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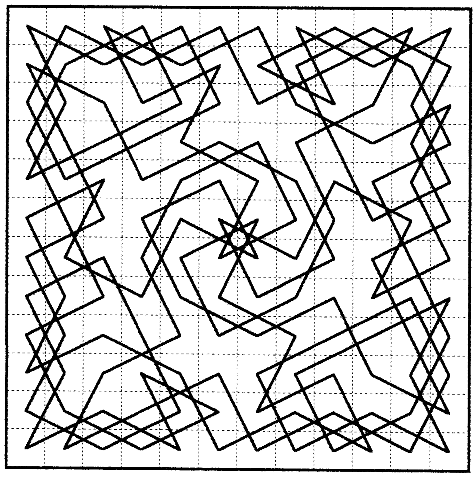
This issue of *The Games and Puzzle Journal* at last completes volume 2, giving Answers to the Problem Questions in issue 17, and providing an Index to the whole volume.

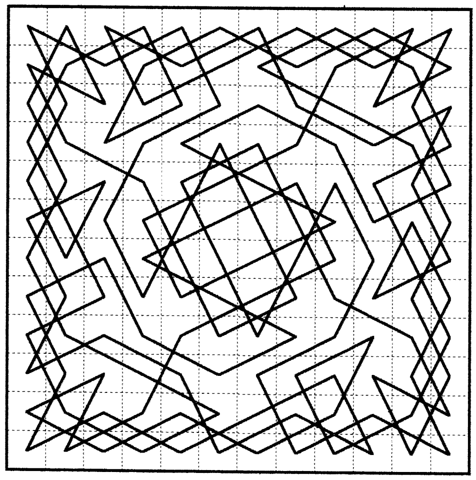
In future *The Games and Puzzles Journal* will appear on the internet. Issues for January and Februay 2001 have already appeared. Visit my site at http://homepages.stayfree.co.uk/gpj for links to these pages. The aim is to publish five items each month; original results are invited. These can be viewed free at present, subject to copyright. A printed paper version will no longer be distributed.

Our cover illustration is of four knight's tours on the 12×12 board showing "mixed quaternary symmetry" constructed by Ernest Bergholt in 1918. See pages 327-341.

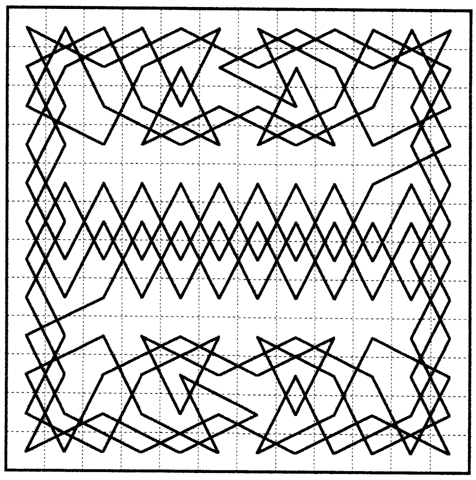
(15)



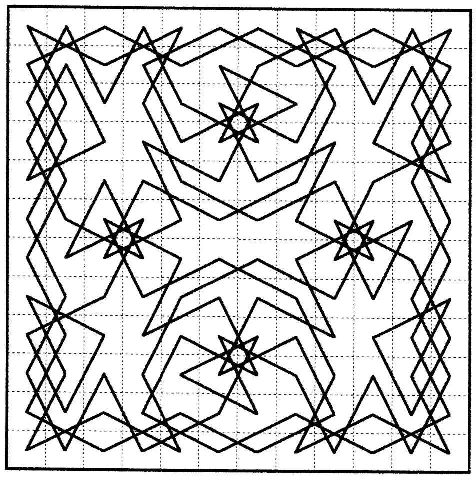
(20) 'Camouflage'



(21) 'The Balcony'



(22) 'The Constellation'



Editorial Meanderings

First some further notes on books on number theory.

A Concise Introduction to the Theory of Numbers by Alan Baker, Cambridge University Press 1984 (reprint 1994). This is very concise and condensed indeed, only 95 pages, but covers a wide range and gives good guidance for further reading. Some of the arguments however take longer leaps than one may be prepared to withstand without vertigo. For example on page 2 the 'greatest common divisor' of a and b is defined to be a number d that divides a and b and such that any common divisor d' of a and b also divides d. This conflates several steps. The name 'greatest common divisor' implies only that every common divisor d' is less than d. The fact that d' divides d needs proof, and the proof usually given relies on the unique factorisation theorem, which Baker proves on page 4. Unfortunately his proof makes use of the gcd — so, unless he has a proof that d' divides d that does not use the unique factorisation theorem, we have a circular argument!

The Nothing that Is: A Natural History of Zero by Robert Kaplan, Allen Lane/Penguin 1999. Like 'Number 9', reviewed last time, this is another small-format book about a single number, which seems to be a current fad. I'm surprised no-one has yet published 'The Little Book of Unity' (perhaps they have). If publishers are commissioning other titles I would like to do 'The Sixfold Way'.

The book is heavy on literary, historical and philosophical speculations but short on actual mathematics. Rather too much space is given to tracing the tenuous history of the symbol 0, which the author likes to think represents the mark left in sand when a counting pebble is removed.

On p.15 he recounts the story from the *Odyssey* of how Odysseus tricked the Cyclops Polyphemos by saying his name was Nobody, so that when he said Nobody had attacked him no help was forthcoming. Surprisingly he does not quote anywhere the similar joke of the White King from *Through the Looking Glass*: "I see nobody on the road," said Alice. "I only wish I had such eyes, the King remarked in a fretful tone. "To be able to see Nobody! And at that distance too! Why, it's as much as I can do to see real people, by this light!".

On p.66 there is an all too brief account of Adelard of Bath (c.1075–1160) who seems to have led an adventurous life, travelling as far as Greece and Syria and returning as an adviser to King Stephen. "But he also brought back mathematical works ('dangerous Saracen magic', William of Malmesbury called it) which he – and later his Irish student, it seems: a certain N. O'Creat – translated from Arabic: the thirteen books of Euclid and the astronomical tables of the great Al-Khowarizmi. In their translations we find three different symbols for zero: ..." On p.208 after quoting from the 12th century 'Salem Codex' (what this is is not explained): "Every number arises from the One, and this in turn from the Zero. In this lies a great and sacred mystery ... He creates all out of nothing, preserves and rules it", the Latin for the underlined phrase is *omnia nihilo creat*, the author wonders "are we dealing instead with some elaborate medieval joke here, a deep pun ..." (viz: N.O'Creat).

On pp.131-6 there is an account of John Napier, whom he credits with the process of solving algebraic problems by 'equation to nothing', i.e. putting all the terms on the same side, then seeking to factorise and using the property that if $a \times b = 0$ then $a = 0$ or $b = 0$ or both. The process of finding the factors also involves zero, in that it often helps to add and subtract the same term. He also (p.203) attributes the binary system of numeration to Napier.

On p.181, in a rather confused discussion of zero as origin he has: "The centre of this centre [it is unclear what 'this' refers to] was called by the early Muslim astronomers 'the cupola of the earth' – by the Hindus, the island of Lanka, their 0° longitude and (charmingly) without latitude. It was there that the demon Râvana built the labyrinthine fortress Yâvana-koti, whose plan is strongly reminiscent of the palace made by the sons of Atlas in Plato's Atlantis." The two designs shown really bear little resemblance other than circularity. The design for the labyrinth is in fact identical to that shown on Minoan coins (circa 1600bc).

A preliminary note says: "You will find the bibliography and notes to the text on the web, at www.oup-usa.org/sc/0195128427/." The start of a new trend in scholarly publication?

Watch

by Derick Green

There are possibly as many board games as there are people who play them, but most are either variants of those played before them or old games with a new theme. Although there is nothing wrong with this as it keeps many good games alive, and in the case of variants many games evolve and some improve, it is very pleasing to find a game that is totally new in play and concept.

I discovered Watch back in the early 1980s and since then it has become one of the few games that I regularly play both face to face and postally. The copy I own, produced by the MPH Games Company (who claim to be the 'Making People Happy' people), is in an A4-size blue flat box covered in red and yellow lettering. The board and pieces are in good quality plastic. Watch, I have been informed was invented and produced in the mid to late 1970s. Unfortunately no designer has been credited. This was a common fault that I am glad to say has been rectified in most games produced over the last few years.

The box contains a 5×5 board that has the square a1 removed and replaced at the top of the board as the extra square b6. I can see no logical reason for this and no explanation is given in the rulebook. The board is white, with the squares outlined in black. Twenty five tiles in ten colours are used to decorate the playing surface; these are 1 dark green, 2 light green, 2 dark purple, 2 light purple, 2 pink, 2 dark blue, 3 orange, 3 yellow, 4 red, 4 light blue. Tile colours do differ slightly between sets, e.g. grey instead of light purple etc. Each player also needs a playing token, e.g. one black and one white pawn. Sets can be made by using painted draughtsmen and two pawns from a chess set, or indeed unpainted black and white draughtsmen.

The game is for two players and after deciding who will begin the game the players must 'dress the board'. To do this each player in turn must place one of the 25 coloured tiles on a square of the playing area in a random manner. While placing tiles each player should be thinking ahead, e.g. to the placing of their pawns and possible positions to capture the opponent's pawn. Once all the coloured tiles are in position, player A places the white pawn on any tile, followed by player B placing the black pawn. The black pawn may not initially be placed next to the white pawn (except diagonally) or on a tile of the same colour.

Each player in turn moves their pawn from one coloured tile to the next in an up, down or sideways direction, but never diagonally. Each time a player moves a playing pawn, the tile from which that pawn has just moved is removed from the board and the game. Moves are made between tiles regardless of the number of empty spaces there may be between those tiles.

A game is won when any of the following is accomplished: (1) A player moves onto the tile occupied by the opponent. (2) A player moves onto a tile of the same colour as that occupied by the opponent. (3) The opponent is unable to move. An interesting sales pitch mentions family tournament play, for which it recommends players buy several additional sets. Although Watch can be played as a single game, it is best played as a series of games with players keeping score from game to game. Each player should play the same number of games as both Black and White.

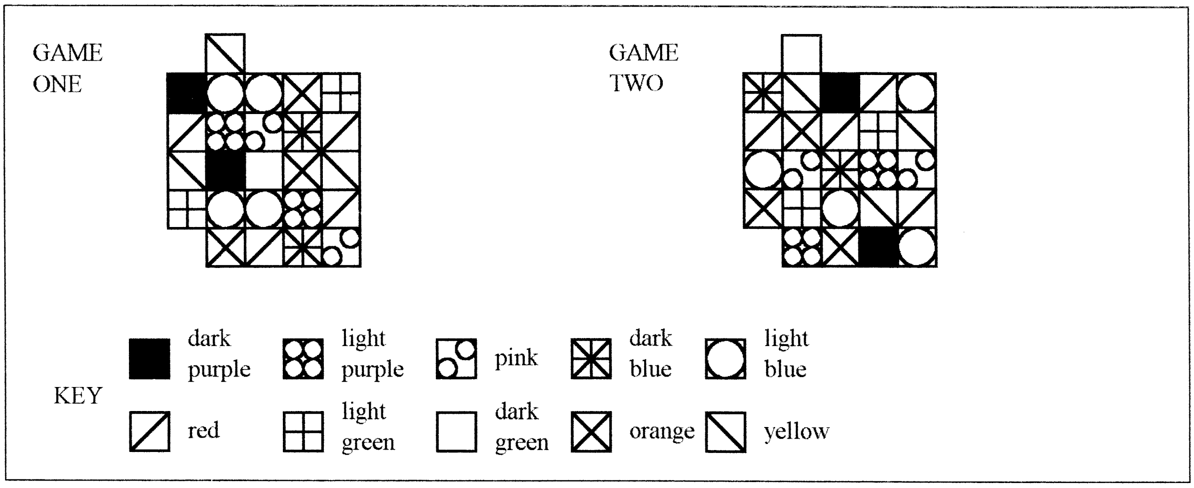
From experience I believe the game to be well balanced. Although the first player gets to place the 25th tile, the placement of this tile is already determined. There may be a small benefit in being able to place the first pawn, but this is offset by the second player being able to react by knowing which tiles surround the first player's pawn.

Thought must be given to tile placement, but a game can be won or lost on the placing of the pawns. For example, try not to place them on the edge of the board and certainly avoid b6. Avoid also having more than one tile of the same colour adjacent to your playing pawn both during initial placement and during game play. Game length varies considerably from two moves to the most I have seen in a game, that of twelve moves.

The scoring system may sound complicated, but means that the players' interest is maintained and that game scores can be carried over to the next tournament. With the aid of a few hand-made sets, I ran a 12-month tournament for a local games club, which worked very well. At the end of a game each player notes down the number of moves they have played, keeping a separate total for games within a series for Black and White. When that series has finished players compare totals. If a player has the lower total as White that player gains one point. The player with the higher total as Black gains one point. If each player has equal points in either Black or White each player gains half a point. However if both players have equal in both Black and White neither score.

Watch: Example Games.

Both the example games are from club play and played with a chess clock set at 10 minutes. Thediagrams show the two arrangements of the coloured tiles.



Game 1.

	tiles:	pawns:
A:	b1, e4, d4, c3, b5, d3, e3, c2, b3, e1, b4, a5, d2	b3, c3, d3, d5, d2 wins
B:	d5, a4, d1, b6, c5, e5, c2, a2, e3, a3, c1, b2	d4, e4, c4, b4

Notes: (1) Set-up: It is often good practice not to place tiles of the same colour adjacent to each other on the board. The two light blue pairs b5/c5 and b2/c2 will act as barriers possibly directing play to the right of the board. (2) Player A achieves several things by moving onto the single dark green (c3). Player B can only move to the red on e4; any other move loses the game. The dark green is the only single tile and so player A is safe there and in the next turn will be able to move to d3 and c2 but not c4 or a3. (3) Player B has lost the game by becoming trapped between the light blue tiles. Player B's mistake was moving back onto the left side of the board. Better, but still losing, would have been :

A: b3, c3, d3, e3, e2, e4, c4, c2, d2 wins
B: d4, e4, e5, d5, c5, b5, b6, b4

Game 2.

	tiles:	pawns:
A:	b6, a4, b2, c5, b4, b3, d5, e5, e4, e3, a3, c4, a5	c2, c1, b1, d1, d4, e4, c4, c5
B:	e2, d4, d3, c1, c3, d2, e1, a2, d1, b1, c2, b5	d3, d2, e2, e3, c3, b3, b2, b4 wins

Notes: (1) Set-up: Both players have attempted during the set-up of the board to avoid placing tiles of the same colour adjacent to each other. (2) A's c2-c1 is a forced move. (3) Forced again is c1-b1. If c1-d1 then d2-d1 wins or if c1-c3 then d2-b2 wins. (4) B's e3-c3 restricts A's possible moves to d4-e4 only. (5) If scoring, the last two moves e.g. either c5-d5, b4-a4 or c5-b5, b4-b5 would be played out.

Regular Polygon Constructions

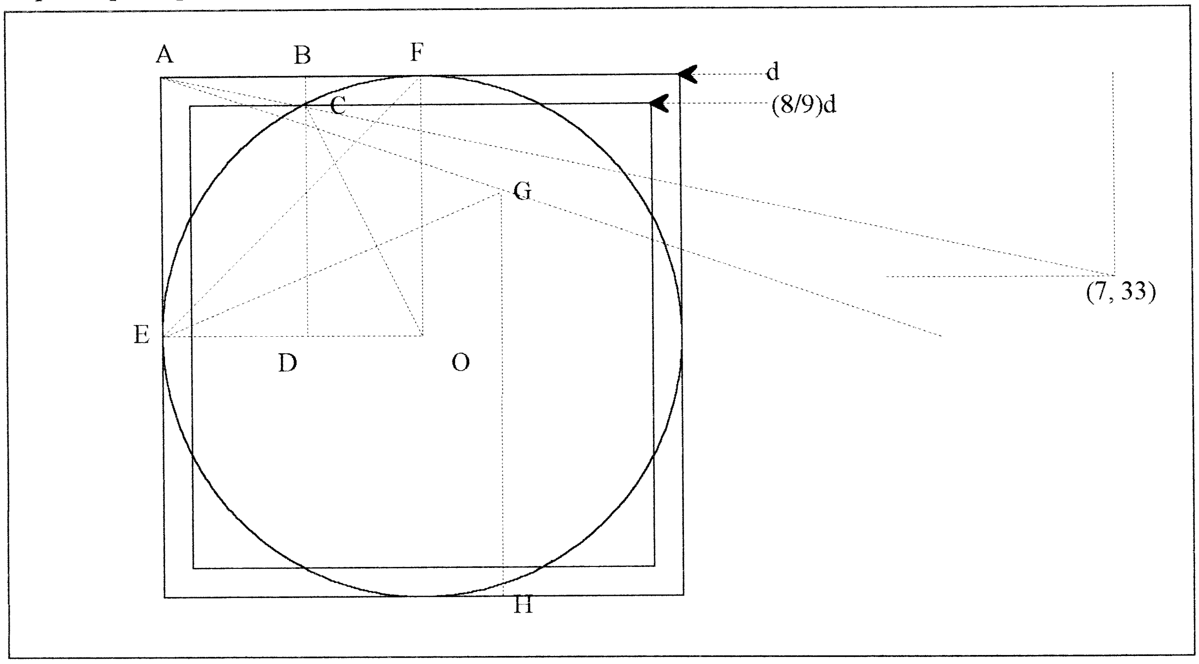
by George Jelliss

The Circular Constant. The ratio of the circumference of a circle to the perimeter of the square containing it is a constant κ (kappa) the same for all circles. The ratio of the area of a circle to that of the square containing it is also a constant, the same for all circles; in fact the same constant κ. To seven places of decimals this **circular constant** is κ = 0.7853981...

To my way of thinking κ seems more physically meaningful than the usual π = 3.1415927... = 4κ. In fact π is the ratio of the area of a circle to that of a square on a radius, or it is the ratio of the circumference to a diameter, neither of which are comparisons of like properties of circle and square.

Leibniz found the neat but slowly converging series: κ = 1 – 1/3 + 1/5 – 1/7 + 1/9 – 1/11 + ...

Squaring the Circle. Here is a possibly new twist on the old problem of constructing, by as few simple steps as possible, of a square nearly equal in area to a given circle.



The length of side of a square equal in area to a circle diameter *d* is (√κ)*d*, and √κ = 0.8862269.... The Rhind Papyrus c.1650bc, approximates the circle area by a square of side (8/9)*d* = 0.8888888...*d*, giving a ratio of 1.003, correct to 3 parts in 1000; very good for such a simple construction.

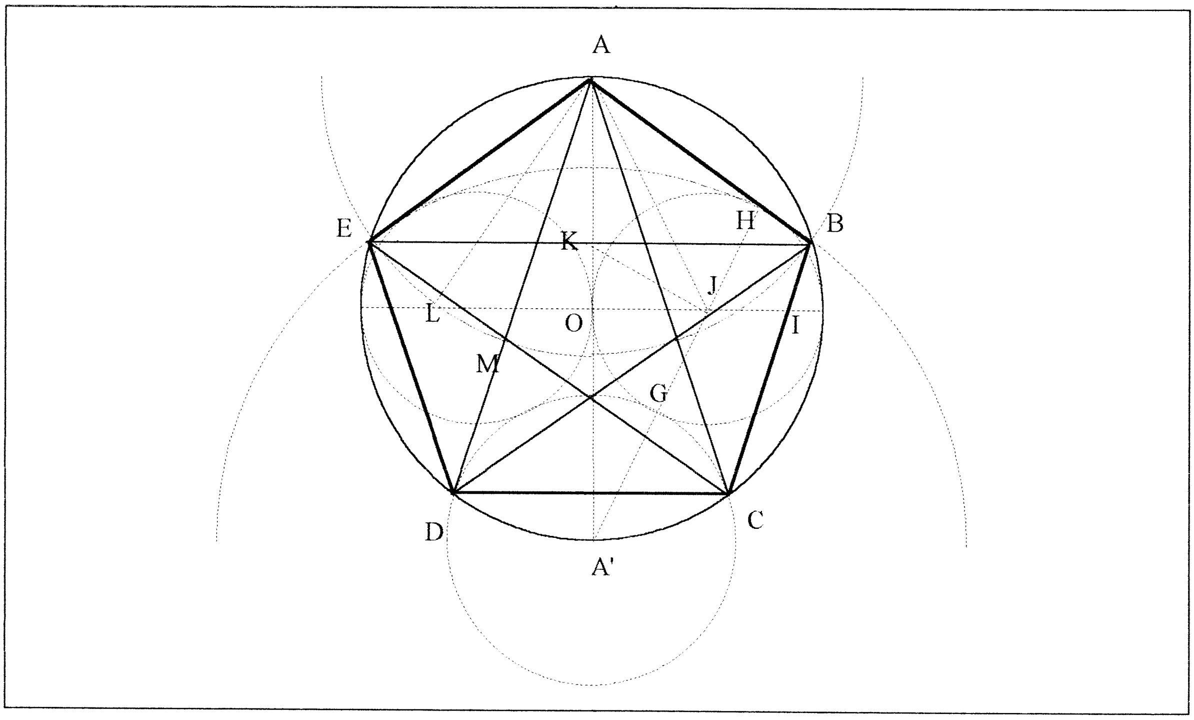
Another way I have found of approximating the equal area square is to draw a line from a corner A of the circumsquare as origin, to a point with coordinates (7, 33). This crosses the circle at a point C on the square. [*Proof*: Draw BCD parallel to the side AE. Take the radius of the circle to be 1. Then AE = EO = OC = 1, and CD = sin θ, where θ = angle COE. The slope of the line AC is CB/BA = (BD – CD)/(EO – DO) = (1– sin θ)/(1 – cos θ) = *m*. Taking CD = √κ we find θ = 62.402886° and *m* = 0.2119672, which can be approximated by 7/33 = 0.2121212. Working back the other way, starting with *m* = 7/33 we get, $(1+m^2)q^2 - 2(1-m)q + (1-2m) = 0$, where *q* = sin θ; a quadratic equation giving *q* = CD = 0.886168. The ratio of this to √κ is 0.9999335, i.e. correct to 7 parts in 100,000.]

The idea for the above construction came in part from the booklet *Ein Dutzend mathematischer Betrachtungen* (1826) by C.R.R.von Schinnern which was mentioned in connection with magic knight's tours (*G & P Journal* #14, pages 238-9). His first 'contemplation' is a method for rectification of the circle which begins by constructing a line from A with slope 1/3. He then finds a point G on this line whose distance from E is equal to EF, and claims the distance GH of this point from the base of the circumsquare approximates to the arc EF. Professor Singmaster finds this length to be (19 – √11)/10 = 1.56833752 which compares with π/2 = 1.5707963, a ratio of 0.9984346; relative error 16 in 10,000.

Some nominally more accurate constructions (for rectification of the circle rather than squaring) are given by Robert Dixon in *Mathographics* (Basil Blackwell Ltd, Oxford, 1987) but because of their complexity I suspect that accuracy is likely to be lost in the number of different steps involved. His first method uses $\pi/4 \simeq (3/10)(1+\Phi)$ where Φ is the larger golden ratio (which he denotes by τ). This approximation is equivalent to $\pi \simeq 0.6\ (3 + \surd 5) = 3.1416408$; discrepancy 15 parts per million. His second method uses the hypotenuse of a right angled triangle with other sides 2 and (3 – tan 30°). This gives a value 3.1415333; discrepancy 20 parts per million.

[The Φ, ϕ notation for the golden ratios was proposed in *G & P Journal* #14 p.245 in the solution to Problem 1. The two values are related by $\Phi - \phi = 1 = \Phi\times\phi$, or $\Phi = 1 + \phi = 1/\phi$.]

Regular Pentagon and Pentagram. Euclid's instructions for inscribing a regular pentagon are complicated, involving first constructing an isosceles triangle with base angles twice the vertex angle (i.e. a 36, 72, 72 triangle) and then scaling it up to fit in the circle, as at ADC. Simpler constructions are given (1) by H. E. Dudeney in *Amusements in Mathematics* (1917, pp.37-38). (2) by H. W. Richmond in W. W. Rouse Ball's *Mathematical Recreations and Essays* (11th edition, 1939, p.95), for comparison with Richmond's construction for a 17-gon and (3) by Robert Dixon in *Mathographics* (p.17), though he does not seem to give explicit instructions for the sequence of operations, and at least two different methods can be seen in his diagram.



All three constructions contain the triangle AOJ or A'OJ, where AA' is a diameter and J bisects the perpendicular radius OI, so that (assuming unit radius) AJ = $\surd 5/2$. Richmond bisects the angle AJO to find the point K on AO, then draws EKB parallel to OI which locates the pentagon vertices E and B, and is a side of the inscribed **pentagram**. Dudeney makes JL = JA, so that LO = $\surd 5/2 - 1/2$ (this is the smaller golden ratio $\phi = 0.618034$), and then locates E and B on the arc with centre A and radius AL. From the triangle AOL we find AL = $\surd(5/2 - \surd 5/2)$ which is the required length of side of a **pentagon**. Dixon extends A'J to H on the circle centre J radius JO, so that AH = $\surd 5/2 + 1/2$ (this is the larger golden ratio $\Phi = 1.618034$). Then he locates E and B on the arc with centre A', radius A'H. Alternatively the circle centre A' and radius AG = $\surd 5/2 - 1/2$ locates C and D. To draw this circle to touch the circle with diameter HG without drawing in the line AGJH is not a proper Euclid-style construction; the point of contact has to be exactly found by intersection.

The above drawing is not as accurately done as I would like, but is the best I could achieve using the limited facilities of the AmiPro drawing program.

Knight's Tour News

Three Memoirs on Knight's Tours

by Ernest Bergholt (1856 - 1925)

Editor's Note: In issue 13 pages 208-217 and issue 14 pages 230-237 and 244 I reproduced six "Memoranda" by Ernest Bergholt that were written in 1916-17 and sent originally to W.W.Rouse Ball. Also included were some historical and biographical notes, and earlier articles published in the magazine *Queen* 1915-16. Bergholt produced three further articles, now termed "Memoirs", which were sent to H.J.R.Murray in 1918. It was planned to include these as a chapter in my proposed book *Knight's Tour Notes*, but since this has not found a publisher I have now decided to reproduce the Memoirs here. They have certainly waited long enough for publication!

The three memoirs consist mainly of Ernest Bergholt's account of what he called 'perfect' or 'complete' quaternary symmetry. Murray proposed the term 'mixed' quaternary symmetry as being preferable, but I have not altered Bergholt's text. Murray's term also is not ideal, since 'mixed quaternary symmetry' is really a type of binary symmetry. Recently I thought of the term 'semi-quaternary symmetry' as an alternative, but am not sure if this is better. The original text is followed closely, though I have ignored Bergholt's over-emphatic style of underlining everything and paragraphing every sentence. Editorial comments are within square brackets [...].

Some isolated results that appear among the 'Miscellaneous Articles and Notes by Ernest Bergholt' among Murray's papers are interpolated in the text where appropriate. The original *Memoirs* are now with Murray's papers on knight's tours in the Bodleian Library, Oxford.

The Method of Terminal Loops

Seventh Memoir on Knight's Tours. (Communicated to Mr. H.J.R.Murray January 16th, 1918.) *An entirely new method for approximate quaternary symmetry (both direct and oblique) over areas of 8m cells*. This method being rather cramped on the 8^2 board (owing to lack of room), it will be clearer if I explain it on the 12^2 board, which [for direct symmetry] I will number as shown below [Figure 1] (where every number is a knight's move distant from the next number in natural order. So also 36 and 1, where the cycle recommences).

I first select two pairs of numbers, each pair in direct symmetry; say 8, 8 and 14, 14. I connect one pair horizontally and the other pair vertically by a cycle twice repeated of an odd number of cells: Suppose we take 8 (15.29.23) (15.29.23) 8 and 14 (6.34.33.32.35) (6.34.33.32.35) 14. The result is shown below [diagram (2)], where the dotted moves depart from quaternary symmetry, forming six pairs in diametral symmetry.

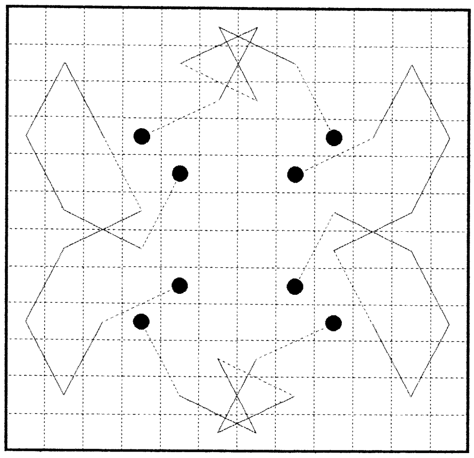
Not counting the 8 and 14, I have now used up eight of my 36 counters. With the remaining 28 counters I form a continuous chain of cells from 8 to 14. Any such chain will form a complete tour of 144 cells, the only departures from perfect symmetry being the six pairs of dotted moves already written down.

Suppose the chain to be 8.13.36.1.2.3.4.5. 11.10.9.20.21.12.25.24. 17.16.18.31. 26.27.28.19.7. 30.22.14. The tour is shown below [5].

(1)

2	27	24	31	16	29	29	16	31	24	27	2
25	32	3	28	23	18	18	23	28	3	32	25
4	1	26	17	30	15	15	30	17	26	1	4
33	12	35	8	19	22	22	19	8	35	12	33
36	5	10	21	14	7	7	14	21	10	5	36
11	34	13	6	9	20	20	9	6	13	34	11
11	34	13	6	9	20	20	9	6	13	34	11
36	5	10	1	14	7	7	14	21	10	5	36
33	12	35	8	19	22	22	19	8	35	12	33
4	1	26	17	30	15	15	30	17	26	1	4
25	32	3	28	23	18	18	23	28	3	32	25
2	27	24	31	16	29	29	16	31	24	27	2

(2)

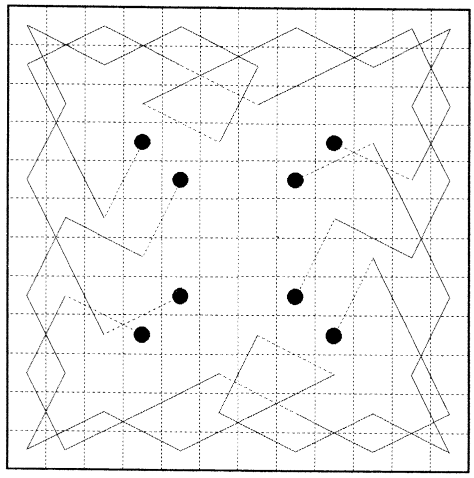


Applied to oblique symmetry, this method leads to some very remarkable results. I will take the numeration shown below [diagram (3)], which is identical with the preceding table except that I have given a diagonal reflection to the second upper and first lower quarters. And, first of all, I will select the same cells (8 and 14) as before, for the purpose of closer comparison with the directly symmetrical scheme. It must be borne in mind that, on an obliquely numbered board every reentrant odd chain will repeat itself four times before returning to its original cell. To connect 8 and 8 we have therefore to take half only of a complete four-fold circuit: so also with 14 and 14. Suppose the connections to be: 8 (13.12.25.24.23) (13.12.25.24.23) 8 and 14 (22.34.16.3.2.1.36.18.17) (22.34.16.3.2.1.36.18.17) 14. Here, instead of making one connection horizontally and one vertically, both connections have been made horizontally; and it is thus that they will be used in the complete tour. But in planning the connections it is not at first necessary to distinguish as to direction, because it will readily be seen that, in every case, the same series of numbers may be used in both ways, by reversal of order of each of the bracketed groups. For example, if we wanted to join 8 and 8 vertically we should only have to write 8 (23.24.25.12.13) (23.24.25.12.13) 8 instead of the order I have written above. Geometrically traced our framework so far is as follows [4].

(3)

2	27	24	31	16	29	11	36	33	4	25	2
25	32	3	28	23	18	34	5	12	1	32	27
4	1	26	17	30	15	13	10	35	26	3	24
33	12	35	8	19	22	6	21	8	17	28	31
36	5	10	21	14	7	9	14	19	30	23	16
11	34	13	6	9	20	20	7	22	15	18	29
29	18	15	22	7	20	20	9	6	13	34	11
16	23	30	19	14	9	7	14	21	10	5	36
31	28	17	8	21	6	22	19	8	35	12	33
24	3	26	35	10	13	15	30	17	26	1	4
27	32	1	12	5	34	18	23	28	3	32	25
2	25	4	33	36	11	29	16	31	24	27	2

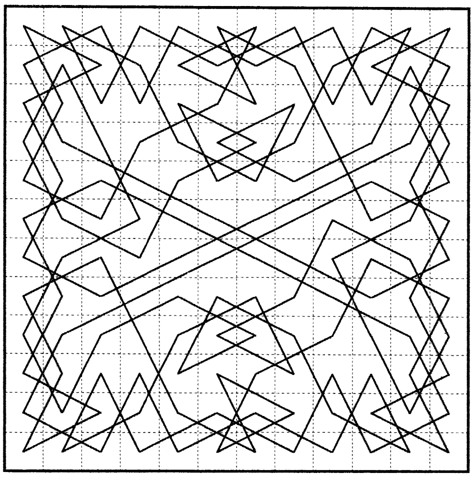
(4)



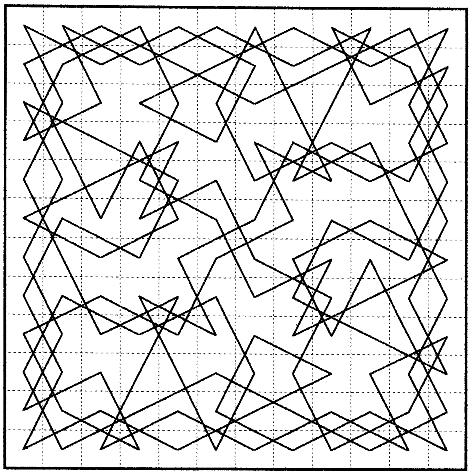
To complete the tour, we may join the 8 and 14 as follows: 8.9.15.11.10.19.6.7. 20.21.30.31.32.33.26.27.28.29.5.4.35.14, but as the result does not exhibit any special elegance, and the example was only chosen at random for didactic purposes, I will not stay to write out a tracing.

[The editor however considers it well worth a diagram (6).]

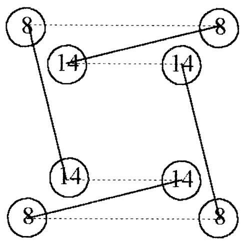
(5) E. Bergholt, Jan 12 1918



(6)

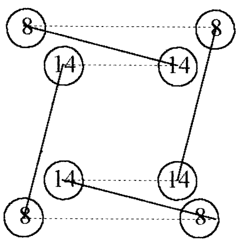


As before, any chain from 8 to 14 which uses the remaining 20 numbers (each once) will complete our tour, the six pairs of dotted moves being the only ones without quaternary counterparts. To prove this — and a similar proof applies to the former case — it is only necessary to observe that, since the 8...14 chain consists of an even number of cells, the 8 can never be connected to the 14 in the same quarter, nor to that in the diagonally opposite quarter; but must always join to the 14 in one of the directly adjacent quarters. The final result must therefore be one of the two schemes diagrammaticaly depicted below [7a,b], each of which obviously represents a complete reentrant circuit of the 144 cells.



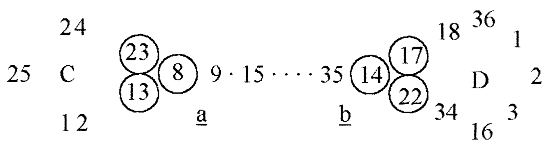
(7a)

(7b)



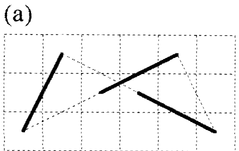
Analytically considered the tour is as follows [diagram 8, combining three diagrams in the ms into one]: being a string with a loop at each end. If we start at a we run twice round the loop C, then follow the string to b, then go twice round the loop D, then return along the string from b to a, etc.

(8)

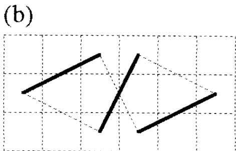


I call it **the method of terminal loops**. The initial step towards forming the loops is to select, for each loop, three cells such as those marked by circles [e.g. 23, 13, 8] in the above diagram. Such a triad of cells I call a **Node**, the necessary and sufficient condition being that each of the three cells shall join by a knight's move to each of the other two. In the case of direct symmetry the three cells of a node taken twice must form a six-sided polygon in direct binary symmetry [9]:

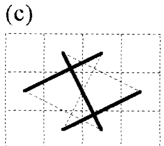
(9) (a)



(b)



(c)

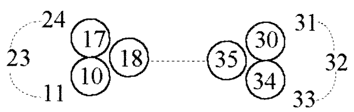


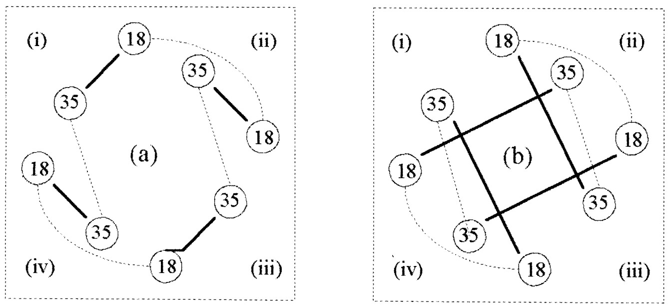
the moves which interrupt the quaternary symmetry of the tour being always three alternate sides of the polygon, as shown above by dotted lines. In the case of oblique symmetry the three cells of a node, continued in cyclical order round the board, form a twelve-sided polygon in oblique quaternary symmetry, and the interrrupting moves fall on alternate sides of half the perimeter of such a polygon.

Having advanced so far it suggested itself to me that, by suitably choosing the nodes it would be possible so to distribute the six pairs of interrupting moves that (in the case of oblique symmetry) the twelve lines should form three groups of four, each group being itself in direct quaternary symmetry. This being done, the whole tour would be completely quaternary (mainly oblique, but partially direct); and an apparently impossible problem would be solved. And this is the only way in which, demonstrably, it can be solved. For it can readily be shown, by the law of parity, that no tour over $8m$ cells can be (1) obliquely quaternary throughout, or (2) directly quaternary throughout. [A condition should be added, such as that the board shall be connected and contain no holes, in order to exclude tours such as the circuit of the 8 outer cells of a 3×3 board, which has both these types of symmetry.]

As regards choice of cells for the two ends of the central string (cells a and b in the diagram above) we are not of course restricted in oblique symmetry to cells on the two great diagonals, such as the 8 and 14 which we took above; but any cell may be chosen that can be suitably linked to the cell bearing the same number in an adjacent quarter of the board. I will select cells 18 and 35, and will form my loops as shown, analytically, below [10]:

(10)

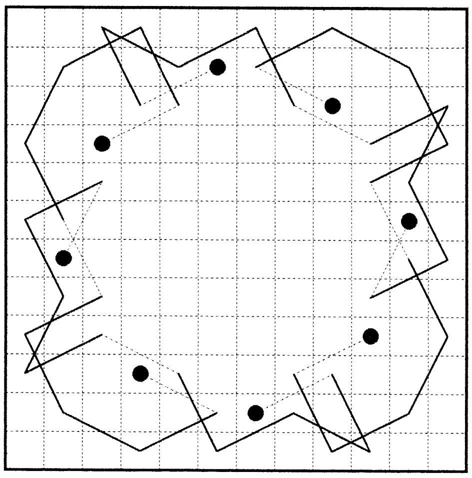




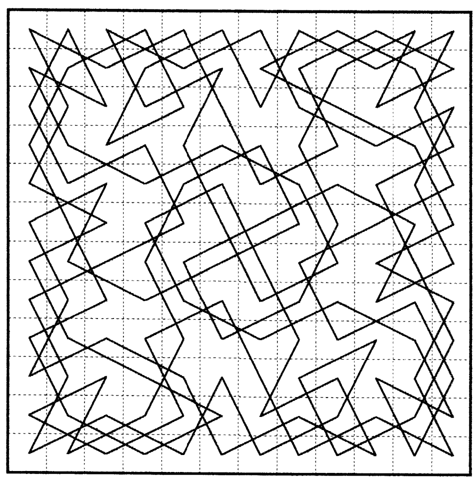
(11)

In the geometrical interpretation of our loops, we have to be careful, after connecting 18 to 18, to see that 35 connects to 35 in the proper direction; otherwise we shall "short-circuit" after traversing one-half of the board. [This is the first use of the term "short circuit" I have encountered in the knight's tour literature.] The scheme will be as [in diagram (11)]. That is to say, after linking 18(i) to 18(ii) we must not go from 35 (ii) back to 35(i) but must go onward from 35(ii) to 35(iii). It will then be immaterial whether, in the central string, we pass from 18(i) to 35(i) or from 18(i) to 35(iii), a complete tour being produced in either case: [11 a, b]. As observed in the case of direct symmetry, to pass from 18(i) to 35(ii) or to 35(iv) is impossible by the law of parity.

(12)



(13)



Translated, then, into geometry, our loops [10] will appear as shown [12], where it will be seen that our 'six pairs' of interrupting moves do, in fact, fall into a perfectly regular quaternary design.

The problem is now reduced to the very simple one of arranging 26 counters (inclusive of 18 and 35) in a chain having those counters for its end links. One way is readily found to be as follows:

18.19.20.7.6.14.22.21.12.25.26.27.4.5.8.9.15.16.3.2.1.36.13.29.28.35,

giving the complete quaternary tour shown [13 above].

Having found one chain from 18 to 35, we may simplify the operation of finding others in the following way. I write out the chain in numbers, on a new Table [not diagrammed], substituting 1 for 18, 2 for 19, ... 25 for 28, 26 for 35, and I proceed to work with this table before me, and counters from 1 to 26. Theoretically, nothing whatever is gained by this manoeuvre. Practically, I find an increase of facility and the growth of new ideas of design I transcribe below my first three attempts.

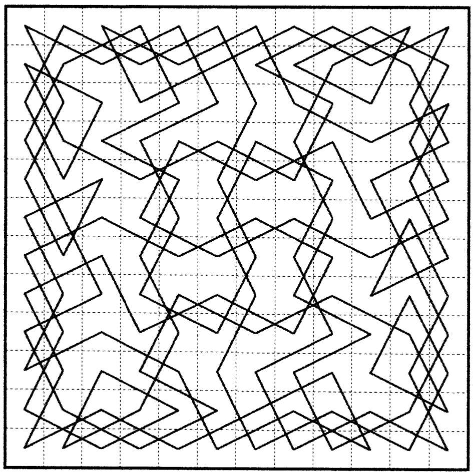
1.22.21.20.19.18.17.15.16.2.5.3.4.23.24.25.12.13.14.11.10.9.8.7.6.26

1.2.16.3.4.5.6.7.8.11.10.9.23.24.25.12.13.14.15.17.18.19.20.21.22.26

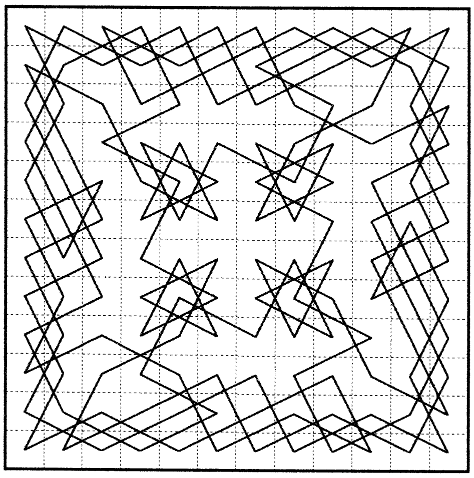
1.22.21.20.19.18.17.14.13.12.25.24.23.9.10.11.8.3.2.5.4.15.16.7.6.26

[We show them also in diagram form: (14) and (16) below; (15) on the front cover.]

(14)



(16)



It is evident that, with a little time and patience the total number of possible chains could be constructed and catalogued; and we should thus learn exactly how many perfectly quaternary tours can be constructed with the two loops that have been postulated. The Method of Terminal Loops is one of which I do not find the remotest hint in any previous writer, and in its full generality it has only been developed by me after considerable cogitation.

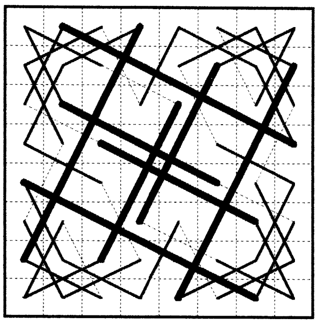
Terminal Loops and Zero Loops on the 8^2 Board

Seventh Memoir continued. It remains to set out an example on the 8^2 board and to point out some remarkable special cases of its application. The numeration I shall use is that of my Table C [see issue 14 page 235]. The analysis of the tour [17] is: ((11.9.7.31.19)) 13.15.17.25.23.21.27.29 ((1.3.5)). [The "loops" are in parentheses and the "nodes" underlined.] It contains the six unrelated pairs marked by the dotted lines. Can the design shown by the heavy lines be obtained in any other way? [A similar but open tour appears on p.353 of Falkener (1892).]

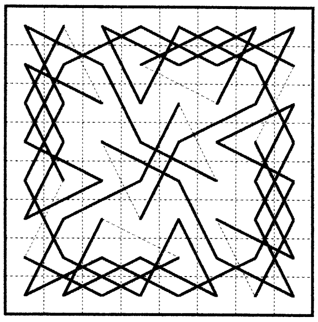
Special cases result from the fact that we may pass direct from 19 to 19 or from 29 to 29 without interposing any loop of actual cells. These are cases of **zero loops**. Whether we go 19 (13.15.17) (13.15.17) 19 for example, or whether we merely go 19.19 does not affect the method in any way. Brede [1844] has several examples (hit upon no doubt by 'trial and error') of these zero loops, two of which I show here [18, 19]. It is to be observed that a zero loop substitutes one 'interrupting move' in the place of three; and therefore reduces by two the total number of unrelated pairs. On the other hand it brings its single unrelated pair disagreeably close to the centre of the design. The former of these is (15.13.17) 25.3.1.5.29.27.21.23.7.9.11.31.19 which has one zero loop, and therefore four unrelated pairs; the latter is 29.27.21.23.25.3.5.1.31.7.9.11.13.15.17.19 which has both loops zero and shows only two unrelated pairs. [Tour (19) is a tour with 60 moves (the maximum) in oblique quaternary symmetry. The editor (GPJ) has studied this type and finds there are 48 in all.]

It was on this principle that I constructed the tour given in *British Chess Magazine* for February 1918; but I introduced the additional artifice of a centre which also exhibited direct quaternary symmetry. The fact of the whole central figure being thus in direct symmetry gives a singularly elegant aspect to the tour, which I think may be regarded as unique.

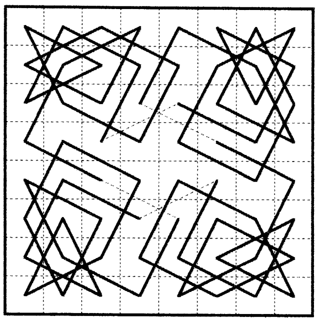
(17) Bergholt 1918



(18) Brede 1844



(19) Brede 1844

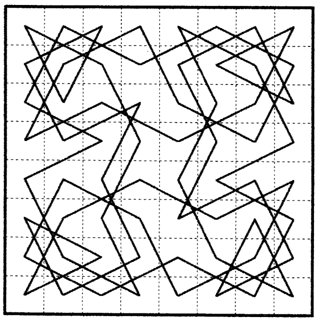


Addendum to the Seventh Memoir. We may of course apply to the quaternary 12^2 an artifice similar to that which I adopted in the 8^2 of the *British Chess Magazine* for February 1918; that is to say, in planning the 1 ... 26 chain we may purposely adopt such a distribution of lines as will heighten the effect of the twelve "interrupting moves" in direct symmetry. If this be skilfully done, all suggestion of an "interrupted" pattern may be banished, and the mathematical regularity may be made very conspicuous to the eye. Using precisely the same loops as before, the result of such a skilful arrangement of the final 26-cell chain may be seen in [diagram (20) "Camouflage" shown on our cover page]. I need not say that the motto of the tour refers to the ingenious arrangement of stripes and other markings whereby the position of guns is disguised to the enemy. To anyone unacquainted with the method, and particularly to a mathematician who has satisfied himself by "proof" that quaternary symmetry on the 12^2 is an impossibility, this tour will be an absolute staggerer!

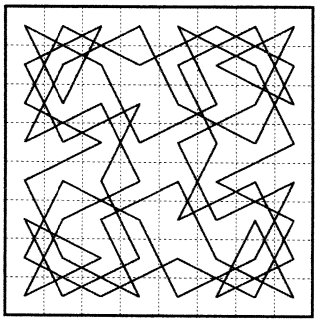
[Diagrams of two other tours, (21) and (22) on the front cover, are among the 'Miscellaneous Articles and Notes' and bear the following comments: 'The Balcony', composed 18 May 1918, shows symmetrical coalescing loops (see later), while 'The Constellation', composed 9 February 1918, has the same nodes as 'The Balcony' but the loops are terminal, and there is another centre by varying the direction of the optional move 14.20.]

Note to Addendum to the Seventh Memoir. (Communicated to H.J.R.Murray, January 22, 1918.) Another example by Brede is cited [23] and is collated with a tour by Mr H. J. R. Murray [24] with which it is seen to be identical [as regards formula].

(23) Brede 1844



(24) Murray 1913

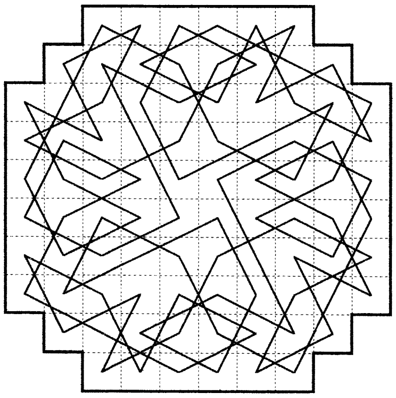


There is one zero loop (at 29) and one loop of 3 cells at 19. The figures (Table C) are: 29.1.3.5.7.9.11.31:25.27.21.23.19.((13.15.17)). The interrupting moves are therefore 29.29, 19.13, 17.13 and 17.19. The chain contains the optional move 31:25, which by Brede is taken in one direction, and by Mr Murray in the other. We have no information as to Brede's method, but may assume that it was based on Käfer (1842), in combination with the square and diamond method. [Notes on history of this method omitted: see our issue 14 pages 238-240.]

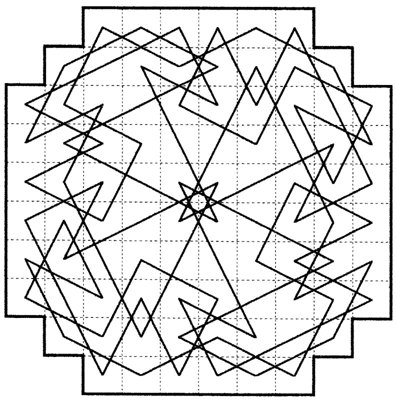
The great interest which both tours have for me personally is that, although planned on a method always held to be applicable to $(4m)^2$ boards, my memoir shows them to be special cases of a method applicable with the greatest ease to every kind of board capable of quaternary symmetry — say the 4-handed chessboard, or the board of 88 cells shown below. This board is quite strange to me, and the tour I have traced [25] is merely drawn at random (it took me 15 minutes at most to construct) in order to illustrate the truth of my remark.

The next time I try this board, I shall do better. Obviously, a second zero loop would improve the design at once. [The miscellaneous notes include another example (26) on this board, with the note:] Not made by method of terminal loops but by another powerful method which I will expound later on. Six unrelated pairs; the rest in pure quaternary symmetry. The unrelated pairs are carefully planned to be remote from the centre.

(25) "Terra Ignota"



(26) "Terra Cognita"



Symmetrical Nodes

Eighth Memoir on Knight's Tours. (Communicated to H.J.R.Murray Esq., February 23rd, 1918.) *Application to the ordinary chessboard of the Method of Terminal Loops with Symmetrical Nodes.*

Let us now inquire to what extent the method of my Seventh Memoir will yield us complete quaternary symmetry on the 8^2 board. First consider Table A (direct symmetry) [issue 14 page 235]. The only symmetrical nodes that I find are the following: (a) 7, 17, 5 and 11, 1, 13; (b) 7, 19, 23 and 11, 29, 27; (c) 7, 19, 17 and 11, 29, 1; (d) others, such as 25, 17, 15 and 25, 1, 3, in which each node contains a cell in common (21, 25 or 31). In the cases (a), (b), (c), 7 must be joined at once to 11 by the corner cell 9; which makes the carrying out of the scheme impracticable; and the duplication of the cell in group (d) has the same result. I therefore conclude that the smallness of the area makes the method, on this Table, inapplicable. (The nodes of group (d) can, however, be used to form coalescing loops, treated later on.)

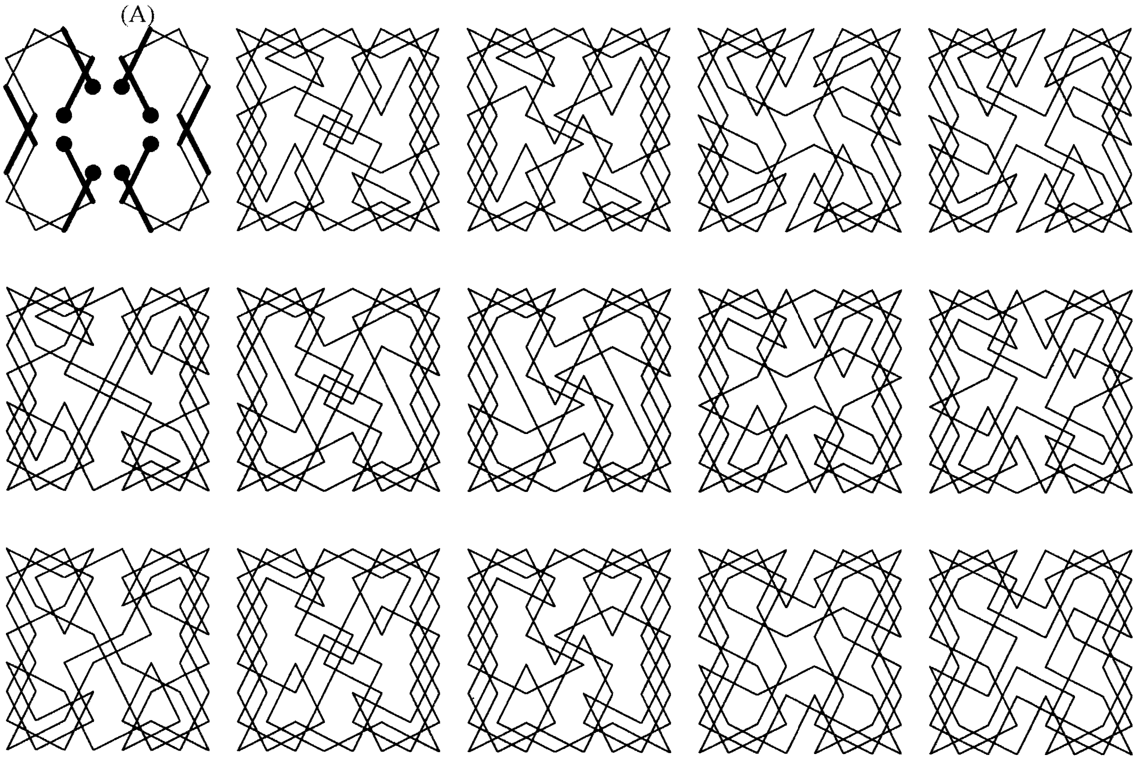
I pass on to Table C (oblique symmetry). Here I find the nodes 19, 13, 17 and 29, 5, 1. The addition of the cells 15 and 3 to these, respectively, will form two odd loops (15, 13, 19, 17) and (29, 5, 3, 1) and the problem is reduced to the very simple one of completing the chain 19...29 by inserting the eight numerals which remain. I have found eight ways in all in which this may be done:

(1) 19.31.7.9.11.27.25.23.21.29;	(2) 19.21.23.25.27.11.9.7.31.29;
(3) 19.31.25.27.11.9.7.23.21.29;	(4) 19.31.25.23.21.27.11.9.7.29;
(5) 19.31.25.23.7.9.11.27.21.29;	(6) 19.21.23.25.31.7.9.11.27.29;
(7) 19.23.25.31.7.9.11.27.21.29;	(8) 19.11.9.7.31.25.23.21.27.29.

It is to be noted, as in all such cases, that (exempli gratia) the sequence 29.31.11.9.7. 23.25.27.21.19 (or its reversal) merely gives a reflection of chain (1) in vertical axis, and does not yield a new solution. Similarly, every other chain will be duplicated, if we form all possible permutations of our counters, and we have to eliminate the duplicates (halving the total number) before we can finally take stock of our result. From the list above, all such duplicates have been excluded.

We have also to note that, in the last six of the chains the move 31:25 occurs, which can be taken in either of two distinct ways. Hence each of chains (3) to (8) yields two different tours. Our total number of distinct tours by this method (all in complete quaternary symmetry) is therefore fourteen. It would be desirable that some skilled enumerator, such as Mr. H.J.R.Murray, should check, independently, the correctness of the above list, as there is always a possibility that one or two may have been overlooked. [A handwritten note, by Murray, states "verified 21.7.18"]

The student should write out the diagrams of all the above. The portion shown below in [A] is constant in each of them; the bold lines show the twelve node-moves which break the oblique symmetry, but themselves form three quaternions of moves in direct symmetry. The remaining part of the tour is filled up by repeating one of the chains given above four times. [We diagram all 14].



It will be seen that four of the tours, [those derived from formulas (1), (2) and (8)] use only the squares and diamonds. This particularity (an accidental circumstance that forms no part of the general method) is not observable in any other tour of the list; nor of course does it occur on other areas to which the method can be applied with equal facility. It must furthermore be pointed out that methods which are essentially distinct (as is proved by the fact that each will yield innumerable results which will not be given by the other methods in question) often run into one another. That is to say, we can obtain from method A a result which is equally obtainable by B; and B will yield other results not obtainable by A. The two methods overlap in special cases of each, just as two categories may partially coincide in logic, although the categories themselves are different. Some kings are musicians, and some artisans are musicians, but kings and artisans are not the same.

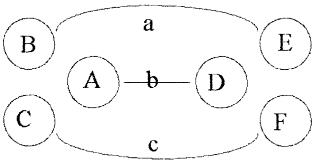
Thus all the results set out in this memoir may also be obtained by another method which I will explain later on, and which I call the Method of Distinctive areas. The reason is to be found in the restricted area which we have selected to work upon. [This method is not further mentioned alas.]

The Method of Coalescing Loops

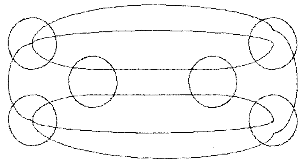
Ninth Memoir on Knight's Tours. (Communicated to Mr. H.J.R.Murray on March 7th 1918.) In my seventh memoir I explained my method of Terminal Loops and showed how, by taking symmetrical nodes, the plan would produce complete quaternary symmetry. I now proceed to explain other forms of loops which are still more important and fruitful, inasmuch as special cases of them can be applied with striking success to obtain [mixed] quaternary symmetry on the 8^2 board. It is necessary to distinguish between areas of 8m cells (where each quarter consists of an even number of cells) and areas of 8m + 4 cells (where each quarter consists of an odd number of cella). In the present memoir I confine myself to the former class, leaving the latter to be illustrated later on. The differences of treatment are only slight; but it will conduce to clearness to handle the two classes separately.

In the seventh memoir I diagrammed a 12^2 tour of which the nodes were 6, 35, 4 and 8, 15, 23, the terminal loops ((6.34.33.32.35)) and ((15.29.23)) and the central chain 14.22.30.7.19. 28.27.26.31.18.16.17.24.25.12.21.20.9.10.11.5.4.3.2.1.36.13.8. Now, instead of forming loops at the two ends of a chain, we may connect our nodes so as to form a kind of figure of eight [(1a)] and the loops thus formed I call **coalescing loops**.

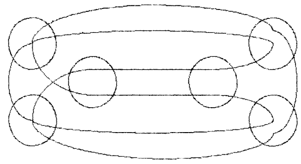
(1a) coalescing loops



(1b) two odd segments



(1c) three even segments



There are here three chains (a), (b), (c). On areas of 8m cells, our total number of counters (for quaternary symmetry) being even, either (I) one chain must be even and the other two odd or (II) all three of the chains must consist of an even number of counters. In the former case, without departure from generality, we may always so depict the loops that the even chain shall be the middle one, (b). In the case where (a) and (c) are both odd chains, the sequence of the complete tour, if we start from A and go towards D, will be as follows:

(I) (A=D)(E–B)(C–F)(D=A)(C–F)(E–B)...

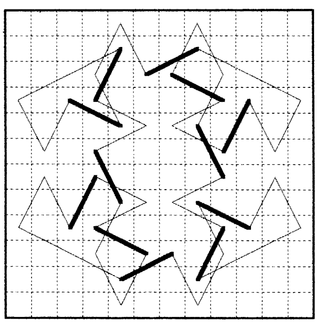
where the even segment is marked with a double line (A=D), and the odd segments with a single line (E–B). This sequence, which takes each segment twice, completes half of the tour; the second half is a duplicate of the first half. Each segment will eventually have been taken four times. Note that the odd segments aways run in the same direction; the even segments never in the same direction twice consecutively. In the case where (a), (b) and (c) are all even chains, the sequence will be:

(II) (A=D)(E=B)(C=F)(D=A)(B=E)(F=C)...

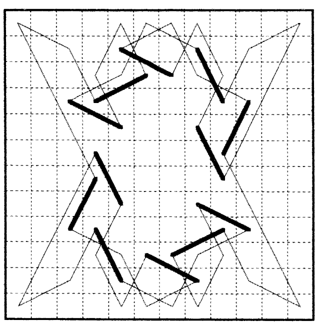
and back to the beginning for the second half as before. All three segments follow the same rule of reversal as the single even segment did in the former case. We may depict the courses of the half tours graphically [as in (1b and c)].

The following skeleton tours on the 12^2 will exemplify the results.

(2a) 48-cell circuit



(2b) 56 cell circuit



The plan of (2a) is: (a) 23.26.33.34.35, (b) 15.16.17.14, (c) 8.7.6 the sequence of moves being (15.16.17.14) (35.34.33.26.23) (8.7.6) (14.17.16.15) (8.7.6) (35.34.33.26.23)... The plan of (2b) is: (a) 23.29.30.35, (b) 15.16.17.14, (c) 8.3.2.1.10.6 and the sequence of moves is: (15.16.17.14) (35.30.29.23) (8.3.2.1.10.6) (14.17.16.15) (23.29.30.35) (6.10.1.2.3.8)... [In the ms this data is given within the figures in the form of diagrams similar to (1a). The (a), (b), (c) refer to the three chains.] The interrupting moves in each case are similarly placed to those in the tour quoted above, some however being reflected.

Having thus explained generally the nature of these coalescing loops, we will now consider their application to [mixed] quaternary symmetry, through the selection of symmetrical nodes. I pointed out in considering terminal loops that the mere employment of such nodes was not always sufficient to ensure [mixed] quaternary symmetry, inasmuch as the direction of rotation round the loops had also to be taken into consideration. It was not then necessary to go more deeply into the matter, because in practice the correct rotation could always be selected by geometrical inspection. This not being the case with coalescing loops, we must now examine the requisite further condition, which is a very simple one.

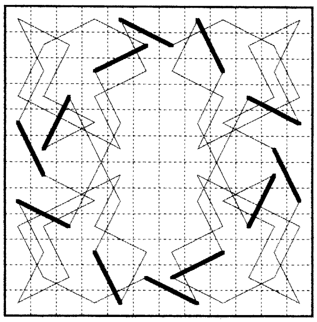
Consider any two cells A, B, one move apart and their obliquely symmetrical cells A', B'. It is clear that from A to A' (and equally from B to B') must always be an even chain, i.e. an even number of cells must intervene between the two terminals. And since B' is one move from A', it follows that from A to B' , or from B to A', must always be an odd chain. In order therefore that the move AB may be followed, in the course of the tour, by its counterpart move A'B' (or B'A', the direction of motion being immaterial) the sequence of the tour must be one of the two following:

(1) ...A.B=B'.A'... or (2) ...A.B–A'.B'...

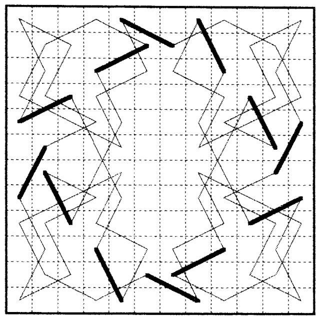
where, as before '=' denotes an even segment of tour and '–' an odd segment. The law that must always be observed then is that when an odd number of cells intervene between the counterpart moves, those moves must be made in similar order (2); but that, when an even number of cells intervene, the moves must be made in contrary order (1).

Take for example the following case of terminal loops: (<u>16</u>.15.8.9.10.33.32.31.<u>18</u>)<u>17</u>.14.26.<u>35</u> (<u>34</u>.21.12.3.2.1.<u>36</u>), where A = !7, B = 16, C = 18, A' = 35, B' = 36, C' = 34. Suppose we trace the tour in the following order: (18–16) (18–16) (17=35) (34–36) (34–36) (35=17) (18...). Here an even number of cells intervene between B.A (16.17) and B'.A' (36.35), the order of the two moves being similar. This is <u>wrong</u>. If however we take the contrary rotation for the right-hand loop, and trace the tour as follows: (18–16) (18–16) (17=35) (36–34) (36–34) (35=17) (18...) an even number of cells intervene, as before, but (not as before) the order of the two moves is dissimilar. This is <u>right</u>.
Reference to the diagrams below [(3a and b)] will verify the accuracy of our reasonings.

(3a) wrong rotation

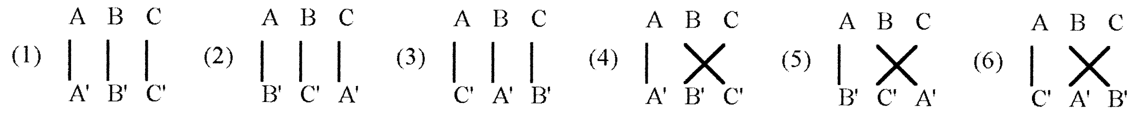


(3b) correct rotation



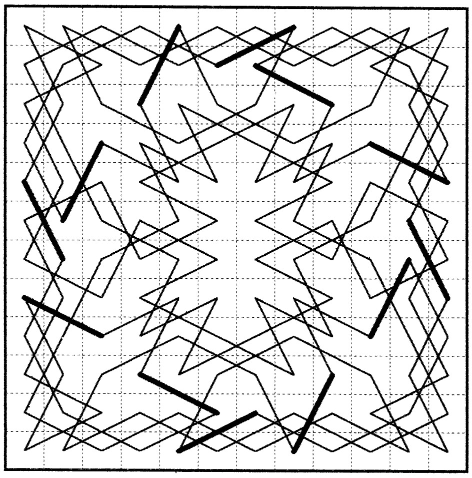
Having now established the law, let us apply it to the two cases of coalescing loops given above. If two of the segments are odd, it will be found that the nodes may be connected as follows: (a) B–B', (b) A=A', (c) C–C' (A=A' being the even segment), from which two more cases may be derived by cyclical permutation of A', B', C' among themselves. But if the segments are all even, the connection must be made differently: (a) B=C', (b) A=A', (c) C=B', with two more cases by cyclical permutation of A', B', C' as before. The connections having been thus settled, the sequence of the whole tour is determined by the formulae (I) and (II) above and the problem is solved.

A simple way of showing the different ways in which the node cells may be correctly connected by chains is given below. When two of the chains are odd we may join as in (1), whence we derive (2) and (3). When all the chains are even, we may join as in (4), whence we derive (5) and (6).

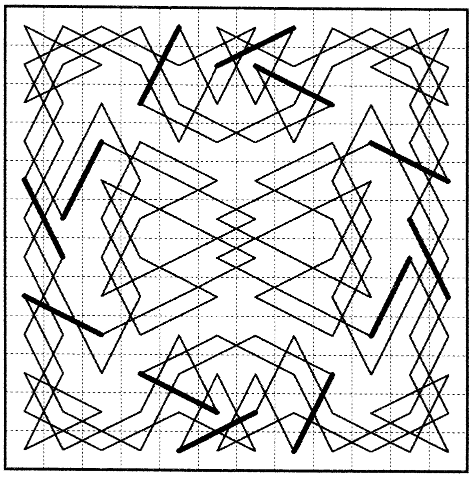


Keeping, for the present, to the same nodes as in the last case of terminal loops, I give the following tours [diagrams 4 – 6] to exemplify the method.

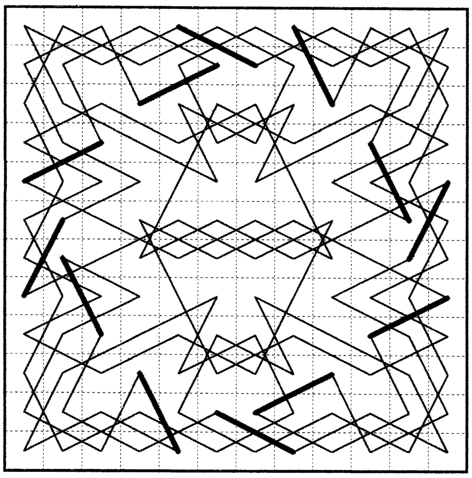
(4) impromptu 1 March 1918



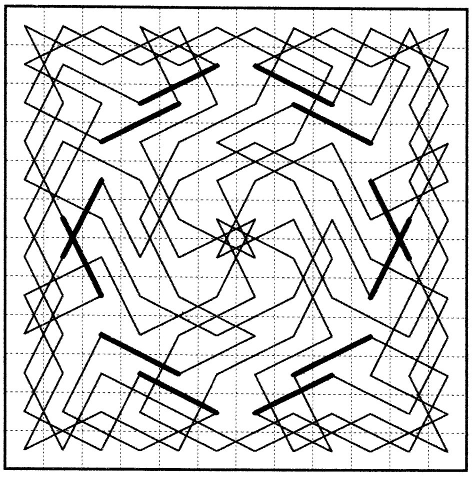
(5) impromptu 1March 1918



(6) 21 February 1918



(7) impromptu 2 March 1918



The bold lines show the 12 interrupting moves in quaternary oblique symmetry. Tour (4): (a) 17.14.35, (b) 16.15.8.9.10.11.5.4.3.2.1.36, (c) 18.31.32.33.34, connections: 16=36, 17–35, 18–34. Tour (5): (a) 17.22.30.31.32.33.34, (b) 16.15.29.23.24.25.12.11.5.22.26.21.13.8.7.6.10.9.20:14.35, (c) 18.19.28.27.4.3.2.1.36. By cyclical permutation of connections in tour (4): 16=35, 17–34, 18–36. Another design may be formed in the centre by taking the optional move 20:14 the other way. Tour (6): (a) 17.24.25.12.21.34, (b) 16.3.2.1.8.7. 30.22.23.29.28.27.4.5.11.10.33.26.19.15.14.6.20.9.13.36, (c) 18.31.32.35, connections: 16=36, 17=34, 18=35 (even segments). [As with figure 2 this data has been simplified from the ms, and some numerical representations of the tours omitted.]

In consequence of the great generality of the method, it is clear that it will apply equally well to a table of numeration in oblique symmetry. In such a case, the main body of the tour will be obliquely symmetrical, and the interrupting moves will form three quaternions directly symmetrical.

On the table of oblique numeration for the 12^2 I base this tour [(7) in the preceding figure]: chains: (a) 18.31.36, (b) 17.32.33.34, (c) 10.22.14.6.19.13.36.1.2.3.16.15.8.9.20.7.21.12.11.23.24.25.26.5. 29.28.27.4.35, connections: 17=34, 18–30, 10–35. This formula will be found a very useful one, as it contains a chain of 29 numbers (10...35) which may be freely permuted in a large number of ways, leaving the rest of the scheme unaltered.

The preceding four examples must, I think, have made the facility and power of the method sufficiently apparent; and I now proceed to call attention to some special cases which will lead to highly important results on the 8^2 board.

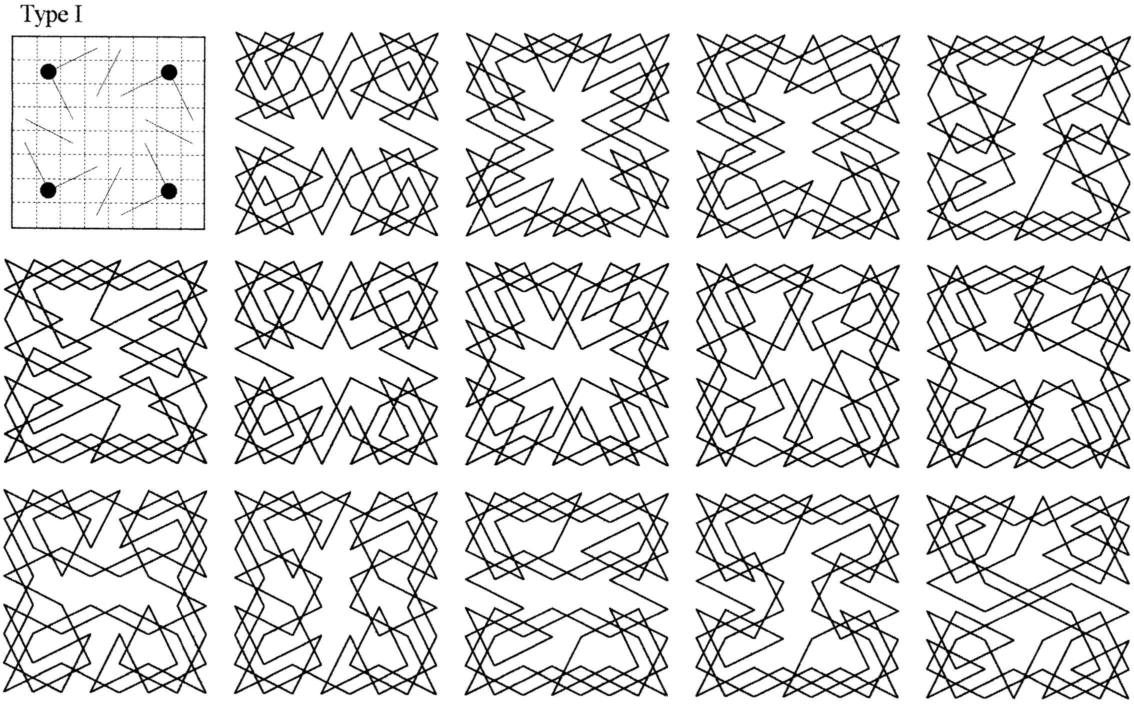
Coalescing Loops on the 8×8 Board

Corollary I. In the case where two segments are odd [Figure (1.b)], one of the odd segments (say B–B') may consist of only a single cell, situate on a principal diagonal. If, in the general formula (A=A')(B'–B)(C–C')(A'=A)(C–C')(B'–B)(A=...) the cells B, B' so coincide, the formula becomes (A=A').B.(C–C') (A'=A)(C–C').B.(A=...), where we may conveniently begin with the cell B, and write: B.(A=A').B.(C–C').(A'=A)(C–C')... &c. We will now apply this formula to the 8^2 (Table A).

The resulting tours fall into three Types: according as cell B is, (I) cell 21 [b2]; (II) cell 25 [c3]; or (III) cell 31 [d4]. Of these, Type I yields the greatest number of tours. I have catalogued them according to the figures formed in the centre by the two moves on each side of cell 31; and for the sake of uniformity, it is desirable to designate these central figures by the same letters [a to k] as are used by the Abbé Jolivald [1882].

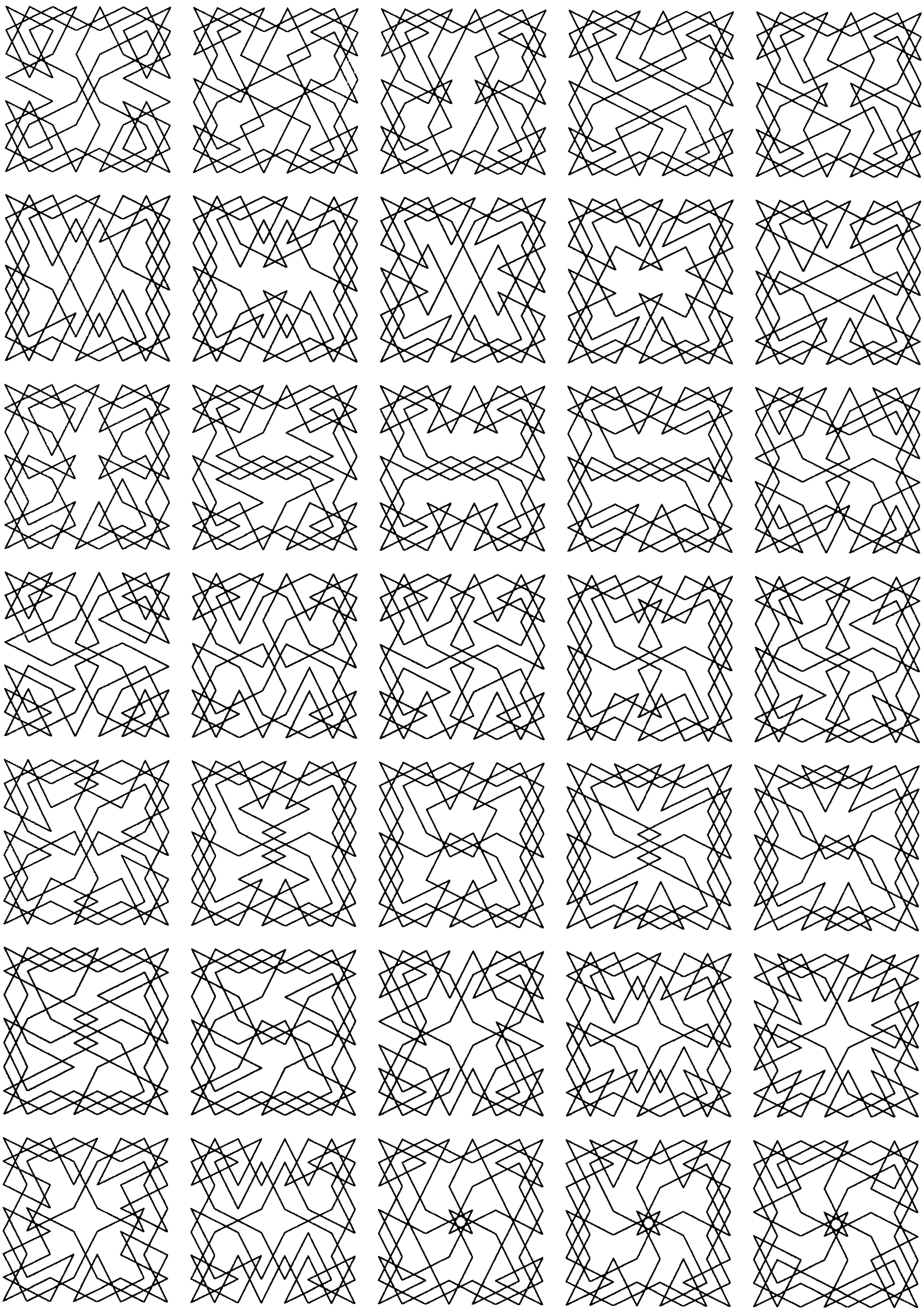
Type I. For quaternary symmetry, the nodes of Type I must be 27,21,29 and 19,21,23, and the formula of each tour must be either: 21.(23=27).21.(19–29) (27=23) (19–29)... or 21.(19=29).21. (23–27) (29=19) (23–27)... according as the chain 23...27 is even or odd. [Here follows a list of quaternary tours of type I in numerical form, but we substitute diagrams below.] Any one of the sequences which contains the move 19:29 or 31:25 will yield two tours; and the first one (containing both these moves), will yield four tours. It is the change in the direction of the optional move or moves that causes the variation in the centre. Hence the total number of tours of Type I is: (a) 5, (b) 4, (c) 4, (d) 1, (e) 1, (f) 10, (g) 3, (h) 7, (i) 6, (j) 5, (k) 3, making 49 in all. All these tours contain precisely the same 12 interrupting moves, two of them (four times repeated) being consecutive.

Type I



continued.

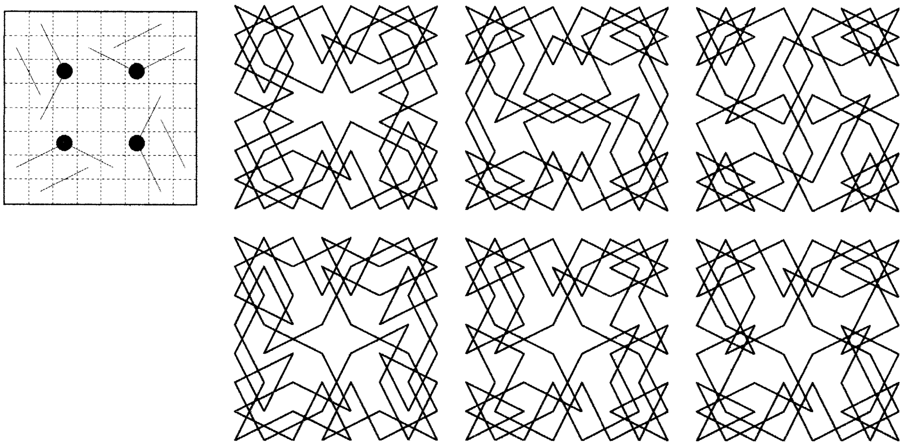
Type I continued.



The above tours are shown in the sequence a to k of the Jolivald letters for the centre patterns. The centres c, f, i occur in two forms.

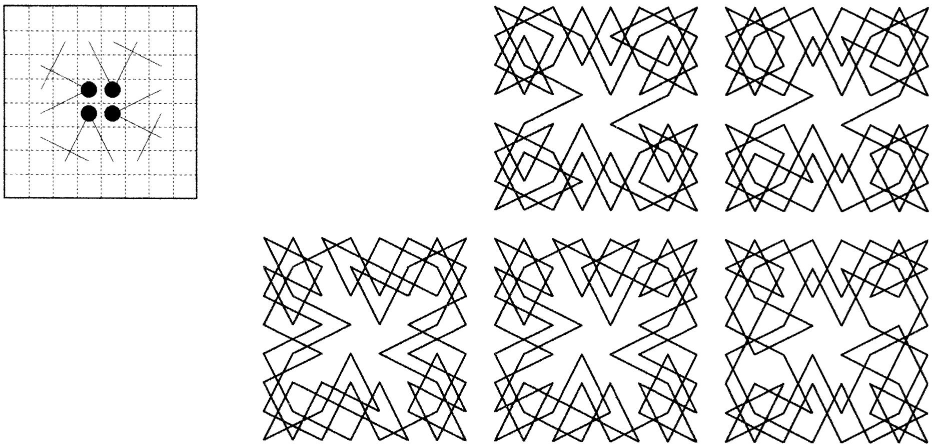
Type II. Nodes 25,3,1 and 17,25,15. Diagram of segments: 25–25, 1–17, 3–15. Formula: 25.(1=17).25.(3–15) (17=1) (3–15)... or 25.(3=15).25.(1–17) (15=3) (1–17)..., according as segment 1...17 is even or odd. I can find only these six, which shows how rare the forms are, and how hopeless to attempt to form them except by a systematic method.

Type II



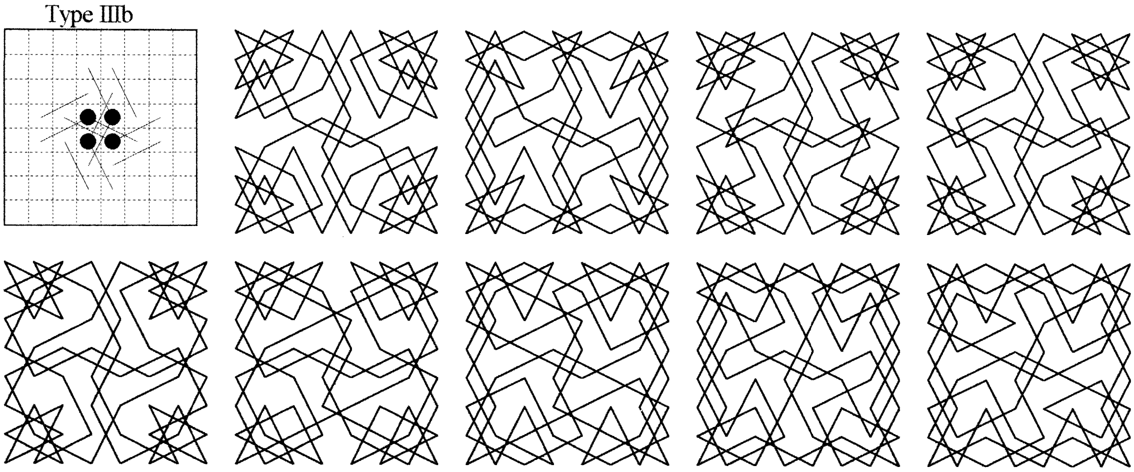
Type IIIa. Nodes 31,11,1 and 31,7,17. Segments: 31–31, 1–17, 11–7. Formula: 31(1=17)31 (11.9.7) (17=1) (11.9.7)... oblique. The first two proceed by quartes only and are half-and-half tours.

Type IIIa



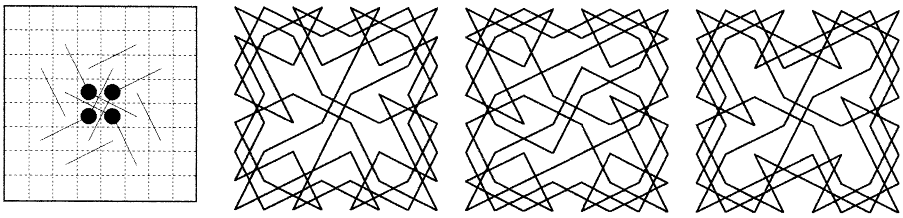
Type IIIb. Nodes 31,29,1 and 31,19,17. Oblique.

Type IIIb



Type IIIc. [The editor found these three further cases using a purely graphical method.]

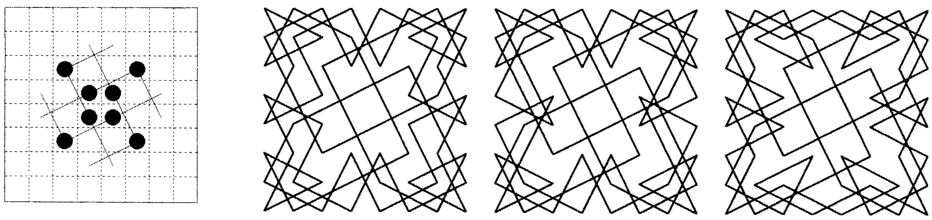
Type IIIc



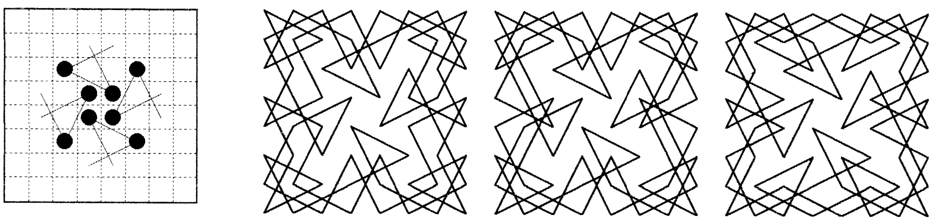
Corollary II. In the general formula for coalescing loops with symmetrical nodes and two odd segments, both the odd segments may reduce to a single cell only, situate on a principal diagonal. On Table A, these single cells can only be 25 and 31. The nodes will be 1,25,31 and 17,25,31 and the formula is 31(1=17)31.25(17=1)25.... It might be thought, on first inspection of this formula, that these tours contain only four interrupting moves (31.1, 17.31, 25.17 and 1.25) but that is not so, because the progression 31.25 stands for two different moves. It is an optional move, and is not taken in the same way twice consecutively. The fact that the three oblique quaternions of moves, in this type, consist each of three consecutive moves, and are thrown into the centre of the design, makes this group of tours a very striking one.

Types IVa and IVb. Each of the formulas represents two different tours by variation of the optional move 31:25. The six tours occur in two pairs of three and each of these diagrams may have the central figure of the other.

Type IVa



Type IVb

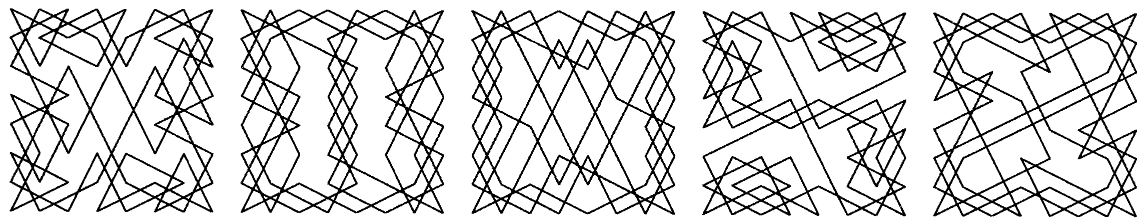


Summary. We have the following mixed quaternary tours (each with three quaternions of moves in oblique symmetry): Type I 49, II 6, IIIa 5, IIIb 9, IIIc 3, IVa 3, IVb 3; Total 78.

Further Work on Mixed Quaternary Symmetry. H.J.R.Murray continued research into mixed quaternary symmetry and in his 1942 ms gives the tabulated figures below which add to 580. He says that each tour is composed of a number of chains alternately in direct and oblique symmetry.

chains:	8	16	24	32	40	total
tours:	145	168	185	67	15	580

The following are examples of tours of this type not included in the special types studied by Bergholt.



The above totals have not been independently checked.

Puzzle Answers

33. A 'Countdown' Curiosity

The problem was to find numbers {A, B, C} such that $(A \times B) + C = (A + B) \times C$, ignoring the trivial case where $B = C = 1$. Since multiplication and addition are commutative we can interchange A and B, so we can assume $A \geq B$.

By simple algebra $C = (A\times B)/(A+B-1)$, so we require to find A and B such that $(A\times B)$ is divisible by $(A+B-1)$. Except when A=1 or B=1, A and B are both less than $(A+B-1)$. So we must have $(A+B-1) = M\times N$ where M divides A and N divides B. By considering successive values for M and N we find there are seven solutions:

{4, 3, 2}: $(4\times3)+2 = (4+3)\times2 = 14$
{6, 5, 3}: $(6\times5)+3 = (6+5)\times3 = 33$
{9, 4, 3}: $(9\times4)+3 = (9+4)\times3 = 39$
{8, 7, 4}: $(8\times7)+4 = (8+7)\times4 = 60$
{10, 6, 4}: $(10\times6)+4 = (10+6)\times4 = 64$
{10, 9, 5}: $(10\times9)+5 = (10+9)\times5 = 95$
{25, 6, 5}: $(25\times6)+5 = (25+6)\times5 = 155$

Since in Countdown a further condition applies (not stated in our problem) namely that the total is always $\geq$ 100, it follows that the last solution is the only one that can in fact occur.

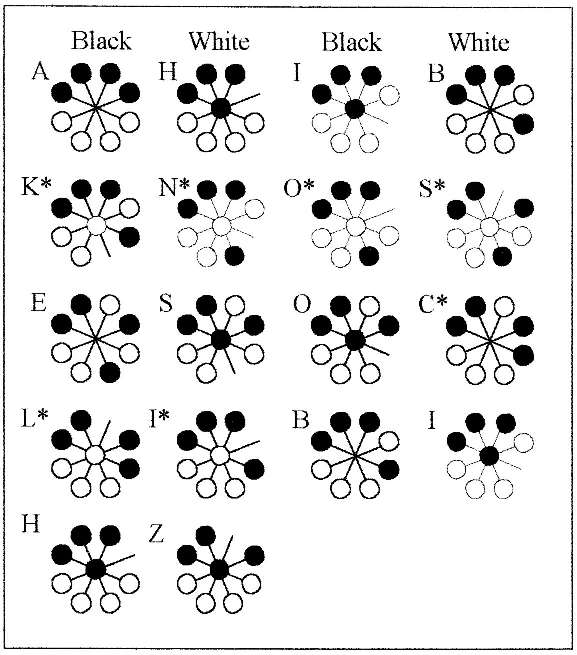
Substantial analyses by Dr C.M.B.Tylor and Prof. D. Singmaster came in, extending the problem, e.g. using negative values, but space prohibits giving more here than the basic solution.

34. Mu Torere

The sequence of positions leading to black's superstalemate of white under reflex conditions is: A, ~H, I, ~B, K*, ~N*, O*, ~S*, E, ~S, O, ~C*, L*, ~I*, B, ~I, H, ~Z as illustrated.

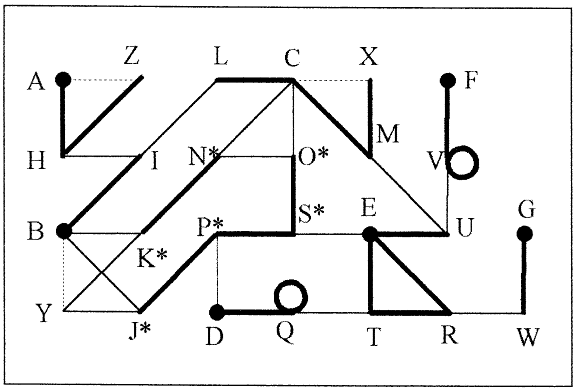
The stalemate is reached on black's 9th move. There are routes via ~J* or ~K* instead of ~I* but these permit white to stalemate by playing to Y* instead of B. The problem is to get back to B with black to move instead of white, i.e. tempo loss, so we seek the shortest odd-length circuit.

Ken Whyld sends details of articles in the US *Mathematics Magazine* by Marcia Ascher (vol.60 1987 pp.90-100), and by Philip D. Straffin Jr (vol.68 1995 pp.382-6, vol.69, 1996 p.65) which analyse Mu Torere. Their results agree with mine except that the rule that 'a man between two of its own men cannot move to the centre' is applied only for each player's first two moves.



This rule makes transitions A-Z, B-Y, C-X for black and A-Z*, B-Y*, C*-X* for white admissible. Since this weasely form of the rule ruins my nice superstalemate problem (the reflex rule forces C*-X*) I'm naturally against it.

The following is a compressed form of the network of moves (more elaborate versions are shown in the above references). Assuming that we start at A with black to move, the heavier lines represent black moves and the lighter lines white moves. However, upon entering any of the six self-complementary nodes A, B, D, E, F, G this convention can, if desired, be reversed (heavy lines then represent white moves and light lines black moves, all the node-names being converted to their complements, that is C becomes C*, O* becomes O, etc). This convention applies until a self-complementary node is reached again.



The dotted lines represent the optional moves, prohibited in my version. The circles at V and Q represent loop moves, in which the turn to play changes but the position is effectively the same.

The Z-Z loop move mentioned in the list in the last issue can in fact never be made since it is not black's move and white is stalemated so the game has ended (similarly for Z*-Z*).

The network diagram indicates that F and G are obvious alternative opening positions.

35. A Subtraction Square

The problem was to arrange the numbers 1 to 25 in a square array so that if we subtract the successive elements in any rank file or diagonal we get the same result. From a line of values a, b, c, d, e the successive subtraction rule produces e–(d–(c–(b–a))) = (a+c+e) – (b+d), so it does not matter from which end of the line we work.

5	4	17	16	11
6	2	1	12	20
7	3	13	23	19
8	14	25	24	18
15	22	9	10	21

The solution shown here was found by placing the first three, middle three and last three numbers in the centre 3×3 in the same formation as for the 3×3 example given by Reichmann. The correct totals in the three middle ranks, three middle files and two diagonals can then be ensured by placing pairs of complements (adding to 26) at their ends. With care the correct totals for the edge ranks and files can then be achieved.

The pattern formed by the even and odd numbers, shown by the bold-bordered squares, has octonary symmetry in this example. In any solution, each line must contain an odd number of odd numbers, and an even number of even numbers. The possible distributions of the 13 odd and 12 even numbers in the five ranks are some permutation of 53311 (02244) or 33331 (22224). The 55111 (00444) distribution in the ranks cannot occur, since it is incompatible with any proper distribution in the files (i.e. it results in an even number of odd numbers in at east two files).

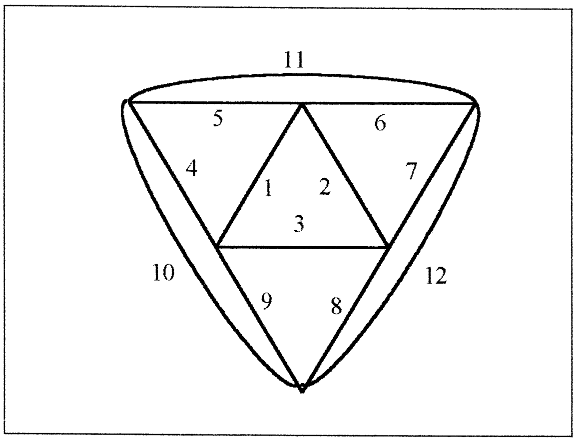
36. Non-Intersecting Tours

Shortage of space prevents us showing the 32×32 non-intersecting open knight's tour with maximum coverage of 854 cells, by VW extension of one of Robin Merson's examples. This should eventually appear on the GPJ web pages.

37. Wire-Framed Octahedron

by T.W.Marlow

The problem was: Given four pieces of wire, all of the same length, you are required to bend the pieces and place them together to form an octahedral frame. How many geometrically distinct ways?



Line segments can be at an angle of 90° or 60° and are called 9 or 6 below. There seem to be four shapes, two having mirror image versions. Shapes can be: 99 (e.g. 9,1,6); 96 R or L (e.g. 11,6,7 or 1,6,11); 66 T (e.g. 1,2,3); 66 R or L (e.g. 1,2,7 or 2,1,4).

The enumeration of how many 'different' ways of putting these together to make the octahedron has not been completed. TWM lists 20 solutions, but believes there are others. I give them according to number of shapes used:

one shape

4×96R: 9,1,2; 12,6,5; 4,3,8; 7,11,10.
4×66T: 1,4,5; 2,6,7; 3,8,9;10,11,12.
4×66R: 1,2,7; 4,5,6; 3,8,12; 9,10,11.

two shapes (3:1)

3×99 + 66T: 9,1,6; 2,5,10; 3,4,11; 7,8,12.
3×99 + 66R: 9,1,6; 5,2,8; 11,4,3; 10,12,7.
3×96R + 66T: 10,5,1; 4,3,8; 9,12,11; 2,6,7.

two shapes (2:2)

2×96R + 2×96L: 9,1,2; 12,6,5; 3,4,10; 11,7,8.
2×96R + 2×66R: 7,3,1; 5,10,12; 11,6,2; 8,9,4.
2×66T + 2×66R: 3,8,9; 10,11,12; 1,2,7; 4,5,6.
2×66R + 2×66L: 1,2,7; 4,5,6; 3,9,10; 8,12,11.

three shapes

99+2×96R+96L: 9,1,6; 5,2,3; 10,8,7; 4,11,12.
99+2×96R+66R: 1,6,12; 5,2,3; 7,11,10; 4,9,8.
2×96R+96L+66R: 10,5,1; 4,3,8; 9,12,7; 2,6,11.

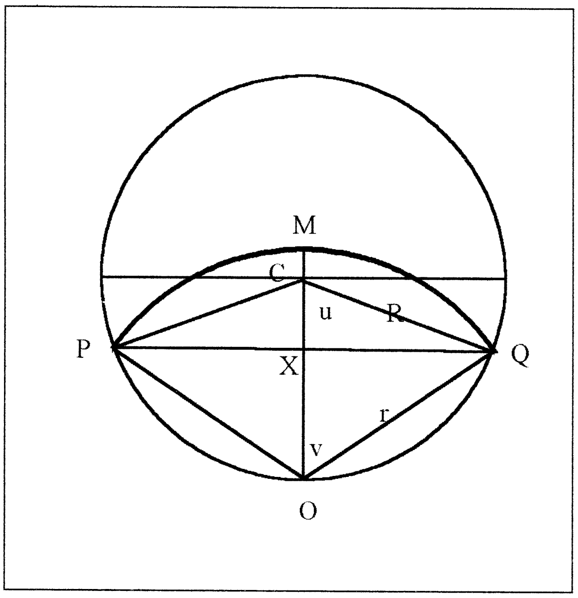
96R+96L+2×66T: 1,6,7; 2,5,4; 3,8,9; 10,11,12.
2×96R+66T+66R: 5,2,3; 1,6,7; 10,11,12; 8,9,4.
2×96R+66T+66L: 1,6,7; 2,5,11; 3,8,9; 4,10,12.
96R+96L+2×66R: 9,12,11; 10,8,3; 1,2,7; 6,5,4.
2×66T+66R+66L: 1,2,3; 10,11,12; 4,5,6;7,8,9.

four shapes (i.e. all different)

99+96R+96L+66T: 9,1,6; 2,5,11; 3,4,10; 7,8,12.
99+96R+96L+66R: 9,1,6; 2,5,4; 7,11,10; 3,8,12.

38. The Tethered Goat

A goat is tethered at the edge of a circular field. What is the length of the tether if the goat has access to exactly half the area of the field?



Let R be the radius of the field, centre C. Let O be the tether point and P, Q the points where the grazing limit meets the field edge. Let u be the angle PCO = QCO. Then area of triangle PQC (half base times height) = R sin u × R cos u, and area of sector OPCQ (ratio of angle PCQ = 2u in radians, to the full cycle 2π, times the area of the circle πR^2) = R^2u, so area of segment POQ = OPCQ less PCQ = $R^2(u - \cos u \sin u)$.

Let r be the length of the tether, and v the angle COP = COQ. Then area of segment PMQ is similarly $r^2(v - \cos v \sin v)$.

From isosceles triangle OCQ, $2v + u = \pi$ hence $v = \frac{1}{2}(\pi - u)$.

Let X be the mid-point of PQ then in the triangle OCQ we have XQ = R sin u = r sin v and CO = CX + XO, that is R = R cos u + r cos v. From these relations, using the identity $(\sin v)^2 + (\cos v)^2 = 1$ we find $r^2 = 2R^2(1 - \cos u)$.

Substituting these values for r and v in the formula for the area of segment PMQ, and using identities $\cos v \sin v = \frac{1}{2} \sin 2v$ and $\sin(\pi - u) = \sin u$, it becomes: $R^2(1 - \cos u)(\pi - u - \sin u)$.

The area grazed is the sum of the two segments, which simplifies to the form:

$$\pi R^2 - [(\pi - u)\cos u + \sin u]R^2$$

We require therefore to find a value of angle u that makes this area $\frac{1}{2}\pi R^2$, i.e. makes the factor in square brackets $\pi/2$. The possible values of u are between 60° (r = R) and 90° ($r = \sqrt{2}R$).

With use of a pocket calculator and 5-figure sine and cosine tables the closest approximation I can reach is u = 70° 49' = 70.81666° = 1.23598 radians, which gives an area of 1.57096 R^2 (whereas $\frac{1}{2}\pi$ is 1.57079), so the result is accurate to three places of decimals. This gives the length of the tether as: r = 1.16374 R, but this cannot be accurate to more than 3 places so r = 1.164 R is the best I can do. People with calculators or tables accurate to more places of decimals can arrive at a more precise figure.

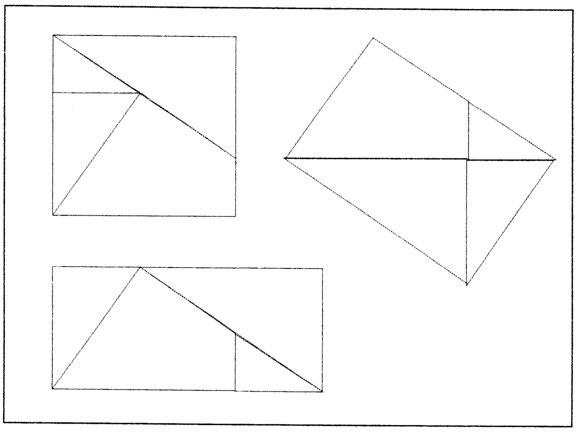
u = 70° 50' gives area 1.57147 R^2 (higher)
u = 70° 48' gives area 1.57042 R^2 (lower)

39. A Double Dissection

by Chris Tylor

To cut a square into four-parts, three of them triangles, that can be put together, in two different ways, to make differently proportioned rectangles.

The solution is illustrated (approximately). If the square is of unit side then the smallest side of the smallest triangle is $x = (1-x)^3$, giving x = 0.317672196 and the hypotenuse of this triangle is $\sqrt{x}$ = 0.563624162.



Chris claims that the relative dimensions are unique if three of the pieces are to be triangular

40. An Incongruent Fallacy

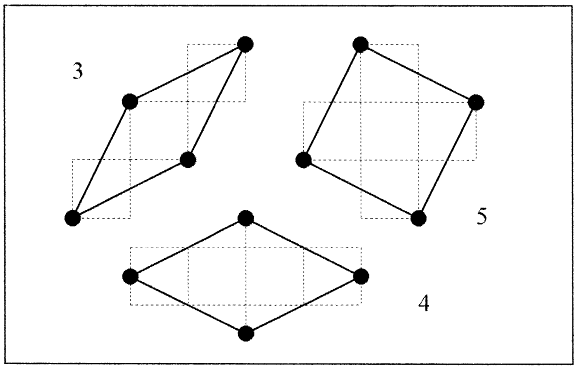
The question was: Where does the following argument go wrong? 32 ≡ 12 (mod 10), that is 2×16 ≡ 2×6 (mod 10), so we can cancel the 2 on each side and obtain 16 ≡ 6 (mod 10), which is true. Similarly this gives 2×8 ≡ 2×3 (mod 10), so as before we cancel the factor 2 on each side finding: 8 ≡ 3 (mod 10), that is 10p + 8 = 10q + 3 for some p and q. But this equates even with odd!

The reason is that although from h ≡ k (mod m) we can deduce that h*n ≡ k*n (mod m) where * can be addition, subtraction, multiplication or raising to a power, the same is not true for division, and cancellation of the factor 2 on each side is equivalent to division by 2.

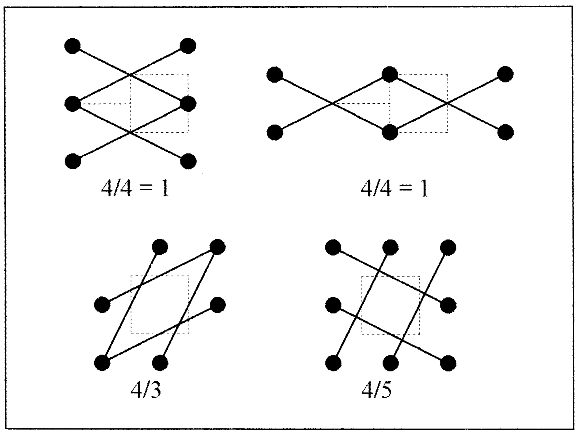
I don't have any rule to determine, before the event, when division <u>is</u> permissible, as from the example it evidently is in some cases.

41. Knightly Quadrangles

This question provides an excuse to present some more results on geometry of knight moves.

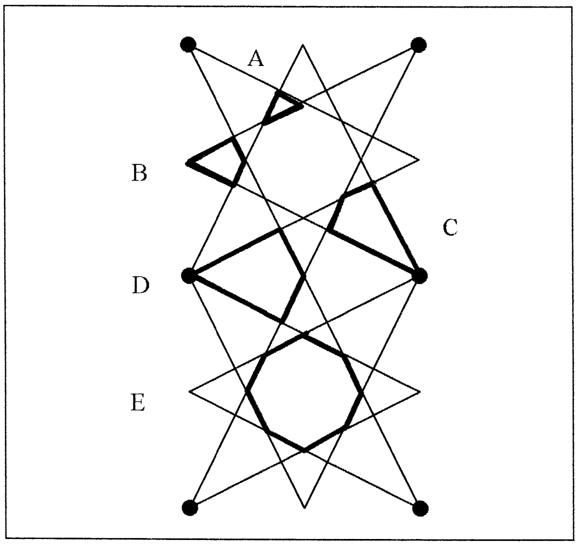


First note that the areas of the three types of rhomb formed by four knight move circuits known as diamond, lozenge and square are 3, 4 and 5 units respectively, as is readily seen from the diagrammed dissections.



Thus the lozenge provides a way of making knight quadrangles with area 1 unit, solving the stated problem. The square gives an area 4/5, and the diamond an area 4/3.

The complete network of knight moves divides up a board (except near an edge) into five basic "irreducible" shapes (in the sense that no knight move can cut across them), namely the smallest triangle (A), three sizes of kite (B, C, D) and an octagon (E). The areas of these are 1/120, 1/40, 1/15, 1/10 and 1/6 respectively, all "aliquot parts" of the unit square.

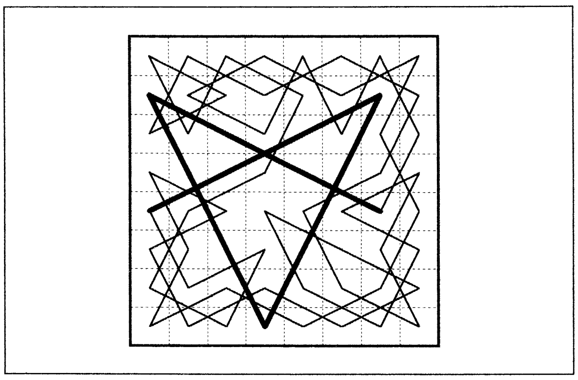


In terms of triangles of size k whose area was proved last time to be $k^2/120$, each kite can be seen to be the difference of two triangles: B = T2 – T1 area (4 – 1)/120 = 1/40; C = T3 – T1 area (9 – 1)/120 = 1/15; D = T4 – T2 area (16 – 4)/120 = 1/10.

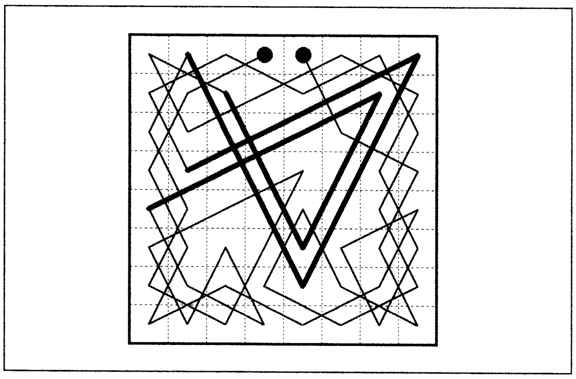
That the area of the octagon is 1/6 can be proved by noting that the square containing it, as proved earlier, is of area 1/5 = 24/120, from which four corners of size 1/120 are removed.

In terms of 120ths the five areas are therefore 1, 3, 8, 12, 20.

Here is an 8×8 tour with a quadrangle of four successive 3-move lines.

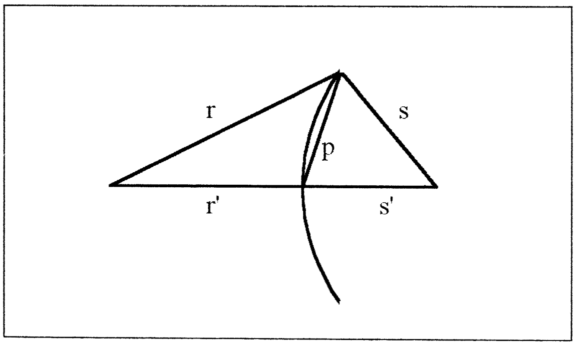


Here is a tour with a couple of large triangles.



42. Touching Spheres

The question was: If two spheres touch, how can the area of contact be calculated? The following analysis is just an approximation.



Suppose the area of contact of two spheres of radii r and s ($r \geq s$) is approximately a spherical indentation in the larger sphere. (It is probably more exactly an ellipsoidal surface.) Then the area of this indentation is πp^2 where p is the distance from the apex of this cap to its rim. [Ref. *CRC Standard Mathematical Tables* 17th edn, p.17.]

Further $p^2 = rs - r's'$. [Ref. J.S.Mackay, *The Elements of Euclid*, 1906, p.338. He cites Schooten 1657, but the figure relates to that for the circle of Apollonius (which is the locus of points whose distances r and s from two fixed points are such that r/s is constant)]

According to Hooke's law it is the relative change in length (strain) that is proportional to the force. That is (along the line of centres) $F = R(r-r')/r = S(s-s')/s$, where R and S are the moduli of elasticity for the two materials. If both spheres are of the same material $(r-r')/r = (s-s')/s$, that is $r'/r = s'/s = k$. (This also implies $r/s = r'/s'$, consistent with the Apollonius assumption)

Thus the area of contact is $\pi rs(1 - k^2)$, where k is related to the force of compression.

43. An Uninteresting Number?

The problem was to prove that 2^(2k–1) + 1 is divisible by 3 (for k = 1, 2, 3, ...), and provide a direct calculation for the quotient.

An answer uses the geometric series formula: [1 + r + r^2 + ... + r^(n–1)](r–1) = (r^n – 1).

Thus 2^(2k–1) + 1 = 2[4^(k–1)] + 1 = 2[4^(k–1) – 1] + 3 = 2[1 + 4 + 16 + 64 + ... + 4^(k–2)](4–1) + 3 = {2[1 + 4 + 16 + ... + 4^(k–2)] + 1}×3, proving the divisibility by 3.

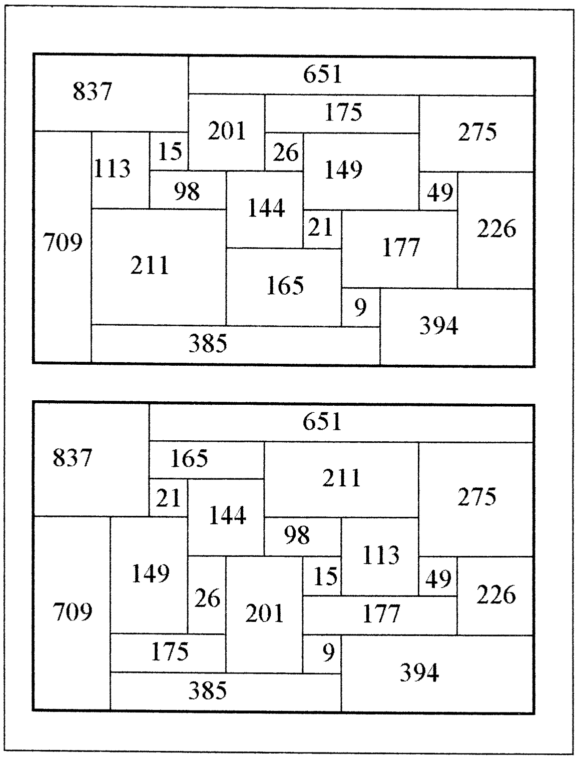
The sequence [2^(2k–1) + 1]/3 runs: 1, 3, 11, 43, 171, 683, 2731, ...

Another sequence including 43 occurred in the answer to the "Ancient Egyptian Camel Tax Problem" (issue 15, p.271).

44. A Double Squaring

by T.H.Willcocks

The problem was: To arrange 21 squares of the following sides into a rectangle in two different ways: 9, 15, 21, 26, 49, 98, 113, 144, 149, 165, 175, 177, 201, 211, 226, 275, 385, 394, 651, 709, 837. His solution follows, in conventional form, since an accurate diagram defeats my drawing capabilities.



The diagrams in the last issue that Dawson attributed to Mr Willcocks are in fact by Messrs Arthur Stone, Cedric Smith, William Tutte, R. Leonard Brooks and Mr Brooks's mother. The story of the problem (but without the diagrams) is told on p.94 of *My Brain is Open: The Mathematical Journeys of Paul Erdos* by Bruce Schechter (Oxford University Press 1998).

45. A Double Tour

The fiveleaper, {0,5}, {3,4}, has four moves at every square. The problem was to find a closed tour in which the unused moves also form a tour. The following is the solution found in Nov. 1991 by Tom Marlow, in his own words:

"The 5-leaper has exactly four moves available on every square of the 8×8 board. In all there are 128 leaps, each being possible from either end. The two closed tours below make use between them of all these leaps. The method of construction was to build a tour starting at a1 and at each leap to mark as unavailable the corresponding leap after 180 degree rotation; e.g. the opening a1-a6 barred h8-h3 and h3-h8. When the tour was complete the same route, rotated 180 degrees, could be travelled using the barred leaps. That tour was then renumbered to start at a1."

20	47	62	55	06	21	46	63
31	42	57	50	11	34	29	44
02	59	16	09	52	03	36	17
13	22	39	26	19	14	23	38
54	05	28	45	64	61	56	07
51	48	35	30	43	58	49	10
32	41	24	37	12	33	40	25
01	60	15	08	53	04	27	18

46	37	60	11	56	49	04	63
39	24	31	52	27	40	23	32
54	15	06	21	34	29	16	13
57	08	03	64	19	36	59	10
26	41	50	45	38	25	42	51
47	28	61	12	55	48	05	62
20	35	30	53	14	07	22	33
01	18	43	58	09	02	17	44

46. Meta-Squares

The problem, quoted from the *Journal of Recreational Mathematics*, was to arrange rectangles of proportions n × (n+1), for n = 1 to 34, in a rectangle 119×120 (or prove it impossible). This problem remains unsolved.

"Meta-Squares" have turned up in an article by Stuart M. Ellerstein: "The Pronic Renaissance: The Ulam Square Spiral (Modified)" in the issue of *Journal of Recreational Mathematics* (vol.29, nr.3, pp.188–189) which came out in February 2000 (but is dated 1998). The name 'pronics' for what I called 'meta-squares' is cited from L.E.Dickson, *History of the Theory of Numbers*, 1952, but was presumably obtained by Dickson from some even earlier source.

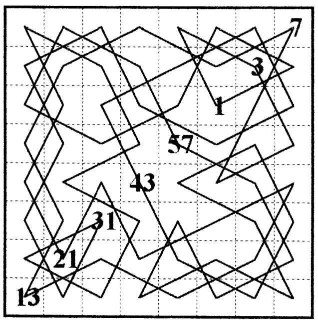
In chessic terminology the Ulam Square Spiral is a numbered Wazir Tour of a square board, beginning at a central square and working outwards. It is noted that, when numbered beginning with 0, the odd and even squares and alternate metasquares lie on radial diagonals.

The next diagram is the Jelliss Metasquare Zigzag invented in response. The numbers along the principal diagonal here are the metasquares (0, 2, 6, 12, ...). The numbers along the first row and column are alternately squares n^2 and near-squares $n^2 - 1 = (n+1)(n-1)$.

0	1	8	9	24	25	48	49	80	81	
3	2	7	10	23	26	47	50	79	82	
4	5	6	11	22	27	46	51	78	83	
15	14	13	12	21	28	45	52	77	84	
16	17	18	19	20	29	44	53	76	85	
35	34	33	32	31	30	43	54	75	86	
36	37	38	39	40	41	42	55	74	87	
63	62	61	60	59	58	57	56	73	88	
64	65	66	67	68	69	70	71	72	89	
99	98	97	96	95	94	93	92	91	90	
00	01	02	03	04	05	06	07	08	09	10

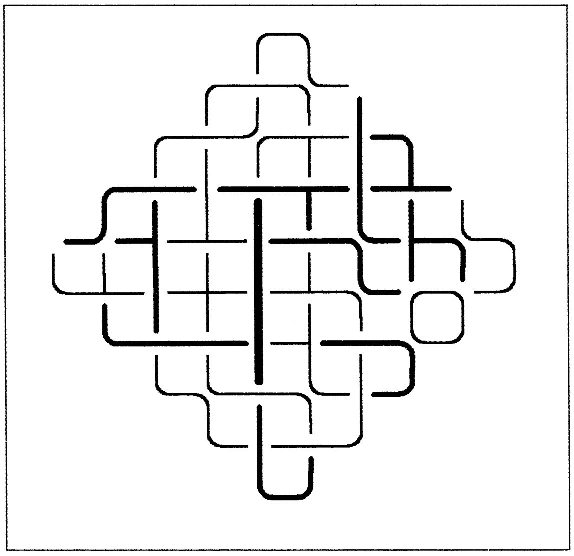
47. Inter-Square Numbers

The question was to construct a figured knight tour showing the eight 'lower intersquare numbers'{1, 3, 7, 13, 21, 31, 43, 57}, which are the numbers of the form n(n–1) + 1 = n² – n + 1, in some regular arrangement. Here is one way, with the sequence, circularly permuted, along a diagonal:

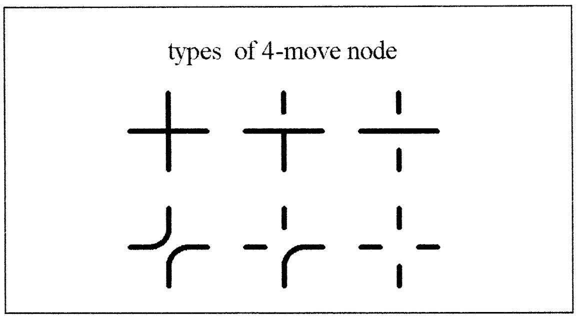


48. Aztec Tetrasticks

The problem was to pack all 25 one-sided tetrasticks into the 'Aztec diamond' shape. A computer-generated solution has been found by Dr Alfred Wassermann, University of Bayreuth. Prof. Knuth tells me this is one of 107 solutions found. Further information available at http://www.mathe2.uni-bayreuth.de/~discreta.



Two facts I established in my attempt at the problem were: (a) the 10 'odd' pieces, i.e. those with 3 moves in one direction and 1 in the perpendicular direction (shown by heavier lines), must be arranged so that 6 are oriented perpendicular to the straight piece and the other 4 parallel to it, to equalise the totals of moves in the two directions; since all the other pieces contribute two moves in each direction. (b) The number of 'breaks' in the 20 outward-pointing corners cannot exceed 6 (the diagram shows 3). This is because there are 59 loose ends on the pieces but 47 of these must be used to complete the 4-move nodes in the interior of the pattern (9 to complete the 3-way junctions, 38 to complete the 19 straight passages) leaving 12, at most, for use round the edges.



Solutions to Crosswords in #17.

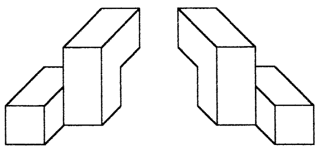
Cryptic Crossword Number 12 by Querculus

D	E	C	A	M	P	■	A	N	T	E	L	O	P	E
■	V	■	U	■	E	■	F	■	Y	■	O	■	A	■
R	E	S	T	O	R	E	R	■	M	A	T	U	R	E
■	N	■	O	■	U	■	I	■	P	■	U	■	T	■
■	■	■	M	U	S	I	C	I	A	N	S	H	I	P
■	C	■	A	■	E	■	A	■	N	■	■	■	C	■
B	O	O	T	Y	■	■	N	A	U	S	E	O	U	S
■	M	■	I	■	P	■	E	■	M	■	X	■	L	■
S	P	E	C	T	R	A	L	■	■	S	T	E	A	L
■	U	■	■	■	O	■	E	■	C	■	R	■	R	■
E	L	E	C	T	R	O	P	L	A	T	E	■	■	■
■	S	■	R	■	O	■	H	■	U	■	M	■	K	■
H	I	K	I	N	G	■	A	U	S	P	I	C	E	S
■	O	■	M	■	U	■	N	■	E	■	T	■	E	■
I	N	T	E	R	E	S	T	■	D	R	Y	I	N	G

Jigsaw Crossword Number 2 by Querculus

■	B	R	A	V	E	■	T	H	R	E	E	S
Q	■	I	■	E	■	X	■	I	■	M	■	E
U	N	D	E	R	■	Y	O	G	H	U	R	T
A	■	G	■	T	■	L	■	H	■	L	■	T
C	R	E	D	I	T	O	R	■	N	A	I	L
K	■	■	■	C	■	P	■	J	■	T	■	E
■	P	A	R	A	P	H	R	A	S	E	D	■
G	■	V	■	L	■	O	■	P	■	■	■	F
L	E	E	K	■	O	N	L	O	O	K	E	R
A	■	R	■	W	■	I	■	N	■	N	■	A
Z	E	A	L	O	U	S	■	I	R	O	N	Y
E	■	G	■	O	■	T	■	C	■	T	■	S
D	R	E	A	D	S	■	M	A	R	S	H	■

Errata. The third pair of polycubes shown in the figure at the top of page 311 is the wrong pair (copied from p.226). The correct diagram is:



I have a vague idea that some of the enumeration totals for closed knight tours of oblong boards on p.265 may be incorrect, but have not found time to check them.

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website: http://homepages.stayfree.co.uk/gpj.

THE END