simply the multiplication table for arithmetic modulo 9, of which he has apparently not heard. (The same table, also without reference to modular arithmetic, appears in Reichmann.)

e | The Story of a Number by Eli Maor, Princeton University Press 1994. This is a history of the exponential number, $e = 2.7182818284...$, from the introduction of logarithms by John Napier in 1614, to proof of its 'transcendence' by Charles Hermite in 1873, taking in the development of calculus [$d(e^x)/dx = e^x$] and the theory of complex functions [$e^{(x + iy)} = e^x(\cos y + i \sin y)$] on the way.

Although Napier is commonly credited with the discovery of 'natural logarithms' (to base e), the Napierian logarithm L of a number N was in effect defined by $N = m(1 - 1/m)^L$, the multiplying factor of $m = 10^7$ being used to avoid decimal fractions, which were still not in customary use at the time. If we put $N/m = n$ and $L/m = l$ then we get $n = (1 - 1/m)^{(ml)}$, and since $(1 - 1/m)^m$ converges to $1/e$ as m increases, we can agree that "Napier's logarithms are virtually logarithms to base $1/e$."

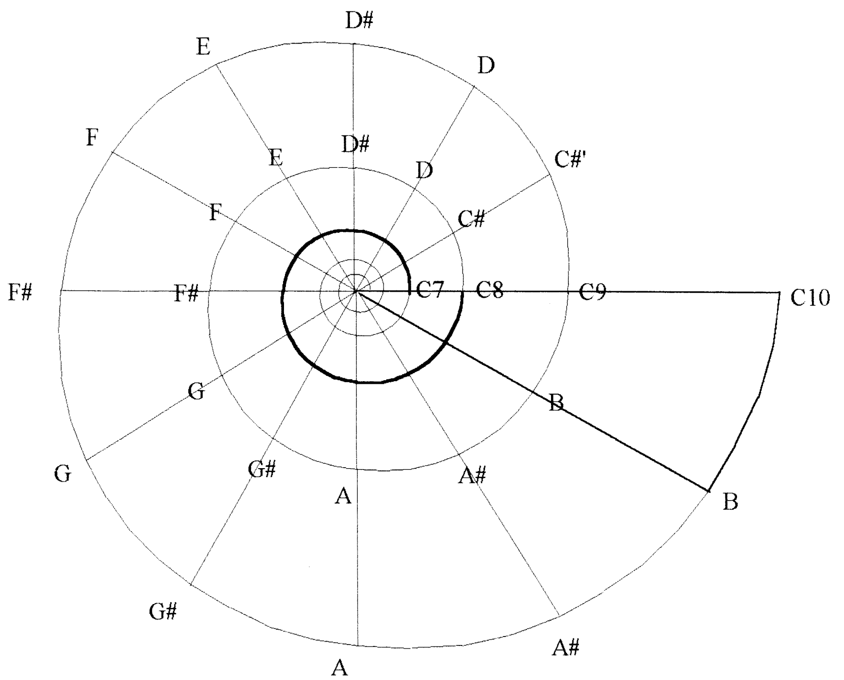
Intriguingly, this book describes yet another fictional encounter with **J. S. Bach**, this time in 1740 with the mathematician **Johann Bernoulli**. Here is an edited extract (the sign /// indicates omissions):

BERNOULLI: /// my interest in music is entirely theoretical; for example, a while ago I and my son Daniel did some studies on the theory of the vibrating string. This is a new field of research involving what we in mathematics call continuum mechanics. /// BACH: As you know, our common musical scale is based on the laws of the vibrating string. The intervals we use in music — the octave, fifth, fourth and so on — are all derived from the harmonics, or overtones, of a string — those feeble higher tones that are always present when a string vibrates. The frequencies of these harmonics are integral multiples of the fundamental (lowest) frequency, so they form the progression 1, 2, 3, 4, ... [a figure is given here showing the notes on a double stave, rising from the C two octaves below middle C, which is the fourth harmonic, to the C two octaves above middle C which is the 16th harmonic]. *The intervals of our scale correspond to ratios of these numbers: 2:1 for the octave, 3:2 for the fifth, 4:3 for the fourth, and so on. The scale formed from these ratios is called the just intonation scale. BERNOULLI: That perfectly fits my love for orderly sequences of numbers. BACH: But there is a problem. A scale constructed from these ratios consists of three basic intervals: 9:8, 10:9 and 16:15* [figure]. *The first two are nearly identical, and each is called a whole tone ///.*

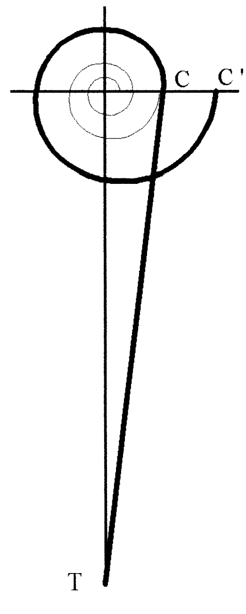


The scale of C-major. The upper numbers indicate the frequency of each note (taking A=440) in cycles/second; the lower numbers are the frequency ratios between successive notes.

The last ratio is much smaller and is called a <u>*semitone*</u>*. /// Not only are there two different kinds of whole tones in use, but if we add up two semitones, their sum will not exactly equal either of the whole tones. /// It's as if 1/2 + 1/2 were not exactly equal to 1, only approximately.* BERNOULLI*: // You're right. To add two intervals, we must multiply their frequency ratios. Adding two semitones corresponds to the product (16:15)(16:15) = 256:225 or approximately 1.138, which is slightly greater than either 9:8 (= 1.125) or 10:9 (= 1.111).* BACH*: You see what happens. The harpsichord has a delicate mechanism that allows each string to vibrate only at a specific fundamental frequency. This means that if I want to play a piece in D-major instead of C-minor — what is known as transposition — then the first interval (from D to E) will have the ratio 10:9 instead of the original 9:8. This is still all right, because the ratio 10:9 is still a part of the scale; and besides, the average listener can barely tell the difference. But the next interval — which must again be a whole tone — can be formed only by going up a semitone from E to F and then another semitone from F to F-sharp. /// And the problem is compounded the farther up I go in the new scale. In short, with the present system of tuning I cannot transpose from one scale to another, unless of course I happen to play one of those few instruments that have a continuous range of notes, such as the violin or the human voice. /// But I have found a remedy: I make all whole tones equal to one another. /// But to accomplish this I had to abandon the just intonation scale in favor of a compromise. In the new arrangement, the octave consists of twelve* <u>*equal semitones*</u>*. I call it the* <u>*equal-tempered scale*</u>*.* (footnote*: Bach was not the first to think of such an arrangement of notes. /// It was owing to Bach, however, that the equal-tempered scale became universally known.*) *The problem is, I have a hard time convincing my fellow musicians of its advantages. They cling stubbornly to the old scale.* BERNOULLI*: /// If there are twelve equal semitones in the octave, then each semitone must have a frequency ratio of* $\sqrt[12]{2} : 1$ *///* (footnote*: The decimal value of this is about 1.059, compared to 1.067 for the ratio 16:15. This slight difference, though still within the range of audibility, is so small that most listeners ignore it.*) BACH*: /// Is there any way you could demonstrate this visually?* BERNOULLI*: I think I can. My late brother Jakob spent much time exploring a curve called the logarithmic spiral. In this curve, equal rotations increase the distance from the pole by equal* <u>*ratios*</u>*. /// To transpose a piece from one scale to another, all you have to do is turn the spiral so that the first tone of your scale falls on the x-axis. The remaining tones will automatically fall into place.*



The twelve notes of the equal-tempered scale arranged along a logarithmic spiral. [Each 30° rotation represents a 12th root of 2 (1.05946) multiplication of the frequency. Thus 360° gives a doubling.]



Length of one turn of the spiral.

The equation of the 'doubling' spiral can be written very simply as $r = 2\^c$, where r is the usual radial coordinate, c is the angle of rotation measured in cycles, and 2 is the magnification factor per cycle. In radian measure it takes the form $r = 2\^(\theta/2\pi)$ or $\theta = 2\pi(\log_2 r)$, whence the name 'logarithmic spiral' for this type of curve. The angle ϕ between the radius and a curve is given by $\tan\phi = r(d\theta/dr)$. So in this case $\tan\phi = r[2\pi(1/r)\log_2 e] = 2\pi\log_2 e = 2\pi/\log_e 2 = 9.06472$, whence $\phi = 83.7047°$. The angle is constant, hence the alternative name 'equiangular spiral'.

In the inset figure the distance CT, cut off on the tangent at C by the y-axis, is equal to the length of the turn of the spiral between C and C ', the octave above. This length is also, paradoxically, the limiting 'length' of the inner part of the spiral, from C towards O. (The drawings of this in the book, on pages 123 and 207, are misleadingly out of scale.) CT = OC.secϕ = 9.12 OC. That the lengths of the inner turns of the spiral also approach this value can be seen since the nth 360° inward turn is of length (secϕ).OC/(2^n), and the sum 1/2 + 1/4 + 1/8 + ... approaches 1.

Readers unfamiliar with geometrical drawing using a computer may be surprised to learn that in the diagram above, the curved lines, which appear reasonably smooth, are in fact entirely made up of straight line segments! This phenomenon supports the finitist thesis of Professor Cranium, as expressed in his article on Pi in *G&PJ* 13, p.218. His views are also supported by the following book:

What is Mathematics, Really? by Reuben Hersh, Vintage (Random House) 1998 (first published by Jonathan Cape 1997). The preface claims: "This book is a subversive attack on traditional philosophies of mathematics." But his thesis that "...mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" hardly seems iconoclastic. He goes on at considerable length against platonism, formalism, intuitionism, foundationism, neo-fregeanism, and other isms, but in the end, although he quotes extensively, and in very small print, from every philosopher from Plato onwards with a view on mathematics, I'm not sure that he presents these philosophies in an unbiased manner.

The following on real numbers from p.175 has some validity: "We use real numbers in physical theory out of convenience, tradition, and habit. For physical purposes we could start and end with finite, discrete models. Physical measurements are discrete, and finite in size and accuracy. To compute with them, we have discretized, finitized models physically indistinguishable from the real number model. The mesh size (increment size) must be small enough, the upper bound (maximum admitted number) must be big enough, and our computing algorithm must be stable. Real numbers make calculus convenient. Mathematics is smoother and more pleasant in the garden of real numbers. But they aren't essential for theoretical physics, and they aren't used for real calculations."

Yet, he does not advocate doing away with real numbers, as a true subversive Finitist like Professor Cranium does, who would argue that continuing to describe the modern finite, uncertain, quantum universe in terms of old fashioned infinitesimalist continuum concepts is akin to the use epicycles in Ptolemaic astronomy.

This book is rather outside our usual remit but an ulterior motive in mentioning it is the presence on page 231 of a diagram relating to **Mu Torere** (see our next page).

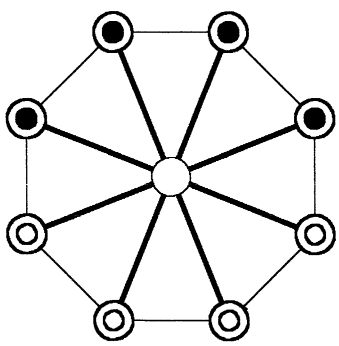
The Journal of Recreational Mathematics. Vol.29, Nr.2, announces the retirement of Joseph S. Madachy from the editorship at the end of Volume 30. He has edited the journal since its inception in 1968 apart from a break in 1976-79. The new general editor is to be Charles Ashbacher (Box 294, 119 Northwood Drive, Hiawatha, Indiana IA 52233, USA) with Colin R. J. Singleton (41 St Quentin Drive, Sheffield, S17 4PN, England) editing the important Problems and Conjectures department.

A particularly interesting problem posed by Peter Raedshelders (Vol.24, Nr.1 and Vol.25, Nr.1) is to tile a rectangle with M 'consecutive rectangles' of sizes 1×2, 2×3, ... , M×(M+1). He gave solutions for up to 8 rectangles, showing that case 6 is insoluble. Edward D. Onstott (Vol.28, Nr.4) reported that the condition that the rectangle so formed also be consecutive results in the Diophantine equation $N\times(N+1) = M(M+1)(M+2)/3$ which has exactly five solutions: N = 1, 4, 15, 55, 119 with M =1, 3, 8, 20, 34. The case 55×56 has now been solved by Allan William Johnson Jr (Vol.29, Nr.2). This leaves the largest rectangle 119×120 as the only unsolved case — get to work! See Problem Question 46.

MU TORERE

Analysis by George Jelliss

This game with very simple rules but complicated play is the distinctive board-game of the Maoris of New Zealand. The game is described in R.C.Bell's *Board and Table Games*, volume 2 (Oxford University Press 1969), but I first encountered it in the Steve Nichols magazine *Games Monthly* (November 1988, pages 32-33) where the rules, some history and a large diagram of the board were given but no analysis. The board as shown there is in the form of an eight-pointed star, with no connecting lines round the circumference, and oriented with the radial lines horizontal, vertical and at 45 degrees. However the symmetry of the opening position and the fact that moves round the circumference are permitted means that the diagram as shown below makes the rules clearer.

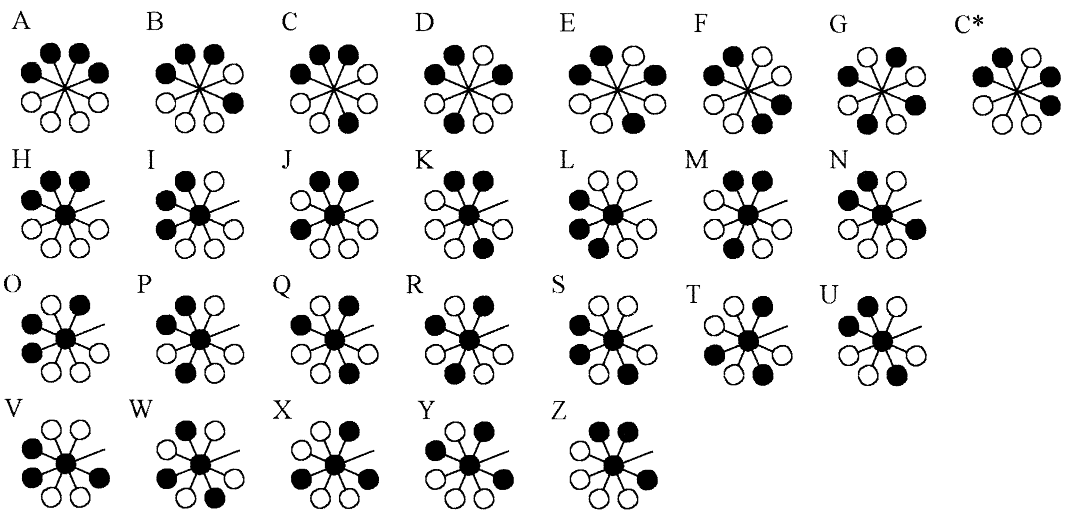


Rules of Mu Torere

There are two players, each with four pieces arranged initially as shown. The player of the black pieces moves first..
All moves are of a single piece along a line of the board to the single vacant cell.
A piece may not move to the centre cell (putahi) unless it is next to an opposing piece.
The turn to move alternates and the objective is to prevent the opponent moving.

The game is mentioned at the end of the history section in the book *What is Mathematics, Really?* (reviewed on the preceding page) where, on p.231, a complex figure showing 'The flow of the game of mu torere' is quoted, without much explanation, from another book, *Ethnomathematics* by Marcia Ascher (Brooks/Cole, San Francisco 1991). This shows 92 numbered circles connected by curved and polygonal lines to form a graph with 180° rotary symmetry.

This stimulated me to revisit my earlier notes and complete the analysis. The number of geometrically distinct positions possible (disregarding rotations and reflections) is 46. Since each of these can have black or white to move we arrive at the total 92 in the above-mentioned diagram. The 46 are made up of 8 with centre vacant, 19 with a black piece in the centre and 19 with white there.



If we also group together positions that are complements of each other, i.e. with black and white interchanged, we arive at 26 basic positions, which are conveniently lettered A to Z.

The above 27 diagrams show all the positions with centre vacant or occupied by black. The complementary positions we denote by the same letters with * attached to indicate interchange of colours. Of the 8 positions with centre vacant two, C and C*, are complements of each other, while the other six, A–F, are self-complementary (A* = A and so on). If we take each letter as indicating the position with black to move (so that A is the opening position) then we can denote the same position but with white to move by the prefix ~. We now have a notation for all possible positions. The stalemate positions to aim for are ~X, X*, ~Y, Y*, ~Z, Z*.

Note that the six positions X, Y, Z and ~X*, ~Y*, ~Z* are in fact impossible to reach in play, since they have no move leading into them. If we move the middle piece in Z, Y, X out to the rim we get positions A, B, C respectively, but the reversals of these moves are illegal.

All transitions by black	A - H	H - A, Z	H* - I*	R - T, E	R* - W*
moves are listed here	B - I, J, K	I - B	I* - H*, L*	S - E	S* - O*, P*
(the ~ s are omitted).	C - L, M	J - B, Y	J* - P*	T - E, R	T* - Q*
To find white move	C* - N, O	K - B, Y	K* - N*	U - E	U* - M*, V*
transitions operate with	D - P, Q	L - C	L* - I*	V - F, V	V* - U*
* on all letters bearing	E - R, S, T, U	M - C, X	M* - U*	W - G	W* - R*
in mind that ** cancels	F - V	N - C*, O	N* - K*	X - C, M	X* - 0
out and that AB DEFG	G - W	O - C*, N	O* - S*	Y - B, J, K	Y* - 0
are unaffected.		P - D	P* - J*, S*	Z - A, H, Z	Z* - 0
(e.g. N* - C, O* and N - K)		Q - Q, D	Q* - T*		

Beginning at A black's only two moves both lead to versions of ~H and white's only reply gives I. Then black's only move gives ~B and white now has three choices, leading to I*, J*, K*. At this point, on his third move, black has his first chance to make an error; moving from I* to ~H* allows white to reach Z* and win (or alternatively white can make another error, taking the position back to A). On the other hand black could win by playing (3) I* to ~L* then white must go to C* and black can play (4) ~N (not ~O) and white must reply K and black has (5) ~Y stalemating white. This analysis indicates that white must avoid I* on his second move and play J* or K*.

Chart of the moves in Mu Torere.

(1)	~	(2)	~	(3)	~	(4)	~	(5)	~	(6)	~	(7)	~	(8)	~	(9)	~
A	H	I	B	I*	H*	A/Z*											
					L*	C*	N	K	B/Y								
							O	S	E	R*	W*	G	W	R	E/T		
										S*	O*	C					
												N*	K*	B/Y*	I	H	A/Z
																L	C
															J	P	
															K	N	C*/O
											P*						
										T*	Q*	D/Q*	T*	E/R*			
										U*	M*	C*/X*					
											V*	F	V				
												V*	U*	E			
				K*	N*	C	L	I									
							M	U	E-								
						O*	S*	E	R	W	G	W*	R*	E/T*			
									S	O	N/C*	L*	I*	B-			
												M*	U*-				
										P	D	Q*-/P*	S*/J*	B-/Y*			
									T	Q	Q/D-						
									U	M	X/C	N*-/O*	S*				
										V	F	V*-					
											V	U					
				J*	P*	D	P	S-/J	B/Y								
							Q	T	E-/R-								

A single underline indicates feedback to an earlier point in the sequence.
A double underline indicates cross-reference to an earlier occurrence in another line.
A hyphen - refers to an entry above in the same column
A stroke / indicates choice of moves to save a line

A whole further sequence continues from the A above which dualises the whole chart, arriving back at A with black to move again

See our Problem Question 34 for a little more on Mu Torere.

Magnetism

by Derick Green (© copyright 1998)

This is a game I have designed based on 'The Colchester Game' which was unearthed in an archaeological dig at Colchester in 1996 and was extensively reported in the press at the time. The find was unusual in that the pieces were found apparently in position for a game that had already started. An account was published in *Variant Chess* issue 23 (Spring 1997).

The rules for **Magnetism** are as follows: (1) A 12×8 board is used, each player has 12 playing pieces and one magnet piece. Each player sets up their pieces as in Figure 1.

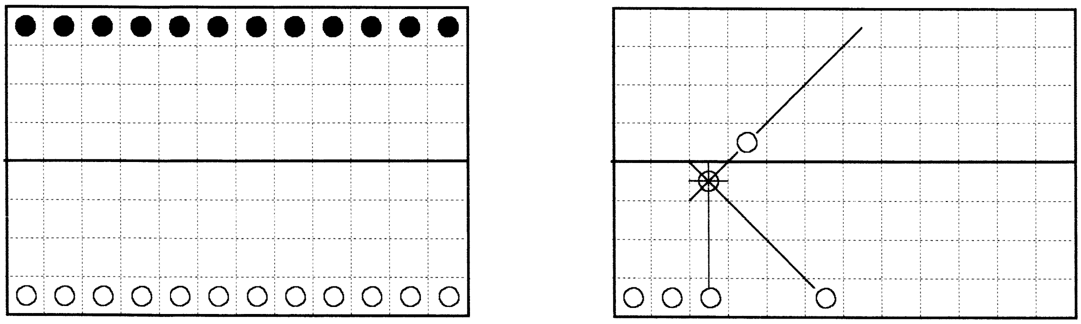


Figure 1 Figure 2

(2) On a piece of paper each player secretly records the square they wish to place their magnet piece on: anywhere in their own half of the board, except on the already occupied back row. The squares are then revealed and the magnets are placed on the board.

(3) The object of the game is simply for a player to be the first to occupy one of the opponent's back rank squares. Once a player achieves this the opponent has one move to place a piece on the 'winning' player's back row. If this occurs the game continues until a player has two pieces on the opponents back row, and so on. However, once a player places all 12 pieces on the opposite back row the game ends and that player wins. There is no capturing and only one piece may ever occupy any square. The final positions of the magnets have no effect on the ending of the game.

(4) All pieces, including magnets, move as the queen in chess, restricted only by the other pieces and magnets and the edge of the board.

(5) A move consists of two parts, both of which must be made. In the first part the player's magnet piece is moved (we show in round brackets the square moved to) and in the second one of the player's twelve pieces must be moved either towards or away from that player's magnet piece (we show the coordinates of the two squares). For example in Figure 2 White's magnet is at c4 and White's piece at d5 could be pushed to e6, f7 or g8 (if these squares are vacant) or White's piece at c1 could be pulled to c2 or c3, and White's piece at f1 could be pulled to e2 or d3. A player who at any time is unable to complete both parts of a move has lost the game.

(6) A magnet's influence may pass through other pieces but not through the opponent's magnet. No piece or magnet however may move through or be pulled or pushed through an occupied square. For example, in Figure 2, if a Black piece was at f7, White's piece could not be pushed to g8 but would have to stop at e6.

Example Partial Game: White magnet g4, Black magnet d5. **1.** (f5) j1-g4 (g2) a8-e4 **2.** (f4) g4-h4 (g6) e8-f7 **3.** (g3) h4-i5 (g5) j8-h6 **4.** (h4) i5-j6 (g6) e4-f5 **5.** (j4) j6-j8 (e6) f5-j1. Each player now has one piece on the opponent's back row. Player B's magnet is slightly better placed. Note White's move **5** shows it is not necessary for the magnet to be adjacent to a piece in order to move it away.

Knight's Tour News

Non-Intersecting Paths by Leapers

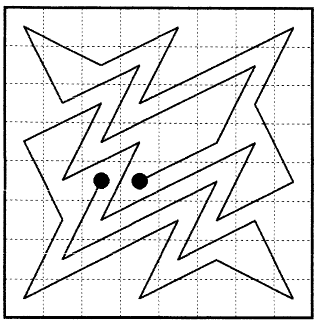
by Robin Merson and George Jelliss

Historical Notes. A simple question, or group of questions, that leads to some interesting convoluted patterns and is difficult to answer with certainty is: *What is the longest journey a given piece can make on a given board without entering any cell twice or crossing its own path?* Though it may not visit all cells of the board, such a path is often called a 'tour' since it never passes through a cell twice and it covers the maximum area possible under the conditions. The 'length' L of a path is usually counted by the number of moves. The area of the board covered however is measured by the number of cells visited C. In a closed tour C = L, but in an open tour C = L+1.

A wazir, {0, 1}-mover, can never cross its own path, so the problem reduces to that of finding the longest path. On a rectangular board $m\times n$ the wazir can tour all the cells.

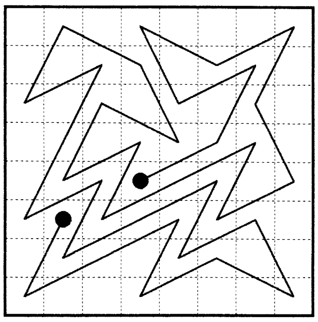
The case of the knight {1, 2} on the 8×8 board was solved in 1930, with an open path of 35 moves by T. R. Dawson and with a closed path of 32 moves by the Romanian chess problemist Wolfgang Pauly (1876-1934) [not to be confused with the Austro-Swiss physicist Wolfgang Pauli (1900-1958)]. These results were reported in *L'Echiquier*, December 1930, though without a diagram of Pauly's result; a diagram appears in H. J. R. Murray's unpublished 1942 knight's tour manuscript.

T. R.Dawson 1930



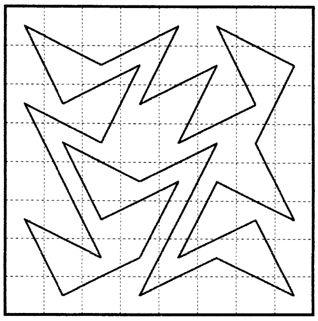
Knight 33 moves, open

T.R.Dawson 1930



Knight 35 moves, reentrant

W.Pauly 1930



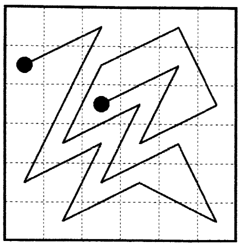
Knight 32 moves, closed

The knight problem for small rectangular boards was rediscovered by L. D. Yarborough in *Journal of Recreational Mathematics* 1968 (vol.1, nr.3, pp.140-142). Some of his results were improved on in letters in the same journal 1969 (vol.2, nr.3, pp.154-157) by R. E. Ruemmler (7×8 and 5×9 to 9×9), D. E. Knuth (5×6, 6×6, 7×8, 8×8, confirming the Dawson/Pauly results, and 5×9) and M. Matsuda (6×6, 6×8, 5×9, 7×9 and 9×9). Their best results up to 9×9, by number of moves, are: 3×3, 2; 3×4, 4; 3×5, 5; 3×6, 6 closed; 3×7, 8; 3×8, 9, 3×9, 10; ... 4×4, 5; 4×5, 7; 4×6, 9 open, 8 closed; 4×7, 11; 4×8, 13 open, 12 closed; 4×9, 15; ... 5×5, 10 open, 8 closed; 5×6, 14; 5×7, 16; 5×8, 19 open, 18 closed; 5×9, 22 open, 20 closed; ... 6×6, 17 open or reentrant; 6×7, 21; 6×8, 25 open, 22 closed; 6×9, 29; ... 7×7, 24 open, 24 closed; 7×8, 30 open, 29 open symmetric, 26 closed; 7×9, 35; ... 8×8, 35 open or reentrant, 32 closed; 8×9, 42;... 9×9, 47, ...

Note on terminology: *J.Rec.Maths.* does not distinguish between 'reentrant' and 'closed'; we use 'reentrant' to describe an open path whose ends are a knight move apart; joining up the ends gives a closed path but causes at least one intersection.

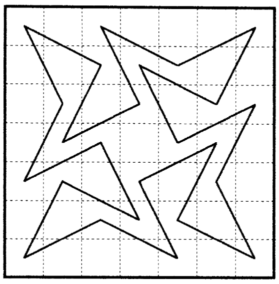
Diagrams of a few interesting small-board tours follow. The 17-move 6×6 path was found by Knuth and Matsuda independently and is unique, apart from rotation or reflection, and is reentrant. It has been quoted many times, e.g. in Martin Gardner's *Scientific American* column (April 1969) and his *Mathematical Circus* and in K. Fabel et al, *Schach und Zahl* (1978), without due acknowledgement. The 7×7 open and closed solutions are symmetrical. The 7×8 open solution is also unique.

Knuth and Matsuda 1969



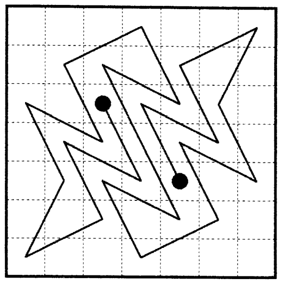
6×6: Knight 17 moves reentrant.

Yarborough 1968



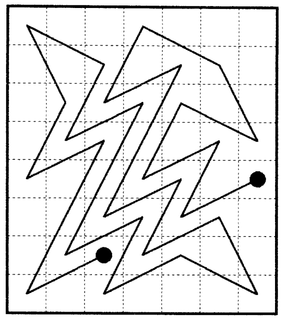
7×7: Knight 24 moves, closed quatersymmetric (90° rot)

Knuth 1969



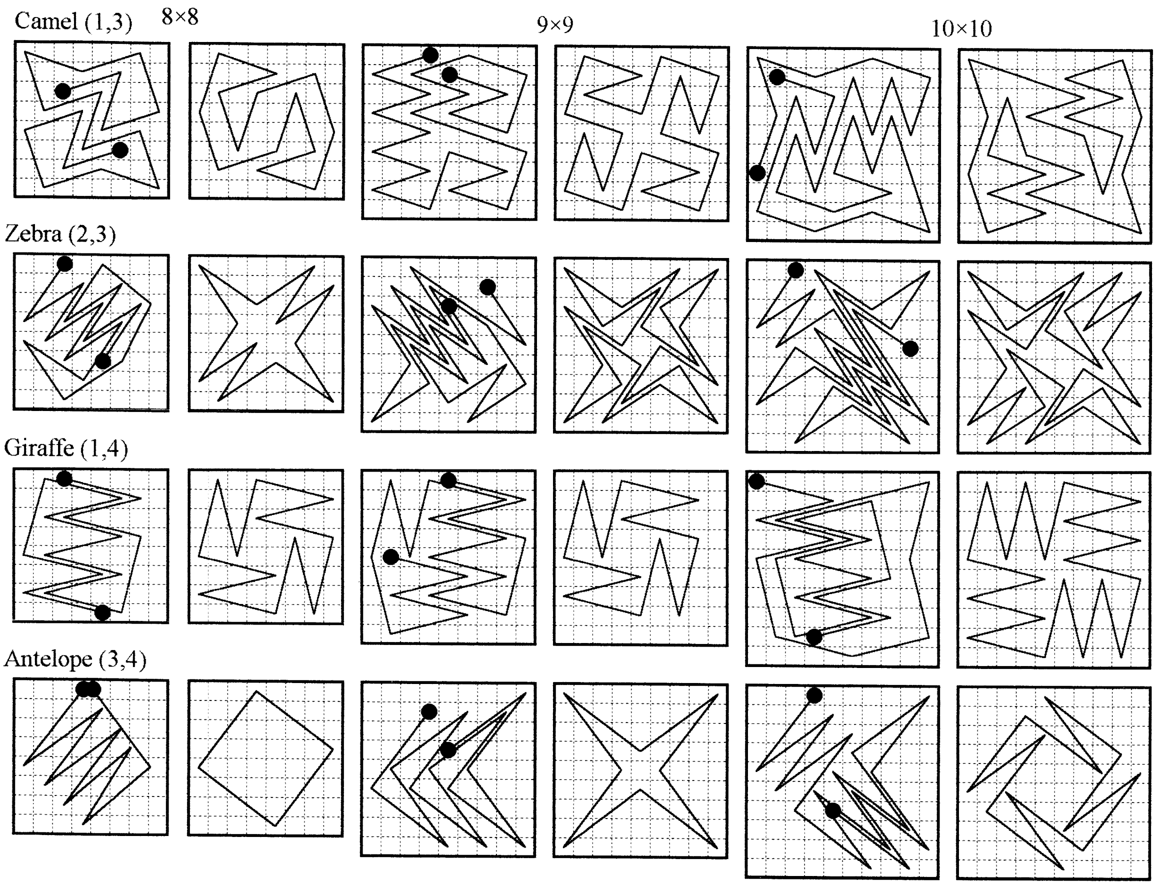
7×7 Knight 24 moves, open symmetric (180° rotation)

Knuth and Ruemmler 1969



7×8: Knight 30 moves, open

The problem for the higher free leapers, giraffe {1,4} antelope {3,4}and zebra {2,3} on the 8×8 board was solved by George Jelliss in *Chessics* (vol.1, issue 9) 1980. Robin Merson, in a letter dated 16 June 1991, accompanied by computer printed diagrams, confirmed these results and extended them to larger boards. One open and one closed path for each of camel, zebra, giraffe and antelope, on boards of side 8, 9 and 10 are shown in the following diagrams. On the 9×9 board the giraffe can do no better in a closed tour than it does on the 8×8 board.



Merson also gave solutions for antelope on 11×11, on which the unique closed path takes the form of a tetraskelion similar to those for the 7×7 knight (1,2) and 9×9 zebra (2,3): an evident progression. These cases were diagrammed in a brief report on his work in *Variant Chess*, (vol.1, nr.6, 1991).

The following is a table of Robin Merson's results on these longer leapers. The figures in brackets give the number of different tours found; thus (1) indicates a unique solution. If we have space we may diagram some more of these results in a subsequent issue.

	8×8		9×9		10×10		11×11	
	open	closed	open	closed	open	closed	open	closed
camel	17 (1)	14 (5)	23 (14)	20 (20+)	29 (1)	26 (1)		
zebra	17 (3)	12 (12)	25 (1)	24 (1)	32 (9)	28 (7)		
giraffe	15 (2)	12 (1)	19 (1)	12 (1)	25 (11)	20 (2)		
antelope	9 (5)	4 (2)	13 (4)	8 (3)	17 (15)	12 (7)	25 (4)	24 (1)

camel = {1,3}, zebra = {2,3}, giraffe = {1,4}, antelope = {3,4}

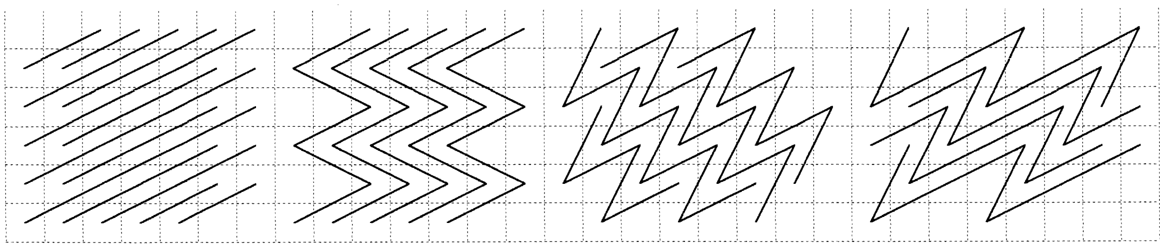
T. R. Dawson *Fairy Chess Review* August 1944 (problem 6038) gave an 8×8 open tour of 52 moves for the gnu (knight + camel), i.e. {1,2} and {1,3} leaper.

The Knight on Larger Square Boards. Robin Merson first became interested in this problem through some items that appeared in *Games & Puzzles* in 1972-3, where he published a letter (issue 9) outlining some results. His later work, reported below, was sent to George Jelliss, for inclusion in a book on tours: results for open paths dated 9 November 1990 and closed paths 23 April 1991.

The table gives the maximum sizes, in number of cells visited, achieved for open and closed non-intersecting paths on square boards of various sizes. His values for open paths up to 9×9 agree with the work of Yarborough and Co. Improvements may still be possible on some of the larger boards.

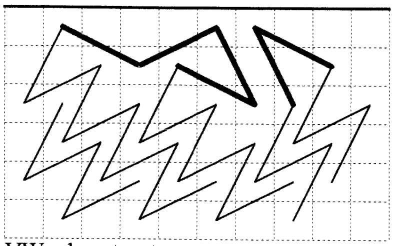
side	open	closed	side	open	closed	side	open	closed
3	2	0	13	113	104	23	414	396
4	5	4	14	134	124	24	453	434
5	10	8	15	158	148	25	498	476
6	17	12	16	181	172	26	541	520
7	24	24	17	210	200	27	588	564
8	35	32	18	237	226	28	638	612
9	47	42	19	268	256	29	689	662
10	61	54	20	302	288	30	742	714
11	76	70	21	337	322	31		768
12	94	86	22	374	360	32		

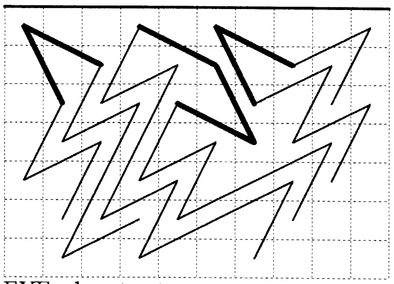
The simplest arrangements of knights moves that cover an area completely are (a) the close-packed parallels (b) lateral zigzags, (c) diagonal zigzags or (d) combinations of these:

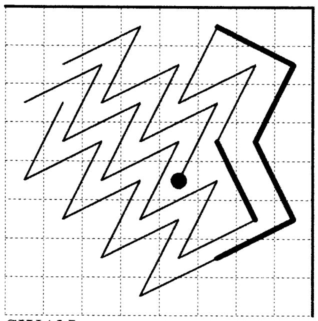


When we consider the ways of joining up these lines in adjacent pairs, using links that fit closely to the edges and corners, the diagonal zigzags (c) or the lengthened type (d) prove the most economical.

Robin Merson draws particular attention to the following cases to which he gives names. It is possible to interpolate a VW structure in each edge of certain tours $n \times n$ to give a tour on an $(n+8)$-side square, though this does not always guarantee that the resulting tour is of maximum length.

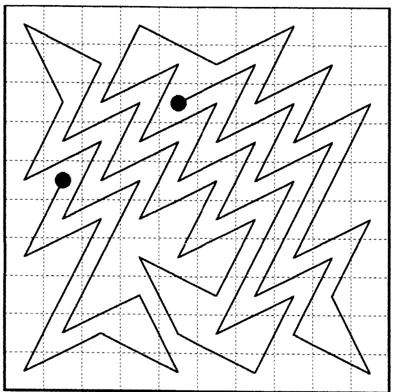

VW edge structure


EYE edge structure

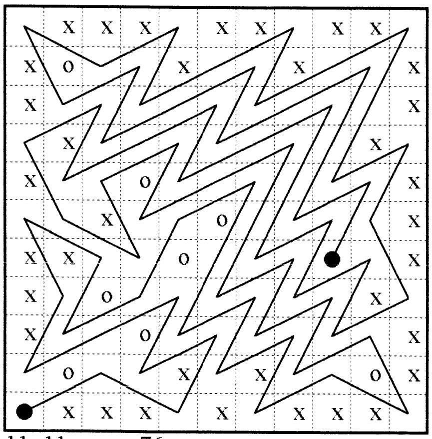

SWAN corner

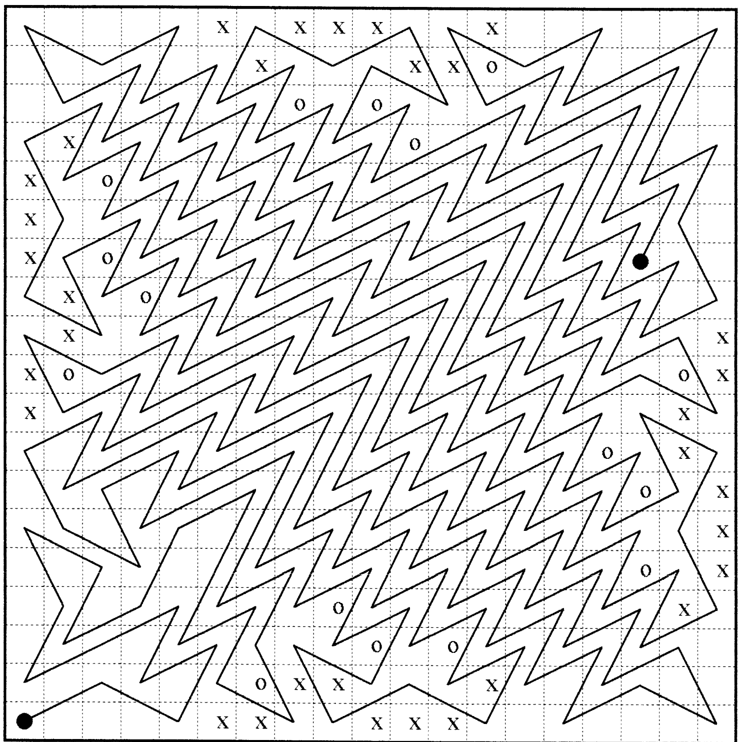
Robin gives the following instructions for determining the length of a tour without counting all lines: Put $(n-2)$ black crosses (x) along each edge on unvisited squares [i.e. one in each row or column perpendicular to the edge, except at the ends]. Put a red blob (o) in each remaining unvisited square, and count the number of such blobs, b, which he calls the '**loss**' of the tour. Then the length in the case of an open tour is $L = n^2 - 4(n-2) - b - 1 = n^2 - 4n + 7 - b$. For example in the 11×11 tour shown below $n = 11$, $b = 8$, $L = 76$. [If instead of the length L we count the coverage C, Merson's formula can be put in the form $C = (n-2)^2 + 4 - b$, true for open or closed tours. The estimate $C \approx (n-2)^2 + 4$ is a slight improvement on the value $(m-2)(n-2)$ conjectured by Yarborough for rectangular boards.]

The following diagrams are some example open tours by Robin Merson, including an illustration of his VW extension method. Note that eight voids (x), one for each extra rank or file, plus four extra voids (blobs, o) are introduced by each VW formation, one inserted in each side.

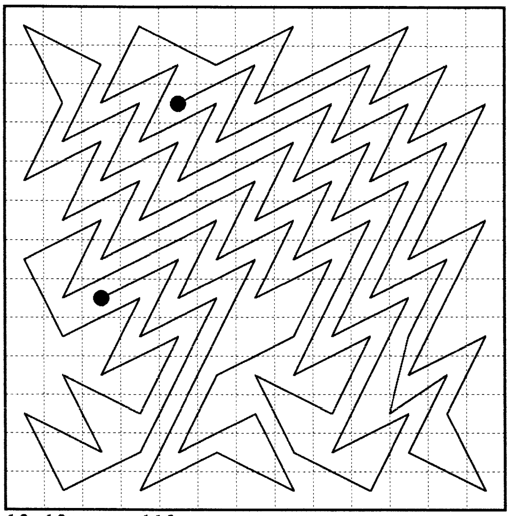

10×10, open, 61 moves

A tour 18×18 can be formed by VW extension of this 10×10 tour, but it covers only 237 cells, and 238 is possible

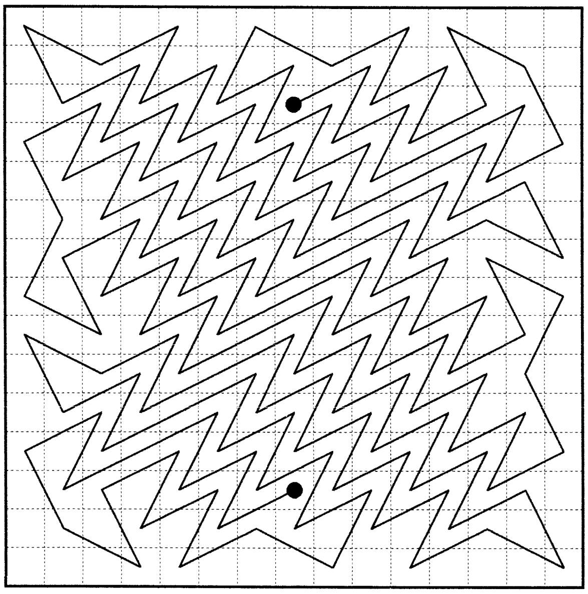

11×11, open, 76 moves


19×19, open, 268 moves formed by VW extension of 11×11

Further we show Merson's 13×13 solution, and a 15×15 solution with its extension to 23×23. These latter are the only symmetric solutions that he produced. The next larger case, that of a 16×16 tour of 181 moves extended to 24×24 tour of 453 moves, is shown on our front cover. He conjectured, from the table, that a 183 path 16×16 ought to be possible but was not able to find one. The 24-size was the largest open tour that he actually diagrammed. The figure for 26 shown in the table was mentioned as an extension from the 237-move 18×18 solution, and the figures for 25 and 27 to 30 are implied by his graph of 'excess' values shown at the end of this article.

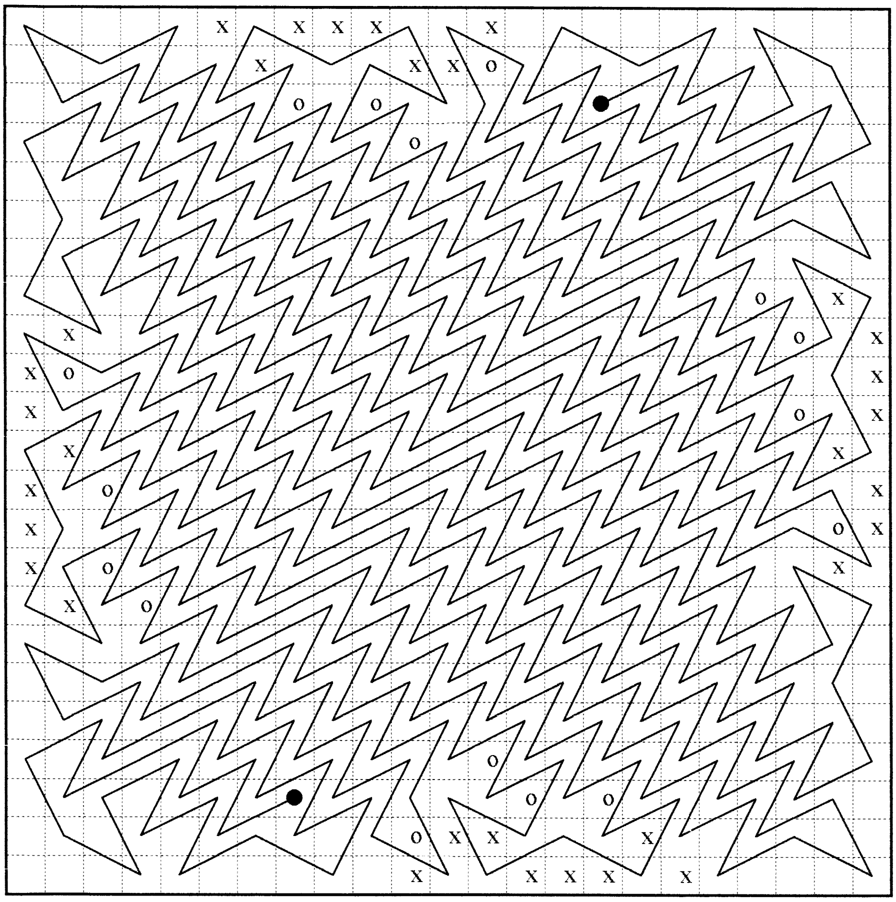


13×13, open, 113 moves

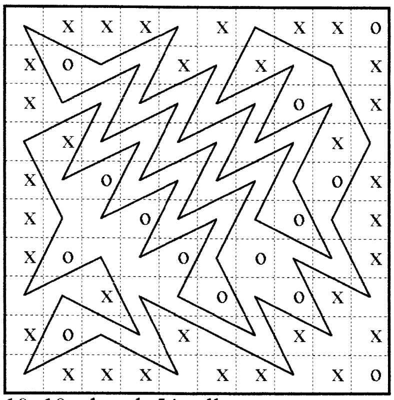


15×15, open, symmetric, 158 moves

23×23, open, symmetric, 414 moves

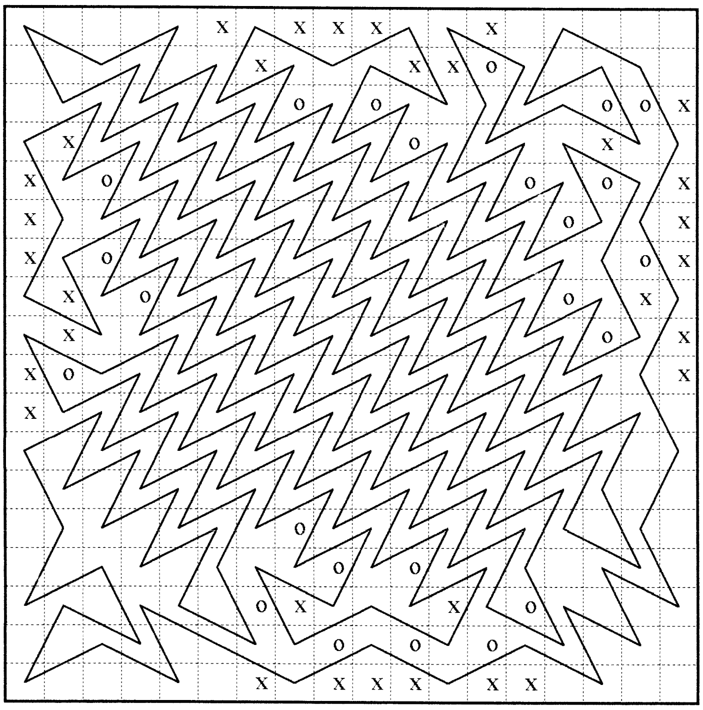


We conclude for the present with a pair of closed tour examples by Merson, showing the VW extension method again. Note that on two edges the VW-formation is moved in one step to allow the extra connecting path to pass round the outside.



10×10, closed, 54 cells

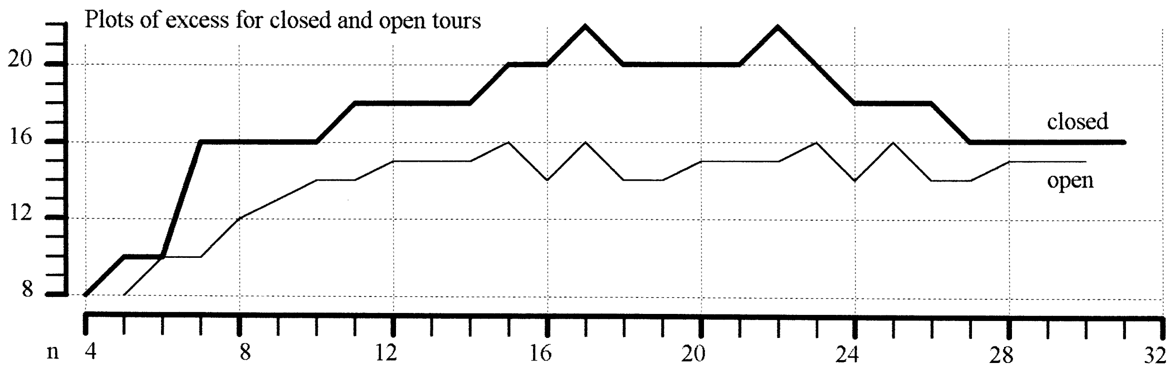
By VW extension this produces the 18×18 tour shown, which in turn can be extended to give a 26×26 tour covering 518 cells, however. a tour covering 520 cells is possible.



18×18, closed, 222 cells

Analysis. What function remains constant during such VW extensions? The board size increases from n to $n' = n + 8$. In the $n \times n$ tour we have $C = n^2 - 4n + 8 - b$. In the open tour case the loss becomes $b' = b + 16$ (4 extra blobs in each side) while in the closed tour case it becomes $b' = b + 24$ (4 extra blobs at top and left, 8 extra at right and bottom). Thus in the open case $2n - b$ remains constant (i.e. $2n' - b' = 2n - b$), while in the closed case $3n - b$ is constant. These numbers can be called the **excess** (E) of the tour. Writing them as $gn - b$ ($g = 2$ for open, 3 for closed) we find the formula: $E = C - n^2 + (4+g)n - 8$, where 4+g equals 6 for open, 7 for closed.

Below are plots of E for the maxima so far found. In the open case for $n \geq 10$, $C \geq n^2 - 6n + 22$. The plot suggests that the maximum value of E for open tours is 16, or does it increase further? For closed tours the excess increases to a peak of 22 and then falls off, and Robin said he would be surprised if it is greater than 16 for any n greater than 31. To summarise: for $7 \leq n \leq 31$ maximum length tours have a length of at least $n^2 - 7n + 24$ and for $n > 31$ have a length of at least $n^2 - 7n + 22$.

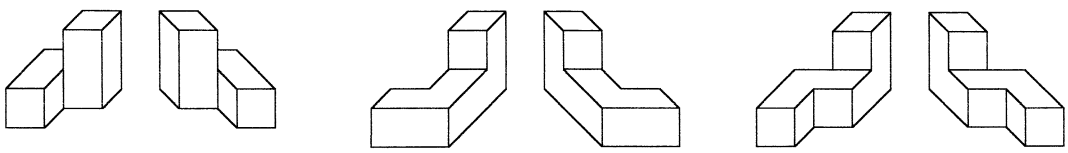


[The above account is simplified slightly from Merson's original version, in which he defined an 'excess' for a closed tour equivalent to $E + 8$, and a 'strength' for an open tour equivalent to $E + 7$ (the difference of 1 resulting from defining it in terms of the length $L = C-1$).]

Polycube Constructions

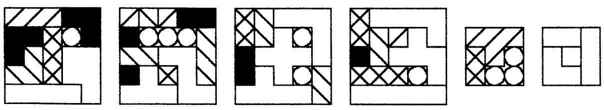
continued, from notes by Walter Stead

First it is necessary to correct an error in my commentary earlier in this series (page 227) where I stated that "The asymmetric cases occur in enantiomorphous pairs". This is not quite correct. It is necessary to say that "The pieces without reflective symmetry occur in enantiomorphous pairs" (which may be a tautology). This is because in fact three (pairs) of the 'irreflectible' pieces are 'rotatable'; by which I mean that they can be superimposed on a copy of themselves by rotation. Namely the following three pairs: one pair of 4 cubes and two pairs of 5 cubes. Visualising these rotations is not at all easy.

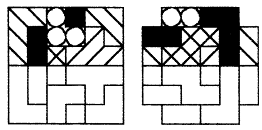


Now we can continue the Frans Hansson series of problems, sent to Walter Stead on 24 June 1954.

(10A): Arrange the reflectible pieces of 1, 2, 3, 4 or 5 cubes into a 5×5×4 crowned with a central 3×3×2. (10B): Arrange the irreflectible 4 and 5-cube pieces in a 6×6×2 with the 4 corner cubes in one of the 6×6 layers omitted. The E-pairs being symmetrically disposed. (This is a type II problem, i.e. using both pieces of each enantiomorphic pair.)



10A 10B



Solutions to the next group of problems (11) to (15) are not illustrated in the notebooks — so anyone who has made a set of pieces may like to try them. Number (15) has only 23 cases to consider!!

(11A): All pieces of 1, 2, 3, 4 and 5 cubes, in 5×8×4 with 8 corner cubes removed. (11B): The 4 or 5-cube pieces not used in (11A) in 3×7×2 with the corner cubes removed. (These are Type I)

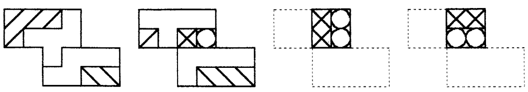
(12): Pack all pieces of 1 to 5 cubes in 7×9×3, omitting centre cubes in each layer. (Type II).

(13): As for (12) in 7×7×4 omitting 8 corner cubes and central cubes in two layers. (Type II).

(14): All 4- and 5-cube pieces in 5×5×5 with 3×3 added centrally at two opposite faces. (Type I)

(15): All 5-cubes and 'flat' 4-cubes can be assembled into a 5-cube shape enlarged 3 times: all cases solved. (Type I) There are 5 'flat' 4-pieces, 12 'flat' 5-pieces and 11 'solid' (i.e. two-decker) 5-pieces, total 27×5; i.e. the right number to form 5 cubes 3×3×3.

(16): Assemble 8 of the 9 'unsymmetrical' 5-cube pieces to form the ninth, twice its linear size. [The text lists the pieces as the four asymmetric 'solid' 5-pieces (in either of their enantiomorphic forms) and the five 'flat' 5-pieces formed from the asymmetric plane pentominoes; but strictly speaking these 'flat' pieces are no longer asymmetric in three dimensions since they can be reflected in the plane through the centres of their cubes. Hence my inverted commas.] One example solution is given:



(17): The 'unsymmetric' flat 5-pieces and symmetric 2-decker 5-pieces in 5×4×3. (i.e. 5 flat, 5 solid reflectible and 2 solid rotatable) (Type I).

(18): The 'unsymmetric' 5-pieces in 5×3×3. (5 flat, 4 solid) (Type I).

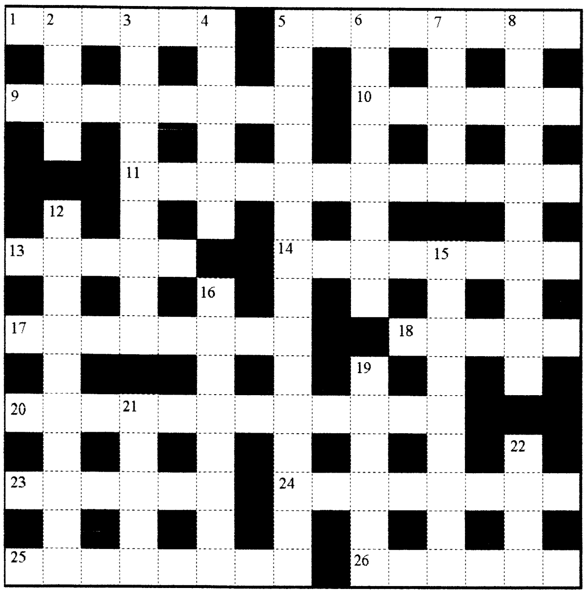
(19): The 'symmetric' 3-, 4- and 5-cube pieces in 10×5×2 (i.e. the rotatable 3-, 4- and 5-pieces and the reflectible 2-decker 5-pieces. (Type I)

(20): The same pieces as for (19) in 5×5×4. (Type I)

to be continued

Word and Letter Puzzles

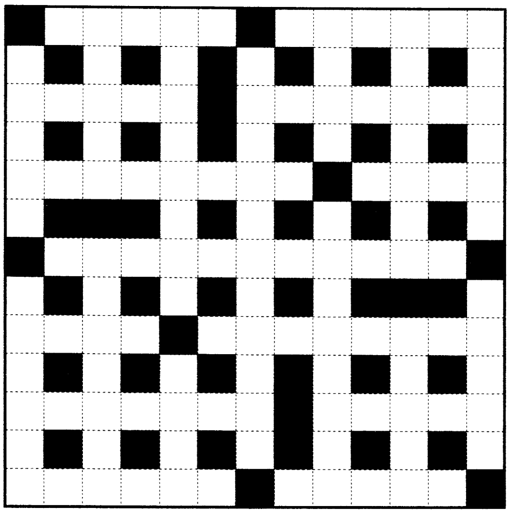
Cryptic Crossword Number 12 by Querculus



ACROSS
1. Make off last month with current amount. (6)
5. Make off after insect or horned creature. (8)
9. Museum worker to set error right? (8)
10. Rum and tea become better when old. (6)
11. Concert skill may produce complaint akin to athlete's foot or housemaid's knee. (12)
13. Made off with, on foot? (5)
14. Sickening North American use Oxford University seconds. (8)
17. Ghostlike rap Celts may create. (8)
18. Make off with least disorder. (5)
20. Vote for king, work late, and find the silver lining. (12)
23. Pulling up to greet ruler. (6)
24. Patronage of gold flavourings. (8)
25. Bury others headless for profit or fun. (8)
26. Dehydrating the Chinese medicine man? (6)

DOWN
2. Dull but poetic time of endless occurrence. (4)
3. Thoughtless Mao makes authentic substitute for chicken. (9)
4. Survey the site of Late Titicaca. (6)
5. The big-eared variety of Nepal hare in fact! (7, 8)
6. Membrane I get tummy pain from. (8)
7. Louts disturb meditative state. (5)
8. Discriminating detail. (10)
12. Scum in pool disturbed by force. (10)
15. An intervention in the end produces extra enmity. (9)
16. Professional villain to bring Parliament to a halt. (8)
19. Brought about a deletion from the fourth amendment. (6)
21. Villainy organised in central America. (5)
22. Make ends meet in sharp point. (4)

Jigsaw Crossword Number 2 by Querculus



The answer to each clue begins with the first letter of the clue.

Assert seniority as normal (7)
Bold enough for second class party (5)
Commend gold owing to one (8)
Deeply fears fourth class books (6)
Equal bird recently departed (7)
Falls apart, sounding like part of P (5)
Given shiny covering the force idled (6)
Hello audibly raised on drugs (4)
Involves heavy or flattening sarcasm? (5)
Juxtapose oriental with harmless mouth organ plant (8)
Know how to remember, like grannies (5)
Liliaceous emblem (4)
Muddy waters lying low (5)
Northern trouble fixer (4)
Observer about beauty (8)
Perhaps 'Who Dared Won' (11)
Qualification lacking in practice (5)
Raised land line (5)
Stay on line to Carlisle (6)
Trembles almost high in trees (6)
Up and in Rugby, but down in Australia (5)
Variable Z direction (8)
Will alter mood to hear but not see trees (4)
X in holy post plays bars of W (11)
Yucky food of that Thor guy (7)
Zero in apathy, but green-eyed and wrong-headed (7)

Just solve the clues and fit the solutions into the grid.

Puzzle Answers

25. Cryptarithms

T.R.Dawson gave full-page solutions to his two cryptarithms but we only have space for a brief sketch. The answers: (a) letters representing the digits 1234567890 spell out SHARP KNIFE. (b) The letters represent 2348 and spell DEAR. The calculation is 4+4 = 8, 8×4 = 32, 32–4 = 28. (Messrs Marlow and Willcocks reported solving these, and I hope other readers tried them.)

In (a) the given ratios can be rearranged in the form of a multiplication calculation: RIA × IR = SFAH + AIKR0 = REPNH. From the addition we see that R = A+1. We want A×R to end in H, and testing 1×2, 2×3, etc we find H is 0, 2 or 6. Trying H=0 with A,R = 4,5 or 5,6 won't work. Similarly H=6 with A,R = 2,3 or 7,8 won't work. This leaves H=2 with A,R = 3,4 or 6,7 or 8,9.

In (b) if the first step is subtraction then R=0 but then a two digit number ED is impossible; if the first step is multiplication A can only be 2 or 3 (else R = A or needs two digits), but trying cases eliminates both; so first step is addition. The second step cannot be subtraction since R–A = 2A–A = A which is not ED. Consideration of + or × shows A=4 and R=8, and the rest follows.

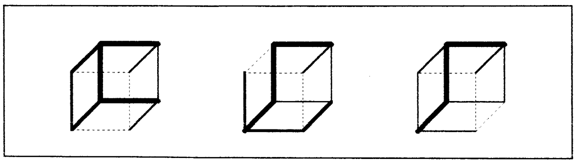
It is surprising what can be deduced from so little information in this sort of problem.

26. Wire-Framed Boxes

(a) To bend four equal pieces of wire to form a cubical frame. How many ways?

My answer was, essentially two, but Tom Marlow counts three, and I think I prefer his analysis, though the discrepancy is partly a matter of definition of what counts as 'different'.

There is a solution with the pieces all in two-dimensional 'C' shape; the Cs follow each other round the cube; there are 6 different orientations.



The second way bends all the pieces in a three-dimensional zigzag shape, its three segments in the directions of height, width and depth; these fit together in pairs in a 'table with two legs at opposite corners' formation. Mr Marlow notes that these zigags occur in left and right-handed forms, and they can either all be of the same handedness, or can be two of each type (the same types fitting together to make the 'two-legged table'), so he separates these solutions into two cases; each is possible in six orientations.

In the diagrams of the three cases the dotted piece is supposed to be at the back.

(b) This part of the question was arithmetical rather than geometrical. Answers:

(1) 13 units: (1,6,6) (2,2,9) volume 36.

(2) 14 units: (1,5,8) (2,2,10) volume 40 and (2,6,6) (3,3,8) volume 72.

(3) 19 units: (2,8,9) (3,4,12) volume 144.

(4) 21 units: any two pairs from (5)

(5) 21 units: (1,8,12) (2,3,16) volume 96 and (2,7,12) (3,4,14) volume 168 and (3,8,10) (4,5,12) volume 240.

27. No Mean Speed!

(a) If I go a given distance at 5 mph and the same distance at 15 mph then my average speed is 7.5 mph. (It is irrelevant whether the route is up or down hill, though this explains the difference in speed.) If the distance is d mile then at 5 mph it takes me $d/5$ hour and at 15 mph it takes me $d/15$ hour. Thus $2d$ miles takes me $d/5 + d/15$ hour = $4d/15$ hour so d miles takes me on average $2d/15$ hour, i.e. 15/2 miles per hour.

The mean of two speeds u and v in the same units over the same distance is $2/(1/u + 1/v) = (2uv)/(u + v)$ which is known as the **harmonic mean**, it is not the arithmetic mean.

(b) If I go up-hill at 5 mph then, to give me an average of 10 mph, I must come downhill in no time at all, i.e. infinitely fast! In other words, it is impossible. This may be why cyclists tend to go at excessive speed down-hill; they are trying to double their mean speed.

28. The Sixfold Way

The first pair of primals (i.e. numbers of the form 6n ± 1) in which both members are composite is (**119**, **121**) since 119 = 7×17 and 121 = 11×11. The composite nature of the first of these numbers is easy to miss.

The product of any two primals is a primal (i.e. the set of primals is closed with respect to multiplication). If we call primals 6n + 1 **upper** and primals 6n – 1 **lower** then the product of two like primals is an upper primal, but the product of two unlike primals is a lower primal.

$(6h+1)(6k+1) = 6(6hk + h + k) + 1$
$(6h-1)(6k-1) = 6(6hk - h - k) + 1$
$(6h+1)(6k-1) = 6(6hk + k - h) - 1$

Or, more generally, a product of primals is upper or lower according as the number of lower primal factors in it is even or odd.

The next case is **(143, 145)** where 143 = 11×13 and 145 = 5×29. Since $(a-1)(a+1) = a^2-1$, when the number 6n is a square ($a^2 = 6^2k^2$) then 6n − 1 is composite. The above case is k=2. The next case k=3 generates **(323, 325)** where 323 = 17×19 and 324 = 18^2 and 325 = 5^2×13.

John Beasley writes: "I suspect that the reason for the non-citation of 'primals' in the literature is two-fold: (a) the result in itself is too elementary to be worthy of comment, and (b) neither the primes of the form 6n + 1 nor those of the form 6n − 1 have any particularly interesting properties. Contrast the striking and by no means obvious result that every prime of the form 4n + 1 can be expressed as the sum of two squares."

John further observes: "The only interesting result that I personally know about primals is that '*m* is a primal' is precisely the condition for it to be possible to place *m* non-attacking queens on an *m*×*m* cylindrical chessboard." (For more on this see J. D. Beasley, *The Mathematics of Games*, Oxford, 1989 pages 82-85).

Against his points I would contend that there are many equally elementary, not to say trivial, results that are found worthy of comment in the literature and that further interesting properties, such as the one he cites, may remain to be found.

29. Birthdates

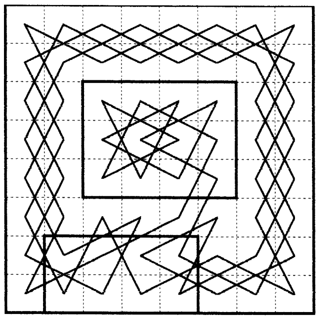
The question, designed to provoke arguments, was: If a baby is born on the first leap day of the next millennium, when will its first birthday occur and how old will it then be?

The answer to the second part of the question is **zero**. Babies are only born once and they start at age zero. They reach age one on the first anniversary of their birthday. Calling the Nth anniversary of one's birthday a 'birthday' is inaccurate shorthand for 'birth anniversary day'.

The first part of the question is calendrical. The first leap day of the next millenium is 29 February 2004. It is true that unlike most end-of century years, 2000 is also a leap year, since it is divisible by 400 but it is the last year of the 20th century, and is not in the next millennium, which begins on 1st January 2001.

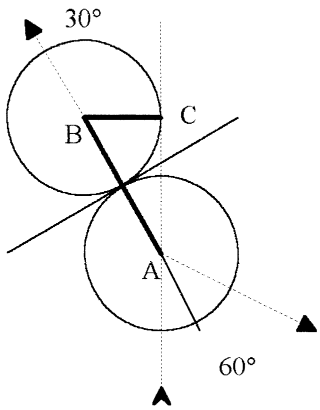
30. Knight's Tour

The second knight's tour with maximum braid and central subtour is:



31. Snooker Shots

As the result of a half-ball shot the object ball should be expected to move forward at 30° to the line of action of the cue and the cue ball to rebound at 60° to the cue line.



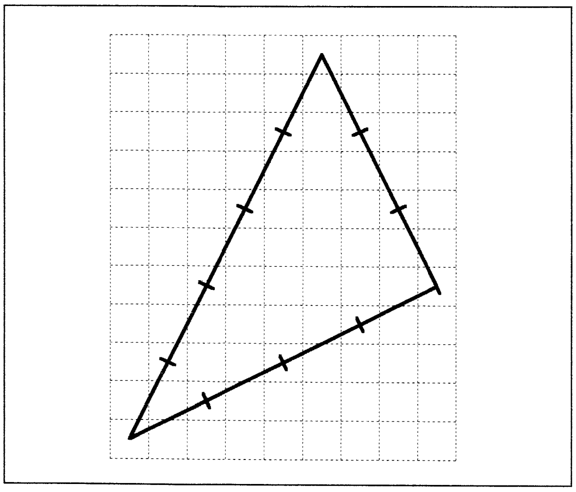
The triangle formed by the centres of the balls A and B and the point C aimed at on the object ball is a 30-60-90 triangle, i.e. half an equilateral triangle, since (a) the line of action is tangent at C and (b) the lengths of the other two sides are 1 and 2 radius units. The cue ball reflects as if from the tangent plane at the point of contact, making equal angles with the normal.

John Beasley, recalling his schoolroom mechanics of over 40 years ago, provided a dynamical argument, resolving the velocity *v* in directions parallel and perpendicular to the plane of contact, and considering elasticity of the material. This led to the same geometry, but I remain unconvinced about his additional results on the subsequent velocities of the balls.

As a special case he maintains that "In the extreme case of zero elasticity and infinite friction, the balls coalesce and continue with velocity *v*/2 in the original direction of motion." Difficult to play snooker under those conditions!

32. Knightly Triangles

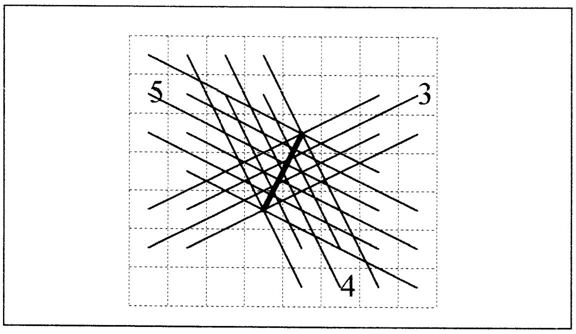
The fact that knight moves make the same angles with each other as the angles in 3:4:5 Pythagorean triangles was noted by H.J.R.Murray on p.3 of his 1942 manuscript about *The Knight's Problem*, in fact he evaluates the acute angles as 36° 52' 11.4" and 53° 7' 48.6". On p.18 he has a diagram of such a triangle formed of 3, 4 and 5 knight moves (as shown below), which is sufficient to prove the point. Since a knight move has length $\sqrt{5}$ (taking the sides of the squares of the board as unit), this triangle has area ½ base × height = $\frac{1}{2}(4\sqrt{5})(3\sqrt{5})$ = 30 square units. To make this complete triangle the knight requires a 9×11 board, somewhat larger than its usual domain.



Any other knight lines drawn on this board will be parallel or perpendicular to one of these three lines, so will not create any new angles. We can therefore say that all triangles formed of lines of knight moves are 3:4:5 triangles.

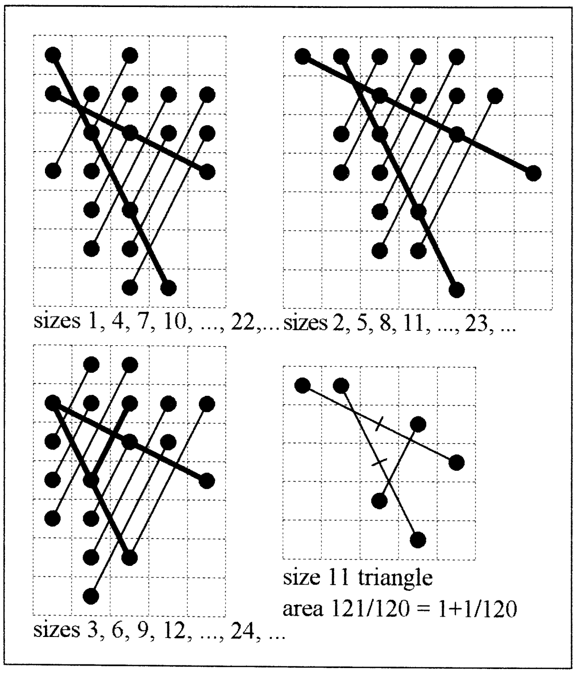
The following is a proof of my more general result, that if we number the triangles from the smallest upwards a size k triangle has area $k^2/120$. I communicated this result (but without a complete proof) to Prof. D. E. Knuth in a letter of 4 December 1992, in response to his idea of a 'celtic tour', which he defined as a tour having 'no three lines nearly concurrent'; in other words no size-one triangles. (For an example celtic tour see p.287 of the last issue.)

A set of close-spaced parallel knight-lines cuts a knight move crossing it into either 5, 4 or 3 equal parts, depending on the angle (see the following diagram). Thus the distance between the two points where knight-lines at different angles cross another knight-line will be a multiple of sixtieths of a knight move, since the least common multiple of 3, 4 and 5 is 60.



Denoting a 60th of a knight move, $\sqrt{5}/60$, by u, the sides of the size k triangle are 3ku, 4ku, 5ku and the area (½ base×height) is thus $6k^2u^2$. Inserting the value for u and a little arithmetic gives the required $k^2/120$.

The diagrams below show all the possible sizes of knight triangles from k = 1 to 24. The long heavy lines are those of the most acute angle.



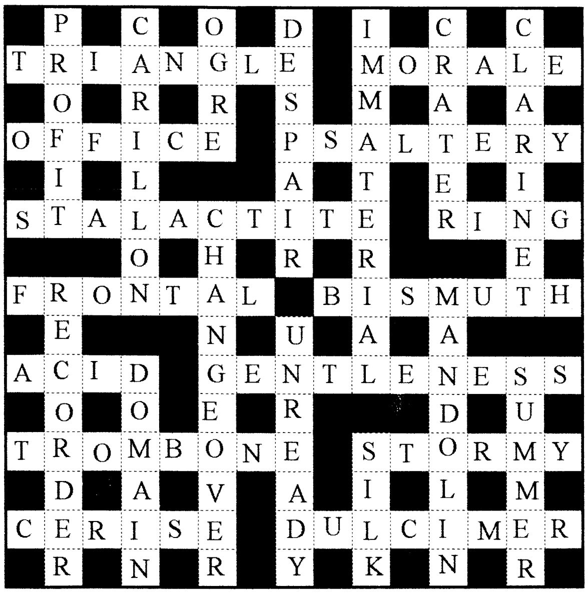
The answers to parts (b) and (c) of the question asked are:

(b) The size 11 triangle is the closest to unit area, since $11^2/120 = 121/120 = 1 + 1/120$.

(c) A triangle of three successive knight moves, being type k = 12, has area 6/5 = 1 + 1/5. (See the bold cross-line in the lower left diagram.)

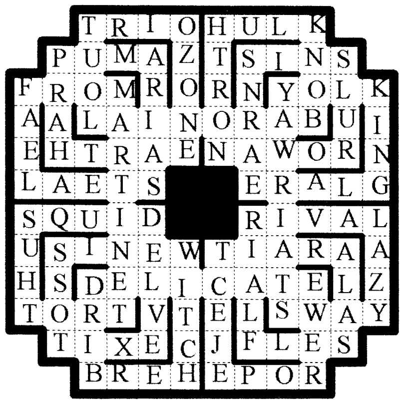
The size number k for the big 3:4:5 triangle shown in Murray's diagram calculated from $k^2/120 = 30$, is k = 60, as might be expected.

The number 60 has other resonances with knight moves. T.R.Dawson's manuscripts in the BCPS Library include a chart of 'The Unit of the Nightrider's Two-Move Domain 60×60' which shows that any square can be reached in one two or three knight-rider moves.



Cryptic Crossword 11

Solutions to Word Puzzles in GPJ 16



Tangleword Puzzle 3.

Puzzle Questions

There are more questions than usual in this issue since number 18 will contain the index to complete volume 2. This will not appear until early in 2000, to give good time for solving, and for solvers' comments to be included. The future shape of the journal should also then be clearer. There are distinct possibilities of getting up to date with a web-site and e-mail.

33. A 'Countdown' Curiosity

On Tuesday 21 September 1999 in the 4.30pm Channel 4 TV Letters and Numbers game 'Countdown' one of the number games required the contestants to form the answer 155 from any subset of the six numbers selected. The first three numbers were 25, 6 and 5. Everyone agreed this was very easy and gave the solution $(25 \times 6) + 5$. This came as a surprise to me, since I had found the solution as $(25 + 6) \times 5$.

Apart from the trivial case involving two ones: $(N \times 1) + 1 = (N + 1) \times 1$ my question is: What other selections of numbers A, B, C make possible $(A \times B) + C = (A + B) \times C$, with interchange of the two operations? Since A and B can be interchanged, we can assume that $A \geq B$.

The numbers used in Countdown are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 each occurring at most twice and 25, 50, 75, 100 which can only occur once. However, more ambitious solvers may like to consider the problem without these restrictions.

34. Mu Torere

The shortest game ending in stalemate follows the unique sequence A, ~H, I, ~B, I*, ~H*, Z* (3 pairs of moves) where on the last move white stalemates black. This is a 'superstalemate' in that all the black pieces are in one group. Our puzzle is: Under the restriction that a player who can play a stalemating move must do so (i.e. 'reflex' play), find the unique sequence leading to black superstalemate of white. Plotting a route through the chart on p.303 should be sufficient.

35. A Subtraction Square

In our review of the book by Reichmann mention is made of a 3×3 'magic subtraction square'. The question that arises is: Can a similar 5×5 construction be made? From a line of values a, b, c, d, e the successive subtraction rule produces $e-(d-(c-(b-a))) = (a+c+e) - (b+d)$, so once again it does not matter from which end of the line we work.

36. Non-Intersecting Tours

There are some challenges still arising from Robin Merson's work on non-intersecting Knight's tours featured in this issue. Can any of his results be improved on? In particular you are challenged to construct 32×32 examples, open or closed, with maximum coverage. According to Robin Merson's formulae the number of cells covered should be at least 854 (open) or 822 (closed).

37. Wire-Framed Octahedron

Tom Marlow notes that a similar problem to 26(a) arises with the regular octahedron. Given four pieces of wire, all of the same length, you are required to bend the pieces and place them together to form an octahedral frame. How many geometrically distinct ways? This looks a good bit more difficult to visualise than the cubical case.

38. The Tethered Goat

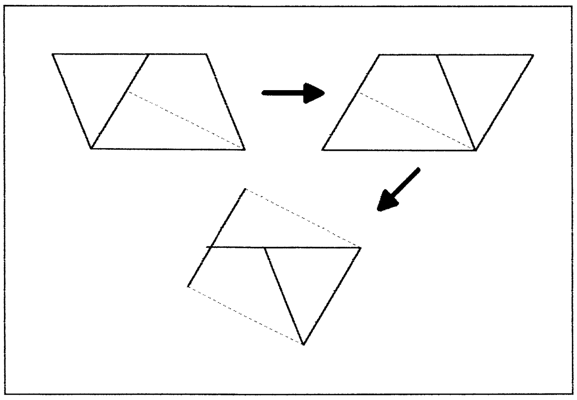
A goat is tethered at the edge of a circular field. What is the length of the tether if the goat has access to exactly half the area of the field?

This problem was mentioned by my brother Edward Jelliss as causing controversy among his colleagues in the Radar establishment at Abu Dhabi Airport. The numerical result is not exciting, but the calculation is interesting.

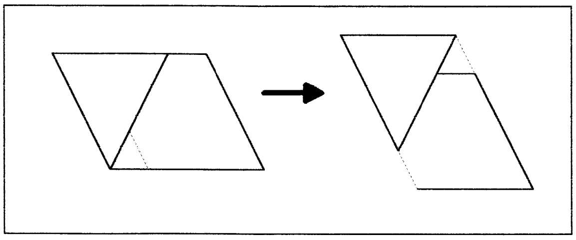
39. A Double Dissection

Euclid, Book I, Proposition 35 says: *Parallelograms on the same base and between the same parallels are equal in area.* This provides a way of converting one parallelogram into another by a two-piece dissection.

Two successive applications of Euclid I.35 gives a three-piece dissection of a parallelogram into another that may have all sides and angles different from the original.



Another type of three-piece dissection of a parallelogram slides one piece along the line of the cut and thus preserves the same angles.



These well known results (e.g. H. Lindgren, *Geometric Dissections* 1964) provide some hints for solving a new problem by Chris Tylor:

To cut a square into four parts, three of them triangles, that can be put together, in two different ways, to make differently proportioned rectangles.

In Chris's solution one rectangle has its ratio of sides slightly less than 1.5 to 1, and the other slightly more than 2 to 1.

40. An Incongruent Fallacy

I favour the notation h ¬ m (read 'h mod m') for the remainder left when h is divided by m (i.e. when m is repeatedly subtracted from h until the number r left is less than m).

Using this, the congruence relation of Gauss, written $h \equiv k \pmod{m}$ means the same as the equality h ¬ m = k ¬ m.

It follows from $h \equiv k \pmod{m}$ that $h*n \equiv k*n \pmod{m}$ where for the sign * we can substitute multiplication, addition, subtraction (of smaller from larger) or raising to a power.

The book by W. J. Reichmann, described on p.298 contains the following demonstration of congruences. Problem: To prove, without calculating the full digital representation of the larger number, that 2^11 − 1 is divisible by 23. Answer: Note that 2^5 = 32 and $32 \equiv 9 \pmod{23}$, so squaring each side $2^{10} \equiv 81 \equiv 12 \pmod{23}$, and multiplying each side by 2: $2^{11} \equiv 24 \equiv 1 \pmod{23}$, and subtracting 1 from each side $2^{11} - 1 \equiv 0 \pmod{23}$, the required result.

The following argument elaborates on an example given by Reichmann. Our question is: Where does the argument go wrong? $32 \equiv 12 \pmod{10}$, that is $2\times16 \equiv 2\times6 \pmod{10}$, so cancelling the 2 on each side $16 \equiv 6 \pmod{10}$, which is true. Similarly this can be written as $2\times8 \equiv 2\times3 \pmod{10}$, so again cancelling the factor 2 on each side $8 \equiv 3 \pmod{10}$, that is 10p + 8 = 10q + 3 for some p and q. But this equates an even with an odd number! Which is impossible, isn't it?

41. Knightly Quadrangles

Draw a quadrilateral of knight's moves which encloses a unit area (i.e. equal to the area of a square of the board).

A more general recreation is to construct knight's tours showing triangles and squares of all possible sizes, alone or in combination. There are many well known examples.

42. Touching Spheres

There used to be a series 'Notes & Queries' in *The Guardian* newspaper, in which reader's questions of all sorts, serious or frivolous, were invited and answers, provided by other readers, were published, usually the next week.

In *The Guardian* of 23 April 1990 William Turner of Hull asked: "If two perfect spheres come into contact, how can the area of touch be calculated?" I sent an answer to this, but never saw any answer published (though may have missed the relevant issue).

My answer began: "Perfect spheres do not exist, except in the fantasy world of infinitesimalist mathematicians, where they can touch at one point, which is of zero area. The area of contact of real spheres depends on their size, material and the force with which they are pressed together."

An answer for the real sphere case is invited. My solution, which was not altogether serious, involved some geometry due to Apollonius.

43. An Uninteresting Number?

In David Wells's compilation *The Penguin Dictionary of Curious and Interesting Numbers* (1987) the first whole number that fails to head an entry is 43. I recently came across a sequence in which it features, so offer it here for his next edition. Can readers provide other examples? (Question 47 has another sequence using 43.)

The Mersenne numbers, of the form: 2^n – 1 = 1 + 2 + 4 + ... + 2^(n–1), giving the sequence {1, 3, 7, 15, 31, 63, 127, ...} are well known and thoroughly treated in all books on number theory, particularly those numbers in the series that are primes (known therefore as Mersenne Primes). Most of the larger primes known are of this type.

But what about numbers of the form 2^n + 1? One more, instead of less, than a power of two. The sequence runs {3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, ...} When n is a power of 2 they are known as Fermat numbers, which have an application to the geometry of polygon constructions, but what of other powers?

It is natural to split the sequence into two according to even and odd powers. When n = 2k then 2^(2k) + 1 = 4^k + 1 giving {5, 17, 65, 257, 1025, 4097, ...} including the Fermat numbers [except for 2^(2^0) + 1 = 3 which is usually counted as a Fermat number, though it is evidently a special case].

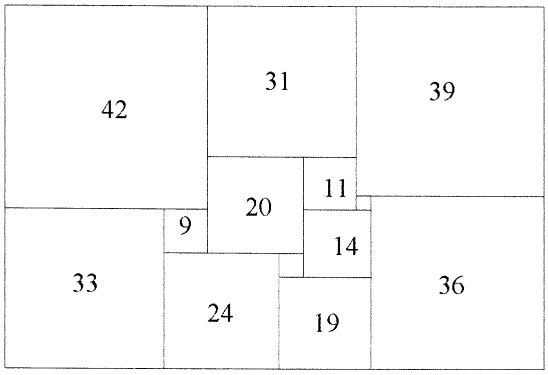
When n is odd however there seems to be no immediate simplification. The sequence is {3, 9, 33, 129, 513, 2049, 8193, ...} which appears to be divisible throughout by 3 giving the sequence {1, 3, 11, 43, 171, 683, 2731, ...} including 43.

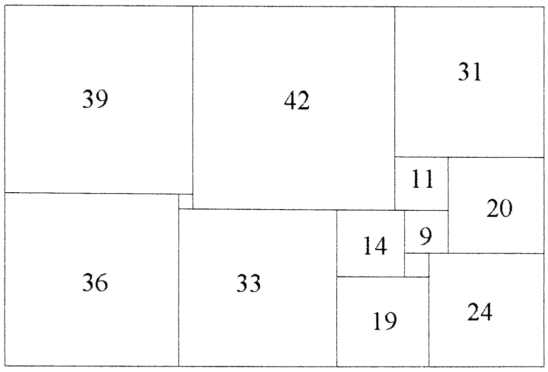
Prove that 2^(2k–1) + 1 is divisible by 3 (for k = 1, 2, 3, ...), and provide a direct calculation for the quotient.

44. A Double Squaring

Mr T. H. Willcocks, sends the latest of his square jigsaw problems: To arrange 21 squares of the following sides into a rectangle in two different ways: 9, 15, 21, 26, 49, 98, 113, 144, 149, 165, 175, 177, 201, 211, 226, 275, 385, 394, 651, 709, 837.

Presented in this form of course the problem is much easier than finding appropriate sizes of the 21 squares in the first place!





To show what is intended, here are the solutions of a similar 13-square problem by Mr Willcocks that appeared in *Fairy Chess Review* February and June issues 1951 (problem 8972). The solutions were presented there 'Forsyth fashion' as a sequence of numbers. The squares used are of sizes 3, 5, 9, 11, 14, 19, 20, 24, 31, 33, 36, 39, 42, and each rectangle is 75 by 112. For comparison I follow the convention of orienting the rectangles to have as large a square as possible in the top left corner.

Diagrams of these solutions appear, though without explanation or source, as Fig. 21 in *Madachy's Mathematical Recreations* by J.S.Madachy (Dover Publications Inc. 1979). This is a reprint of Madachy's *Mathematics on Vacation* (Charles Scribner's Sons, 1966) with additions and corrections; I do not know if the diagrams appeared in this earlier version too.

45. A Double Tour

At the AGM of the British Chess Variants Society in May 1999 John Beasley reminded me of a fiveleaper tour question that I proposed in *Variant Chess* (vol.1, nr.6, p.75, 1991).

Non-chessist readers may need reminding that just as the knight makes moves of length √5 that have coordinates {1,2}, a five-leaper is a type of generalised knight that makes moves of length 5 units, with coordinates either {0,5} or {3,4}.

Since the fiveleaper has four moves at every square, it follows that in a closed tour the unused moves are also two to every square and therefore form either a tour or a pseudo-tour (i.e. a set of closed circuits). The question is: is such a double tour possible?

This question was in fact answered in the affirmative by Tom Marlow in a letter to me of 17 November 1991. I thought I had published his result before, either in *VC* or *G&PJ*, but it seems not. It will now appear in the next issue, with apologies to Mr Marlow for the hiatus.

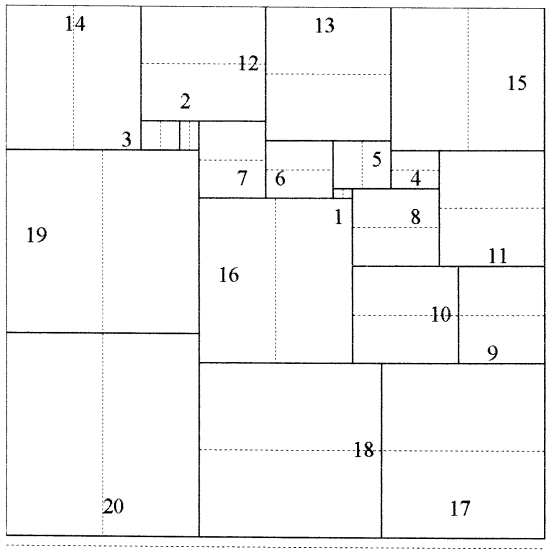
46. Meta-Squares

It is well known that summing the successive odd numbers gives the **square** numbers. The numbers obtained by summing the successive even numbers I call the **meta-square** numbers: general form $n(n+1)$, i.e. a product of successive numbers. The series runs: {2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, ...} Usually we divide by 2 throughout to give the triangular numbers, however meta-square numbers have at least two interesting properties of their own.

The *n*th meta-square is $n^2 + n$ and also $(n+1)^2 - (n+1)$, i.e. a square plus or minus the number squared. The meta-squares are thus distributed in between the squares in a regular manner (hence the name — 'meta' meaning 'between').

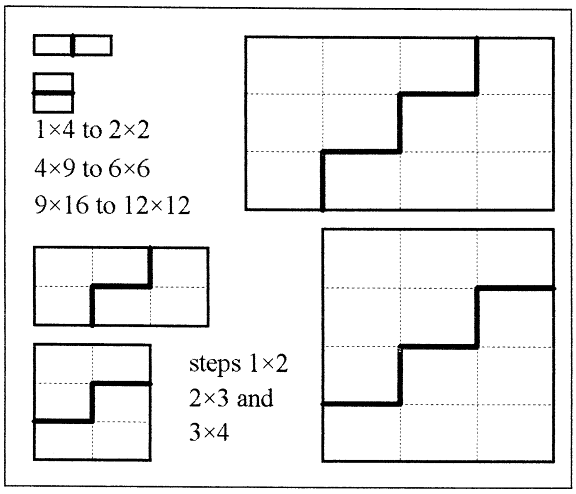
The relationship between the squares and the metasquares however is strangely asymmetrical. Another way of expressing the well known result that a square number is the sum of two triangular numbers is that each square is the arithmetic mean of two successive meta-square numbers: $n^2 = [(n-1)n + n(n+1)]/2$. On the other hand each metasquare is the geometric mean of two successive squares: $n(n+1) = \sqrt{[n^2\times(n+1)^2]}$.

Meta-squares, under the name of 'consecutive rectangles' appear in the dissection problem mentioned in our news item about *J. Rec. Maths* on p.301. Here is a diagram of A. W. Johnson's magnificent 55×56 dissection given there.



I have put in dotted lines bisecting the even edges to make the orientation clearer, and give only the length of the shortest side. Thus rectangle 17 is 17×18. As mentioned earlier the 119×120 case remains to be solved.

Meta-squares also turn up in the much simpler dissection puzzle of cutting a rectangle into two pieces to form a differently proportioned rectangle by cuts along lines parallel to the sides.



The cuts form a 'staircase' with k risers and k+1 treads, or vice versa, so if each step is a×b the rectangles must be ka × (k+1)b and (k+1)a × kb. To convert a rectangle to a square by this method the ratio a : b must be k : (k+1) or vice versa. In other words the squares must be of side n(n+1) and the rectangles of sides $n^2 \times (n+1)^2$ in some unit. This result is mentioned, without proof, in the introduction to *Geometric Dissections* (1964) by Harry Lindgren. Each step of the staircase is a metasquare, n by (n+1).

47. Inter-Square Numbers

Counting 0 as a meta-square (actually it is a square), the meta-square numbers plus one form the sequence: {1, 3, 7, 13, 21, 31, 43, 57, ...} (there is 43 again). These are numbers of the form $n(n-1) + 1 = n^2 - n + 1$.

Similarly the meta-squares minus one form the sequence: {1, 5, 11, 19, 29, 41, 55, ...}. These are numbers of the form $n(n+1) - 1 = n^2 + n - 1$.

I call these lower and upper **inter-square** numbers respectively. The following property of intersquare numbers, though without any special terminology, is mentioned in *The Fascination of Numbers* by W. J. Reichmann reviewed earlier.

The sum of the n odd numbers commencing with the nth lower intersquare and ending with the nth upper intersquare is the nth **cube** number n^3. The series of cubes runs: {1, 8, 27, 64, 125, ...} and the sums take the form: $1^3 = 1$, $2^3 = 3 + 5$, $3^3 = 7 + 9 + 11$, $4^3 = 13 + 15 + 17 + 19$, $5^3 = 21 + 23 + 25 + 27 + 29$ and so on.

The above results can be generalised. Any power of the form n^x can be built up from n consecutive odd numbers, irrespective of the value of x. When n is odd the central number in the odd series for n^x is n^(x–1). For example 5^4 = 121 + 123 + 125 + 127 + 129 = 625, and 125 = 5^3. When n is even the central pair of numbers in the series are n^(x–1) ± 1. For example 4^4 = 61 + 63 + 65 + 67 = 256, where 63 and 64 are 4^3 ±1. [This is also from Reichmann.]

Our puzzle question is tangential to the above discussion. As there are eight lower intersquare numbers less than 64 they offer an opportunity for a Figured Knight's Tour. Since they are all odd they will appear on squares of the same colour, so I suggest a diagonal be tried, but any other formation will be acceptable.

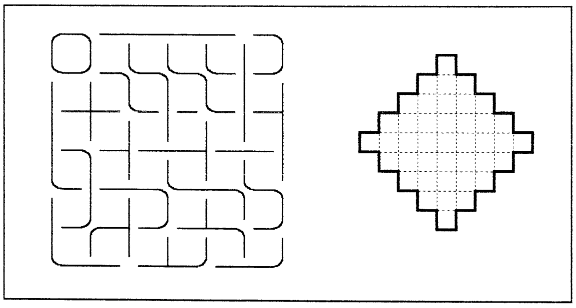
Construct a (preferably closed) knight's chessboard tour with the squares numbered 1, 3, 7, 13, 21, 31, 43, 57 in a regular formation.

48. Aztec Tetrasticks

'Lattice Dissections' of the type shown in the first diagram below were first studied by H. D. Benjamin around 1946-8. A few results by him and T.R.Dawson were recorded in W. Stead's notebooks but they don't seem to have been published in print at the time.

This subject was recently rediscovered by Brian Barwell under the felicitous name of 'Polysticks' (*J. Rec. Maths* vol.22, nr.3, 1990). A polystick is formed, analogously to a polyomino, by attaching unit sticks end to end either straight or at right-angles, or by bending or welding together pieces of wire.

The number of geometrically distinct polysticks of 1, 2, 3 and 4 units are 1, 2, 5 and 16 respectively. We show Benjamin's 1948 solution using the 24 to form a 6×6 lattice square.



If the pieces are perfectly made and fitted together they form a perfect lattice and become indistinguishable, so it is necessary to show them with rounded corners. This also allows their corners to come together without crossing. It will be seen from the diagram that 6 of the tetrasticks and 1 tristick are 'welded'.

In connection with the above Prof. D. E. Knuth has sent a preprint of a paper of his titled 'Dancing Links' which is about a programming technique but is illustrated by recreational questions such as polyomino and chess-piece arrangements. He mentions that there is a 'color' version of this paper available http://www-cs-faculty.stamford.edu/~knuth/preprints.html.

Towards the end of this paper he presents the unsolved problem of packing all 25 one-sided tetrasticks into the 'Aztec diamond' shape shown above. I was able to fit in all but one of the pieces, but a complete solution looks impossible to me, though I could be wrong. If impossible, a proof is required. By 'onesided' polysticks Knuth means that he adds mirror images of the non-axially symmetric pieces to the set, and requires such pieces to appear in their two distinct forms.