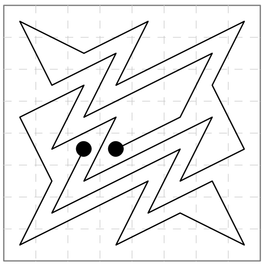
⌘ 11

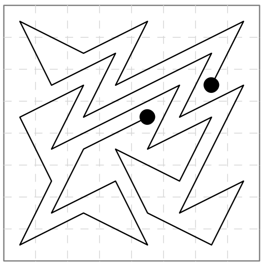
# Alternative Worlds

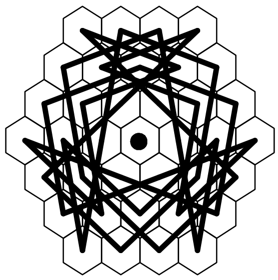
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 55 | 6 | 33 | 10 | 15 | 44 | 31 |
| 5 | 60 | **9** | 56 | 43 | 32 | 11 | 14 |
| 54 | 7 | 34 | 59 | **16** | 13 | 30 | 45 |
| 61 | **4** | 17 | 42 | 57 | 46 | **25** | 12 |
| 18 | 53 | 58 | 35 | 24 | 29 | 40 | 47 |
| **1** | 62 | 3 | 50 | 41 | **36** | 23 | 26 |
| 52 | 19 | **64** | 37 | 28 | 21 | 48 | 39 |
| 63 | 2 | 51 | 20 | **49** | 38 | 27 | 22 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 20 | 47 | 38 | 5 | 10 | 45 | 40 |
| 48 | 51 | **4** | 19 | 46 | 39 | 8 | 11 |
| 21 | 2 | 37 | 50 | **9** | 6 | 41 | 44 |
| 52 | **49** | 32 | **1** | 18 | 43 | 12 | 7 |
| 63 | 22 | 53 | **36** | 31 | **16** | 59 | 42 |
| 54 | 33 | **64** | 17 | 60 | 13 | 30 | 27 |
| 23 | 62 | 35 | 56 | **25** | 28 | 15 | 58 |
| 34 | 55 | 24 | 61 | 14 | 57 | 26 | 29 |

by G. P. Jelliss

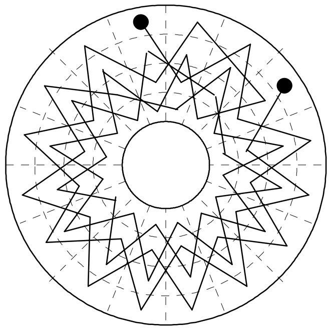






2019

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| B | D | D | M | & | F | F | K |
| E | M | A | E | G | L | Z | G |
| C | C | F | L | N | H | J | E |
| S | **K** | **N** | **I** | **G** | **H** | **T** | Y |
| N | B | **T** | **O** | **U** | **R** | H | I |
| U | R | J | O | K | X | X | S |
| A | M | P | W | Z | V | Q | I |
| Q | V | & | L | P | J | Y | W |



A

| | | | |
|---|---|---|---|
| 63 | 12 | 25 | 34 |
| 28 | 35 | 64 | 15 |
| 39 | 24 | 11 | 52 |
| 10 | 49 | 36 | 23 |

B

| | | | |
|---|---|---|---|
| 26 | 41 | 54 | 13 |
| 59 | 16 | 29 | 46 |
| 6 | 53 | 40 | 19 |
| 37 | 30 | 1 | 50 |

C

| | | | |
|---|---|---|---|
| 5 | 62 | 33 | 18 |
| 38 | 21 | 8 | 51 |
| 27 | 48 | 61 | 14 |
| 58 | 9 | 22 | 45 |

D

| | | | |
|---|---|---|---|
| 42 | 17 | 4 | 55 |
| 7 | 56 | 43 | 20 |
| 60 | 3 | 32 | 47 |
| 31 | 44 | 57 | 2 |

**Title Page Illustration**
Figured Tour with Square Numbers touring a Square by Dawson 1932.
Figured Tour with Square Numbers touring a Cube by Jelliss 1986.
Non-Crossing 8×8 Knight Tours by Dawson 1930.
Hex Knight Tour of a 36-cell hexagon board with three axes of symmetry by Jelliss 2011.
Tour lettered E...Z&A...Z&A...N spelling KNIGHT TOUR by Jelliss 1978.
Knight Tour of a Circular Board by Twiss 1789.
Space Knight Tour of a 4×4×4 Cube by Vandermonde 1771.

## Contents




http://www.mayhematics.com/
Knight's Tour Notes, volume 11, Alternative Worlds.

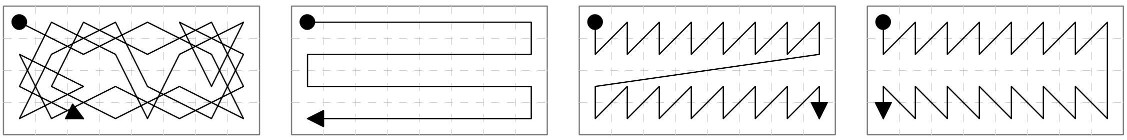
# Lettered Tours

It seems that no published knight tours were shown by means of geometrical diagrams until dalla Volpe (1766), though there are signs of symmetry in some, e.g. Suli (c.900) which suggest that diagrams must have been used in their construction (see History section in ⌘ 6).

At least four different methods of presentation of tours using letters occur in the mediaeval manuscripts: 1) Numbering the cells, using the methods of numeration then in use, based on the successive letters of the alphabet. 2) Listing the coordinates of the cells, again using letters as coordinates rather than specialised numerals. The ‘algebraic’ system of notation now used for chess preserves this in part, using letters for files and numerals for ranks. 3) Usng the literal coordinates of the cells as the first syllables in an acrostic poem. 4) The ‘verse tour’ method of Rudrata.

## Verse Tours

In H. J. R. Muray's History of Chess, pages 53-55, there is an account of three tours, by Knight, Rook and Elephant, given by the Kashmirian poet Rudrata around the year 900. In visual form the tours are as shown below, together with an alternative interpretation of the Elephant tour..



Apparently Rudrata presented these tours in verse form. “The principle of construction is as follows: certain syllables are placed in the various squares of a half-chessboard in such a way that whether the syllables be read straight on as if there were no chessboard or be read in accordance with the moves of a particular piece, the same verse is obtained.” According to my calculations this means that the patterns of syllables in the four tours are of the forms:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | C | C | C | B | B | C |
| C | B | B | B | B | C | C | C |
| B | C | B | C | D | B | C | B |
| C | C | C | B | B | B | B | C |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H |
| I | J | K | L | L | K | J | I |
| M | N | O | P | Q | E | S | T |
| U | V | W | X | X | W | V | U |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | B | C | B | D | C | E |
| B | C | D | E | C | E | E | F |
| G | H | H | I | H | J | I | K |
| H | I | J | K | I | K | K | K |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | B | C | B | D | C | E |
| B | C | D | E | C | E | E | F |
| G | G | H | G | H | H | H | G |
| G | H | H | H | G | H | G | G |

If poems can be written to fit the first and third of these patterns then Sanskrit must indeed be a strange language! Murray argues that the Elephant tour is split at the halfway point because of the difficulty of fitting a verse to the alternative pattern. which “allows the use of only two syllables in the third and fourth lines” and “such a task approaches sufficiently near to impossibility to justify abandonment of the chess condition in part.” However I do not find Murray's argument convincing, since in the Knight tour, apart from the 1st and 21st squares the others use only two syllables, and this was apparently achieved.

It is natural to wonder whether it is possible to fit English verses to tours in the fashion described above. My own efforts (Jelliss 1985) have extended only as far as the following four rather absurd little Rook-tour Reverse-verses.

```
LOVEMADE   YORKDONS   GOODLETS   SWIMFANS
EROSSORE   EBORROBE   EVILLIVE   LOOPPOOL
```

The following is an even more absurd Elephant tour verse. I presume punctuation, pronunciation and spelling of the syllables are open to poetic licence.

```
THE    DO    DO    TOO    DO    SO    TOO   CAN
DO     TOO   SO    CAN    TOO   CAN   CAN   CAN
```

[This account first appeared in *Chessics* #22 1985 p.99]

## Coordinates

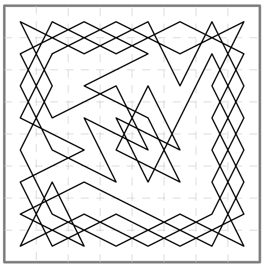
The use of coordinates to record positions and moves in chess goes back at least to the time of al-Adli (c.840), as we have described in the earlier historical sections. One manuscript labels files with arabic letters equivalent to y, k, l, m, n, r, sh, t; and ranks: a, b, j, d, h, w, z, hh. [Murray 1902]

A similar consonant-vowel system of coordinates as in the Somesvara ms (see ⌘ 6) was used in the telegraphic code for transmission of chess games, known as the Gringmuth notation in which the files a–h are lettered BCDFGHKL on White's half of the board and MNPRSTWZ on Black's side, while the ranks 1–4 and 8–5 are lettered AEIO, so that each cell has a two-letter designation. For example castling king-side with the White king is shown as GAKA and with the Black king as SAWA. Beverley's tour in this notation, divided into four-cell sections, is almost an incantation:

MA PE NO RI - SA WE ZO TI - GO KI LA HE - FA CE DO BI
DE BA CI FO - GE KA LI HO - SI WO ZE TA - RE NA MI PO
BO DI CA FE - HA LE KO GI - TO ZI WA SE - RO NI PA ME
PI MO NE RA - TE ZA WI SO - HI LO KE GA - FI CO BE DA

One can of course easily 'demonstrate' the knight's tour just by memorising a simple tour, preferably closed so that the knight can begin on any cell. The coordinate type of presentation can be adapted to serve as a method of memorising a tour. An elaborate verbal scheme of this type was described by George Walker in an article on 'Chess without a Chessboard by a Chess Player' in *Fraser's Magazine* March 1840.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Len | Het | Fen | Bet | Dix | Bor | Doo | Gun |
| Koo | Mun | Lee | Kun | Moo | Kee | Goo | Dun |
| Bee | Div | Ben | Fet | Hen | Let | Mix | Lor |
| Hee | Kiv | Gor | Hix | Liv | Men | Ket | Gen |
| Kix | Giv | Fee | Hor | Gix | For | Hiv | Gee |
| Fiv | Den | Biv | Dee | Bun | Foo | Hun | Loo |
| Mor | Lix | Met | Ken | Get | Fix | Det | Bix |
| Dor | Boo | Fun | Hoo | Lun | Mee | Kor | Miv |



The above article was referenced by Roget (1840) who described the method as "designating each square by a different syllable composed of certain consonants and vowels, indicating the horizontal and vertical columns in which it stands. The whole series of these 64 arbitrary syllables, joined into 16 words, to be learned by heart."

It was only recently that I saw the original reference [in Google Books]. As suspected, the method is the same as that described by William Mason in the *Good Companions Chess Problem Club Folders* (1917) as: "A stunt I used to do over fifty years ago: moving the knight to each square of the board, with my eyes blindfolded or my back turned to it." The ranks are named: un, oo, ee, or, iv, ix, en, et and the files: M, L, K, H, G, F, D, B. This provides monosyllabic names for the 64 squares, which can then be memorised, in the order of the tour, as a piece of nonsense doggerel. The tour is represented as above.

The tour in the original source is the same, except that the verse starts at Den Biv Dee Bun ...

## Acrostics

Some of the tours in the Arabic manuscripts, such as the Suli tour of two halfboards, were presented by means of an acrostic verse, the first two letters of the 64 lines of verse giving the coordinates of the cells. Poems of this type are attributed by Murray to al Basri and ibn Duraid. However, there are no modern examples of this type of acrostic composition because of the modern use of semi-numerical coordinates. Perhaps some poet will undertake the task of completing an appropriate set of 16 quatrains for the Beverley tour, or some other, using the letter-coordinates of the Gringmuth notation, as listed above, as the initial pairs of letters!

The following 'kooky gibberish' is the best I have managed so far. The coordinates are the first two letters of the words:

**Ma**gical **Pe**gasus **No**bly **Ri**de, **Sa**gely **We**nding **Zo**nal **Ti**les,
**Go** **Ki**ndly **La**byrinthine **He**ro, **Fa**bulous **Ce**ntaurs **Do**ubly **Bi**nd,
**De**lineate **Ba**sic **Ci**rcuits **Fo**urfold, **Ge**nerate **Ka**leidoscopic **Li**vely **Ho**rsemanship,
**Si**xtyfour **Wo**rlds **Ze**alously **Ta**rget, **Re**aching **Na**tural **Mi**ddle **Po**ints,
**Bo**ldly **Di**rect **Ca**reful **Fe**et, **Ha**rmoniously **Le**ap **Ko**oky **Gi**bberish,
**To**ur **Zi**gzag **Wa**nder **Se**rpentine, **Ro**ve **Ni**ghtmarish **Pa**lamedean **Me**anders,
**Pi**rouetting **Mo**tion **Ne**atly **Ra**nged, **Te**legraph **Za**ny **Wi**ndings **So**lo,
**Hi**nge **Lo**ck **Ke**y **Ga**te, **F**inally **Co**mpleting **Be**verley's **Da**nce.

---

## Alphabetical Numbering

Many of the earliest tours were presented simply by lettering the successive squares visited by the knight in alphabetical order, since at that time the symbols for numbers were not separate from those for the sounds of speech. I do not know if any knight's tours incorporated words in this way, but Singmaster (1991) reports that al Buni (c.1200) presented some 4×4 magic squares in alphabetical form with the top rank spelling a word.

**PUZZLE 1**: Using A to P for 1 to 16 construct 4×4 magic squares with a word in one rank.

For some solutions see p.65.

The earliest examples I know of that apply this principle to knight's tours are the dedicatory lettered tours employed by T. R. Dawson on the title pages of four of his books in the C. M. Fox series, which began in 1935. Dawson's tours are all lettered A...Z&A...Z&A...J and are all closed. The ampersands are inserted between Z and A to give an odd number of symbols, so that a letter can occur on cells of either colour. We show his WILD ROSES example from Caissa's Wild Roses in Clusters 1937. It gives more scope for showing difficult words or phrases if one allows the tour to be open and to start at any letter of the alphabet. Here is my tour, from *Chessics* 1978, that spells out KNIGHT TOUR and runs from E(ast) to N(orth).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Z | J | M | X | A | F | K | H |
| N | Y | & | **I** | **L** | I | B | E |
| A | Y | **W** | Z | C | **D** | G | J |
| V | O | B | E | H | & | D | C |
| X | K | U | **O** | **S** | **E** | G | A |
| P | N | **R** | D | F | P | **S** | F |
| J | W | L | T | H | U | B | Q |
| M | Q | I | V | C | R | G | T |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| B | D | D | M | & | F | F | K |
| E | M | A | E | G | L | Z | G |
| C | C | F | L | N | H | J | E |
| S | **K** | **N** | **I** | **G** | **H** | **T** | Y |
| N | B | **T** | **O** | **U** | **R** | H | I |
| U | R | J | O | K | X | X | S |
| A | M | P | W | Z | V | Q | I |
| Q | V | & | L | P | J | Y | W |

In *Chessics* #15 (1983) I used a tour lettered Q to Z spelling out CHESSAYS along the top rank to advertise a series of booklets with that title. A similar tour spelling out CHESSICS is impossible because of two Cs on one colour and two Ss on the other colour. Nevertheless Clive Grimstone, on the same page, managed to show CHESSICS along a diagonal by the drastic device of joining each pair of opposite sides of the board like a cylinder but with a Moebius twist – the result of which is known as an 'RP-Board' – he manages this also without employing ampersands.

The tours themselves of course tend to be extremely irregular, since they are constructed to fit the given letters. There is much scope for amusement in trying to get a tour to spell out a phrase of your own choosing. The initial and final letters can be any of the following pairs: AJ, BK, CL, DM, EN, FO, GP, HQ, IR, JS, KT, LU, MV, NW, OX, PY, QZ, R&, SA, TB, UC, VD, WE, XF, YG, ZH, &I. To find which letters appear on which colour it helps to write the alphabet in a zigzag form.

The interest in these tours is in constructing them rather than tracing the tour from the given letters, which is usually quite easy.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **C** | **H** | **E** | **S** | **S** | **A** | **Y** | **S** |
| F | R | B | B | Q | T | R | & |
| I | D | G | T | T | Z | R | X |
| Q | A | C | Z | U | W | I | Q |
| D | J | U | U | H | Y | V | K |
| & | P | G | M | V | J | P | X |
| K | E | N | Y | V | N | L | W |
| O | Z | L | F | M | X | W | O |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C | **K** | E | Y | A | M | **K** | W |
| F | Z | B | L | J | X | K | N |
| B | D | **N** | & | L | **N** | V | L |
| O | **G** | A | **I** | M | J | O | O |
| & | C | R | **I** | P | **G** | M | U |
| S | P | **H** | F | X | U | P | S |
| D | Z | R | Q | **H** | R | T | V |
| Q | **T** | E | Y | S | W | **T** | Q |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| U | Q | W | F | S | S | Y | U |
| H | G | T | R | X | V | R | Q |
| P | V | E | **J** | **O** | E | T | Z |
| H | **I** | O | D | W | & | **P** | Q |
| N | D | **I** | **N** | **K** | **P** | F | A |
| **J** | A | **L** | X | C | **M** | I | **O** |
| C | M | Z | **L** | **M** | K | B | G |
| & | **K** | B | L | Y | H | **N** | J |

Alphabetical tours can also be constructed to show other tricks. The second diagram above (a K...T tour) depicts the 6-cell routes (lettered KNIGHT) of the two black knights in visiting their white cousins. Although the routes the two knights take may seem somewhat arbitrary, in fact they are the only such routes possible under the alphabetical conditions. The Hs and Is have to cross-connect. One of the attractions of these alphabetical tours is the ambiguity that sometimes arises, that can send you off on the wrong track when trying to trace the tour. For example in the middle tour above one starts b8(K)-d7(L) but then has a choice of two Ms at f8 or e5. The scheme shown on the right is an attempt to maximise such ambiguities. The tour runs from H to Q and can be done in 16 different ways, there being two choices at each of the bold-printed letters I, K, M, and O.

**PUZZLE 2**: “Revolver Practice” Construct a tour spelling out REVOLVER on the top rank. This was suggested by C. J. Morse who claimed that this is the only word whose letters form the same pattern as the chessmen on the back row in the opening position! (Solution at end.)

------------------------------------------------------------------------------------------------------------------------

## Cryptotours

The presentation of a knight's tour as a puzzle by means of a verse written on the cells, to be read in the sequence of the knight's moves I call a **cryptotour**. The puzzle is both to reveal the path of the tour and to read the verse. In the earlier examples the verse was written one word to a cell, but later it was more usual to split the words into their syllables. Another form of the puzzle simply uses a single letter on each cell (in which case it is termed **polygraphy**). Often the verses used are specially composed, consisting of eight lines each of eight words or syllables, usually relating to chess or to knight's quests. Other examples extract passages from well known writers. Verses with repeated words often make the unravelling more puzzling because there are choices of route.

### French Polygraphie

This type of puzzle appears to have begun in France in the 1840s by Jules de Poilly in the early chess magazine *Le Palamède* Paris Jun 1842, and others in Oct/Nov 1844 and Sep/Oct 1846).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| qu'au | d'allure | pourtant | et | coursier | fuis | fois | tous |
| mais | blanc | meme | et | quatre | en | ne | ta |
| toujours | tremble | noir | peut | trace | ton | sens | seize |
| monde | faire | change | point | galoper | pas | propre | repasse |
| parvenir | abime | cheval | franchir | marque' | tu | course | du |
| qu'un | d'un | toi | s'ouvrir | doivent | l'ourse | ainsi | verrais |
| un | veux | chaque | voila | tes | au | midi | sa |
| degre' | ce | pieds | qui | jusqu'a | borner | sous | but |

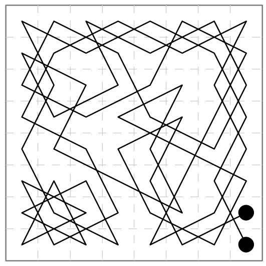
The 1842 example uses the Anderson squares and diamonds tour from the Troupenas article in the same magazine. The poems are specially composed, to consist of eight lines each of eight words. The solution is: ‘Franchir chaque degré d’un monde blanc et noir / Galoper en tous sens du midi jusqu’a lourse / Voila ce qu’un cheval peut faire; mais pourtant / Quatre fois seize pas doivent borner sa course /Au but ainsi marqué toi qui veux parvenir / Tremble qu’au meme point ton coursier ne repasse / Tu verrais sous tes pieds un abime s’ouvrir / Change toujours d’allure et fuis ta propre trace.’ This problem is quoted by Lucas (1882) and in *Cahiers de L’Echiquier Français* (1928).

The 1844 example uses the 1725 Moivre tour (rotated 180°). The verse reads: ‘Il faut que ton cheval sente toujours l’étreinte, / Prends soin de modérer sa trop bouillante ardeur, / Car tu ne pourrais pas sortir du labyrinthe / Si tu ne conservais le sang-froid protecteur; / Le sort de Phaéton ou le destin d’Icare / Viendront t’atteindre, ami, si tu descends trop bas; / En remontant trop haut par malheur on s’égare; / Que le juste-milieu te sauve du trépas!’

The 1846 example uses the 1725 Mairan tour. The verse reads: ‘Galopez en tous sens, en avant, en arrière; / Mais ne repassez pas par le mème chemin. / Gagnez tantôt le cente, et tantôt la frontière, / Si vous voulez mener l’affaire à bonne fin. / Il faut, vous le savez, visiter chaque cae, / Sans en oublier une; et si votre Pégase / Se montre, en bondissant, rétif à votre voix. / Dormez, vous le ferez courir une autre fois.’

A later French example is by A. Canel *Recherches sur les jeux d`esprit* (1867). See the Bibliography for full title and publication details.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| tres | ce | ca- | bruit | guer- | pour | te | sim- |
| tout | un | re | sur- | sci- | qu'un | la | j'ai |
| ren- | peu | tra- | ce | pre | re | ple | ber- |
| re | par- | fe- | la | tout | plo- | truit | me |
| c'est | vail | gloi- | de- | co- | paix | li- | de- |
| plus | l'en | la | qui | re | de- | j'ai | arts |
| ter- | que | vaut | la | ne | de- | boi- | mais |
| jeu | que | re | de | re | ja- | qu'on | Si |



The cryptotour verse reads: ‘Si j'aime la guerre / Tres peu / C'est que de la terre / L'enjeu / Vaut plus que la gloire,- / Un bruit / Qu'un simple deboire / Detruit. / J'ai pour preference / Surtout / Liberte, science / Partout, / Travail qui decore / La Paix, / Arts qu'on ne deplore / Jamais.’ Like the verse the open tour is irregular.

From 1876 onwards, as I have recently found, cryptotours became popular again in France. They appeared weekly in puzzle columns in several newspapers. The tours in most cases are magic knight tours of one, two or four knight paths, where the ends of the knight paths are joined by other types of move, mainly rook moves. Others are king tours (see ⌘ 2).

In particular the column ‘Un Problème Par Jour’ run by **A. Feisthamel** in *Le Siècle* from 30 Oct 1876 to 30 Apr 1894 published most of the new magic tour discoveries as they appeared. The tours appear once a week usually on Fridays, making a total of around 900. *Le Siècle* can now be accessed online through the Gallica (BnF) website. Feisthamel also notes around the start of each year the numbers of magic tours that have been published in other French periodicals. Titles he lists that include many tours are *Le National, Le Gaulois* and *Gil Blas*, found online, and also: *Globe*, *Soir*, *Telegraph*e, *Clairon*, *Etoile francaise* which I have not seen, and others that include single examples.

**E. Framery** is the editor of a puzzle series ‘Passe-Temps Quotidien’ in *Gil Blas*, and the editor the ‘Jeu D'Esprit’ column in *Le Gaulois* signs his name as **Edouard Fy**. Both these columns contain work under the pen-name ‘Mme Celina Fr’ who is identified by Murray as **Edouard Francony**. It seems possible that all four may one and the same person, though I have no definite proof of this.

Similarly **Felix André** is the editor of a puzzle series ‘Divertissements Quotidiens’ in *Le National* and **Felicien Alexis** is the editor of a puzzle series called variously ‘Passe-Temps Du Dimanche’ and ‘Passe-Temps Hebdomadaire’ in *Le Gaulois*, and both feature contributions from a certain ‘M. F. A.’ (M being for Monsieur). In *Le Gaulois* ‘Un Probleme Quotidien’ was later run by **Francois Accloque**. This seems likely another case of multiple personality!

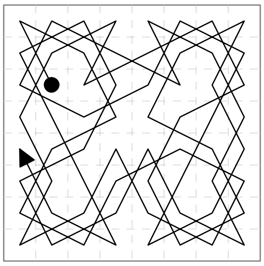
A search for ‘Polygraphie’ in these newspapers online listed results, but not in date sequence, the earliest in *Le Gaulois* being number 42 on 30 Dec 1879. And in *Le National* number 128 by ‘M.F.A.’ 7 Sep 1880, and in *Gil Blas* number 69 by ‘Adsum’ 2 Oct 1881.

The difference from the earlier French examples, besides using magic tours, is that the wording encoded by the tour, instead of being a verse with some connection to chess, is instead a word puzzle in itself, the solution being an anagram, or word-square, or charade or logogriff, or anything of that nature, making it effectively three problems in one, although most of these word puzzles seem to me to be rather trivial. They have had the effect of hiding some good material from clear view.

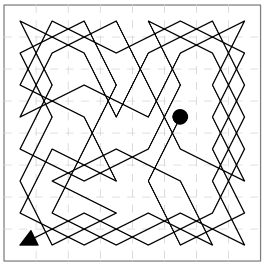
KNIGHT'S TOUR NOTES

Here are two ezamples. The first is ¶772 in *Le Siècle* 25 Apr 1879. No name of the designer is given, but the Anagramme puzzle is by ‘M. Jacquemin-Molez à Reims’. The second is a cryptotour example by ‘Mlle Cartouche à Rennes’ ¶307 in the ‘Passe-Temps Quotidien’ column conducted by Monsieur E. Framery in *Gil Blas* 21 Sep 1880 (solution 2 Oct).This shows an array with one letter per cell and the name MEPHISTO along the top rank. [See also Staunton 1870 prob V below].

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| O | T | V | I | H | N | I | U |
| E | O | U | R | M | P | O | U |
| U | S | T | N | E | E | I | S |
| V | N | A | S | E | E | S | N |
| E | U | O | O | E | O | L | S |
| C | E | A | F | R | U | N | R |
| N | L | N | E | E | T | S | B |
| M | T | E | B | O | A | M | E |



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **M** | **E** | **P** | **H** | **I** | **S** | **T** | **O** |
| E | E | A | S | A | F | M | E |
| H | E | O | E | S | S | R | O |
| R | T | C | L | S | U | P | P |
| S | R | R | P | E | I | P | T |
| E | P | S | I | N | O | L | U |
| E | T | T | O | E | M | C | E |
| E | L | D | O | I | D | S | E |

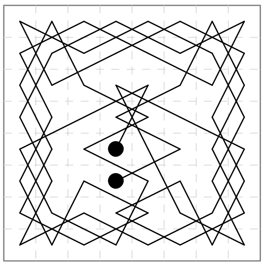


The solution to the first is “Sous une forme on me voit miserable, Puis une autre, honnete et convenable” which is a clue to the anagram CRIME / MERCI. The tour is magic tour 00i (see ⌘ 9).

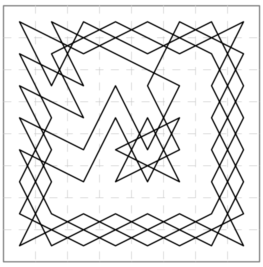
The solution for the second is: “Un dépôt - A la messe / Simple mot répété / L'opposé de richesse / Puis héros fort cite.” which provides clues to the word square GAGE / AMEN / GENE / ENEI. Among the solvers are listed ‘M. M. Méphisto, à Besançon’.

***Cahiers de l'Echiquier Francais*** ‘Le Problème du Cavalier’. Quotes a cryptotour p.168 (sol p.256) from *Le Palamède* 1842 (see Poilly above), and the following lettered tour problem ‘Polygraphie du Cavalier’ solved by a quotation from Barbey d’Aurevilly: ‘La beauté tend a l’unité tandis que la laideur est multiple: Barbey d’Aurevilly’ giving an open tour with approximate axial symmetry, though spoilt by having the end-points internal.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| E | Y | L | E | A | N | A | T |
| T | R | T | I | U | S | N | U |
| E | T | D | E | L | E | M | U |
| U | I | L | B | A | L | L | D |
| I | B | U | Y | U | A | T | A |
| P | E | E | L | R | N | I | A |
| R | V | I | E | B | Q | L | E |
| D | L | A | E | A | T | D | S |



Another use of cryptotours is in encryption. **Gérard de Sède** *L'Or de Rennes* (1967) describes its use in the Rennes le Chateau mystery. The tour used is Euler's first closed tour beginning from c3.



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| X | G | P | U | C | D | E | P |
| R | Q | D | S | F | E | L | E |
| O | A | A | I | S | R | O | L |
| E | D | N | E | E | G | T | X |
| R | I | N | E | E | A | C | U |
| E | T | B | P | R | R | X | E |
| T | A | I | T | T | I | S | A |
| N | N | A | P | S | L | N | X |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| O | D | I | U | M | E | I | A |
| G | E | M | N | D | J | E | C |
| D | P | E | E | C | O | C | M |
| D | A | I | A | X | R | H | S |
| S | E | A | O | M | T | A | H |
| R | E | L | E | C | I | B | E |
| D | E | E | I | A | E | E | N |
| C | D | L | U | V | E | V | L |

Following the tour over the two boards reveals the mysterious message: “Bergere pas de tentation que Poussin teniers gardent la clef pax DCLXXXI par / la croix et ce cheval de Dieu J Acheve ce daemon de gardien a midi pommes bleues”. The cypher was supposedly hidden in *Le Grand Parchemen* discovered by the Abbé Sauniere in 1891. This is an extract from St John's Gospel with 128 additional letters inserted. These apparently need to be decoded (using a Vigenère cypher) before they can be entered into a chessboard grid and read along the knight path.

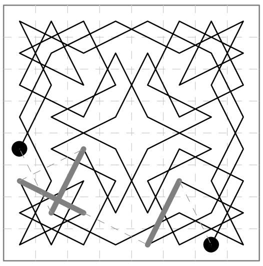
What it all means, or if it is all one big hoax, has been the subject of numerous books.

## Dutch Cryptotours

Cryptotours came in fashion in Holland for a while in the chess magazine *Sissa*: ‘De oplossing van den paardensprong’ (1848 p.319, 341) a cryptotour is posed using words and some capital letters on the cells. The solution is in the form of an eight-line verse in rhyming couplets. Some of the letters refer to the chess pieces in Dutch notation. No tour diagram is given but it is a distinctive pattern with some diagonal symmetry (see ⌘ 7).

Another (1849) is by **E. A. Schmitt** ‘Paardensprong-raadsel’ {Knight's tour enigma} *Sissa* p.330. This cryptotour has a single letter in each cell, some of them being capitals. The solution given (1850 p.45) reads: “A. D. Philidor war Componist und des geschickteste Schachspieler seiner Zeit.” From this I deduce the near-symmetric tour shown, which derives from one of the octonary pseudotours having two 12-move and two 20-move circuits, one move being deleted from each and three links inserted. I show the linkage polygon in grey. No diagram is shown with the solution in the magazine.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| e | w | d | e | r | h | h | d |
| o | s | Z | C | e | d | e | c |
| a | i | r | i | s | a | n | c |
| g | l | r | t | o | S | p | n |
| t | s | l | i | i | P | m | u |
| h | e | c | e | p | i | i | D |
| k | e | c | e | h | s | t | n |
| s | i | t | r | s | o | A | e |

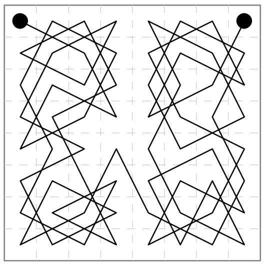


In the original cryptotour some of the letters are wrong, corrected in diagram above: Tracing the tour, even given the solution, was puzzling because of places where alternative choices occur.

## German Cryptotours

The earliest example I know of in German is from: *Leipziger Illustrirte Zeitung* #111 (16 Aug 1845), shown on the Elke Rehder website.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Schach | wacht, | ter | lopp | ni- | dran | chin | Matt |
| im | vor- | tont | er- | Scha- | ich, | Ko- | chen |
| Ihr | Feld- | Ga- | Rit- | und | gin, | und | biet' |
| an, | Ihr | auf! | der | Schach | Schach | Ro- | Auf |
| ruf, | Man- | Ihr | an, | brich | drauf! | lem | hat, |
| Lau- | die | Schlacht, | zur | Al- | noch | auf, | trum |
| ner | Hei- | ke | greift | Ins | sie- | ben | was |
| Flan- | fer | all | ssen | Le- | Mit | en- | gend |



The verse reads: Schach tönt der Feldruf, auf! erwacht, / Ihr Männer all' zur heißen Schlacht, / Ihr Ritter, im Galopp voran, / Ihr Läufer greift die Flanke an, / Ins Zentrum, Rochen, dran und drauf! / Auf, Königin, brich siegend auf, / Mit Allem, was noch Leben hat, / Schach, Schächin biet' ich, Schach und Matt. The solution is a corner-to-corner tour (a8-h8) in two halves, almost completely (14/16) of squares and diamonds.

The *Illustrierte Zeitung* published a series of cryptotours on the Four Handed Chess board apparently beginning in 1852. Photocopies of cuttings in the J. G. White collection showing 10 such tours were sent to me by Cleveland Public Library (Ohio, USA). Unfortunately the hand-written publication details are difficult to read with any certainty. See the section on Shaped Tours (p.188).

There appear to be many other German examples in **A. Herma**, *Rösselsprünge aus deutschen Dichtern*: *Auflösungen* {Knight’s tours from the German Poets. With solutions} published in Frankfurt 1889 (and possibly 1849). This is a source I have not seen, but it appears to consist of poems in German presented in the form of knight tours, as in the earlier Leipzig example. According to Google Books, New York Library has *Illustrirter familienschatz: ein universum für die deutsche familie*, Volume 14, Part 2 (Verlag des Illustrirten Familienschatz, 1889) which has a section with the same title. The Google snippets show part of a cryptotour.

Two other groups of cuttings from newspapers sent to me from the Cleveland (Ohio) Public Library, show cryptotours on the 8×8 board from around this date. These, judging from the printing, appear to be from two different sources. See the section on Approximate Axial Symmetry (p.460) for diagrams. One set of 24 problems, where the paper has evidently discoloured with age, are just headed ‘Rösselsprung’ while in the other set they are numbered 1 to 17. In each case the typeface is a style of German blackletter (Fraktur) with long esses. The quotations used for the tours are mostly from well known German poets of the time including: E. M. Arndt (1769-1860), F. Rückert (1788-1866), H. Heine (1797-1856), G. F. Daumer (1800-1875), K. E. Ebert (1801-1882). A couple quote Byron (1788-1824) in German translation (1864/5) by Otto Gildemeister (1823-1902).

## English Cryptotours

Cryptotours only came to Britain in the 1870s where they were popularised by **Howard Staunton** in his chess column in *Illustrated London News* 1870–74. The column began 25 Jun 1842 and was edited by Staunton, the leading British chess player of his time, from 1845. In the 31 Dec 1870 issue he began a series of 16 cryptotours (numbered I – XVII, inadvertently omitting XII) which attracted much attention, judging by the long lists of solvers. This was before the introduction of crossword puzzles. These are the first English examples I have found. The 27 Jun 1874 issue, p.619, announced the death of Mr Staunton ‘who for many years had charge of this column’.

**PUZZLE 3**. (Solution at end). This is his first cryptotour problem.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| sor | to | king | good | say | luck | loy | eth |
| and | moth | a | soon | dis | our | to | bad |
| place | ry | church | his | force | is | hat | al |
| er | queen | him | wight | he | to | may | truth |
| man | his | and | and | chess | es | knight | op's |
| a | sneer | the | and | un | lawn | of | tates |
| cas | that | at | less | pawn | no | bish | lant |
| eth | faith | tles | hath | the | gal | in | love |

“In the diagram above the composer has endeavoured to popularise the problem by rendering the solution poetical. The reader is to discover the first syllable (which marks the starting point of the knight) and then to copy all the syllables one by one, according to the knight's march over the sixty four squares. The result will be certain sentences with which most chess players will agree. We shall be obliged to anyone conversant with the subject who will favour us with an explanation.” The solution has one of the most amusing chess-related verses I have come across, given at the end of this book. The author is not mentioned; perhaps it was Staunton himself (he was a literary man; a one-time actor, and editor of an edition of Shakespeare).

His examples employ a wide range of literary quotations, especially from Walter Scott and Shakespeare, and even include examples in French (de Musset), Italian (Dante) and Macaronic verse (Porson). Any connection between the verses used and knight's tours is slight and whimsical, except in the three cases I, V, XIV, where specifically chess related text is used. The series begins with the customary mention of Euler and Moivre. Then comes Problem I (vol.57, 1870, p.691 with solution in vol.58, 1871, p.67).

Problem II (p.139, 187) uses a verse from Robert Herrick (*Night Piece to Julia* - evidently a pun). The tour is of squares and diamonds.

Problem III (p.267, 322) uses a verse from Shakespeare (*The Tempest* Act I, scene 2, Ariel's song, containng the phrase ‘Foot it featly here and there’). The tour is Jaenisch magic (12o).

Problem IV (vol.59, 1871, p.139, 235) uses a verse from Christopher Marlowe (Come live with me and be my love ...). The tour is Beverley (27a).

Problem V (p.259, 339) is of different form, with single letters on the cells instead of syllables.

“Knight’s Tour No. V, by an old problem composer, respectfully dedicated to Mr Staunton. The letters, taken continuously in the order of the knight’s route over the board, form a descriptive sentence which is ‘Bagman’s (or Piper’s) News’ to a chessplayer.”

This is offered for solution here.

**PUZZLE 4** : (Problem V) For solution see end.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **S** | **T** | **A** | **U** | **N** | **T** | **O** | **N** |
| R | A | I | O | N | O | A | R |
| H | O | O | G | Q | E | E | M |
| R | E | E | T | P | T | E | X |
| N | E | P | C | L | S | O | C |
| V | E | K | O | H | O | I | N |
| T | I | O | S | E | F | T | R |
| S | E | H | G | M | F | R | S |

Problem VI (p.363, 435) uses two related verses by Edmund Waller (Go lovely rose ...) and Kirke White (Yet though thou fade ...). The tour is similar to Euler's first closed tour.

Problem VII (p.575 and vol.60, 1872, p.187) uses a verse from Alexander Pope (All nature is but art unknown to thee ...). This is presented with three letters on each cell (all nat ure isb uta rtu nkn own tot hee ...) which is noted as being ‘more difficult than preceding tours’. Nevertheless the tour attracted 132 solvers (88 + 24 + 20 listed in the January 6, 13, 20 issues). The tour is Jaenisch (12n).

Problem VIII (p.251, 315) uses a verse from Walter Scott (*The Fair Maid of Perth* containing the phrase ‘Thy life its final course has found’). The tour is symmetric formed of two half-board tours.

Problem IX (p.411, 555) uses an anonymous verse, ending: ‘An hour of joy you know not / Is winging its silent flight’). The tour is the one from Tomlinson (1845) with constant difference 16.

Problem X (p.587 and vol.61, 1872, p.22) uses a version of the *Elegy* said to have been written by Chedicke [Chidiock] Tichborne in the Tower of London on the night before his execution for high treason in 1586: ‘My spring is past, and yet it hath not sprung; / The fruit is dead, and yet the leaves are green / My youth is past, and yet I am but young / I saw the world, and yet I was not seen / My thread is cut, and yet it is not spun / And now I live, and now my life is done.’ The tour includes a 15-cell tour of the 4×4 centre area.

Problem XI (p.71, 143) uses a hymn said to have been written by Walter Scott at age 12. The tour is the ‘Roget Card’ tour from Tomlinson (1845). There is no Problem XII.

Problem XIII (p.239, 335) is in French, using a verse by Alfred de Musset. The tour is approximately axially symmetric.

Problem XIV (p.575 and vol.62, 1873, p.67) uses an anonymous verse composed for the purpose, similar to the French verses used by de Poilly (1842): “From right to left and to and fro / Caught in a labyrinth you go / And turn and turn and turn again / To solve the mystery but in vain / Stand still and breathe and take from me / A clue that soon shall set you free / You entered easily find where / And make with ease your exit there.” The tour is irregular.

Problem XV (p.187, 259) is in Italian, the first verse from Dante's *Inferno*. The tour is open and irregular featuring a central knot.

Problem XVI (p.92 and vol.63, 1873, p.183) uses a Macaronic verse by Porson, a mixture of English and Latin. The tour is Jaenisch (12n).

Problem XVII (vol.64, 1874, p.115, 163 verbal solution, 307 geometrical solution) uses a verse by ‘the late Sheriff Bell’ [Henry Glassford Bell, whose obituary appeared in the Jan 24 issue]. The tour consists of two 32-move knight paths (forming a biaxial pseudotour) connected by a rook move. This last diagram is very attractively drawn, the tour being shown by white lines, with white dot ends, on a board of black and grey squares, though the rook link is not shown.

## Other English Examples

An article by **J. B. D—** ‘The Knight's Tour’ *The Leisure Hour, A Family Journal of Instruction and Recreation* (vol.22 1873) #1133 p.587-590, 752, includes a cryptotour encoding ‘O County Guy’ by Walter Scott. The lines are presented using three letters to a cell, as in Staunton's Problem VII.

The last item in *Poems and Problems* (1882) by **John Augustus Miles** is a cryptotour bearing the title ‘The Retrospect’ which uses a passage from *Ossian* (J. Macpherson) [cited by N. M. Gibbins in *FCR*]. He has another in *British Chess Magazine*, vol.4, p.72. This is a cryptotour using the Beverley magic tour [cited by Murray 1930]

The compilation *Chess Fruits* (1884) by a husband and wife team **Thomas B. Rowland** and **Frideswide F. Rowland**, includes four good cryptotours, one apparently sent to a *British Chess Magazine* knight's tour prize tourney, others from *Sussex Chess Magazine* and *Brighton Guardian*. [BCPS Library]. The verse of the second, which may owe something to Jules de Poilly's French examples, is: "With nerve of steel and heart of fire / A gallant knight did once aspire / To roam the land of black and white / And prove to all his powerful might. / He started from the King's domain / And back again to it he came / Without missing a single square / Nor resting twice on any there."

**PUZZLE** 5: Here is the first one to solve. (Solution at end.)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Now | irt | ght | our | heb | seo | pat | red |
| Kni | veils | bri | hun | Our | cke | ier | fs |
| mb | tray | day | ast | tyf | tot | cour | hwa |
| ness | he | Fro | ght | che | yne | ix | Wej |
| das | ima | dark | Our | ard | our | nda | ver |
| Till | but | yst | ge | the | O'er | our | Thr |
| lgr | lea | lif | ave | hwa | onw | cle | rkw |
| eis | lon | api | hat | ar | etr | oug | ney |

The third quotes a verse from Shakespeare (*Merchant of Venice* Act 3, Scene 2, lines 131-8). The last is a seasonal greeting.

One of the oddest cryptotours, and certainly the longest is by **H. Eschwege** (1896). Byron's poem *Mazeppa* is presented on a series of chessboards, one word to a square over 48 boards. See the Bibliography p.757 for more details.

In 'The Knight's Tour on the Half-Chessboard' *The British Chess Magazine* (vol.37, 1917, p.305–9, 355, 392) **H. J. R.Murray** makes interesting use of a cryptotour p.309: "My most recent investigations have dealt with the tour on the whole board in general. I have made some advance, but not sufficient to make it desirable to give any account of the lines upon which I am working. Under similar circumstances mathematicians of the sixteenth and seventeenth century were wont to publish anagrams which concealed the key to their work, so that they could prove their claim to priority if a rival investigator forestalled them in publication. May I adopt their custom and give a tour in the puzzle form of older chess magazines, which illustrates three maxima in a theory of the tour which promises to reveal the total number of tours possible? No excuse is necessary in this magazine for borrowing for the purpose a poem from one of its most welcome contributors in its early days — the late Professor Tomlinson."

**PUZZLE 6:** (Solution at end.)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| he | fall | dom | ne | the | thinks | then | he |
| frm | man | speaks | the | wis | may | think | ly |
| and | fore | him | wise | ver | hap | or | first |
| game | fol | thinks | of | fool | at | chess | our |
| be | words | tious | same | the | in | speaks | all |
| ly | less | out | dom | cau | is | roy | of |
| its | plays | and | rule | the | game | so | life |
| hour | frets | thought | a | wis | of | still | al |

To conclude this section here is a problem by **G. E. McGuffey** *Fairy Chess Review* (vol.3 #8 Oct 1937 p.86 ¶2932, Sol #9 Dec 1937 p.96). A cryptotour with one letter per cell. The answer is from Ben Jonson. No-one solved it, so perhaps a clue would help that it was intended as a tribute to P. C. Taylor and most of the I's are the first person singular.

**PUZZLE 7**. (Solution at end.)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| G | I | D | E | O | R | E | O |
| L | K | O | A | E | Y | E | T |
| E | N | L | L | N | R | U | N |
| I | E | C | F | M | N | A | H |
| I | A | O | I | I | W | S | I |
| H | T | H | K | O | G | H | U |
| T | U | E | M | H | O | T | O |
| H | U | E | R | A | T | G | R |

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## Crossword Puzzle Tours

It seems there is a long-established tradition of the literary application of knight’s tours to the construction of special crossword puzzles. The examples 10×10 by ‘Alban’ (see ⌘ 5) and 12×12 by ‘Wolfram’ (see ⌘ 7) are the most interesting examples I have seen.

Will Scotland alias ‘Alban’ in *Crossword* the magazine of the Crossword Club 1988 gives a puzzle using a 10×10 knight's quaternary tour and the first verse of Lewis Carroll’s *Jabberwocky* (which happens to have 100 letters) entered along the knight path. The minor point that as a result there are no actual words across and down was overcome by arranging that one letter be omitted from each answer before being entered on the diagram. This is a standard ploy among crossword buffs, known as ‘Letters Latent’. Thus for example the fourth clue across was “Cask of wine the French knocked back around noon — boring!” The answer to this is TUN-N-EL, from which the Ns are omitted, giving TUEL to enter. Some letters remained unclued. The omitted letters, astonishingly enough, spell out the title of the book from which the quotation is taken.

As the result of a request in *Crossword* in 1992 I received several replies about crosswords with chess-move themes that appeared in the BBC Radio magazine *The Listener*. Derek Willan particularly recalled one that appeared “about 15 years ago and involved tracing a knight’s move around the six faces of a cube”. D. G. Mockford mentioned crossword 1020 by Pipeg (13 Oct 1949) in which clues were entered by following routes taken by knight, rook and bishop, and 1024 by Cocos with moves in knight paths. This had the title ‘Knight’s Move – IX’ presumably the ninth in a series!

H. J. Godwin listed 18 later examples. I reproduce here his list of numbers, dates and composers for any reader who may wish to follow up the references (1306 May 55 Gib, 1348 Mar 56 Wray, 1378 Oct 56 Halezfax, 1386 Dec 56 Sugden, 1468 Jul 58 Halezfax, 1590 Nov 60 Wray, 1632 Sep 61 Wray, 1672 Jun 62 Aeschylus, 1677 Jul 62 Wray, 1709 Feb 63 Doghouse, 1714 Apr 63 Chabon, 1783 Jul 64 Wray, 2150 Aug 71 Sabre, 2407 Jan 77 Leiruza, 2428 Jul 77 Leiruza, 2512 Aug 79 Leiruza, 2900 May 87 Adam, 2980 Dec 88 Casein).

Though the magazine has closed, ‘The Listener Crossword’ still appears in *The Times* on Saturdays. David Pritchard sent me a cutting of 3353 by ‘Wolfram’ (13 Apr 1996) that uses a 12×12 knight’s tour (see ⌘ 7). Words are entered along the tour, commencing at the top left corner, one letter to each square. Further, by reflecting the tour in the principal diagonal a second series of words is spelt out along the tour!

How some of these remarkable feats of word manipulation are accomplished I’m not too sure, but they certainly carry on and advance the tradition in an admirable way.

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# Non-Crossing Knight Tours

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## Historical Introduction

A simple question leading to interesting convoluted patterns and difficult to answer with certainty is: *What is the longest journey a piece can make on a given board without crossing its own path?*

Such a path is known as a non-crossing or non-intersecting 'tour' though it does not necessarily visit every cell of the board. The simple case of non-crossing paths by the wazir is treated in the booklet ⌘ 2 on Walker tours, since two wazir moves cannot intersect, and the wazir can in fact make complete non-crossing tours of rectangular boards.

The knight problem for the 8×8 board was solved by T. R. Dawson and W. Pauly in the 1930s. The problem for small rectangular boards was rediscovered by L. D. Yarbrough in the *Journal of Recreational Mathematics* 1968 (vol.1 #3 p.140-142) who gave solutions on all board sizes up to 9×9. Some of his results were improved on in letters in the same journal 1969 (vol.2 #3 p.154-157) by R. E. Ruemmler (7×8, 5×9 to 9×9), D. E. Knuth (5×6, 6×6, 7×8, 8×8, confirming Dawson/Pauly, and 5×9) and M. Matsuda (6×6, 6×8, 5×9, 7×9 and 9×9).

The best results of these authors, up to 9×9, by number of moves, are:

3×3 = 2; 3×4 = 4; 3×5 = 5; 3×6 = 6 closed; 3×7 = 8; 3×8 = 9; 3×9 = 10.
4×4 = 5; 4×5 = 7; 4×6 = 9 open, 8 closed; 4×7 = 11; 4×8 = 13 open, 12 closed; 4×9 = 15.
5×5 = 10 open, 8 closed; 5×6 = 14; 5×7 = 16; 5×8 = 19 o, 18 c; 5×9 = 22 o, 20 c.
6×6 = 17 open or reentrant; 6×7 = 21; 6×8 = 25 open, 22 closed; 6×9 = 29.
7×7 = 24 open, 24 closed; 7×8 = 30 open, 29 open symmetric, 26 closed; 7×9 = 35.
8×8 = 35 open or reentrant, 32 closed; 8×9 = 42.
9×9 = 47.

Apart from one new result on the 18×18 board (published on the Knight's Tour Notes website 22 July 2003) the results reported here were first published in *The Games and Puzzles Journal* issue 17 (1999), which was based mainly on work that **Robin Merson** sent to me in 1990-91. It was only in 2002 that I learnt that Robin had died in August 1992. Robin had reported that he first became interested in this problem through some items that appeared in *Games & Puzzles* magazine in 1972-3, where he published a letter (issue 9) outlining some results. His later work, reported below, was sent to me (see letters on website) open path results 9 Nov 1990 and closed paths 23 Ap 1991).

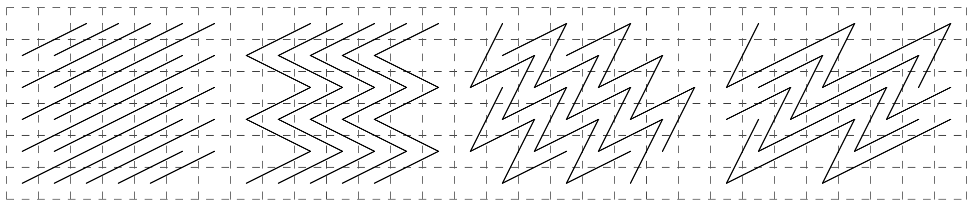
The table below gives the maximum sizes, in number of **moves**, achieved for open and closed non-intersecting paths on square boards of various sizes. Merson's values for open paths up to 9×9 agreed with the work of Yarbrough & Co. Improvements may still be possible on the larger boards.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Side:** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** |
| Closed: | 0 | 4 | 8 | 12 | 24 | 32 | 42 | 54 | 70 | 86 |
| Open: | 2 | 5 | 10 | 17 | 24 | 35 | 47 | 61 | 76 | 94 |
| **Side:** | **13** | **14** | **15** | **16** | **17** | **18** | **19** | **20** | **21** | **22** |
| Closed: | 106 | 126 | 148 | 172 | 200 | 228 | 256 | 288 | 322 | 360 |
| Open: | 113 | 135 | 158 | 183 | 211 | 238 | 268 | 302 | 337 | 375 |
| **Side:** | **23** | **24** | **25** | **26** | **27** | **28** | **29** | **30** | **31** | **32** |
| Closed: | 396 | 434 | 478 | 520 | 564 | 612 | 662 | 714 | 768 | |
| Open: | 414 | 455 | 499 | 542 | 588 | 638 | 689 | 743 | 798 | |

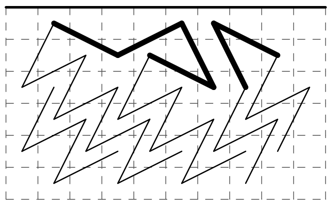
The following improvements since 2000 have been included. **Open tours**: G. P. Jelliss (2003) side 18 to 238. Alexander Fischer (2006) size 14 to 135 and 16 to 183 open. Alex Chernov (2011) size 22 to 375, size 24 to 455, size 26 to 542, size 30 to 743. Bernard Lemaire (2015) size 17 to 211 and 25 499 open. N. Makarova (2011) new result size 31 of 798 open. **Closed tours**: Chernov (2011) size 13 to 106. Derek Kisman (2018) size 14 to 126, size 18 to 228, size 25 to 478.

## Methods of Construction

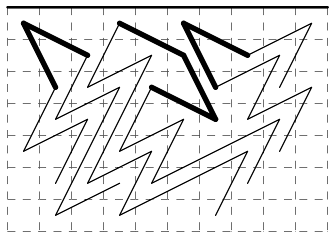
The simplest arrangements of knights moves that cover an area completely are (a) the close-packed parallels (b) lateral zigzags, (c) diagonal zigzags or (d) combinations of these. When we consider the ways of joining up these lines in adjacent pairs, using links that fit closely to the edges and corners, types (c) and (d) prove the most economical.



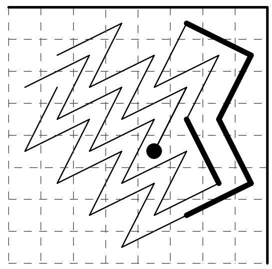
Robin Merson drew attention to the following cases to which he gave names. It is possible to interpolate a VW structure in each edge of certain tours $n \times n$ to give a tour on an $(n + 8)$-side square, though this does not always guarantee that the resulting tour is of maximum size.



VW edge structure



EYE edge structure



SWAN corner

Robin gives the following instructions for determining the length of a tour without counting all lines: Put $(n - 2)$ black crosses (x) along each edge on unvisited squares, i.e. one in each row or column perpendicular to the edge, except at the ends. Put a red blob (o) in each remaining unvisited square, and count the number of such blobs, b, which he calls the loss of the tour. Then the length in the case of an open tour is $L = n^2 - 4(n - 2) - b - 1 = n^2 - 4n + 7 - b$. For example in the $11 \times 11$ tour shown below $n = 11$, $b = 8$, $L = 76$. If instead of the length L we count the coverage C (number of cells used) Merson's formula can be put in the form $C = (n - 2)^2 + 4 - b$, true for open or closed tours. The estimate $C \approx (n - 2)^2 + 4$ is a slight improvement on the value $(m - 2)(n - 2)$ conjectured by Yarbrough for rectangular boards.

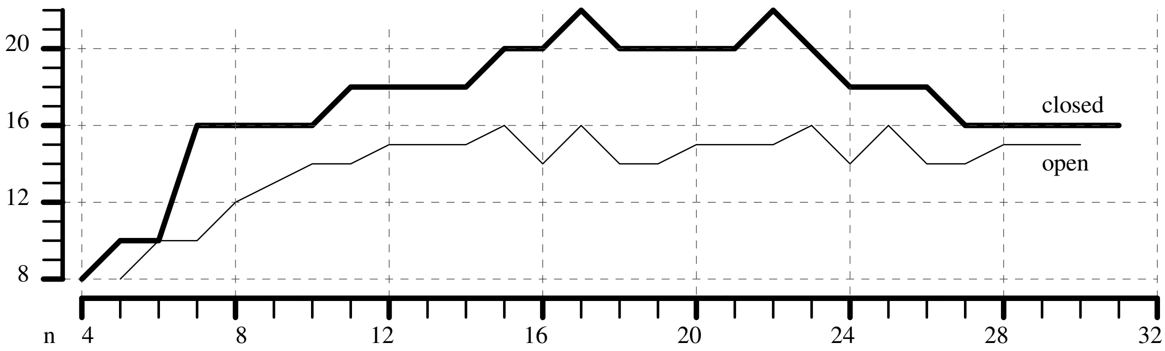
The following diagrams are some example open tours by Robin Merson, including an illustration of his VW extension method, and the new $18 \times 18$ result by Jelliss. Note that eight voids (x), one for each extra rank or file, plus four extra voids (blobs, o) are introduced by each VW formation, one inserted in each side. The paths of side 15 and 23 are the only symmetric solutions that he produced.

## Some Analysis

The length L of a path is counted by the number of moves and its coverage C by the number of cells visited. In a closed tour $C = L$, but in an open tour $C = L + 1$. What function remains constant during the VW extensions? The board size increases from n to $n' = n + 8$. In the $n \times n$ tour we have $C = n^2 - 4n + 8 - b$. In the open tour case the loss becomes $b' = b + 16$ (4 extra blobs in each side) while in the closed tour case it becomes $b' = b + 24$ (4 extra blobs at top and left, 8 extra at right and bottom). Thus in the open case $2n - b$ remains constant (i.e. $2n' - b' = 2n - b$), while in the closed case $3n - b$ is constant. These numbers can be called the excess (E) of the tour. Writing them as $gn - b$ ($g = 2$ for open, 3 for closed) we find the formula: $E = C - n^2 + (4 + g)n - 8$, where $4 + g$ equals 6 for open, 7 for closed.

Below are plots of E for the maxima so far found. In the open case for $n > 10$, $C > n^2 - 6n + 22$. The plot suggests that the maximum value of E for open tours is 16, or does it increase further? For closed tours the excess increases to a peak of 22 and then falls off, and Robin said he would be surprised if it is greater than 16 for any n greater than 31.

To summarise: for $7 < n < 31$ maximum length tours have a length of at least $n^2 - 7n + 24$ and for $n > 31$ have a length of at least $n^2 - 7n + 22$. Plots of excess for closed and open tours:




The above account is simplified slightly from Merson's original version, in which he defined an ‘excess’ for a closed tour equivalent to E + 8, and a ‘strength’ for an open tour equivalent to E + 7 (the difference of 1 resulting from defining it in terms of the length L = C – 1).

The 24-size was the largest open tour that Robin actually diagrammed in his letters to me. The figure for 26 in the table was mentioned as an extension from the 237-move 18×18 solution, and the figures for 25 and 27 to 30 are implied by his graph of excess values.

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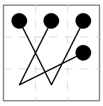
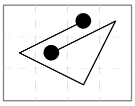
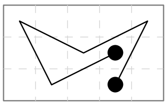
# Knight Open Path Solutions

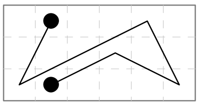
Diagrams of the maximum length open paths by number of moves. Two examples are given in some cases. Symmetric solutions are preferred where possible.

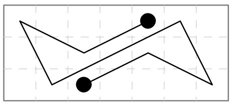
Three, Four and Five Ranks:

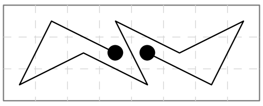

3x3 ~ 2

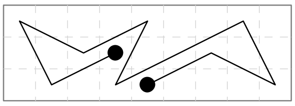

3x4 ~ 4

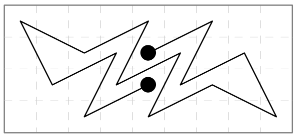

3x5 ~ 5

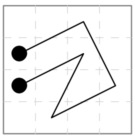

3x6 ~ 6

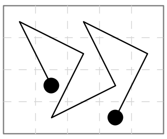

3x7 ~ 8

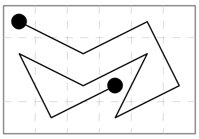

3x8 ~ 9

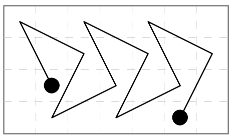

3x9 ~ 10

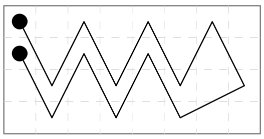
4x9 ~ 15


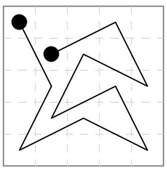

4x4 ~ 5

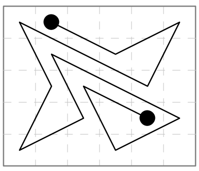

4x5 ~ 7

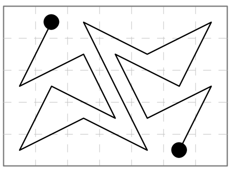

4x6 ~ 9

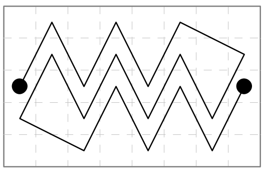

4x7 ~ 11

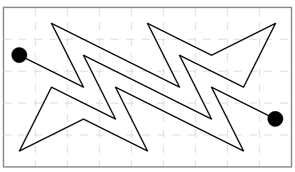
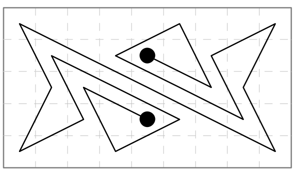

4x8 ~ 13


5x5 ~ 10

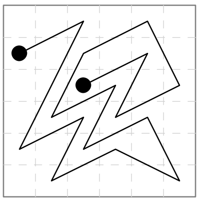

5x6 ~ 14

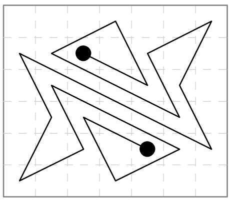

5x7 ~ 16

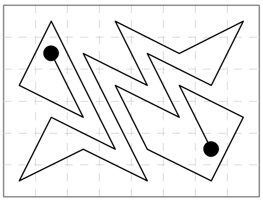

5x8 ~ 19

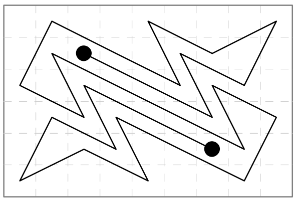


5x9 ~ 22

Six Ranks


6x6 ~ 17

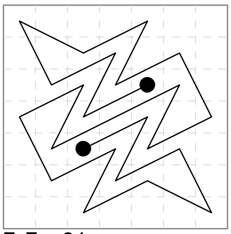

6x7 ~ 21

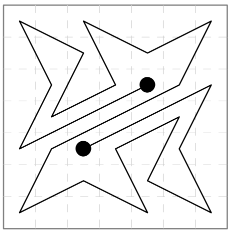

6x8 ~ 25

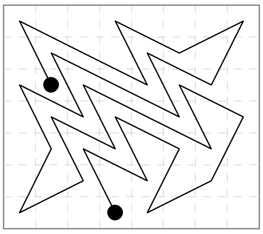

6x9 ~ 29

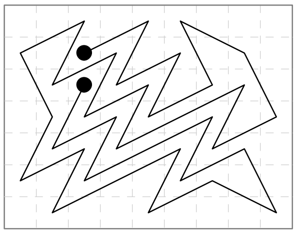
The 17-move 6×6 path was found independently by D. E. Knuth and M. Matsuda 1969 and is unique, apart from rotation or reflection, and is reentrant. It has been quoted many times, e.g. in Martin Gardner's *Scientific American* column (April 1969) and his *Mathematical Circus* and in K. Fabel et al *Schach und Zahl* (1978), without acknowledgement.

Seven ranks


7x7 ~ 24

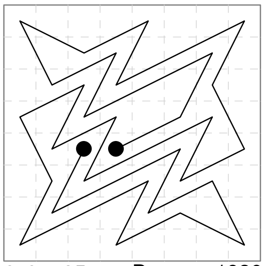


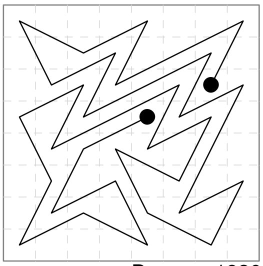

7x8 ~ 30

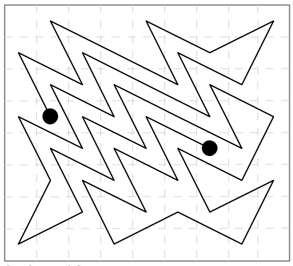

7x9 ~ 35

The 7×7 non-intersection knight solutions are symmetrical the first open solution by D. E. Knuth 1969. Second solution Jelliss 2015 The 7×8 open solution is unique, found independently by D. E. Knuth and R. E. Ruemmler 1969.

Eight ranks: the original problem, posed and solved by T. R. Dawson 1930.


8x8 ~ 35 Dawson 1930

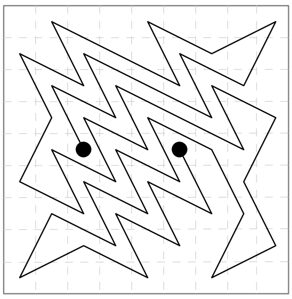

Dawson 1930

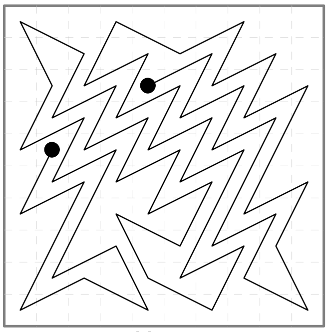

8x9 ~ 42

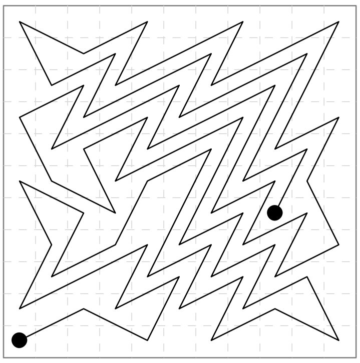
Diagrammed in orientation as published in *L'Echiquier* 1930 p.1150.

Robin Merson made a comprehensive study of these paths on larger boards in 1991, published in *Games and Puzzles Joutrnal* 1999.

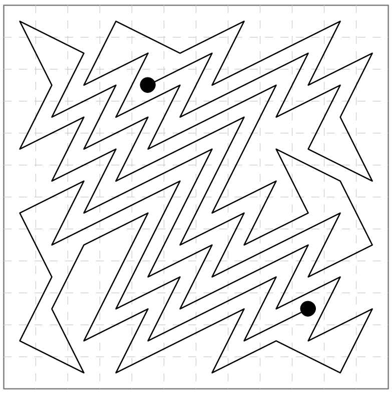
Larger square boards: 9, 10, 11


9x9 ~ 47 Matsuda 1969

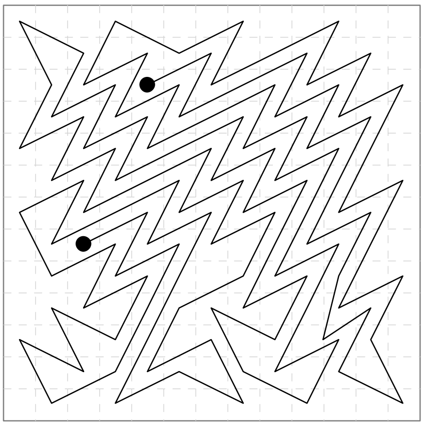

10x10 ~ 61 ~ Merson 1991


11x11 ~ 76 ~ Merson 1991

12 and 13

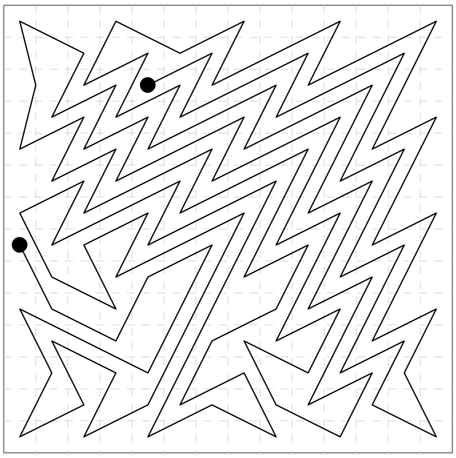


12x12 ~ 94 ~ Merson 1991

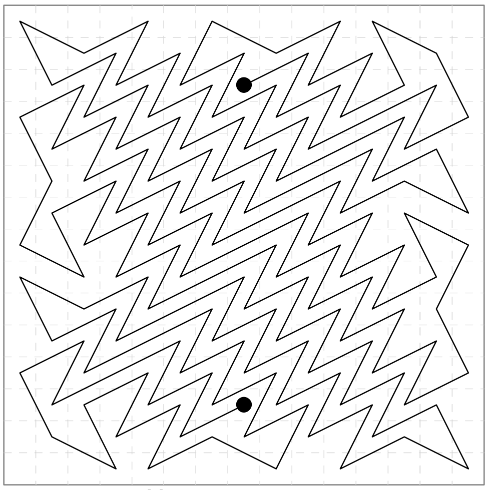


13x13 ~ 113 ~ Merson 1991

14 and 15

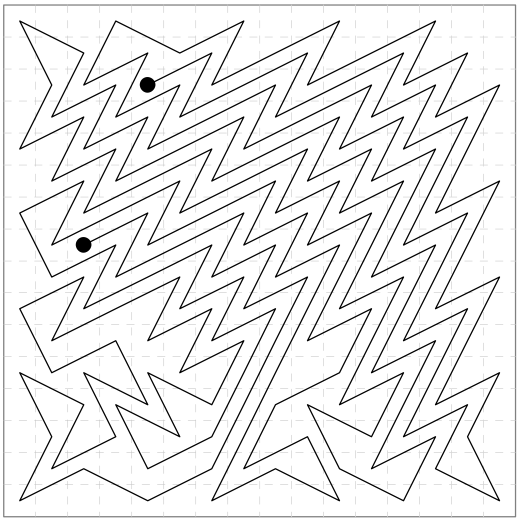


14x14 ~ 135 ~ Fischer

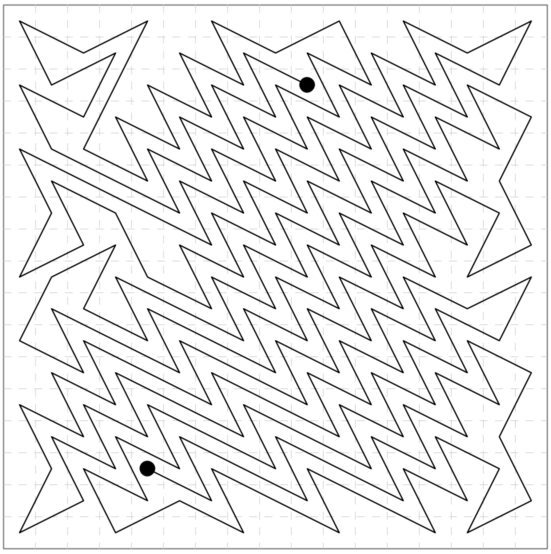


15x15 ~ 158 ~ Merson 1991

16 and 17. Robin Merson had conjectured, from the analysis, that a 183 path 16×16 ought to be possible and this was achieved by Alexander Fischer (2006).

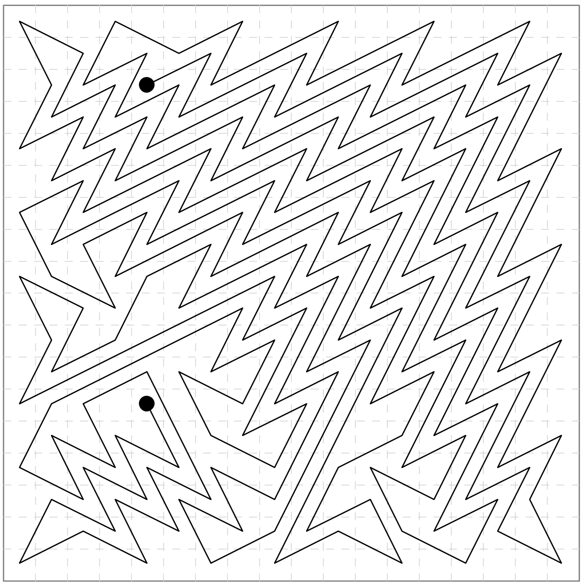


16x16 ~ 183 ~ Fischer

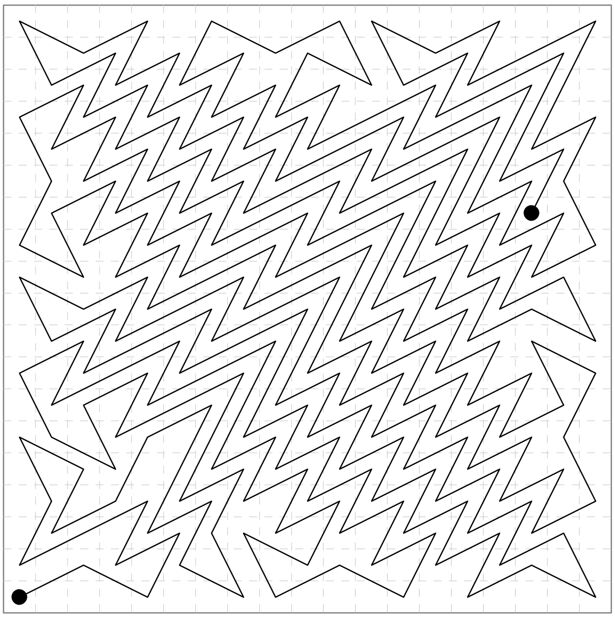


17x17 ~ 211 ~ Lemaire

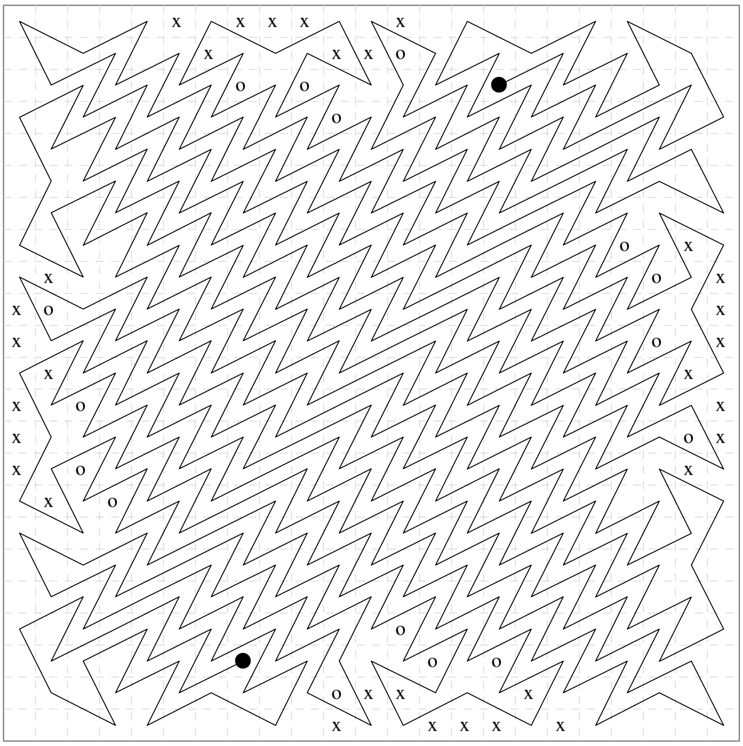
18×18 open. This 238-move open tour (Jelliss 2003) differs only in removal of two and addition of three moves in an earlier 237-move solution by Robin Merson.



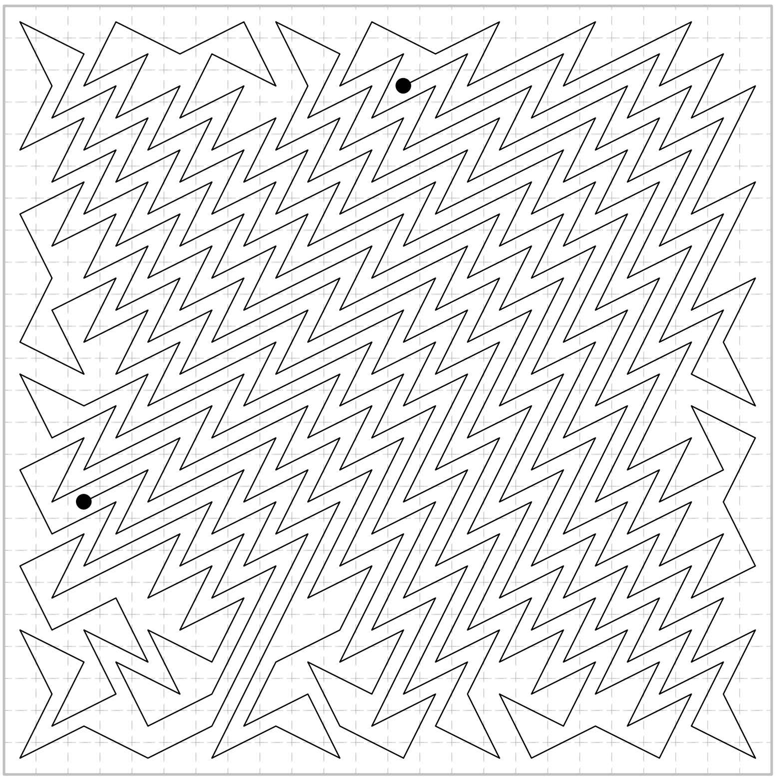
19×19 open. Non-Intersecting Tour, 268 moves, by Robin Merson 1991.Formed by VW extension of an 11×11 solution.



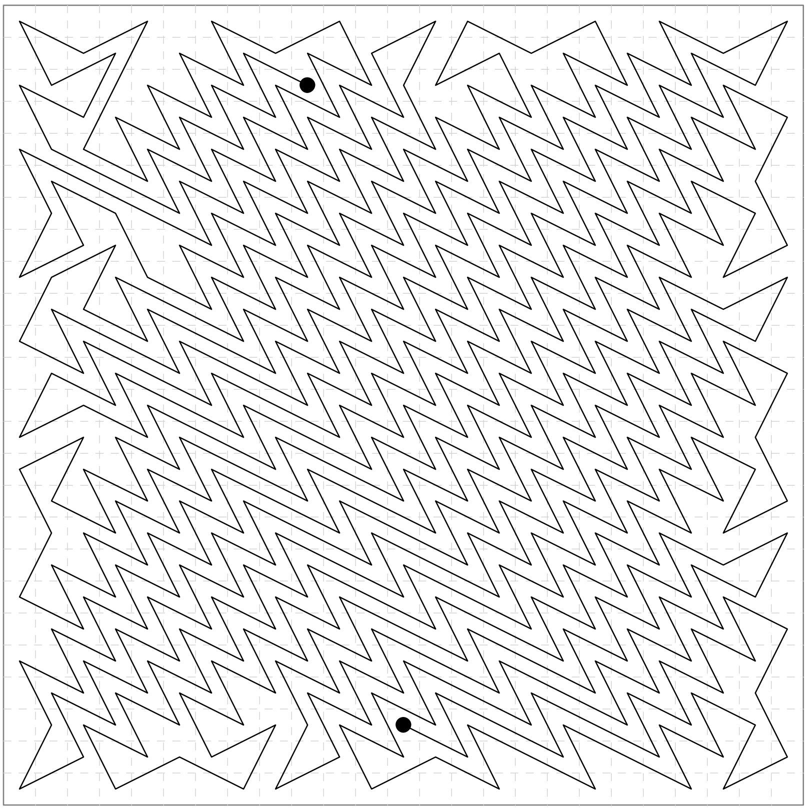
Non-intersecting tour of 23×23 board, open, 414 moves, symmetric, by Robin Merson 1991, by VW extension from 15×15 example.



Merson's 453 solution to 24×24, used as the cover illustration for *The Games and Puzzles Journal* issue 17 (1999), is surpassed by Alex Chernov (2011) with 455 moves:



Lemaire 25×25 open 499 moves



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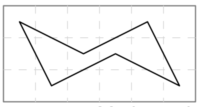
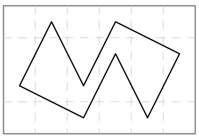
## Knight Closed Path Solutions

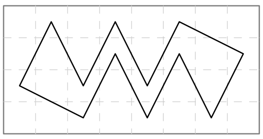
The 3×6 equals the open path maximum. The 7×7 solution by Yarbrough has quaternary symmetry, so is in the next section, a solution with less symmetry seems impossible.



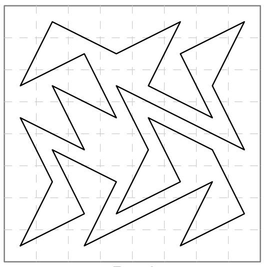
3x6 ~ 6 ~ Yarbrough



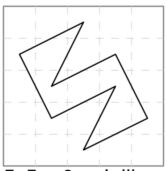
4x6 ~ 8 ~ Knuth



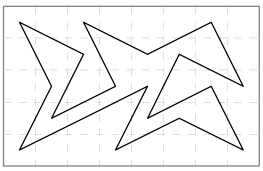
4x8 ~ 12 ~ Knuth



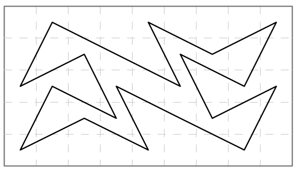
8x8 ~ 32 ~ Pauly 1930



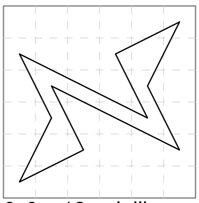
5x5 ~ 8 ~ Jelliss



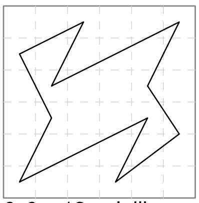
5x8 ~ 18 ~ Knuth



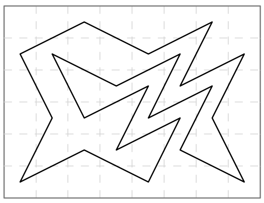
5x9 ~ 20 ~ Knuth



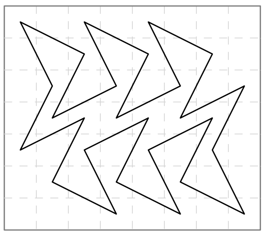
6x6 ~ 12 ~ Jelliss



6x6 ~ 12 ~ Jelliss



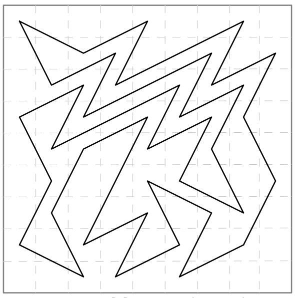
6x8 ~22 ~ Knuth

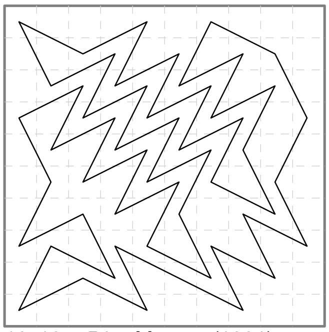


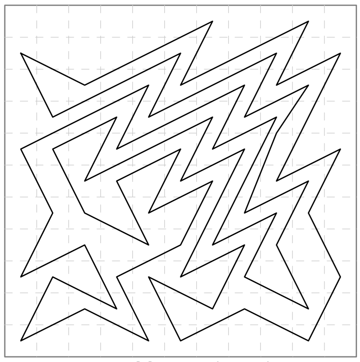
7x8 ~ 26 ~ Knuth

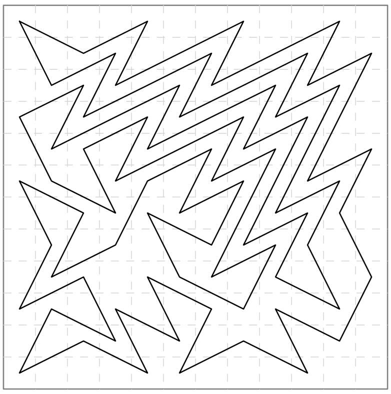
The Pauly solution is diagrammed in the orientation as published in *L'Echiquier* 1930 p.1150.

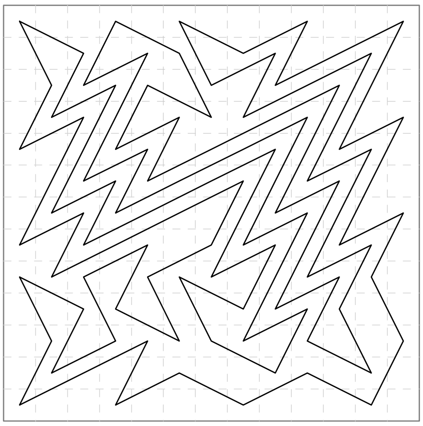
## Larger closed tours


9x9 ~ 42 ~ Merson (1991)

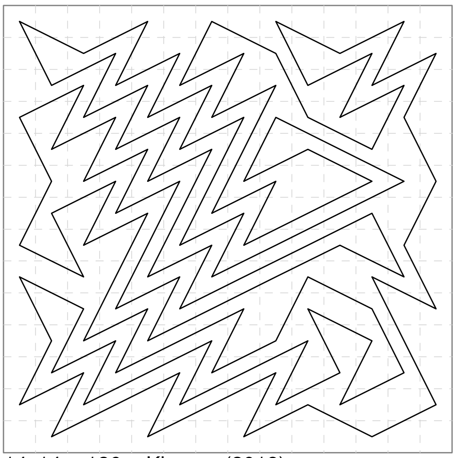

10x10 ~ 54 ~ Merson (1991)

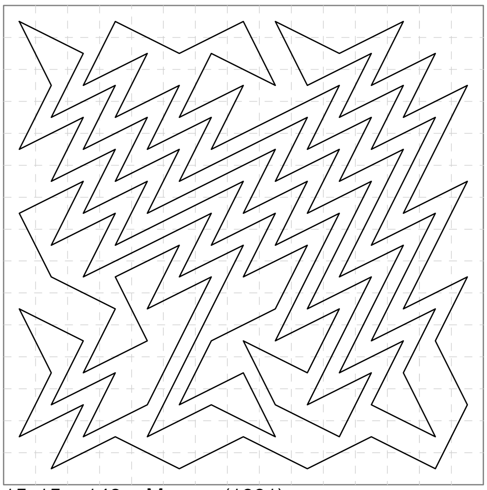

11x11 ~ 70 ~ Merson (1991)


12x12 ~ 86 ~ Merson (1991)

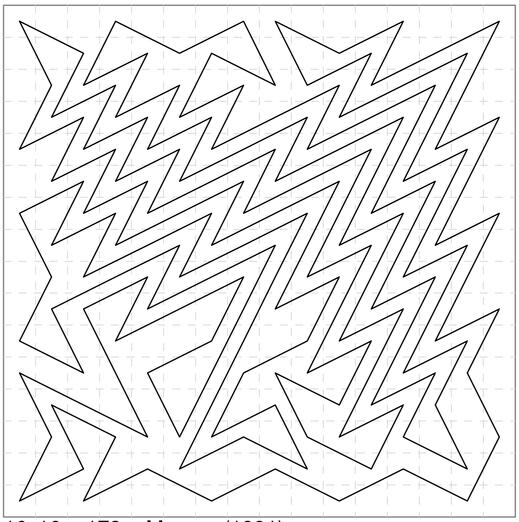

13x13 ~ 106 ~ Chernev (2011)

## Kisman (2018) 14×14 closed 126 moves


14x14 ~ 126 ~ Kisman (2018)

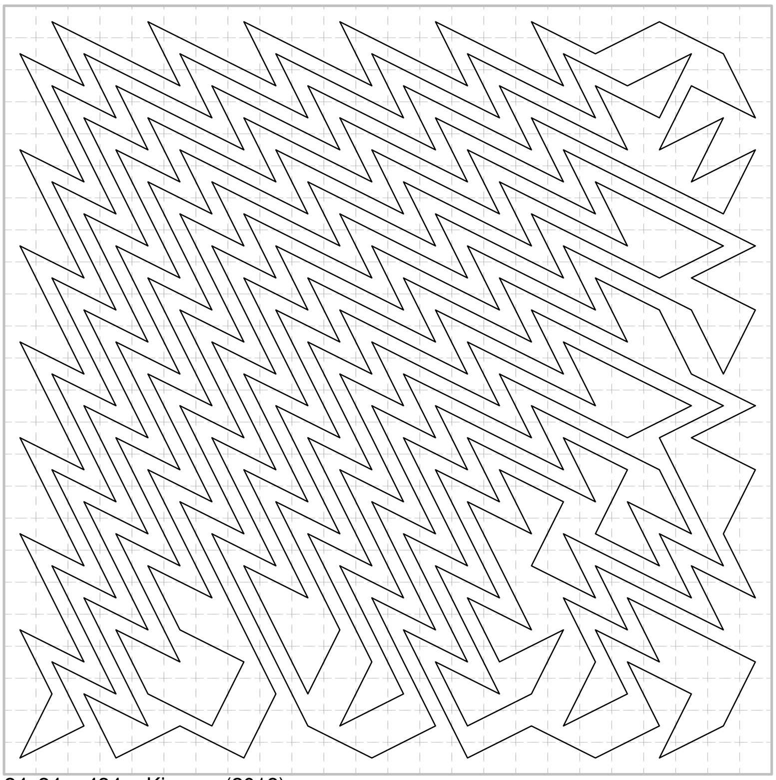

15x15 ~ 148 ~ Merson (1991)

Merson (1991) 16×16 closed 172 moves



16x16 ~ 172 ~ Merson (1991)

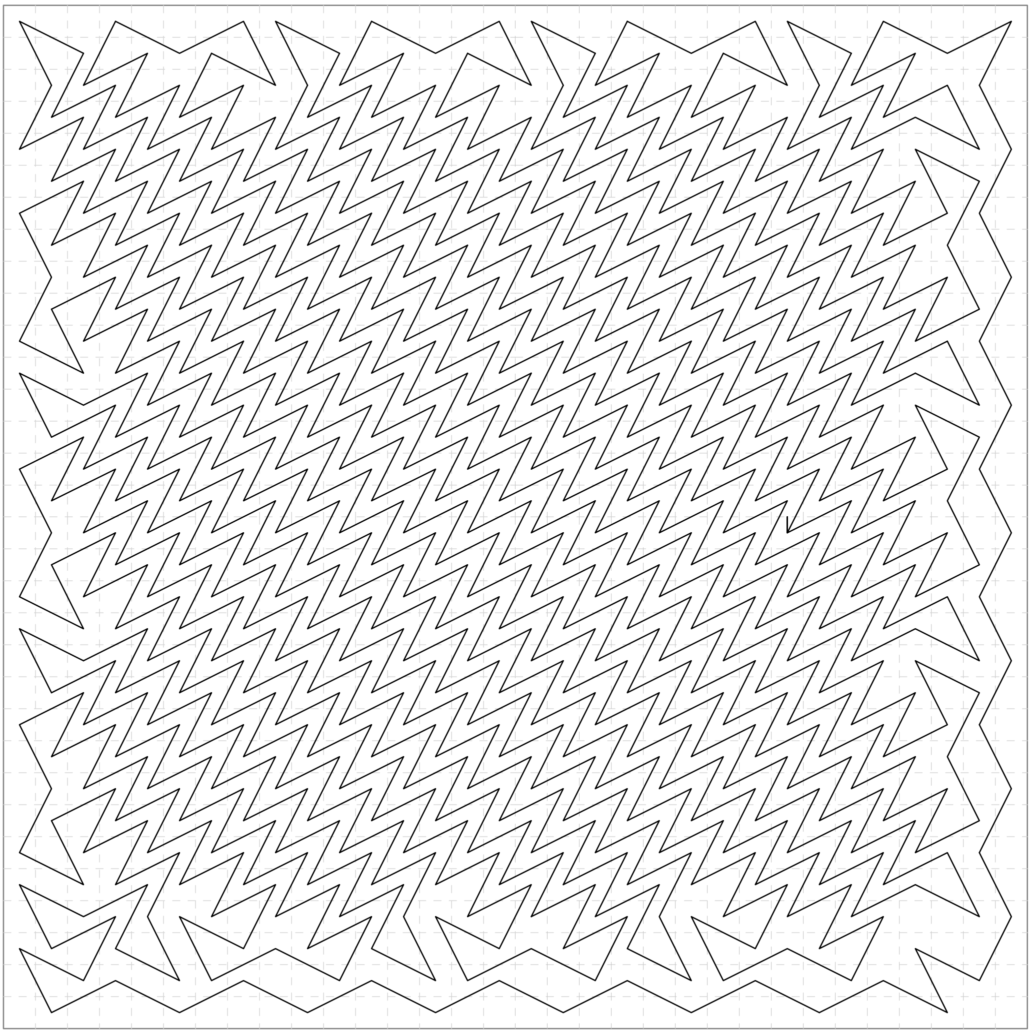
Kisman (2018) 24×24 closed 434 moves



24x24 ~ 434 ~ Kisman (2018)

Alex Chernov maintains a website with results for open and closed non-crossing tours up to side 31: http://ukt.alex-black.ru/.

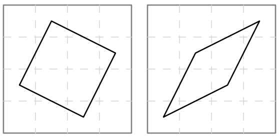
Here is an attempt at the 32×32 closed path, constructed (22 Sep 2019) while preparing this monograph, based on the Merson 28×28 example (as shown on the Chernov site).



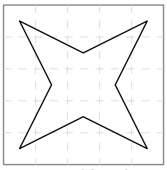
This uses 820 moves giving a coverage of 820/1024 = 80.08%

# Knight Solutions with Quaternary Symmetry

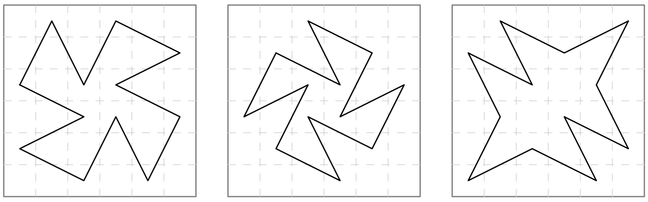
Bernard Lemaire and I recently did some work on longest non-crossing paths.
Here are examples of the smaller sizes.



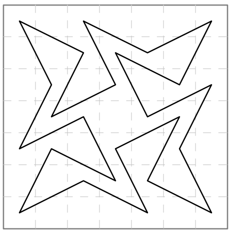
4x4 ~ 4



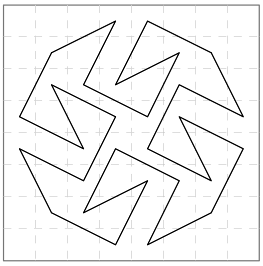
5x5 ~ 8 Knuth



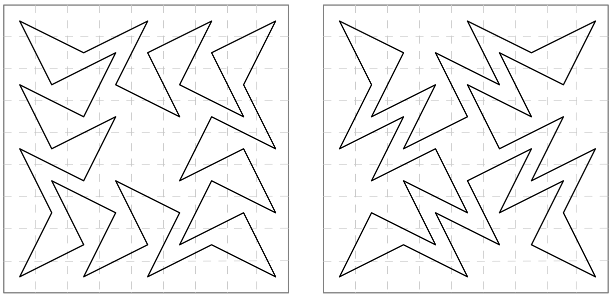
6x6 ~ 12



7x7 ~ 24 ~ Yarbrough

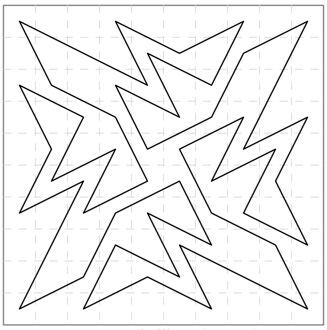


8x8 ~ 28 ~ Jelliss

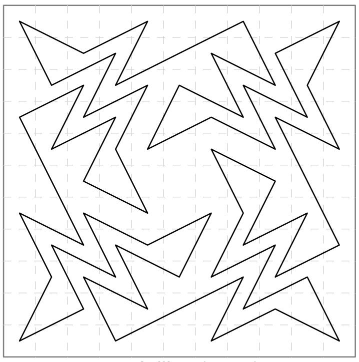


9x9 ~ 40 ~ Lemaire

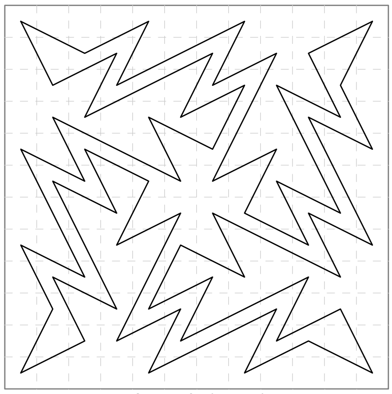
The 12×12 example here is one of half a dozen we constructed. Maybe 84 is possible? The 14×14 example is one of a dozen we found. Is 116 possible? (112 with binary symmetry is).



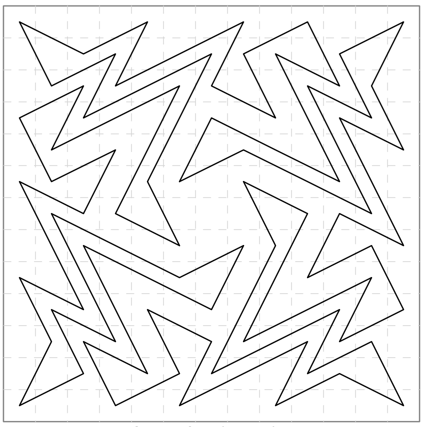
10x10 ~ 52 ~ Jelliss (2015)



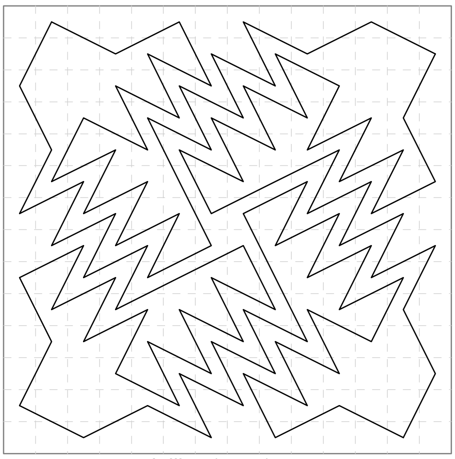
11x11 ~ 64 ~ Jelliss (2015)



12x12 ~ 76 ~ Lemair (2015)

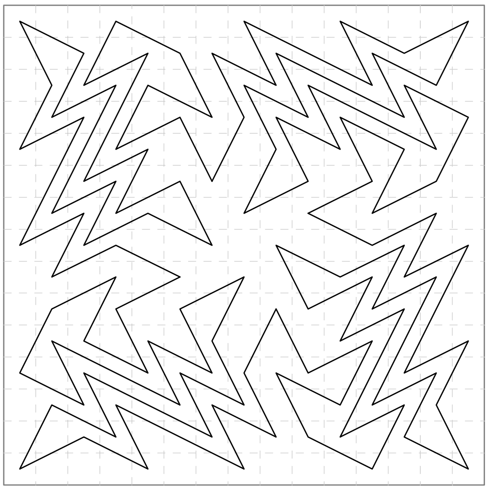


13x13 ~ 96 ~ Lemaire (2015)

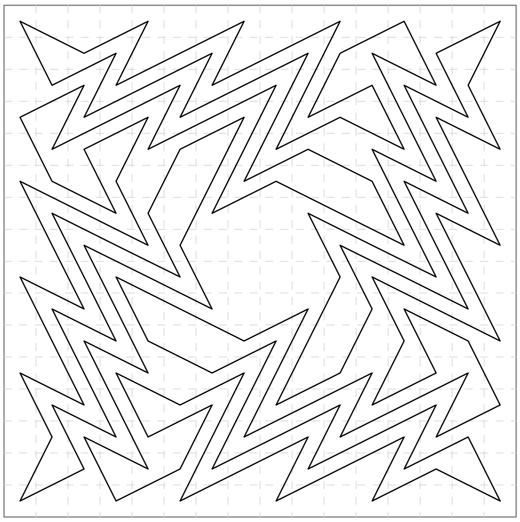


14x14 ~ 108 ~ Jelliss (2015)

Examples 15×15 and 16×16.



15x15 ~ 128 ~ Lemaire (2015)

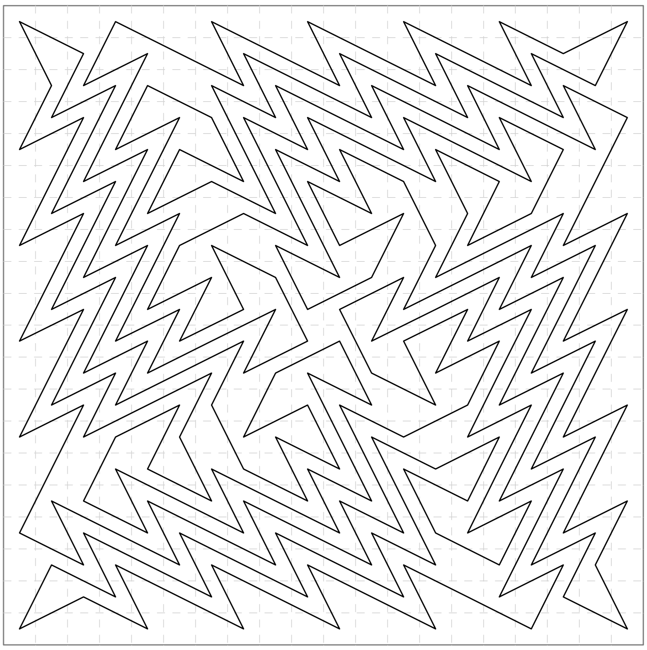


16x16 ~ 164 ~ Jelliss (2015)

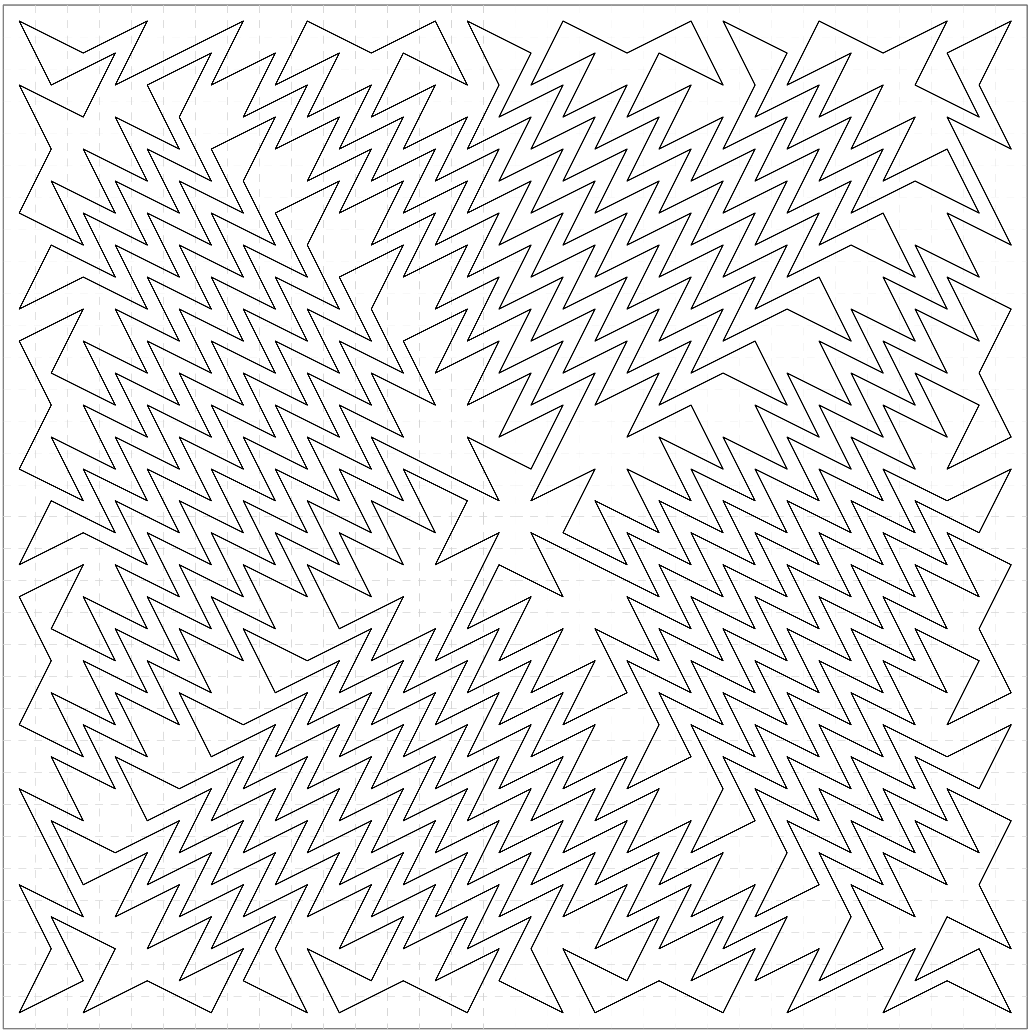
The ratios of the cells covered to the size of the board are; side 14: 108/196 ~= 55%, side 15: 128/225 ~= 57%, side 16: 164/256 ~= 64%. One would expect the percentage to increase with the size of the board, since larger areas of dense coverage can be achieved.

We constructed tours of this type up to 32×32 (and a few larger) but lack the space to show them all here, some more will be found on the KTN website. Many of the results can probably be bettered. We found the following: Side 17 - 184 moves (~64%), side 18 - 220 moves ~68%, side 19 - 240 moves ~ 66%, side 20 - 284 moves ~71%, side 21 - 296 moves ~67%, side 22 - 340 moves ~70%, side 23 - 376 moves ~71%, side 24 ~ 412 moves ~71%, side 25 - 448 moves ~72%, side 26 - 500 moves ~74%, side 27 - 520 moves ~71%, side 28 - 580 moves ~74%, side 29 - 632 moves ~75%, size 30 - 676 moves ~75%, side 31 - 720 moves ~75%, side 32 - 772 moves ~75%.

20×20 Jelliss (9 April 2015) 4·71 = 284 coverage 284/400 = 71% exactly.



32×32 Jelliss (20 April 2015) coverage 772/1024 ~= 75.39%.



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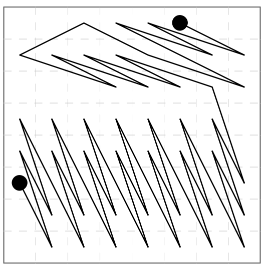
## Non-crossing Leaper Paths

In the following sections we solve the non-crossing problem for Camel, Zebra, Giraffe and Antelope. Here is one by the Gnu (Knight+Camel)



**8×8** T. R. Dawson (*Fairy Chess Review* Aug 1944 prob 6038) gave an 8×8 open non-crossing tour of 52 moves by the gnu (knight + camel), shown above.

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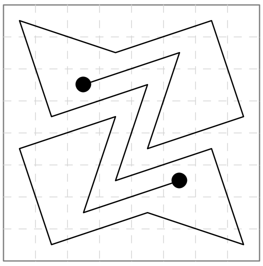
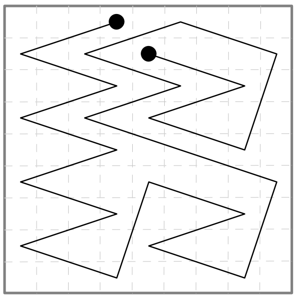
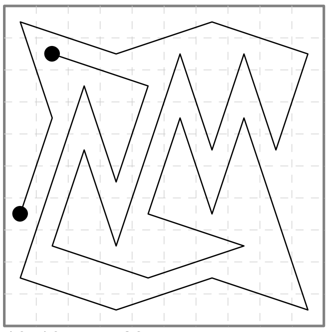
# Non-crossing Camel Paths



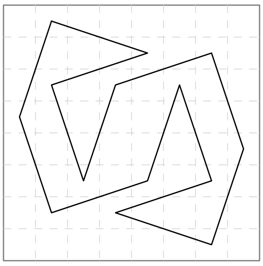
8x8 open 17 moves



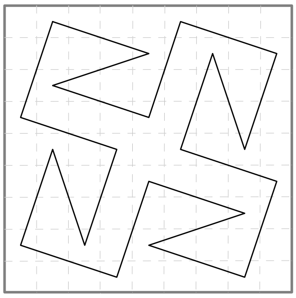
9x9 open 23 moves



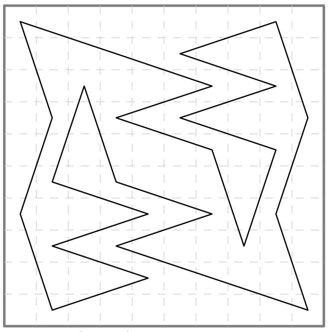
10x10 open 29 moves



8x8 closed 14 moves



9x9 closed 20 moves



10x10 closed 26 moves

This problem for the single-pattern leapers, camel {1,3}, zebra {2,3}, giraffe {1,4} and antelope {3,4} on the 8×8 board was solved by George Jelliss (*Chessics* vol.1, issue 9, 1980). Robin Merson, in a letter dated 16 June 1991, accompanied by computer printed diagrams, confirmed these results and extended them to larger boards.

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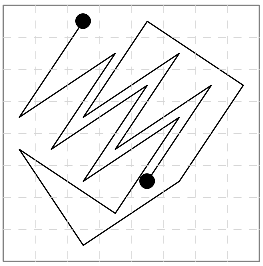
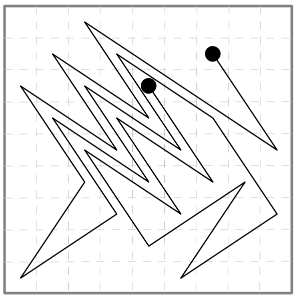
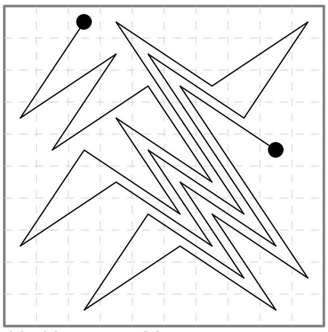
# Non-crossing Zebra Paths



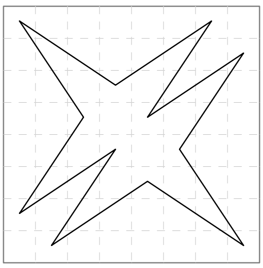
8x8 open ~ 17



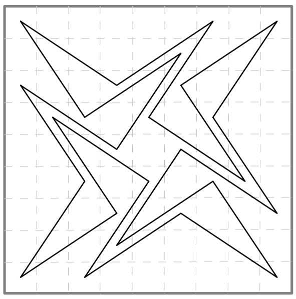
9x9 open ~ 25



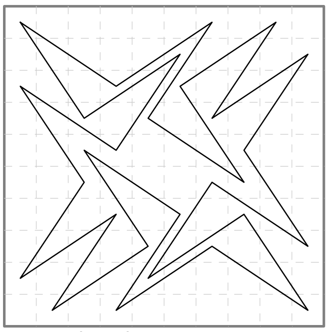
10x10 open ~ 32



8x8 closed ~ 12



9x9 closed ~ 24



10x10 closed ~ 28

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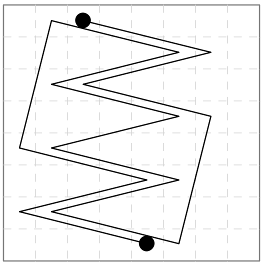
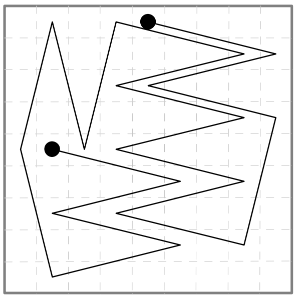
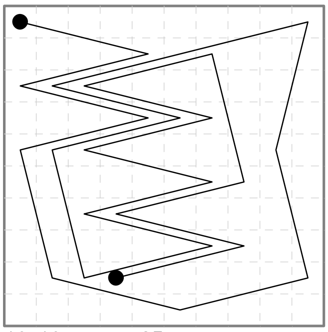
## Non-crossing Giraffe Paths



8x8 open ~ 15



9x9 open ~ 19



10x10 open ~ 25

Closed on the 8×8 Giraffe can do 12 moves a4-e3-a2-e1 repeated four times by 90 rotation.
On the 9×9 board the Giraffe can do no better in a closed tour than it does on the 8×8.
Closed on the 10×10 Giraffe can do 20 moves a6-e5-a4-e3-a2-e1, repeated four times.

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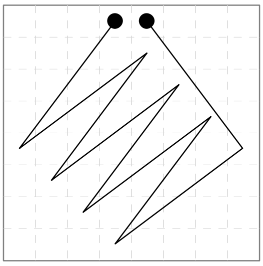
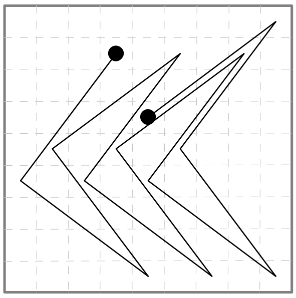
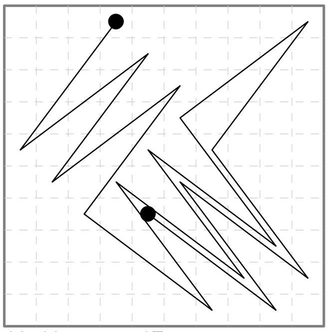
## Non-crossing Antelope Paths



8x8 open ~ 9



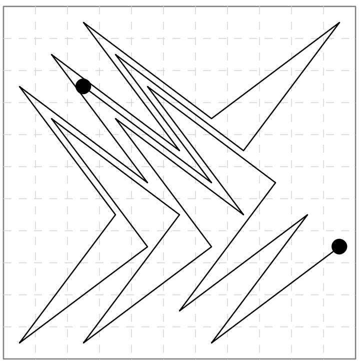
9x9 open ~ 13

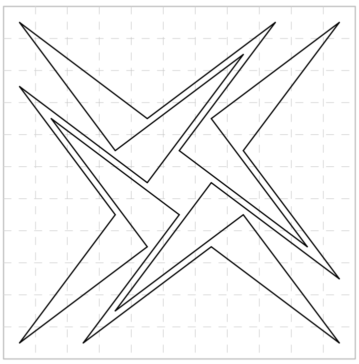


10x10 open ~ 17

The 8×8 9×9 and 10×10 closed cases are all solved by quaternary paths, the quarters being a4-e1; d5-a1-e4; a5-e8-b4-f1. The 8×8 cases solved by Jelliss 1980, others by Merson 1991.

Robin also gave solutions for antelope on 11×11, 25 open (one of four solutions), 24 closed (unique). The closed path takes the form of a tetraskelion similar to those for the 7×7 knight and 9×9 zebra, an evident progression. These cases were diagrammed in a brief report on his work in *Variant Chess* (vol.1, nr.6, 1991).





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# Figured Tours

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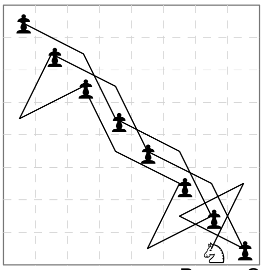
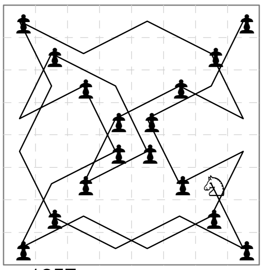
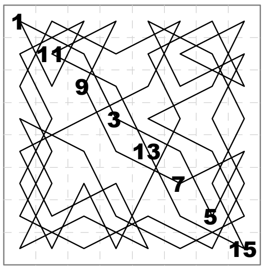
## Two Mediaeval Puzzles

An Arabic ms dated 1257, in the British Museum, contains two problems involving capture of pawns by a knight in minimum moves. They lead to intricate knight-move paths. Murray (*BCM* 1902 p.1-7) noted: “There is no explanatory text or comment to the first of these problems, but the second is to be solved in thirty moves.” The first has Ng1, Ps on a8-h1 diagonal, and is solved in 17 moves. The second Ng3 and Ps along both main diagonals. The ingenious routes can be varied.

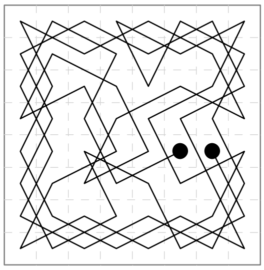




Bonus Socius 1257




Jackson 1993



Jelliss 2019

The scheme of moves in the first diagram was rediscovered independently by D. E. Jackson (*J. Rec Math* 1993) in solving the problem posed by Eliot W. Collins of constructing a knight tour with minimum diagonal sum. This is in effect a Figured Tour with the first eight odd numbers on the diagonal. While writing this note I also realised that versions of the double diagonal route can also be incorporated into a tour as in the example on the right.

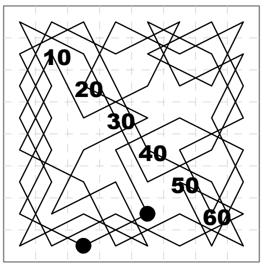
----------------------------------------------------------------------------------------------------------------------------

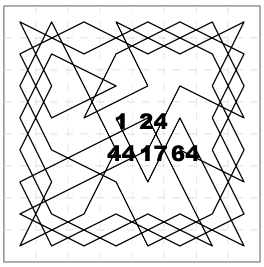
## History of Figured Tours

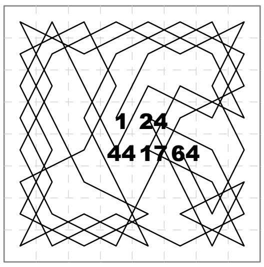
A tour or group of tours presented in arithmetical form may have related numbers forming a pattern, making it a **figured tour**. In retrospect figured tours showing arithmetic progressions can be seen in small board tours 3×4 and 5×5 by **Leonhard Euler** (1759). See ⌘ 4 and ⌘ 5 for these and similar results. There are figured tours on shaped boards in ⌘ 3 and by other pieces in ⌘ 2 and ⌘ 10. Here however we deal only with figured knight tours on evenly even square boards.

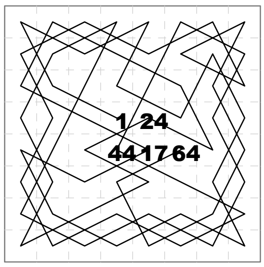
The first deliberately composed figured tours are probably those given on the 8×8 board by **Ernst II von Sachsen-Gotha** (*Reichs-Anzeiger*, 18 Sep 1797 p.366-368). In this article he gives a tour showing the multiples of 10 in sequence along a diagonal. At the end of the article is a diagram showing the numbers 1, 17, 24, 44, 64 on the cells d5, e4, e5, d4, f4, and in his library, which still exists, there is a manuscript with the title *Auflösung einer systematische Aufgabe der sogenannter Roesselsprungs auf dem Schachbrete* (dated 1798), which enumerates tours with the five numbers in these given positions, all naturally rather irregular. I have however only seen an online image of the title page which shows 27 tours of this type. Why this choice of numbers is unclear.

We show the first three:



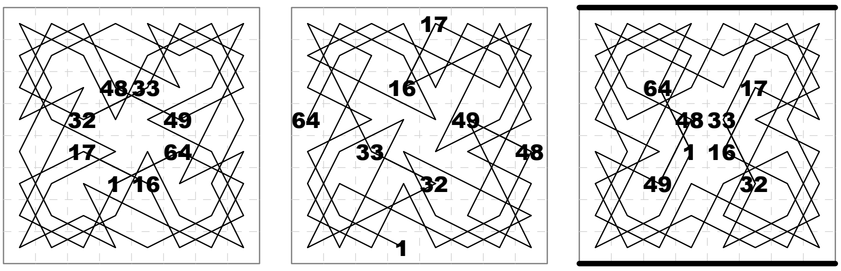



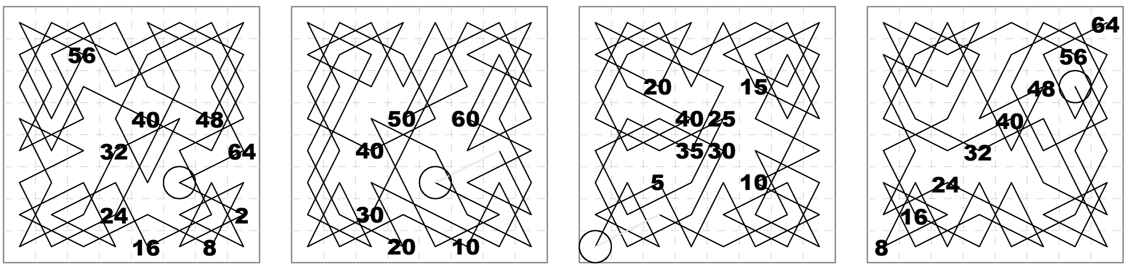
Similar examples are in **H. C. von Warnsdorf** (1823) but with fewer fixed points. His Figs 74-83 show tours with 1 and 64 on given cells, the third number being 8, 24 or 36. Figs 84-91 specify the numbers 2, 18, 36, 64 on ranks 2, 4, 6, 8 in all possible files a-h.

A special type of figured tour resulting from linking of four equal circuits occurs in **Jaenisch** (1862) where the numbers 1, 16, 17, 32, 33, 48, 49, 64 mark the ends of the quarters. These normally form a knight circuit but can also appear in other formations, some of which are shown below. In the first tour they circle round the centre in a circuit of alternating knight and wazir moves. This tour is demi-magic: four ranks and files add to 252 and the other four add to 268.



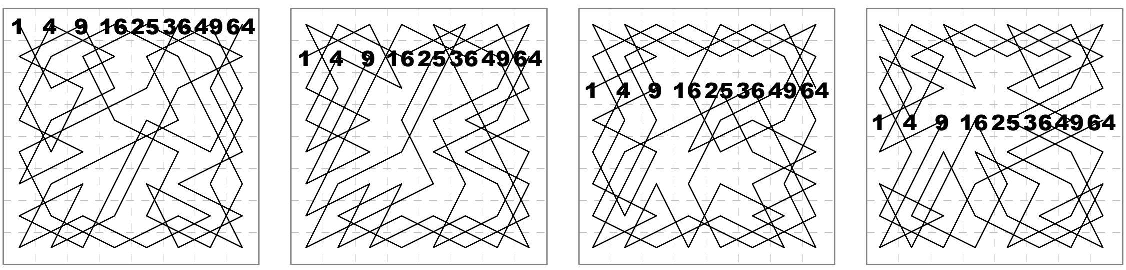

In the second tour the circuit alternates rook and knight moves and has a difference of 16 between numbers in diametrally opposite cells. The third tour is semi-magic adding to 260 in the files. Several magic tours have these Jaenischian properties (see ⌘ 9).

The **Harikrishna** (1871) manuscript includes six figured tours by the **Rajah of Mysore**, who appears to have originated the idea of having the specified numbers arranged in a knight wheel around the initial cell. He gives two 12×12 examples in his Figs 1 and 10 (see later) as well as 8×8 in Figs 35, 61 and 62. He also gave one (Fig 67) with the multiples of eight arranged along a diagonal.

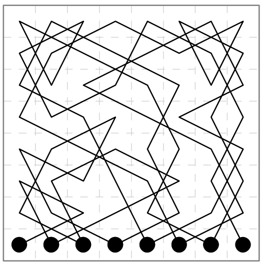


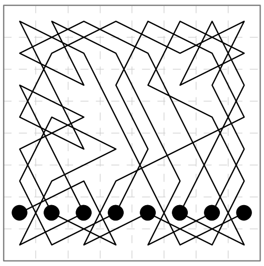

His #63 and #65 which include two 3×4 components have APs of difference 3 (1-22) and (3-24) along the third rank and #63 an AP of difference 5 (26-61) along the fifth rank.

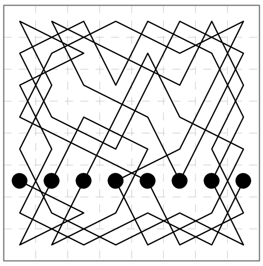
The problem set by **G. E. Carpenter** and solved by **S. Hertzsprung** in *Brentano's Chess Monthly* (May and Sep 1881) of arranging the square numbers along a rank was taken to be the origin of the subject by T. R. Dawson.

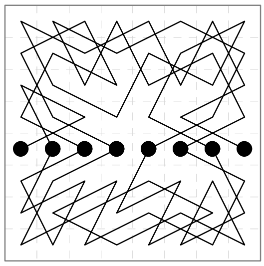


An alternative set of solutions to the Carpenter problem were given by S. H. Hall (*FCR* 1937). In his 4th rank example the move-segments are successively above and below the line.

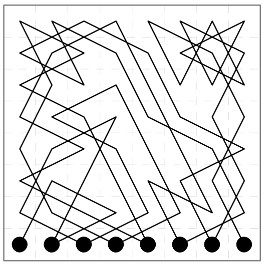




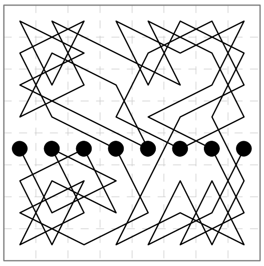




Others have given part solutions: **G. Fuhlendorf** (*Fairy Chess Review* 1937) on first rank, and **A. S. M. Dickins** (*Chessics* 1978) on fourth rank.



Fuhlendorf



Dickins

However, the main development of figured tours must be attributed to **T. R. Dawson** who first used the term (*FCR*, Jun 1944, p.96) and published a comprehensive collection of such tours, by himself and other contributors, in the *Problemist Fairy Chess Supplement* (1930-1936) and its continuation the *Fairy Chess Review* (1936-1958) of which he was founder-editor. I must thank John Beasley for lending me (in November 1989) his complete set of these now rare magazines. A number of results by Dawson also first appeared in an article in the *Comptes Rendus du Premier Congres International de Récréation Mathématique* (CIRM), Brussels 1935, edited by M. Kraitchik.

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## Dawsonian Figured Tours

Dawson invented the type of figured tour in which the square numbers form a knight chain. This was introduced in *Problemist Fairy Chess Supplement* (vol.1 #10 Feb 1932 ¶329 p.53, 58) and he seems to have shown examples to other members of the British Chess Problem Society before that date, since examples by P. C. Taylor were given in a lecture to the Society given in the same month on 26 Feb 1932 as mentioned in the *British Chess Magazine* (vol.52 1932 p182).

In the first article in *PFCS* Dawson stated: “No.329 is the first published solution of a question given by G. E. Carpenter in Brentano's, May 1881, to construct a knight tour with the square numbers in a knight chain.” However, five years later (in *FCR* vol.3 #7 Aug 1937 p.77) he wrote: “The famous chess historian, H. J. R. Murray, a few months ago asked me to verify this reference, and to my astonishment it was quite incorrect — G. E. C. proposed no such problem. It appears that I must have unconsciously modified his actual proposition — S-tours with the integral square numbers all on one rank — into our present S-chain proposition. I now accept this child of my fantasy as my own, thanks to H. J. R. M.” [S is the symbol used for Knight in *PFCS/FCR*]

In particular he made a complete set of 100 tours with the square numbers in symmetric knight chains, showing all possible eight-move symmetric circuits (see ⌘ 1) that will fit in a tour. They were published in *PFCS/FCR* over seventeen years 1932-1948. The 69 axial solutions were published 1932-1937, but the 31 other cases desultorily: 2 in 1932, 3 in 1938, 6 in 1944, 10 in 1945, 3 in 1946, 4 in 1947, 3 in 1948. [For fuller references see the Bibliography] It is of interest that, contrary to our terminology, he calls only the axial tours ‘symmetric’ while the rotary tours are ‘pseudosymmetric’.

Dawson notes at the end of the collection (*FCR* #18 Jun 1948 p.138) that there are in fact 106 symmetric circuits, but 3 require a 9-rank board and 3 will not form part of a complete tour. These six shapes are modally related to each other in pairs.

In *PFCS* (vol.2 #2 Aug 1933 p.4) he wrote: “The Knight's tour with the integral squares also in a S-chain continues to interest several members of our Fairy Ring and a number of examples have accumulated to enrich our collection in these pages” and (#3 Oct 1933 p.14): “I was asked recently why I printed so many of these tours. The answer is that I like them and like PFS to be a source of reference to a very complete collection of them.”

We gather the tours together here in two batches.

---

## Square Numbers in Axial Circuits

No. 329 is the first published solution. There are three with skew axes of symmetry:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 55 | 6 | 33 | 10 | 15 | 44 | 31 |
| 5 | 60 | **9** | 56 | 43 | 32 | 11 | 14 |
| 54 | 7 | 34 | 59 | **16** | 13 | 30 | 45 |
| 61 | **4** | 17 | 42 | 57 | 46 | **25** | 12 |
| 18 | 53 | 58 | 35 | 24 | 29 | 40 | 47 |
| **1** | 62 | 3 | 50 | 41 | **36** | 23 | 26 |
| 52 | 19 | **64** | 37 | 28 | 21 | 48 | 39 |
| 63 | 2 | 51 | 20 | **49** | 38 | 27 | 22 |

329

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 14 | 21 | 54 | 7 | **16** | 19 | 56 | 27 |
| 53 | 24 | 15 | 20 | 55 | 26 | **9** | 18 |
| 22 | 13 | 6 | **25** | 8 | 17 | 28 | 57 |
| 35 | 52 | 23 | 12 | 29 | **4** | 41 | 10 |
| 50 | 47 | **36** | 5 | 40 | 11 | 58 | 3 |
| 37 | 34 | 51 | 48 | **1** | 30 | 61 | 42 |
| 46 | **49** | 32 | 39 | 44 | 63 | 2 | 59 |
| 33 | 38 | 45 | **64** | 31 | 60 | 43 | 62 |

450

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 13 | 26 | 59 | 38 | 11 | 18 | 5 | 22 |
| 60 | 37 | 12 | **25** | 58 | 21 | 10 | 17 |
| 27 | 14 | 39 | 42 | 19 | **16** | 23 | 56 |
| 40 | 61 | **36** | 15 | 24 | 43 | 20 | **9** |
| 35 | 28 | 41 | 48 | **1** | 8 | 55 | 44 |
| 62 | **49** | 34 | 31 | 52 | 47 | **4** | 7 |
| 29 | 32 | 51 | **64** | 5 | 2 | 45 | 54 |
| 50 | 63 | 30 | 33 | 46 | 53 | 6 | 3 |

1452

Among the axial solutions there are 4 that have three-unit lines, and 14 that have two-unit lines, and another 12 with two-unit lines among the rotary examples.

In *PFCS* vol.2 #10 Feb 1935 p.104 it is noted: “There is a pretty relationship, in the octagons formed by the integral square numbers, between those with orthogonal axes of symmetry and the closely similar shapes with diagonal axes of symmetry. Nos 493 and 492 are a good case in point, the orthogonal star of 493 being rotated 45 degrees into the diagonal star of 492.” [In his booklet *Caissa's Wild Roses in Clusters* (1937) Dawson calls this relationship ‘modal transformation’ and applies it more generally to chess problem themes.]

There are 66 with lateral or diagonal axes, of which 4 are octonary (with 4 axes):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 62 | 3 | 50 | 17 | 38 | 15 | 6 | 47 |
| 51 | 18 | 61 | 2 | 5 | 48 | 39 | 14 |
| 24 | 63 | **4** | **49** | **16** | 37 | 46 | 7 |
| 19 | 52 | **25** | 60 | **1** | 8 | 13 | 40 |
| 26 | 23 | **64** | **9** | **36** | 41 | 58 | 45 |
| 53 | 20 | 29 | 32 | 59 | 44 | 35 | 12 |
| 30 | 27 | 22 | 55 | 10 | 33 | 42 | 57 |
| 21 | 54 | 31 | 28 | 43 | 56 | 11 | 34 |

377

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 20 | 5 | 52 | 63 | 54 | 31 | 28 |
| 6 | 51 | 2 | 19 | 32 | 29 | 62 | 55 |
| 21 | **4** | 7 | 50 | 53 | **64** | 27 | 30 |
| 8 | 47 | 18 | **1** | 26 | 33 | 56 | 61 |
| 17 | 22 | **9** | 46 | **49** | 60 | 37 | 40 |
| 10 | 45 | 48 | **25** | 14 | 39 | 34 | 57 |
| 23 | **16** | 43 | 12 | 59 | **36** | 41 | 38 |
| 44 | 11 | 24 | 15 | 42 | 13 | 58 | 35 |

492

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 62 | 21 | 2 | 53 | **64** | 11 | 50 | 31 |
| 3 | 54 | 63 | 22 | 51 | 32 | 35 | 12 |
| 20 | 61 | 52 | **1** | 10 | **49** | 30 | 33 |
| 55 | **4** | 41 | 60 | 23 | 34 | 13 | **36** |
| 42 | 19 | 56 | **9** | 40 | **25** | 48 | 29 |
| 5 | 8 | 59 | 24 | 45 | 28 | 37 | 14 |
| 18 | 43 | 6 | 57 | **16** | 39 | 26 | 47 |
| 7 | 58 | 17 | 44 | 27 | 46 | 15 | 38 |

493

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 29 | 12 | 55 | **4** | 31 | 6 | 43 |
| 13 | 56 | **9** | 30 | 7 | 44 | **1** | 32 |
| 28 | 11 | 14 | 3 | 54 | 5 | 42 | 45 |
| 57 | **16** | 53 | 8 | 39 | 2 | 33 | **64** |
| 24 | 27 | 20 | 15 | 52 | 63 | 46 | 41 |
| 17 | 58 | **25** | 38 | 21 | 40 | **49** | 34 |
| 26 | 23 | 60 | 19 | **36** | 51 | 62 | 47 |
| 59 | 18 | 37 | 22 | 61 | 48 | 35 | 50 |

608

Four of the 66 are quaternary with 2 axes:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 52 | 55 | 48 | 61 | 22 | 57 | 20 | 37 |
| 47 | 62 | 53 | 56 | **49** | 38 | 23 | 58 |
| 54 | 51 | **64** | 39 | 60 | 21 | **36** | 19 |
| 63 | 46 | 41 | 50 | 35 | 18 | 59 | 24 |
| 42 | 5 | 14 | **1** | 40 | **25** | 34 | 17 |
| 13 | 2 | 45 | 8 | 15 | 10 | 31 | 28 |
| 6 | 43 | **4** | 11 | 26 | 29 | **16** | 33 |
| 3 | 12 | 7 | 44 | **9** | 32 | 27 | 30 |

660

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 30 | 45 | 6 | **1** | 56 | 47 | 58 | 63 |
| 5 | 8 | 29 | 46 | 61 | **64** | 55 | 48 |
| 44 | 31 | **4** | 7 | 2 | 57 | 62 | 59 |
| **9** | 28 | 43 | 32 | 11 | 60 | **49** | 54 |
| 42 | 33 | 10 | 3 | **36** | 53 | 12 | 21 |
| 27 | **16** | 35 | 18 | 13 | 22 | 37 | 50 |
| 34 | 41 | 14 | **25** | 52 | 39 | 20 | 23 |
| 15 | 26 | 17 | 40 | 19 | 24 | 51 | 38 |

1675

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 11 | 54 | 21 | 40 | 13 | 56 | 23 | 44 |
| 20 | 39 | 12 | 55 | 22 | 43 | 14 | 57 |
| 53 | 10 | 37 | 18 | 41 | 58 | 45 | 24 |
| 38 | 19 | **64** | **9** | **36** | 15 | 42 | 59 |
| **1** | 52 | 17 | 34 | 63 | 8 | **25** | 46 |
| 30 | 33 | **4** | **49** | **16** | 35 | 60 | 7 |
| 51 | 2 | 31 | 28 | 5 | 62 | 47 | 26 |
| 32 | 29 | 50 | 3 | 48 | 27 | 6 | 61 |

1053

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 41 | 54 | 61 | 18 | 43 | 52 | 47 |
| 55 | 58 | 3 | 42 | 53 | 46 | 19 | 44 |
| 40 | **1** | 62 | 57 | 60 | 17 | 48 | 51 |
| 63 | 56 | 59 | **4** | **49** | 32 | 45 | 20 |
| 8 | 39 | **64** | 31 | **16** | 21 | 50 | 33 |
| 11 | 14 | **9** | **36** | 5 | 30 | 27 | 24 |
| 38 | 7 | 12 | 15 | 22 | **25** | 34 | 29 |
| 13 | 10 | 37 | 6 | 35 | 28 | 23 | 26 |

1813

The remaining 58 have a single axis of symmetry.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15 | 46 | **25** | 38 | 13 | 44 | 27 | 56 |
| 24 | 37 | 14 | 45 | 26 | 57 | 12 | 43 |
| 47 | **16** | 39 | **36** | 41 | 10 | 55 | 28 |
| 34 | 23 | 48 | **9** | 58 | 29 | 42 | 11 |
| 17 | 8 | 35 | 40 | **49** | **4** | 59 | 54 |
| 22 | 33 | 20 | 5 | 30 | 51 | 62 | **1** |
| 7 | 18 | 31 | 50 | 3 | **64** | 53 | 60 |
| 32 | 21 | 6 | 19 | 52 | 61 | 2 | 63 |

1451

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 54 | 5 | 40 | 19 | 56 | 23 | 38 |
| 6 | 3 | 18 | 55 | 24 | 39 | 20 | 57 |
| 53 | **16** | 47 | **4** | 41 | 22 | 37 | 30 |
| 46 | 7 | 2 | **25** | 48 | 31 | 58 | 21 |
| 15 | 52 | **9** | 42 | **1** | **36** | 29 | 32 |
| 8 | 45 | 12 | **49** | 26 | 33 | 62 | 59 |
| 51 | 14 | 43 | 10 | 61 | **64** | 35 | 28 |
| 44 | 11 | 50 | 13 | 34 | 27 | 60 | 63 |

1595

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 51 | 46 | **1** | 42 | 5 | 18 | 63 | 40 |
| 2 | 43 | 50 | 53 | **64** | 41 | 6 | 17 |
| 47 | 52 | 45 | **4** | 19 | 8 | 39 | 62 |
| 44 | 3 | 54 | **49** | 14 | 21 | **16** | 7 |
| 55 | 48 | 13 | 20 | **9** | **36** | 61 | 38 |
| 12 | 31 | 28 | 35 | 58 | 15 | 22 | **25** |
| 29 | 56 | 33 | 10 | 27 | 24 | 37 | 60 |
| 32 | 11 | 30 | 57 | 34 | 59 | 26 | 23 |

1596

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 52 | 11 | 38 | 59 | 40 | 13 | 46 | 61 |
| 37 | 58 | 51 | 12 | 45 | 60 | 41 | 14 |
| 10 | 53 | **64** | 39 | 50 | 43 | 62 | 47 |
| 57 | **36** | **9** | 44 | 63 | 48 | 15 | 42 |
| 54 | **1** | 56 | **49** | **16** | 29 | 22 | 31 |
| 35 | 8 | **25** | **4** | 23 | 32 | 19 | 28 |
| 2 | 55 | 6 | 33 | 26 | 17 | 30 | 21 |
| 7 | 34 | 3 | 24 | 5 | 20 | 27 | 18 |

2637

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 26 | 41 | 12 | 37 | **16** | 33 | 14 | 31 |
| 11 | 38 | **25** | 28 | 13 | 30 | 17 | 34 |
| 42 | 27 | 40 | **9** | **36** | 15 | 32 | 51 |
| 39 | 10 | 3 | 24 | 29 | 50 | 35 | 18 |
| 2 | 43 | 8 | **49** | **4** | 23 | 52 | 61 |
| 7 | 46 | **1** | 58 | 55 | 62 | 19 | 22 |
| 44 | 57 | 48 | 5 | **64** | 21 | 60 | 53 |
| 47 | 6 | 45 | 56 | 59 | 54 | 63 | 20 |

2638

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 56 | 53 | 8 | **1** | 20 | 51 | 10 | 23 |
| 7 | 2 | 55 | 52 | **9** | 22 | 17 | 50 |
| 54 | 57 | **4** | 21 | **64** | 19 | 24 | 11 |
| 3 | 6 | 47 | 58 | 35 | **16** | **49** | 18 |
| 46 | 33 | 42 | 5 | 48 | 63 | 12 | **25** |
| 41 | 30 | 59 | 34 | 43 | **36** | 15 | 62 |
| 32 | 45 | 28 | 39 | 60 | 13 | 26 | 37 |
| 29 | 40 | 31 | 44 | 27 | 38 | 61 | 14 |

2639

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 20 | 15 | 22 | 47 | 52 | 33 | 60 |
| 14 | 23 | 18 | 63 | 34 | 61 | 46 | 51 |
| 19 | **16** | 21 | 48 | 53 | **64** | 59 | 32 |
| 8 | 13 | 24 | 35 | 62 | 31 | 50 | 45 |
| **25** | 38 | **9** | 54 | **49** | 44 | **1** | 58 |
| 12 | 7 | **36** | 39 | **4** | 55 | 30 | 43 |
| 37 | 26 | 5 | 10 | 41 | 28 | 57 | 2 |
| 6 | 11 | 40 | 27 | 56 | 3 | 42 | 29 |

2871

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 2 | 59 | 46 | 41 | 44 | 27 | 22 |
| 58 | 47 | 62 | **1** | 28 | 21 | 40 | 43 |
| 3 | **64** | 57 | 60 | 45 | 42 | 23 | 26 |
| 48 | 61 | **4** | 37 | 24 | 29 | 20 | 39 |
| 5 | 10 | **49** | 56 | 19 | 38 | **25** | 30 |
| 52 | 55 | 6 | **9** | **36** | 33 | 18 | 15 |
| 11 | 8 | 53 | 50 | 13 | **16** | 31 | 34 |
| 54 | 51 | 12 | 7 | 32 | 35 | 14 | 17 |

2872

The above 8 have no cell on the axis, which Dawson calls a Virtual Axis, and I call Sulian.
The next twelve with cells on the axis include lines of two or three moves.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 50 | 53 | 62 | 23 | 48 | 43 | 26 | 29 |
| 63 | 22 | **49** | 52 | **25** | 28 | 45 | 42 |
| 54 | 51 | 24 | 61 | 44 | 47 | 30 | 27 |
| 21 | **64** | 55 | **36** | 31 | **16** | 41 | 46 |
| 2 | 35 | 60 | 17 | 56 | 37 | 10 | 16 |
| 59 | 20 | **1** | 32 | **9** | 12 | 7 | 40 |
| 34 | 3 | 18 | 57 | 38 | 5 | 14 | 11 |
| 19 | 58 | 33 | **4** | 13 | 8 | 39 | 6 |

1303

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 51 | 60 | 47 | 58 | 29 | 6 | 45 |
| 61 | 48 | 3 | 54 | 5 | 46 | 57 | 28 |
| 52 | **1** | 50 | 59 | 30 | 55 | 44 | 7 |
| **49** | 62 | 53 | **4** | 43 | 10 | 27 | 56 |
| 22 | 39 | **64** | 35 | 26 | 31 | 8 | 11 |
| 63 | **36** | 23 | 42 | **9** | 14 | 17 | 32 |
| 40 | 21 | 38 | **25** | 34 | 19 | 12 | 15 |
| 37 | 24 | 41 | 20 | 13 | **16** | 33 | 18 |

1204

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 46 | 43 | 12 | 41 | 8 | 27 | 14 | 31 |
| 11 | 40 | 47 | 44 | 13 | 30 | 7 | 28 |
| 50 | 45 | 42 | **9** | 26 | 5 | 32 | 15 |
| 39 | 10 | 51 | 48 | 33 | **16** | 29 | 6 |
| 52 | **49** | 34 | **25** | **4** | 21 | 56 | 17 |
| 35 | 38 | 61 | **64** | 55 | 18 | 3 | 22 |
| 62 | 53 | **36** | 59 | 24 | **1** | 20 | 57 |
| 37 | 60 | 63 | 54 | 19 | 58 | 23 | 2 |

1305

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 20 | 23 | 18 | 45 | 2 | 33 | 56 | 47 |
| 17 | 44 | 21 | **64** | 57 | 46 | 31 | 34 |
| 22 | 19 | 24 | 3 | 32 | **1** | 48 | 55 |
| 43 | **16** | 63 | 58 | **49** | 54 | 35 | 30 |
| 10 | 59 | 12 | **25** | **4** | 29 | 50 | 37 |
| 15 | 42 | **9** | 62 | 53 | **36** | 5 | 28 |
| 60 | 11 | 40 | 13 | 26 | 7 | 38 | 51 |
| 41 | 14 | 61 | 8 | 39 | 52 | 27 | 6 |

1306

continued:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 27 | 42 | 23 | 6 | 33 | 44 | 31 |
| 61 | 24 | 3 | 28 | 43 | 30 | 7 | 34 |
| 26 | **1** | 62 | 41 | 22 | 5 | 32 | 45 |
| 63 | 60 | **25** | **4** | 29 | 40 | 35 | 8 |
| 56 | 15 | **64** | 21 | **36** | **9** | 46 | 19 |
| 59 | 50 | 57 | **16** | 53 | 20 | 39 | 10 |
| 14 | 55 | 52 | **49** | 12 | 37 | 18 | 47 |
| 51 | 58 | 13 | 54 | 17 | 48 | 11 | 38 |

1525

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 55 | 42 | 33 | 44 | 53 | 14 | 51 |
| 41 | 34 | **1** | 54 | 15 | 50 | 45 | 48 |
| 56 | 3 | 32 | 43 | **64** | 47 | 52 | 13 |
| 31 | 40 | 35 | **4** | 29 | **16** | **49** | 46 |
| 6 | 57 | 30 | 63 | **36** | 21 | 12 | 17 |
| 39 | 60 | 5 | 28 | **9** | 18 | **25** | 22 |
| 58 | 7 | 62 | 37 | 24 | 27 | 20 | 11 |
| 61 | 38 | 59 | 8 | 19 | 10 | 23 | 26 |

1526

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 56 | 19 | 32 | 61 | 14 | 47 | 34 |
| 18 | 31 | 62 | 57 | 48 | 33 | 60 | 13 |
| 55 | **64** | 29 | 20 | 15 | 58 | 35 | 46 |
| 30 | 17 | 54 | **49** | 38 | 45 | 12 | 59 |
| 53 | 28 | **1** | **16** | 21 | **36** | 39 | 44 |
| 2 | 5 | 50 | 37 | 42 | **9** | 24 | 11 |
| 27 | 52 | 7 | **4** | **25** | 22 | 43 | 40 |
| 6 | 3 | 26 | 51 | 8 | 41 | 10 | 23 |

1527

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 42 | 53 | 58 | 19 | 60 | 23 | 38 |
| 52 | 55 | 18 | 41 | 24 | 39 | 34 | 61 |
| 43 | **16** | 57 | 54 | 59 | 20 | 37 | 22 |
| 56 | 51 | 44 | **25** | 40 | 33 | 62 | 35 |
| 15 | 26 | **9** | 50 | 45 | **36** | 21 | 32 |
| 10 | 5 | 12 | 29 | 8 | **1** | 48 | 63 |
| 27 | 14 | 7 | **4** | **49** | 46 | 31 | 2 |
| 6 | 11 | 28 | 13 | 30 | 3 | **64** | 47 |

1528

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 12 | 15 | 54 | 21 | 26 | 33 | 30 |
| 14 | 55 | 18 | **25** | 32 | 29 | 22 | 27 |
| 11 | **16** | 13 | 20 | 53 | 24 | 31 | 34 |
| 56 | 19 | 10 | 39 | **36** | 51 | 28 | 23 |
| **9** | 38 | 7 | 52 | 45 | 62 | 35 | 50 |
| 6 | 57 | **4** | 37 | 40 | **49** | 44 | 63 |
| 3 | 8 | 59 | 46 | **1** | 42 | 61 | 48 |
| 58 | 5 | 2 | 41 | 60 | 47 | **64** | 43 |

1449

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 24 | 33 | 40 | 59 | 26 | 55 | 38 |
| 34 | 41 | 6 | **25** | 56 | 39 | 58 | 27 |
| 23 | 8 | 35 | 32 | 29 | 60 | 37 | 54 |
| 42 | 31 | **16** | 5 | **36** | 57 | 28 | 61 |
| **9** | 22 | 43 | 30 | 15 | 62 | 53 | 50 |
| 44 | 17 | **4** | 21 | 52 | **49** | 14 | 63 |
| 3 | 10 | 19 | 46 | **1** | 12 | 51 | 48 |
| 18 | 45 | 2 | 11 | 20 | 47 | **64** | 13 |

1450

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 48 | 19 | 32 | 51 | 14 | 21 | 30 |
| 18 | 33 | 62 | **49** | 20 | 31 | 52 | 13 |
| 47 | **64** | 35 | 58 | 15 | 50 | 29 | 22 |
| 34 | 17 | 46 | 61 | **36** | 23 | 12 | 53 |
| **1** | 6 | 59 | **16** | 57 | 28 | 37 | 24 |
| 42 | 45 | **4** | 7 | 60 | **25** | 54 | 11 |
| 5 | 2 | 43 | 40 | **9** | 56 | 27 | 38 |
| 44 | 41 | 8 | 3 | 26 | 39 | 10 | 55 |

1593

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 47 | 62 | 43 | 60 | 41 | 14 | 31 | 34 |
| 44 | 59 | 46 | **49** | 32 | 35 | 40 | 13 |
| 63 | 48 | 61 | 42 | 15 | 50 | 33 | 30 |
| 58 | 45 | **64** | 5 | **36** | 29 | 12 | 39 |
| **1** | 6 | 57 | **16** | 51 | 38 | 55 | 28 |
| 20 | 17 | **4** | 37 | 56 | **25** | 52 | 11 |
| 7 | 2 | 19 | 22 | **9** | 54 | 27 | 24 |
| 18 | 21 | 8 | 3 | 26 | 23 | 10 | 53 |

1594

That leaves 38.

Thirty two occur in 16 lateral / diagonal pairs:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 14 | 19 | 8 | 3 | 12 | 55 | 58 |
| 20 | 7 | **16** | 13 | 54 | 57 | **4** | 11 |
| 15 | 18 | 21 | 6 | **9** | 2 | 59 | 56 |
| 22 | **25** | 34 | 53 | 60 | 5 | 10 | **1** |
| 33 | 46 | 23 | 26 | 35 | 52 | 61 | 42 |
| 24 | 27 | **36** | 45 | 30 | 41 | **64** | 51 |
| 47 | 32 | 29 | 38 | **49** | 62 | 43 | 40 |
| 28 | 37 | 48 | 31 | 44 | 39 | 50 | 63 |

661

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 48 | 59 | 32 | 61 | 50 | 27 | 34 |
| 58 | 31 | 62 | **49** | 28 | 33 | 54 | 51 |
| 47 | **64** | 29 | 60 | 53 | 20 | 35 | 26 |
| 30 | 57 | 2 | 11 | **36** | 55 | 52 | 19 |
| **1** | 46 | 5 | 56 | 21 | 12 | **25** | 38 |
| 6 | 43 | 8 | 3 | 10 | 37 | 18 | 15 |
| 45 | **4** | 41 | 22 | 13 | **16** | 39 | 24 |
| 42 | 7 | 44 | **9** | 40 | 23 | 14 | 17 |

1674

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19 | 10 | 7 | 54 | 21 | 24 | 15 | 26 |
| 8 | 55 | 20 | 11 | **16** | 27 | 52 | 23 |
| 43 | 18 | **9** | 6 | 53 | 22 | **25** | 14 |
| 56 | 5 | 44 | 17 | 12 | 51 | 28 | 35 |
| 45 | 42 | 57 | **4** | 29 | **36** | 13 | 50 |
| 58 | 3 | 30 | 47 | **64** | 61 | 34 | 37 |
| 41 | 46 | **1** | 60 | 39 | 32 | **49** | 62 |
| 2 | 59 | 40 | 31 | 48 | 63 | 38 | 33 |

716

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 38 | 23 | 50 | 57 | 32 | 59 | 48 | 63 |
| 51 | 56 | 37 | 22 | **49** | 62 | 33 | 60 |
| 24 | 39 | 52 | 31 | 58 | 35 | **64** | 47 |
| 55 | 30 | 21 | **36** | 53 | 2 | 61 | 34 |
| 40 | **25** | 54 | 3 | 20 | 11 | 46 | **1** |
| 29 | 6 | 27 | **16** | 43 | **4** | 19 | 12 |
| 26 | 41 | 8 | 5 | 14 | 17 | 10 | 45 |
| 7 | 28 | 15 | 42 | **9** | 44 | 13 | 18 |

1676

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 39 | 2 | 53 | 60 | 11 | 6 | 55 | 62 |
| 52 | 59 | 40 | **1** | 54 | 61 | 12 | 7 |
| 3 | 38 | 51 | 58 | 5 | 10 | 63 | 56 |
| 50 | 47 | **4** | 41 | **64** | 57 | 8 | 13 |
| 37 | 28 | **49** | 46 | **9** | 42 | 17 | 22 |
| 48 | 31 | 34 | **25** | 20 | 23 | 14 | 43 |
| 27 | **36** | 29 | 32 | 45 | **16** | 21 | 18 |
| 30 | 33 | 26 | 35 | 24 | 19 | 44 | 15 |

981

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 42 | 5 | 24 | 3 | 26 | 39 | 18 | 15 |
| 23 | 2 | 41 | 8 | 19 | **16** | 27 | 38 |
| 6 | 43 | **4** | **25** | 40 | 45 | 14 | 17 |
| **1** | 22 | 7 | 44 | **9** | 20 | 37 | 28 |
| 60 | 57 | **64** | 21 | **36** | 29 | 46 | 13 |
| 51 | 54 | 61 | 58 | 63 | 10 | 33 | 30 |
| 56 | 59 | 52 | **49** | 32 | 35 | 12 | 47 |
| 53 | 50 | 55 | 62 | 11 | 48 | 31 | 34 |

1677

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 52 | 11 | 24 | 47 | 26 | 13 | 38 |
| 10 | 23 | 8 | 51 | 12 | 39 | 48 | 27 |
| 53 | 6 | 21 | 46 | **25** | 50 | 37 | 14 |
| 22 | **9** | 54 | 5 | 40 | 15 | 28 | **49** |
| 55 | 20 | 41 | **16** | 45 | **36** | 61 | 32 |
| 42 | 17 | **4** | 35 | 58 | 31 | **64** | 29 |
| 3 | 56 | 19 | 44 | **1** | 62 | 33 | 60 |
| 18 | 43 | 2 | 57 | 34 | 59 | 30 | 63 |

1054

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 43 | 14 | 57 | 8 | 41 | 10 | 59 |
| 63 | 56 | **1** | 42 | 13 | 58 | 7 | 40 |
| 44 | 3 | 62 | 15 | 38 | **9** | 60 | 11 |
| 55 | **64** | 45 | **4** | 61 | 12 | 39 | 6 |
| 46 | 51 | 20 | 37 | **16** | 5 | 22 | 31 |
| 19 | 54 | **49** | 34 | 21 | 30 | **25** | 28 |
| 50 | 47 | 52 | 17 | **36** | 27 | 32 | 23 |
| 53 | 18 | 35 | 48 | 33 | 24 | 29 | 26 |

1814

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 24 | 27 | 38 | 43 | **36** | 45 | 50 | 53 |
| 39 | 42 | **25** | 28 | 55 | 52 | 35 | 46 |
| 26 | 23 | 40 | 37 | 44 | **49** | 54 | 51 |
| 41 | **16** | 21 | 56 | 29 | 34 | 47 | **64** |
| 22 | 7 | 12 | 17 | 48 | **1** | 60 | 33 |
| 15 | 20 | **9** | 6 | 57 | 30 | 63 | 2 |
| 8 | 11 | 18 | 13 | **4** | 61 | 32 | 59 |
| 19 | 14 | 5 | 10 | 31 | 58 | 3 | 62 |

1834

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 41 | 44 | 3 | 18 | 59 | 10 | 5 | 8 |
| 2 | 19 | 42 | 45 | **4** | 7 | 60 | 11 |
| 43 | 40 | **1** | 58 | 17 | 62 | **9** | 6 |
| 20 | 51 | 46 | 63 | 22 | 53 | 12 | 61 |
| 39 | **64** | 21 | 52 | 57 | **16** | 35 | 24 |
| 50 | 47 | 30 | 27 | **36** | 23 | 54 | 13 |
| 29 | 38 | **49** | 32 | 15 | 56 | **25** | 34 |
| 48 | 31 | 28 | 37 | 26 | 33 | 14 | 55 |

1835

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 23 | 26 | 43 | 32 | 45 | 28 | 51 | 2 |
| 42 | 33 | 24 | 27 | 50 | 3 | 46 | 29 |
| **25** | 22 | 35 | 44 | 31 | 48 | **1** | 52 |
| 34 | 41 | **16** | **49** | **4** | 53 | 30 | 47 |
| 15 | **36** | 21 | 40 | 17 | **64** | 57 | 6 |
| 20 | 39 | 12 | **9** | 54 | 5 | 60 | 63 |
| 11 | 14 | 37 | 18 | 61 | 58 | 7 | 56 |
| 38 | 19 | 10 | 13 | 8 | 55 | 62 | 59 |

1836

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 60 | 3 | 58 | 23 | 62 | 5 | 30 | 7 |
| 57 | 22 | 61 | **4** | 31 | 8 | 27 | 34 |
| 2 | 59 | 20 | 63 | 24 | 33 | 6 | 29 |
| 21 | 56 | **1** | 32 | **9** | 28 | 35 | 26 |
| 54 | 19 | 52 | 39 | **64** | **25** | 10 | 13 |
| 43 | 40 | 55 | **16** | 51 | 12 | 47 | **36** |
| 18 | 53 | 42 | 45 | 38 | **49** | 14 | 11 |
| 41 | 44 | 17 | 50 | 15 | 46 | 37 | 48 |

1837

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 20 | 53 | 46 | 7 | 22 | 13 | 44 | 11 |
| 47 | 8 | 21 | 54 | 45 | 10 | 23 | 14 |
| 52 | 19 | 6 | **9** | 50 | 17 | 12 | 43 |
| 5 | 48 | 51 | 18 | 55 | 42 | 15 | 24 |
| 34 | 59 | **4** | **49** | **16** | 39 | 28 | 41 |
| **1** | 62 | 35 | 58 | 31 | 56 | **25** | 38 |
| 60 | 33 | **64** | 3 | **36** | 27 | 40 | 29 |
| 63 | 2 | 61 | 32 | 57 | 30 | 37 | 26 |

1917

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 43 | 2 | 19 | 8 | 45 | 10 | 51 | 54 |
| 18 | 5 | 44 | 3 | 50 | 53 | 46 | 11 |
| **1** | 42 | 7 | 20 | **9** | 48 | 55 | 52 |
| 6 | 17 | **4** | **49** | 56 | 21 | 12 | 47 |
| 35 | **64** | 41 | **16** | 37 | 24 | 29 | 22 |
| 40 | 61 | **36** | 57 | 32 | 15 | 26 | 13 |
| 63 | 34 | 59 | 38 | **25** | 28 | 23 | 30 |
| 60 | 39 | 62 | 33 | 58 | 31 | 14 | 27 |

1918

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 14 | 59 | 18 | 35 | 62 | 37 | 32 | 41 |
| 17 | 20 | 15 | 60 | 33 | 40 | 63 | 38 |
| 58 | 13 | 34 | 19 | **36** | 61 | 42 | 31 |
| 21 | **16** | 57 | 48 | 43 | 30 | 39 | **64** |
| 12 | 55 | 6 | **25** | **4** | **49** | 44 | 29 |
| 7 | 22 | **9** | 56 | 47 | 26 | **1** | 50 |
| 54 | 11 | 24 | 5 | 52 | 3 | 28 | 45 |
| 23 | 8 | 53 | 10 | 27 | 46 | 51 | 2 |

1919

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 42 | 45 | 32 | 63 | 34 | 47 | 38 | 19 |
| 31 | 62 | 43 | 46 | 39 | 20 | 35 | 12 |
| 44 | 41 | **64** | 33 | 48 | 13 | 18 | 37 |
| 61 | 30 | 5 | 40 | 21 | **36** | 11 | 14 |
| 28 | **1** | 60 | **49** | 56 | 15 | 22 | 18 |
| 59 | 6 | 29 | **4** | **25** | 10 | 55 | 52 |
| 2 | 27 | 8 | 57 | 50 | 53 | **16** | 23 |
| 7 | 58 | 3 | 26 | **9** | 24 | 51 | 54 |

1920

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 2 | 45 | 60 | 51 | 56 | 47 | 58 |
| 44 | 61 | 6 | **1** | 46 | 59 | 30 | 55 |
| 3 | 8 | 63 | 52 | 5 | 50 | 57 | 48 |
| 62 | 43 | **4** | 13 | **64** | 29 | 54 | 31 |
| **9** | 12 | 17 | 42 | 53 | 32 | **49** | 28 |
| 20 | 23 | 10 | **25** | 14 | 41 | 38 | 35 |
| 11 | **16** | 21 | 18 | 33 | **36** | 27 | 40 |
| 22 | 19 | 24 | 15 | 26 | 39 | 34 | 37 |

2288

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 54 | 51 | 26 | 17 | 24 | 19 | 58 | 35 |
| 27 | **16** | 53 | 50 | 57 | **36** | 23 | 20 |
| 52 | 55 | 10 | **25** | 18 | 21 | 34 | 59 |
| **9** | 28 | 15 | 56 | **49** | 60 | 37 | 22 |
| 40 | 11 | **4** | 29 | 38 | 33 | 48 | 61 |
| 3 | 8 | 39 | 14 | 5 | **64** | 45 | 32 |
| 12 | 41 | 6 | **1** | 30 | 43 | 62 | 47 |
| 7 | 2 | 13 | 42 | 63 | 46 | 31 | 44 |

2289

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 45 | 56 | 3 | 34 | 5 | 58 | 11 | 32 |
| 2 | 53 | 46 | 57 | 10 | 33 | 8 | 59 |
| 55 | 44 | 37 | **4** | 35 | 6 | 31 | 12 |
| 52 | **1** | 54 | 47 | 38 | **9** | 60 | 7 |
| 43 | 22 | 39 | **36** | 61 | 20 | 13 | 30 |
| **64** | 51 | 42 | 21 | 48 | 27 | **16** | 19 |
| 23 | 40 | **49** | 62 | **25** | 18 | 29 | 14 |
| 50 | 63 | 24 | 41 | 28 | 15 | 26 | 17 |

2352

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 17 | 8 | 43 | 26 | 15 | 38 | 41 |
| 7 | 44 | 11 | **16** | 39 | 42 | 27 | 14 |
| 18 | **9** | 2 | 5 | 12 | **25** | 40 | 37 |
| 3 | 6 | 45 | 24 | **1** | 60 | 13 | 28 |
| 46 | 19 | **4** | 59 | 52 | 29 | **36** | 61 |
| 55 | 58 | 23 | **64** | 35 | 32 | 51 | 30 |
| 20 | 47 | 56 | 53 | 22 | **49** | 62 | 33 |
| 57 | 54 | 21 | 48 | 63 | 34 | 31 | 50 |

2353

continued

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 42 | 5 | 24 | 3 | 26 | 39 | 18 | 15 |
| 23 | 2 | 41 | 8 | 19 | **16** | 27 | 38 |
| 6 | 43 | **4** | **25** | 40 | 45 | 14 | 17 |
| **1** | 22 | 7 | 44 | **9** | 20 | 37 | 28 |
| 56 | 61 | 52 | 21 | **36** | 29 | 46 | 13 |
| 51 | **64** | 55 | 58 | 53 | 10 | 33 | 30 |
| 60 | 57 | 62 | **49** | 32 | 35 | 12 | 47 |
| 63 | 50 | 59 | 54 | 11 | 48 | 31 | 34 |

2468

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 35 | 6 | 55 | 50 | 3 | 48 | 39 | 58 |
| 54 | 51 | **36** | 5 | 38 | 57 | 2 | 47 |
| 7 | 34 | 53 | 56 | **49** | **4** | 59 | 40 |
| 52 | **25** | 8 | 37 | 60 | 41 | 46 | **1** |
| 15 | 20 | 33 | 24 | **9** | **64** | 31 | 42 |
| 26 | 23 | **16** | 61 | 32 | 43 | 10 | 45 |
| 19 | 14 | 21 | 28 | 17 | 12 | 63 | 30 |
| 22 | 27 | 18 | 13 | 62 | 29 | 44 | 11 |

2469

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 11 | 52 | 5 | 46 | 13 | 18 | 3 | 62 |
| 6 | 47 | 12 | 51 | **4** | 63 | 14 | 17 |
| 53 | 10 | 45 | **64** | 19 | **16** | 61 | 2 |
| 48 | 7 | 32 | **9** | 50 | **1** | 20 | 15 |
| 33 | 54 | **49** | 44 | 21 | 60 | **25** | 28 |
| 40 | 43 | 8 | 31 | **36** | 27 | 22 | 59 |
| 55 | 34 | 41 | 38 | 57 | 24 | 29 | 26 |
| 42 | 39 | 56 | 35 | 30 | 37 | 58 | 23 |

2544

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 11 | 18 | 35 | 46 | 13 | 22 | 33 | 54 |
| 48 | 45 | 12 | 17 | 34 | 55 | 14 | 23 |
| 19 | 10 | 47 | **36** | 21 | **16** | 53 | 32 |
| 44 | **49** | 20 | **9** | 56 | 51 | 24 | 15 |
| 63 | 8 | 37 | 50 | **25** | **4** | 31 | 52 |
| 40 | 43 | **64** | 5 | 60 | 57 | 28 | 3 |
| 7 | 62 | 41 | 38 | **1** | 26 | 59 | 30 |
| 42 | 39 | 6 | 61 | 58 | 29 | 2 | 27 |

2545

Tour 2703, the second below, is of squares and diamonds except from 53 to 60. Many of the tours make use of squares and diamonds where possible.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 35 | 2 | 37 | 52 | 15 | 32 | 23 | 62 |
| 38 | 51 | 34 | **1** | 24 | 63 | 14 | 31 |
| 3 | **36** | 53 | 50 | 33 | **16** | 61 | 22 |
| 48 | 39 | **4** | **25** | **64** | 21 | 30 | 13 |
| 5 | 54 | **49** | 40 | **9** | 26 | 17 | 60 |
| 44 | 47 | 8 | 57 | 20 | 41 | 12 | 29 |
| 55 | 6 | 45 | 42 | 27 | 10 | 59 | 18 |
| 46 | 43 | 56 | 7 | 58 | 19 | 28 | 11 |

2702

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 27 | 62 | 37 | 6 | 35 | 42 | 31 |
| 63 | 38 | **1** | 28 | 41 | 32 | 7 | 34 |
| 26 | 3 | 40 | 61 | **36** | 5 | 30 | 43 |
| 39 | **64** | **25** | **4** | 29 | 44 | 33 | 8 |
| 24 | 15 | 60 | **49** | 20 | **9** | 58 | 45 |
| 53 | 50 | 21 | **16** | 59 | 48 | 19 | 10 |
| 14 | 23 | 52 | 55 | 12 | 17 | 46 | 57 |
| 51 | 54 | 13 | 22 | 47 | 56 | 11 | 18 |

2703

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 43 | 32 | 35 | 28 | 53 | 30 | 39 | 14 |
| 34 | 27 | 44 | 31 | 40 | 15 | 54 | 51 |
| 63 | 42 | 33 | **36** | 29 | 52 | 13 | 38 |
| 26 | 45 | **64** | 41 | **16** | 37 | 50 | 55 |
| 23 | 62 | **25** | **4** | **49** | 6 | 17 | 12 |
| 46 | **1** | 22 | 59 | 20 | **9** | 56 | 7 |
| 61 | 24 | 3 | 48 | 5 | 58 | 11 | 18 |
| 2 | 47 | 60 | 21 | 10 | 19 | 8 | 57 |

2704

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 27 | 30 | 3 | 42 | 15 | 40 | 17 | 34 |
| 2 | 43 | 28 | 31 | 18 | 33 | 14 | 39 |
| 29 | 26 | 63 | **4** | 41 | **16** | 35 | 20 |
| 62 | **1** | 44 | **25** | 32 | 19 | 38 | 13 |
| 45 | 58 | 5 | **64** | **9** | **36** | 21 | 52 |
| 6 | 61 | 48 | 55 | 24 | 51 | 12 | 37 |
| 57 | 46 | 59 | 8 | **49** | 10 | 53 | 22 |
| 60 | 7 | 56 | 47 | 54 | 23 | 50 | 11 |

2705

the last of the 16 pairs:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 55 | 14 | 57 | 24 | 29 | 26 | 43 | 38 |
| 22 | 59 | 54 | 15 | 44 | 37 | 30 | 27 |
| 13 | 56 | 23 | 58 | **25** | 28 | 39 | 42 |
| 60 | 21 | **16** | 53 | 40 | 45 | **36** | 31 |
| 7 | 12 | 61 | 20 | **1** | 32 | 41 | 46 |
| 62 | 17 | 6 | **9** | 52 | **49** | 2 | 35 |
| 11 | 8 | 19 | **64** | 33 | **4** | 47 | 50 |
| 18 | 63 | 10 | 5 | 48 | 51 | 34 | 3 |

2785

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 45 | 30 | 3 | 28 | 5 | 54 | 41 | 22 |
| 2 | 27 | 46 | 31 | 42 | 23 | 6 | 55 |
| 47 | 44 | 29 | **4** | 53 | 8 | 21 | 40 |
| 26 | **1** | 32 | 43 | 24 | 39 | 56 | 7 |
| 33 | 48 | **25** | 52 | **9** | 20 | 13 | 38 |
| **64** | 51 | 62 | 19 | **36** | 17 | 10 | 57 |
| 61 | 34 | **49** | **16** | 59 | 12 | 37 | 14 |
| 50 | 63 | 60 | 35 | 18 | 15 | 58 | 11 |

2786

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 11 | 18 | 37 | 62 | 13 | 60 | 51 | 54 |
| 38 | 35 | 12 | 17 | 52 | 55 | 14 | 59 |
| 19 | 10 | 63 | **36** | 61 | **16** | 53 | 50 |
| 34 | 39 | 20 | **9** | 56 | **49** | 58 | 15 |
| 21 | 8 | 33 | **64** | **25** | **4** | 29 | 48 |
| 40 | 43 | 24 | 5 | 32 | 57 | 26 | 3 |
| 7 | 22 | 45 | 42 | **1** | 28 | 47 | 30 |
| 44 | 41 | 6 | 23 | 46 | 31 | 2 | 27 |

2787

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 27 | 34 | 3 | 52 | 29 | 56 | 47 | 58 |
| 2 | 51 | 28 | 33 | 48 | 59 | 30 | 55 |
| 35 | 26 | **49** | **4** | 53 | 32 | 57 | 46 |
| 50 | **1** | 12 | **25** | 60 | 45 | 54 | 31 |
| 11 | **36** | **9** | **64** | 5 | 42 | 17 | 44 |
| 8 | 13 | 24 | 61 | **16** | 63 | 20 | 41 |
| 37 | 10 | 15 | 6 | 39 | 22 | 43 | 18 |
| 14 | 7 | 38 | 23 | 62 | 19 | 40 | 21 |

2788

There remain six that occur only in diagonal form:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 14 | 51 | 44 | 57 | 60 | 53 | 18 | 55 |
| 45 | 48 | 15 | 52 | 17 | 56 | 59 | 62 |
| 50 | 13 | 46 | 43 | 58 | 61 | 54 | 19 |
| 47 | 42 | **49** | **16** | 5 | 20 | 63 | 26 |
| 12 | 35 | **4** | 41 | **64** | **25** | 6 | 21 |
| 3 | 38 | 33 | **36** | **9** | 22 | 27 | 30 |
| 34 | 11 | 40 | **1** | 32 | 29 | 24 | 7 |
| 39 | 2 | 37 | 10 | 23 | 8 | 31 | 28 |

381

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 61 | 42 | 33 | 54 | 59 | 44 | 3 | 18 |
| 34 | 31 | 60 | 43 | 2 | 19 | 58 | 45 |
| 41 | 62 | 53 | 32 | 55 | **4** | 17 | 20 |
| 30 | 35 | 40 | **1** | **16** | 21 | 46 | 57 |
| 63 | 52 | 29 | **36** | 5 | 56 | **9** | 22 |
| 28 | 39 | **64** | 15 | 12 | **25** | 6 | 47 |
| 51 | 14 | 37 | 26 | **49** | 8 | 23 | 10 |
| 38 | 27 | 50 | 13 | 24 | 11 | 48 | 7 |

910

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 56 | 35 | 58 | 37 | 54 | 15 | 18 |
| 34 | 59 | 6 | 55 | **16** | 19 | 38 | 51 |
| 5 | 8 | 57 | **36** | 53 | 50 | 17 | 14 |
| 60 | 33 | **64** | **9** | 20 | **25** | 52 | 39 |
| 63 | **4** | 61 | 26 | **49** | 40 | 13 | 24 |
| 32 | 29 | 48 | **1** | 10 | 21 | 44 | 41 |
| 3 | 62 | 27 | 30 | 43 | 46 | 23 | 12 |
| 28 | 31 | 2 | 47 | 22 | 11 | 42 | 45 |

1815

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 54 | 29 | 52 | 31 | 56 | 11 | 22 |
| 28 | 51 | 6 | 55 | 10 | 23 | 32 | 57 |
| 5 | 8 | 53 | 30 | 63 | 34 | 21 | 12 |
| 50 | 27 | **64** | **9** | 24 | 45 | 58 | 33 |
| 39 | **4** | **25** | 46 | 35 | 62 | 13 | 20 |
| 26 | **49** | 38 | **1** | **16** | 19 | 44 | 59 |
| 3 | 40 | 47 | **36** | 61 | 42 | 17 | 14 |
| 48 | 37 | 2 | 41 | 18 | 15 | 60 | 43 |

1816

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 47 | 22 | 31 | 20 | 39 | 18 | 35 | 14 |
| 30 | 41 | 48 | 45 | 32 | 15 | 38 | 17 |
| 23 | 46 | 21 | 40 | 19 | **36** | 13 | 34 |
| 42 | 29 | 24 | **49** | 44 | 33 | **16** | 37 |
| 55 | 50 | 43 | **4** | **25** | 6 | 63 | 12 |
| 28 | 3 | 56 | 53 | **64** | **9** | 60 | 7 |
| 51 | 54 | **1** | 26 | 5 | 58 | 11 | 62 |
| 2 | 27 | 52 | 57 | 10 | 61 | 8 | 59 |

2869

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 39 | 46 | 59 | 30 | 41 | 20 | 61 | 18 |
| 58 | 27 | 40 | 45 | 60 | 17 | 42 | 21 |
| 47 | 38 | 29 | 26 | 31 | 44 | 19 | 62 |
| 28 | 57 | 32 | 37 | **16** | 63 | 22 | 43 |
| 33 | 48 | **25** | **64** | 23 | **4** | 15 | 6 |
| 56 | 51 | 54 | **9** | **36** | 7 | 12 | 3 |
| 53 | 34 | **49** | 24 | **1** | 10 | 5 | 14 |
| 50 | 55 | 52 | 35 | 8 | 13 | 2 | 11 |

2870

---

# Square Numbers in Rotary Circuits

There are 31 centrosymmetric non-axial forms, 16 centred at a point where four cells meet.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 46 | 63 | 50 | 21 | 32 | 19 | 34 | 23 |
| 51 | 2 | 47 | 42 | **49** | 22 | 31 | 18 |
| 62 | 45 | **64** | 3 | 20 | 33 | 24 | 35 |
| **1** | 52 | 43 | 48 | 41 | **36** | 17 | 30 |
| 44 | 61 | **4** | 57 | 10 | 29 | 40 | **25** |
| 53 | 56 | 7 | 28 | 37 | **16** | 11 | 14 |
| 60 | 5 | 54 | **9** | 58 | 13 | 26 | 39 |
| 55 | 8 | 59 | 6 | 27 | 38 | 15 | 12 |

566

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 42 | 39 | 8 | 3 | 22 | 15 | 10 | 17 |
| 7 | 2 | 41 | 38 | **9** | 18 | 21 | 14 |
| 40 | 43 | **4** | 23 | 20 | 13 | **16** | 11 |
| **1** | 6 | 29 | 44 | 37 | 24 | 19 | 54 |
| 30 | 61 | 48 | 5 | 50 | 55 | 12 | **25** |
| 47 | **64** | 45 | 28 | 33 | **36** | 53 | 56 |
| 60 | 31 | 62 | **49** | 58 | 51 | 26 | 35 |
| 63 | 46 | 59 | 32 | 27 | 34 | 57 | 52 |

607

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 29 | 26 | 31 | 34 | 13 | **16** | 59 | 10 |
| 32 | 35 | 28 | **25** | 60 | 11 | 14 | 17 |
| 27 | 30 | 33 | 12 | 15 | 24 | **9** | 58 |
| 48 | 39 | **36** | 61 | 46 | 41 | 18 | 23 |
| 37 | 62 | 47 | 40 | 19 | **4** | 57 | 8 |
| 52 | **49** | 38 | 45 | 42 | 7 | 22 | 3 |
| 63 | 44 | 51 | 54 | **1** | 20 | 5 | 56 |
| 50 | 53 | **64** | 43 | 6 | 55 | 2 | 21 |

3324

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 38 | 59 | 56 | 47 | **36** | 23 | 26 |
| 58 | 55 | **64** | 37 | 24 | 27 | 48 | 35 |
| 39 | 62 | 57 | 60 | **49** | 46 | **25** | 22 |
| 54 | **1** | 40 | **9** | 28 | 21 | 34 | 45 |
| 5 | 10 | 61 | 50 | 33 | **16** | 29 | 20 |
| 2 | 53 | **4** | 41 | 8 | 19 | 44 | 15 |
| 11 | 6 | 51 | 32 | 13 | 42 | 17 | 30 |
| 52 | 3 | 12 | 7 | 18 | 31 | 14 | 43 |

6230

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 60 | 51 | 40 | 27 | 38 | 35 | 32 |
| 52 | 41 | 62 | 59 | 50 | 33 | 26 | 37 |
| 61 | **64** | 43 | 28 | 39 | **36** | 31 | 34 |
| 42 | 53 | 58 | **49** | 44 | 29 | 8 | **25** |
| **1** | 48 | 13 | 54 | **9** | 24 | 19 | 30 |
| 12 | 57 | **4** | 45 | 14 | 21 | **16** | 7 |
| 47 | 2 | 55 | 10 | 5 | 18 | 23 | 20 |
| 56 | 11 | 46 | 3 | 22 | 15 | 6 | 17 |

6025

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 23 | 14 | 45 | 12 | 27 | **36** | 53 | 48 |
| 44 | 11 | 24 | 37 | 52 | 47 | 26 | 35 |
| 15 | 22 | 13 | 46 | **25** | 28 | **49** | 54 |
| 10 | 43 | **16** | 21 | 38 | 51 | 34 | 29 |
| 17 | 2 | 39 | 6 | 33 | **64** | 55 | 50 |
| 42 | **9** | 20 | **1** | 58 | 61 | 30 | 63 |
| 3 | 18 | 7 | 40 | 5 | 32 | 59 | 56 |
| 8 | 41 | **4** | 19 | 60 | 57 | 62 | 31 |

6026

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 47 | 56 | 3 | 52 | 5 | 44 | 17 | 14 |
| 2 | 51 | 46 | 55 | 18 | 15 | 8 | 43 |
| 57 | 48 | 53 | **4** | 45 | 6 | 13 | **16** |
| 50 | **1** | 40 | 19 | 54 | **9** | 42 | 7 |
| 39 | 58 | **49** | 10 | 41 | 32 | **25** | 12 |
| **64** | 61 | 20 | 33 | **36** | 11 | 28 | 31 |
| 59 | 38 | 63 | 22 | 29 | 26 | 35 | 24 |
| 62 | 21 | 60 | 37 | 34 | 23 | 30 | 27 |

6027

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 38 | 7 | 54 | 63 | 22 | 35 | 52 | 11 |
| 55 | 62 | 37 | 8 | 53 | 10 | 21 | 34 |
| 6 | 39 | **64** | 23 | **36** | 33 | 12 | 51 |
| 61 | 56 | 5 | 40 | **9** | 50 | **25** | 20 |
| 42 | **1** | 60 | **49** | 24 | 19 | 32 | 13 |
| 57 | 48 | 41 | **4** | 45 | **16** | 29 | 26 |
| 2 | 43 | 46 | 59 | 18 | 27 | 14 | 31 |
| 47 | 58 | 3 | 44 | 15 | 30 | 17 | 28 |

6094

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 45 | 8 | 61 | 10 | 59 | 56 | 23 | 40 |
| 62 | 11 | 44 | 7 | 42 | 39 | 58 | 55 |
| 5 | 46 | **9** | 60 | 57 | 22 | 41 | 24 |
| 12 | 63 | 6 | 43 | 38 | **25** | 54 | 21 |
| 47 | **4** | 35 | **16** | 53 | 20 | 37 | 26 |
| 34 | 13 | **64** | 3 | **36** | 29 | 52 | 19 |
| **1** | 48 | 15 | 32 | 17 | 50 | 27 | 30 |
| 14 | 33 | 2 | **49** | 28 | 31 | 18 | 51 |

6306

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 60 | **9** | 12 | 5 | 30 | 17 | 14 |
| 10 | 3 | 6 | 61 | **16** | 13 | 28 | 31 |
| 59 | 8 | 11 | **4** | 29 | 26 | 15 | 18 |
| 2 | 39 | 62 | **25** | 54 | 19 | 32 | 27 |
| 63 | 58 | **1** | 40 | 45 | 24 | 53 | 20 |
| 38 | 43 | **36** | 55 | 50 | 21 | 46 | 33 |
| 57 | **64** | 41 | 44 | 35 | 48 | 23 | 52 |
| 42 | 37 | 56 | **49** | 22 | 51 | 34 | 47 |

6581

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 51 | 12 | 45 | 38 | 47 | **64** | 33 | 2 |
| 44 | 39 | 50 | 11 | 34 | 3 | 48 | 63 |
| 13 | 52 | 37 | 46 | **49** | 60 | **1** | 32 |
| 40 | 43 | 10 | 21 | **4** | 35 | 62 | 59 |
| 53 | 14 | 41 | **36** | 61 | 20 | 31 | 6 |
| 42 | **25** | 22 | **9** | 28 | 5 | 58 | 19 |
| 15 | 54 | 27 | 24 | 17 | 56 | 7 | 30 |
| 26 | 23 | **16** | 55 | 8 | 29 | 18 | 57 |

6582

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 22 | 19 | 38 | **25** | 42 | 59 | 40 |
| 20 | 37 | **16** | 23 | 58 | 39 | 26 | 43 |
| 15 | 18 | 21 | **36** | 45 | 24 | 41 | 60 |
| 50 | 3 | 14 | **9** | 48 | 57 | 44 | 27 |
| 5 | 10 | **49** | 46 | 35 | 28 | 61 | 56 |
| 2 | 51 | **4** | 13 | 8 | 47 | 32 | 29 |
| 11 | 6 | 53 | **64** | 31 | 34 | 55 | 62 |
| 52 | **1** | 12 | 7 | 54 | 63 | 30 | 33 |

6637

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 38 | 41 | 62 | 7 | 10 | 27 | 24 | 59 |
| 63 | 8 | 37 | 40 | 61 | 58 | 11 | 26 |
| 42 | 39 | 6 | **9** | 28 | **25** | 60 | 23 |
| 5 | **64** | 43 | **36** | 57 | 22 | 17 | 12 |
| 50 | 35 | **4** | 29 | **16** | 19 | 56 | 21 |
| **1** | 32 | **49** | 44 | 53 | 46 | 13 | 18 |
| 34 | 51 | 30 | 3 | 48 | 15 | 20 | 55 |
| 31 | 2 | 33 | 52 | 45 | 54 | 47 | 14 |

6933

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 47 | 6 | 45 | 32 | 53 | 50 | 43 | 2 |
| 34 | 31 | 48 | 5 | 44 | **1** | 54 | 51 |
| 7 | 46 | 33 | 30 | **49** | 52 | 3 | 42 |
| 28 | 35 | 8 | 17 | **4** | 41 | **64** | 55 |
| 61 | **16** | 29 | **36** | 63 | 10 | 19 | 40 |
| 24 | 27 | 62 | **9** | 18 | 37 | 56 | 11 |
| 15 | 60 | **25** | 22 | 13 | 58 | 39 | 20 |
| 26 | 23 | 14 | 59 | 38 | 21 | 12 | 57 |

7474

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 33 | 42 | 39 | 48 | 61 | 44 | 51 |
| 41 | 38 | **1** | 34 | 43 | 50 | 47 | 60 |
| 32 | 3 | 40 | **49** | 62 | 59 | 52 | 45 |
| 37 | **64** | 31 | **4** | 35 | 46 | 17 | 58 |
| 10 | 5 | **36** | 63 | **16** | 57 | 22 | 53 |
| 27 | 30 | **9** | 56 | 23 | 20 | 15 | 18 |
| 6 | 11 | 28 | **25** | 8 | 13 | 54 | 21 |
| 29 | 26 | 7 | 12 | 55 | 24 | 19 | 14 |

7475

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 43 | 62 | 17 | 2 | 41 | 52 | 37 | 48 |
| 18 | 3 | 42 | 63 | 46 | **49** | 40 | 51 |
| 61 | 44 | **1** | **16** | 53 | 38 | 47 | **36** |
| **4** | 19 | 14 | 45 | **64** | **25** | 50 | 39 |
| 13 | 60 | **9** | 26 | 15 | 54 | 35 | 24 |
| 8 | 5 | 20 | 57 | 10 | 23 | 32 | 29 |
| 59 | 12 | 7 | 22 | 27 | 30 | 55 | 34 |
| 6 | 21 | 58 | 11 | 56 | 33 | 28 | 31 |

7718

There are 11 centred at the centre of a cell.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 54 | 41 | 52 | 37 | 48 | 57 | 24 | 35 |
| 51 | 38 | 55 | 44 | 23 | **36** | 47 | 58 |
| 42 | 53 | 40 | **49** | 56 | 45 | 34 | **25** |
| 39 | 50 | 43 | 22 | 17 | 26 | 59 | 46 |
| 2 | 21 | **64** | 27 | 60 | 11 | **16** | 33 |
| 63 | 28 | 3 | 18 | 5 | 14 | 7 | 10 |
| 20 | **1** | 30 | 61 | 12 | **9** | 32 | 15 |
| 29 | 62 | 19 | **4** | 31 | 6 | 13 | 8 |

3388

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12 | 47 | 58 | 31 | 14 | 33 | 26 | 37 |
| 57 | 30 | 13 | 48 | 27 | **36** | 23 | 34 |
| 46 | 11 | 28 | 59 | 32 | 15 | 38 | **25** |
| 29 | 56 | 45 | 10 | **49** | 24 | 35 | 22 |
| 2 | 51 | **64** | 55 | 60 | 21 | **16** | 39 |
| 63 | 44 | 3 | 50 | **9** | 18 | 7 | 20 |
| 52 | **1** | 42 | 61 | 54 | 5 | 40 | 17 |
| 43 | 62 | 53 | **4** | 41 | 8 | 19 | 6 |

3465

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 34 | 23 | 56 | 11 | 52 | 7 | 38 | 63 |
| 55 | 12 | 35 | 22 | 57 | 62 | 51 | 6 |
| 24 | 33 | 10 | 53 | 8 | 37 | **64** | 39 |
| 13 | 54 | 21 | **36** | 61 | 58 | 5 | 50 |
| 32 | **25** | 60 | **9** | 20 | **49** | 40 | **1** |
| 17 | 14 | 31 | 28 | 59 | **4** | 43 | 46 |
| 26 | 29 | **16** | 19 | 48 | 45 | 2 | 41 |
| 15 | 18 | 27 | 30 | 3 | 42 | 47 | 44 |

6231

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 39 | 14 | 43 | 10 | 29 | 32 | 45 | 26 |
| 42 | 11 | 40 | 15 | 44 | 27 | 34 | 31 |
| 13 | 38 | **9** | 28 | 33 | 30 | **25** | 46 |
| 8 | 41 | 12 | 37 | **16** | 47 | 52 | 35 |
| 63 | **4** | 17 | 48 | 53 | **36** | 59 | 24 |
| 18 | 7 | **64** | 3 | 60 | 21 | 56 | 51 |
| **1** | 62 | 5 | 20 | **49** | 54 | 23 | 58 |
| 6 | 19 | 2 | 61 | 22 | 57 | 50 | 55 |

6305

continued

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 46 | 3 | 6 | 59 | 50 | 27 | 30 |
| 2 | 5 | **64** | 45 | 26 | 29 | 58 | 51 |
| 47 | 62 | 7 | **4** | **49** | 60 | 31 | 28 |
| 8 | **1** | 48 | 61 | 44 | **25** | 52 | 57 |
| 19 | 40 | **9** | **36** | 21 | 56 | 15 | 32 |
| 10 | 37 | 20 | 43 | **16** | 35 | 24 | 53 |
| 41 | 18 | 39 | 12 | 55 | 22 | 33 | 14 |
| 38 | 11 | 42 | 17 | 34 | 13 | 54 | 23 |

6381

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 41 | 12 | 37 | 62 | 29 | 14 | 27 | 24 |
| 38 | 63 | 40 | 13 | 22 | **25** | 30 | 15 |
| 11 | 42 | 61 | **36** | **9** | 28 | 23 | 26 |
| **64** | 39 | 10 | 43 | 56 | 21 | **16** | 31 |
| 47 | 60 | **49** | **4** | 35 | 8 | 55 | 20 |
| 50 | **1** | 46 | 57 | 44 | 17 | 32 | 7 |
| 59 | 48 | 3 | 52 | 5 | 34 | 19 | 54 |
| 2 | 51 | 58 | 45 | 18 | 53 | 6 | 33 |

6382

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 18 | 11 | 40 | 7 | 38 | 13 | 34 | 55 |
| 41 | 8 | 17 | 12 | 33 | 54 | 37 | 14 |
| 10 | 19 | 6 | 39 | **16** | 35 | 56 | 53 |
| 5 | 42 | **9** | 20 | 57 | 32 | 15 | **36** |
| 62 | 21 | 58 | **1** | 46 | **25** | 52 | 31 |
| 43 | **4** | 63 | 24 | 59 | 28 | **49** | 26 |
| 22 | 61 | 2 | 45 | **64** | 47 | 30 | 51 |
| 3 | 44 | 23 | 60 | 29 | 50 | 27 | 48 |

6461

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 28 | 51 | 40 | 45 | 32 | 61 | 2 | 63 |
| 41 | 46 | 29 | 50 | 3 | **64** | 33 | 60 |
| 52 | 27 | 48 | 39 | 44 | 31 | 62 | **1** |
| 47 | 42 | 37 | 30 | **49** | **4** | 59 | 34 |
| 26 | 53 | 24 | 43 | 38 | 35 | 10 | 5 |
| 19 | 22 | 17 | **36** | **9** | 6 | 13 | 58 |
| 54 | **25** | 20 | 23 | 56 | 15 | 8 | 11 |
| 21 | 18 | 55 | **16** | 7 | 12 | 57 | 14 |

6462

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 34 | 55 | 26 | 15 | 42 | 57 | 24 | 11 |
| 27 | 14 | 35 | 56 | **25** | 12 | 43 | 58 |
| 54 | 33 | **16** | 13 | 60 | 41 | 10 | 23 |
| 17 | 28 | 53 | **36** | **9** | 22 | 59 | 44 |
| 52 | 3 | 32 | 21 | 48 | 61 | 40 | 7 |
| 29 | 18 | **49** | **4** | 37 | 8 | 45 | 62 |
| 2 | 51 | 20 | 31 | **64** | 47 | 6 | 39 |
| 19 | 30 | **1** | 50 | 5 | 38 | 63 | 46 |

6636

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 58 | 39 | 62 | 7 | 26 | 33 | 28 | 31 |
| 63 | 8 | 59 | 40 | 61 | 30 | **25** | 34 |
| 38 | 57 | 6 | **9** | **36** | 27 | 32 | 29 |
| 5 | **64** | 37 | 60 | 41 | **16** | 35 | 24 |
| 56 | 53 | **4** | **49** | 10 | 23 | 42 | 17 |
| **1** | 48 | 11 | 54 | 45 | 20 | 15 | 22 |
| 52 | 55 | 46 | 3 | 50 | 13 | 18 | 43 |
| 47 | 2 | 51 | 12 | 19 | 44 | 21 | 14 |

6775

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 42 | 51 | 40 | 37 | 32 | 35 | 18 | 15 |
| 39 | 2 | 43 | 50 | 19 | **16** | 31 | 34 |
| 52 | 41 | 38 | 3 | **36** | 33 | 14 | 17 |
| **1** | 60 | **49** | 44 | **9** | 20 | **25** | 30 |
| 48 | 53 | **4** | 61 | 24 | 29 | 8 | 13 |
| 59 | **64** | 57 | 54 | 45 | 10 | 21 | 26 |
| 56 | 47 | 62 | 5 | 28 | 23 | 12 | 7 |
| 63 | 58 | 55 | 46 | 11 | 6 | 27 | 22 |

6934

Finally four with centre at the mid-point of the edge of a cell (i.e. Bergholtian symmetry).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 24 | 13 | 22 | 11 | 38 | 57 | 46 |
| 14 | 21 | **16** | 37 | 58 | 47 | 60 | 39 |
| **25** | 18 | 23 | 12 | 41 | 10 | 45 | 56 |
| 20 | 15 | **36** | **9** | 48 | 59 | 40 | 61 |
| 3 | 26 | 19 | 42 | 35 | 8 | 55 | 44 |
| 32 | 29 | **4** | **49** | 52 | 43 | 62 | 7 |
| 27 | 2 | 31 | 34 | 5 | **64** | 51 | 54 |
| 30 | 33 | 28 | **1** | 50 | 53 | 6 | 63 |

7535

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 32 | 3 | 44 | 57 | 30 | 59 | 46 | 11 |
| 43 | 56 | 31 | 34 | 45 | 10 | 29 | 60 |
| 2 | 33 | **4** | 55 | 58 | 35 | 12 | 47 |
| 53 | 42 | 23 | **64** | **9** | 48 | 61 | 28 |
| 22 | **1** | 54 | 5 | 24 | 63 | **36** | 13 |
| 41 | 52 | 19 | **16** | **49** | 8 | 27 | 62 |
| 18 | 21 | 50 | 39 | 6 | **25** | 14 | 37 |
| 51 | 40 | 17 | 20 | 15 | 38 | 7 | 26 |

7536

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 29 | 6 | 31 | 46 | 63 | 42 | 33 |
| 7 | 50 | 3 | **64** | 5 | 32 | 45 | 62 |
| 28 | **1** | 30 | 47 | 20 | 43 | 34 | 41 |
| 51 | 8 | **49** | **4** | 35 | 40 | 61 | 44 |
| 10 | 27 | 52 | 21 | 48 | 19 | 58 | 39 |
| 53 | 24 | **9** | **36** | 57 | 60 | 15 | 18 |
| 26 | 11 | 22 | 55 | **16** | 13 | 38 | 59 |
| 23 | 54 | **25** | 12 | 37 | 56 | 17 | 14 |

7716

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 27 | 54 | 29 | 50 | 7 | 2 | 41 | 44 |
| 30 | 51 | 26 | 3 | 40 | 43 | 6 | **1** |
| 55 | 28 | 53 | 8 | **49** | **4** | 45 | 42 |
| 52 | 31 | 48 | **25** | 46 | 39 | **64** | 5 |
| 15 | 56 | 19 | 32 | **9** | **36** | 23 | 38 |
| 18 | 59 | **16** | 47 | 24 | 63 | 10 | 35 |
| 57 | 14 | 61 | 20 | 33 | 12 | 37 | 22 |
| 60 | 17 | 58 | 13 | 62 | 21 | 34 | 11 |

7717

Dawson concluded: “These notes complete a 16-year piece of work -- nothing like sticking to it!”

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## Other Dawsonian Figured Tours

Here we look at other figured tours with the square numbers in chains. These are presented in roughly chronological sequence. We begin with squares in open knight paths: The first example, from *British Chess Magazine* (vol.52 1932 p182 Prob.3044) not in my *Figured Tours* booklet is one of the first such tours. In *PFCS* Dawson comments: “The apparently pointless 451 is given to emphasise the usefulness of Ciccolini's squares and diamonds in constructions of this kind - the whole tour being made up of these units, with squares in S-chain.”.

Dawson 1932

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 5 | 10 | 13 | 26 | 3 | 58 | 23 |
| 11 | 14 | 7 | **4** | 57 | 24 | 27 | 2 |
| 6 | **9** | 12 | **25** | 54 | **1** | 22 | 59 |
| 15 | **36** | 41 | 56 | 17 | **64** | 53 | 28 |
| 34 | 45 | **16** | **49** | 40 | 55 | 60 | 21 |
| 37 | 48 | 35 | 42 | 63 | 18 | 29 | 52 |
| 44 | 33 | 46 | 39 | 50 | 31 | 20 | 61 |
| 47 | 38 | 43 | 32 | 19 | 62 | 51 | 30 |

BCM 3044

Dawson 1932

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 29 | 6 | 61 | **16** | 27 | **64** | 47 |
| 5 | 60 | **9** | 28 | 63 | 48 | 17 | 26 |
| 30 | 7 | 62 | 15 | 10 | **25** | 46 | **49** |
| 59 | **4** | 11 | 24 | 35 | 50 | 41 | 18 |
| 12 | 31 | 14 | 51 | 40 | 43 | **36** | 45 |
| **1** | 58 | 3 | 34 | 23 | 54 | 19 | 42 |
| 32 | 13 | 56 | 39 | 52 | 21 | 44 | 37 |
| 57 | 2 | 33 | 22 | 55 | 38 | 3 | 20 |

PFCS note to 329

Dawson 1932

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 27 | 2 | 57 | 62 | 15 | 30 | 23 | **64** |
| 58 | 61 | 28 | **1** | 24 | 63 | 14 | 31 |
| 3 | 26 | 69 | 56 | 29 | **16** | **49** | 22 |
| 60 | 55 | **4** | **25** | 50 | 21 | 32 | 13 |
| 5 | 38 | 51 | 44 | **9** | **36** | 17 | 48 |
| 54 | 41 | 8 | 37 | 20 | 45 | 12 | 33 |
| 39 | 6 | 43 | 52 | 35 | 10 | 47 | 18 |
| 42 | 53 | 40 | 7 | 46 | 19 | 34 | 11 |

PFCS 451

Dawson 1932

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 23 | 62 | 19 | 46 | 15 | 12 | 37 | 48 |
| 20 | 45 | 22 | 63 | **36** | 47 | 14 | 11 |
| 61 | 24 | 35 | 18 | 13 | **16** | **49** | 38 |
| 34 | 21 | 44 | **25** | **64** | 39 | 10 | 15 |
| 3 | 60 | 33 | 40 | **9** | 50 | **27** | 54 |
| 32 | 43 | **4** | 57 | 26 | 53 | 6 | 51 |
| 59 | 2 | 41 | 30 | 5 | **8** | 55 | 28 |
| 42 | 31 | 58 | **1** | 56 | 29 | 52 | 7 |

PFCS 380

[In mis-attributing the squares and diamonds to Ciccolini he is following earlier authorities such as Lucas and Ahrens.] The tour is not strict squares and diamonds since two moves are deleted from the top right diamond, The fourth is an open tour in which the cubes also form a path (cf ¶381 above).

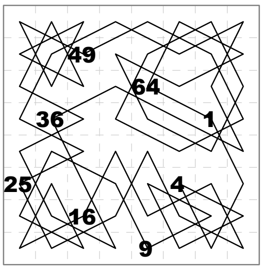
Two symmetric Dawsonian tours by H. A. Adamson and G. Fuhlendorf and a non-symmetric example by A. H. Haddy with a matching example by Dawson. The middle diagrams here use an alternative style of presentation.

Adamson 1932

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 30 | 45 | 6 | 1 | 52 | 47 | 40 | 63 |
| 7 | 4 | 29 | 46 | 41 | 64 | 53 | 48 |
| 44 | 31 | 8 | 5 | 2 | 51 | 62 | 39 |
| 9 | 28 | 3 | 42 | 35 | 60 | 49 | 54 |
| 32 | 43 | 10 | 59 | 50 | 55 | 38 | 61 |
| 27 | 16 | 19 | 34 | 11 | 36 | 23 | 56 |
| 18 | 33 | 14 | 25 | 58 | 21 | 12 | 37 |
| 15 | 26 | 17 | 20 | 13 | 24 | 57 | 22 |

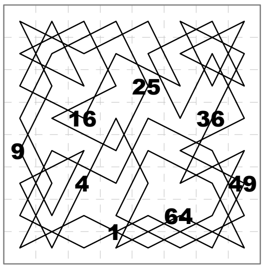
PFCS 376

Fuhlendorf 1932




PFCS 378

Haddy 1932




PFCS 565

Dawson 1932

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 33 | 30 | 37 | 50 | 21 | 52 | 19 | 26 |
| 38 | 63 | 32 | 29 | 36 | 27 | 22 | 53 |
| 31 | 34 | 49 | 62 | 51 | 20 | 25 | 18 |
| 64 | 39 | 44 | 35 | 28 | 17 | 54 | 23 |
| 43 | 48 | 1 | 10 | 61 | 24 | 7 | 16 |
| 2 | 11 | 40 | 45 | 6 | 9 | 58 | 55 |
| 47 | 42 | 13 | 4 | 57 | 60 | 15 | 8 |
| 12 | 3 | 46 | 41 | 14 | 5 | 56 | 59 |

PFCS 567

A lecture to the British Chess Problem Society 26 Feb 1932 by P. C. Taylor is mentioned in *British Chess Magazine* (vol.52 p182 1932). This contained a large number of knight's tours, either of the squares in a chain, or monogram types (see ⌘ 6). Most of them were later published in *PFCS/FCR* over the years. We begin with two symmetric circuits of the same shape and numbering but differently positioned, and an open tour with square numbers delineating D for Dawson.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 18 | 57 | 8 | 23 | 16 | 59 | 14 | 11 |
| 7 | 22 | 17 | 58 | 13 | 10 | 25 | 60 |
| 56 | 19 | 6 | 9 | 24 | 15 | 12 | 37 |
| 5 | 44 | 21 | 48 | 3 | 36 | 61 | 26 |
| 20 | 55 | 4 | 45 | 62 | 47 | 38 | 35 |
| 43 | 52 | 63 | 2 | 49 | 32 | 27 | 30 |
| 54 | 1 | 50 | 41 | 46 | 29 | 34 | 39 |
| 51 | 42 | 53 | 64 | 33 | 40 | 31 | 28 |

Taylor PFCS 379

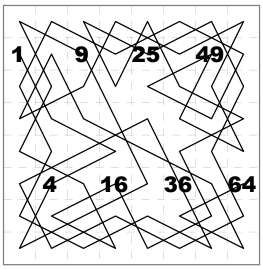
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 41 | 38 | 29 | 16 | 19 | 14 | 27 | 24 |
| 30 | 17 | 40 | 37 | 28 | 25 | 20 | 13 |
| 39 | 42 | 9 | 18 | 15 | 58 | 23 | 26 |
| 8 | 31 | 50 | 3 | 36 | 21 | 12 | 59 |
| 43 | 4 | 35 | 10 | 51 | 60 | 57 | 22 |
| 32 | 7 | 2 | 49 | 46 | 11 | 54 | 61 |
| 1 | 44 | 5 | 34 | 63 | 52 | 47 | 56 |
| 6 | 33 | 64 | 45 | 48 | 55 | 62 | 53 |

Taylor PFCS 448

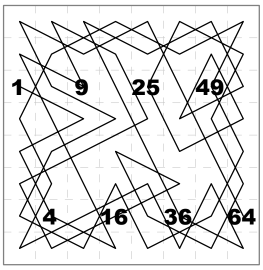
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 33 | 2 | 61 | 64 | 43 | 46 | 51 | 48 |
| 60 | 63 | 34 | 1 | 52 | 49 | 42 | 45 |
| 3 | 32 | 59 | 62 | 35 | 44 | 47 | 50 |
| 58 | 17 | 4 | 53 | 24 | 15 | 36 | 41 |
| 31 | 6 | 57 | 16 | 37 | 28 | 23 | 14 |
| 18 | 9 | 30 | 5 | 54 | 25 | 40 | 27 |
| 7 | 56 | 11 | 20 | 29 | 38 | 13 | 22 |
| 10 | 19 | 8 | 55 | 12 | 21 | 26 | 39 |

Taylor PFCS 591

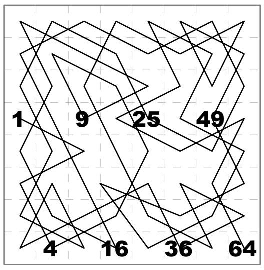
Three more from Taylor's 1932 lecture (*PFCS* 1933) showing squares in open Giraffe chains.




Taylor PFCS 717




Taylor PFCS 718




Taylor PFCS 719

Two more Dawsonian tours, one by F. Dignal the other by A. H. Kniest *PFCS* 1933.

Dignal 1933

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 62 | 35 | 56 | 51 | 40 | 37 | 18 | 15 |
| 55 | 50 | 63 | 36 | 17 | 14 | 41 | 38 |
| 64 | 61 | 34 | 57 | 52 | 39 | 16 | 18 |
| 33 | 54 | 49 | 60 | 25 | 20 | 13 | 42 |
| 48 | 1 | 26 | 53 | 58 | 9 | 24 | 7 |
| 29 | 32 | 59 | 4 | 21 | 6 | 43 | 12 |
| 2 | 47 | 30 | 27 | 10 | 45 | 8 | 23 |
| 31 | 28 | 3 | 46 | 5 | 22 | 11 | 44 |

PFCS 1052

Kniest 1933

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 39 | 14 | 3 | 58 | 5 | 12 | 33 | 56 |
| 2 | 59 | 38 | 13 | 34 | 57 | 6 | 11 |
| 15 | 40 | 35 | 4 | 17 | 8 | 55 | 32 |
| 60 | 1 | 16 | 37 | 54 | 31 | 10 | 7 |
| 41 | 36 | 61 | 30 | 9 | 18 | 53 | 28 |
| 64 | 47 | 44 | 25 | 50 | 29 | 22 | 19 |
| 45 | 42 | 49 | 62 | 21 | 24 | 27 | 52 |
| 48 | 63 | 46 | 43 | 26 | 51 | 20 | 23 |

PFCS 909

Eight examples by F. Dignal in *PFCS* 1933-34.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 26 | 63 | 24 | 7 | 28 | 61 | 30 | 19 |
| 23 | 8 | 27 | 62 | 3 | 20 | 55 | 60 |
| **64** | **25** | **4** | 21 | 6 | 29 | 18 | 31 |
| **9** | 22 | **49** | 2 | 17 | 54 | 59 | 56 |
| **36** | **1** | **16** | 5 | 48 | 57 | 32 | 53 |
| 13 | 10 | 37 | 50 | 33 | 42 | 45 | 58 |
| 38 | 35 | 12 | 15 | 40 | 47 | 52 | 43 |
| 11 | 14 | 39 | 34 | 51 | 44 | 41 | 46 |

PFCS 907

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 22 | 43 | 52 | 61 | 6 | 45 | 54 |
| 42 | 51 | 62 | **1** | 44 | 53 | 60 | 5 |
| 23 | **64** | 21 | 48 | 7 | **4** | 55 | 46 |
| 50 | 41 | 24 | 33 | 2 | 47 | 8 | 59 |
| 37 | 32 | **49** | 20 | **9** | 58 | 3 | 56 |
| 40 | 29 | 38 | **25** | 34 | 11 | 14 | 17 |
| 31 | **36** | 27 | 10 | 19 | **16** | 57 | 12 |
| 28 | 39 | 30 | 35 | 26 | 13 | 18 | 15 |

PFCS 908

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 43 | 46 | 31 | 64 | 41 | 60 | 33 | 62 |
| 30 | 51 | 42 | 45 | 32 | 63 | 40 | 59 |
| 47 | 44 | 49 | 52 | 1 | 58 | 61 | 34 |
| 50 | 29 | 4 | 57 | 36 | 15 | 2 | 39 |
| 5 | 48 | 53 | 16 | 3 | 38 | 35 | 14 |
| 28 | 9 | 6 | 37 | 56 | 25 | 22 | 19 |
| 7 | 54 | 11 | 26 | 17 | 20 | 13 | 24 |
| 10 | 27 | 8 | 55 | 12 | 23 | 18 | 21 |

PFCS 980

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 13 | 52 | 3 | 50 | 15 | 46 | 39 | 42 |
| 2 | 55 | 14 | 45 | 38 | 41 | 24 | 47 |
| 53 | 12 | 51 | **4** | **49** | **16** | 43 | 40 |
| 56 | **1** | 54 | 37 | 44 | 23 | 48 | **25** |
| 11 | 60 | 5 | **64** | **9** | **36** | 17 | 22 |
| 6 | 57 | 10 | 31 | 20 | 33 | 26 | 35 |
| 61 | 30 | 59 | 8 | 63 | 28 | 21 | 18 |
| 58 | 7 | 62 | 29 | 32 | 19 | 34 | 27 |

PFCS 1051

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12 | 15 | 18 | **25** | 46 | 35 | 32 | 27 |
| 17 | 24 | 11 | 14 | 19 | 26 | 47 | 34 |
| 10 | 13 | **16** | 45 | **36** | 33 | 28 | 31 |
| 23 | 2 | 37 | 20 | 53 | 30 | 61 | 48 |
| 38 | **9** | 22 | **1** | 44 | **49** | 54 | 29 |
| 3 | 6 | 41 | 52 | 21 | 62 | 57 | 60 |
| 8 | 39 | **4** | 43 | **64** | 59 | 50 | 55 |
| 5 | 42 | 7 | 40 | 51 | 56 | 63 | 58 |

PFCS 1132

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 22 | 51 | 62 | **1** | 6 | 57 | 28 | 55 |
| 63 | 2 | 23 | 52 | 27 | 54 | 5 | 58 |
| 50 | 21 | **64** | 61 | **4** | 7 | 56 | 29 |
| 39 | 24 | 3 | 8 | 53 | 26 | 59 | 10 |
| 20 | **49** | 40 | **25** | 60 | **9** | 30 | 33 |
| 41 | 38 | 47 | 18 | 45 | 32 | 11 | 14 |
| 48 | 19 | **36** | 43 | **16** | 13 | 34 | 31 |
| 37 | 42 | 17 | 46 | 35 | 44 | 15 | 12 |

PFCS 1133

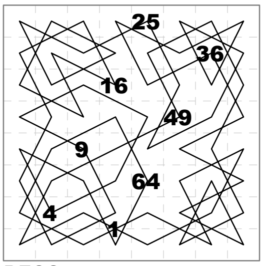
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 34 | 37 | 10 | 17 | 8 | 21 | 30 | 19 |
| 11 | **16** | 35 | 26 | 31 | 18 | 7 | 22 |
| **36** | 33 | 38 | **9** | 24 | 5 | 20 | 29 |
| 15 | 12 | **25** | 32 | 27 | 40 | 23 | 6 |
| 56 | **49** | 14 | 39 | **4** | 43 | 28 | 41 |
| 13 | 52 | 55 | **64** | 59 | 62 | 3 | 44 |
| 48 | 57 | 50 | 53 | 46 | **1** | 42 | 61 |
| 51 | 54 | 47 | 58 | 63 | 60 | 45 | 2 |

PFCS 1134

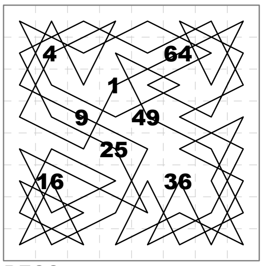
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 45 | 10 | 47 | 34 | 43 | 8 | 3 | 18 |
| 48 | 51 | 44 | **9** | 2 | 19 | 42 | 7 |
| 11 | 46 | 35 | 50 | 33 | **4** | 17 | 20 |
| 52 | **49** | 32 | **1** | **16** | 21 | 6 | 41 |
| 31 | 12 | 29 | **36** | 5 | 40 | 57 | 22 |
| 28 | 53 | **64** | 15 | 56 | **25** | 60 | 39 |
| 13 | 30 | 55 | 26 | 37 | 62 | 23 | 58 |
| 54 | 27 | 14 | 63 | 24 | 59 | 38 | 61 |

PFCS 1135

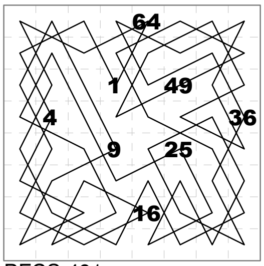
Eight examples by A. H. Haddy.



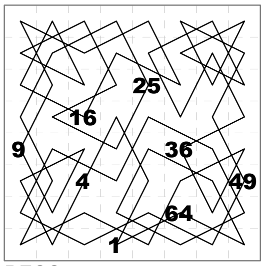

PFCS 449



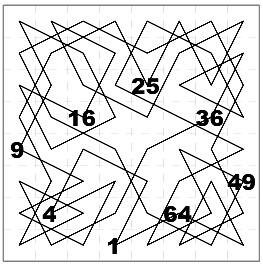

PFCS 490



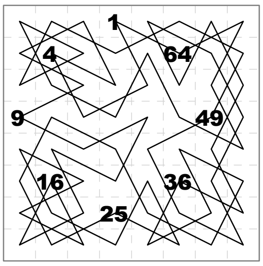

PFCS 491



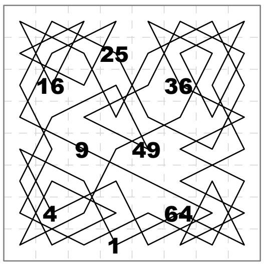

PFCS 564



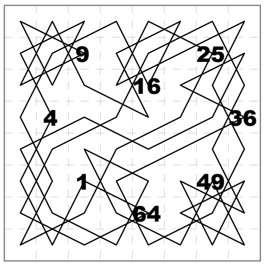

PFCS 605




PFCS 606




PFCS 658




PFCS 659

Three tours by Frans Hansson *FCR* 1938, and one by T. R. Dawson *BCM* 1942 with squares and cubes in closed chains. In the first three the cube path is centred on the board. (Others of this type by Dawson are *PFCS* ¶380 and ¶381 above.)

Hansson 1938

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 10 | 19 | 28 | **25** | 32 | 51 | 30 |
| 20 | 23 | **16** | 11 | 50 | 29 | 26 | 33 |
| **9** | 18 | 21 | 24 | $\boxed{27}$ | **36** | 31 | 52 |
| 22 | 15 | $\boxed{8}$ | **49** | 12 | 59 | 34 | 37 |
| 43 | **4** | 13 | 60 | 35 | $\boxed{\mathbf{64}}$ | 53 | 58 |
| 14 | 7 | 44 | $\boxed{\mathbf{1}}$ | 48 | 55 | 38 | 63 |
| 3 | 42 | 5 | 46 | 61 | 40 | 57 | 54 |
| 6 | 45 | 2 | 41 | 56 | 47 | 62 | 39 |

FCR 3108

Hansson 1938

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 24 | **9** | 54 | 7 | 56 | 21 | 18 |
| 10 | 53 | 6 | 23 | **16** | 19 | 28 | 57 |
| 37 | **4** | **25** | $\boxed{8}$ | 55 | 22 | 17 | 20 |
| 52 | 11 | 38 | 15 | 40 | $\boxed{27}$ | 58 | 29 |
| 3 | **36** | $\boxed{\mathbf{1}}$ | 26 | 59 | 30 | 41 | 46 |
| 12 | 51 | 14 | 39 | $\boxed{\mathbf{64}}$ | 45 | 60 | 31 |
| 35 | 2 | **49** | 44 | 33 | 62 | 47 | 42 |
| 50 | 13 | 34 | 63 | 48 | 43 | 32 | 61 |

FCR 3180

Hansson 1938

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 59 | 42 | 13 | 46 | 61 | 28 | 31 |
| 41 | 14 | 11 | 60 | 43 | 30 | 47 | 62 |
| 58 | **9** | 18 | 45 | 12 | $\boxed{27}$ | 32 | 29 |
| 15 | 40 | 5 | $\boxed{8}$ | 17 | 44 | 63 | 48 |
| **4** | 57 | **16** | 19 | $\boxed{\mathbf{64}}$ | 7 | 26 | 33 |
| 39 | 20 | $\boxed{\mathbf{1}}$ | 6 | **25** | 34 | **49** | 52 |
| 56 | 3 | 22 | 37 | 54 | 51 | 24 | 35 |
| 21 | 38 | 55 | 2 | 23 | **36** | 53 | 50 |

FCR 3251

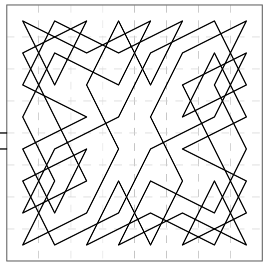
Dawson 1942

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 54 | 19 | 50 | 3 | 48 | 23 | 34 | 37 |
| 51 | 2 | 53 | 18 | 33 | **36** | 47 | 24 |
| 20 | 55 | **4** | **49** | 22 | 31 | 38 | 35 |
| $\boxed{\mathbf{1}}$ | 52 | 21 | 32 | 17 | 40 | **25** | 46 |
| 56 | 5 | $\boxed{\mathbf{64}}$ | **9** | 26 | 13 | 30 | 39 |
| 63 | $\boxed{8}$ | 61 | 58 | 41 | **16** | 45 | 12 |
| 60 | 57 | 6 | $\boxed{27}$ | 10 | 43 | 14 | 29 |
| 7 | 62 | 59 | 42 | 15 | 28 | 11 | 44 |

BCM 5674

We conclude this section with a remarkable symmetric tour with squares in knight chain by Valeriu Onitiu *FCR* 1939: “VO notes that he examined 144 dispositions of the squares, all that are possible for diametral symmetry, and this is the only case leading to a tour. Moreover every move of the tour is determined, so that the tour is UNIQUE in all the millions possible.”

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 56 | 7 | 54 | 3 | 58 | 5 | 46 | 31 |
| 53 | 60 | 57 | 6 | 47 | 32 | 43 | 18 |
| 8 | 55 | 2 | 59 | **4** | 19 | 30 | 45 |
| 61 | 52 | **9** | 48 | 33 | 44 | 17 | 42 |
| 10 | **49** | 12 | **1** | **16** | 41 | 20 | 29 |
| 13 | 62 | 51 | **36** | 27 | 34 | 23 | 40 |
| 50 | 11 | **64** | 15 | 38 | **25** | 28 | 21 |
| 63 | 14 | 37 | 26 | 35 | 22 | 39 | 24 |



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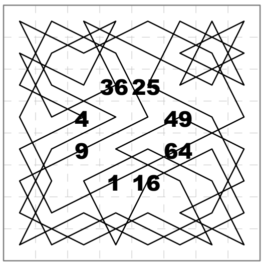
## Square Numbers in Other Formations

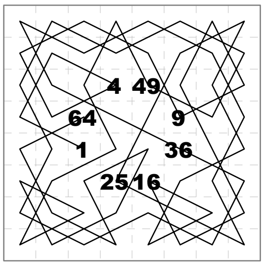
Tours with the square numbers in a ring by S. H. Hall and T. R. Dawson *PFCS* 1935. TRD wrote: “I tried this same theme as long ago as July 1932”. His example is an open tour.

Hall 1935



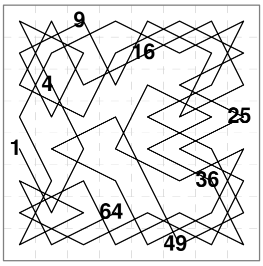

PFCS 1705

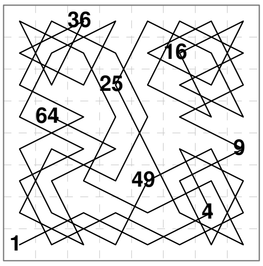
Dawson 1935

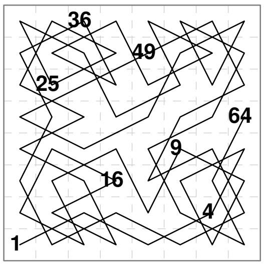



PFCS 1706

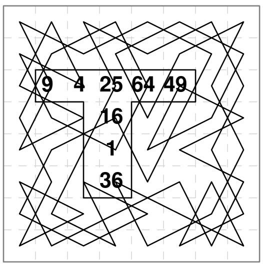
**1943 T. R. Dawson** *Comptes Rendus du Premier Congres International de Recreation Mathematique* (*CIRM* 1935, two tours) *British Chess Magazine*: (vol.63, 1943 p.23 ¶5970 sol p.70, one tour). Open tours with squares forming an 8-Queens position.

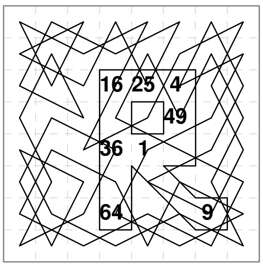


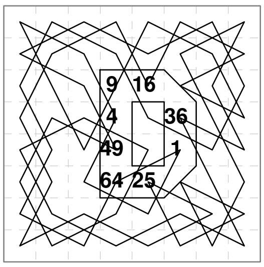




**1946: Dennison Nixon** *Fairy Chess Review* (vol.6 #4 Feb 1946 p.23 ¶6661-6663, sol. #5 Apr 1946 p.33). Put the square numbers on given cells to show letters T, R, D, and complete the tours.



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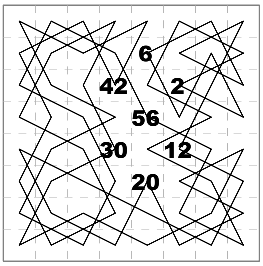
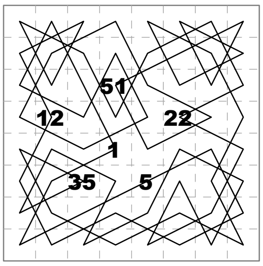
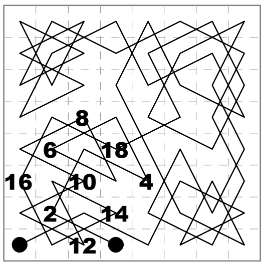
## Other Numbers in Figured Tours

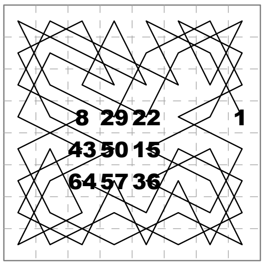
Various examples by T. R. Dawson (London) *Evening Standard* 1932. Tour with double triangle numbers n·(n+1) namely 2, 6, 12, 20, 30, 42, 56 in a figure of eight formation. *British Chess Magazine*: 1941. Tour with pentagonal numbers n·(3·n – 1)/2 in pentagon: 1 d4, 5 e3, 12 b5, 22 f5, 35 c3, 51 d6. *Vie Riennaise* 19 Nov 1932 The first nine even numbers forming a magic square (see in the Rotary section of ⌘ 3 for my version of this on a 19-cell shaped board).

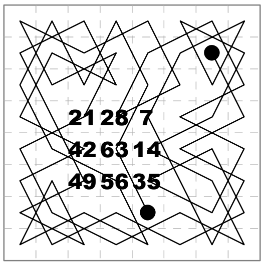


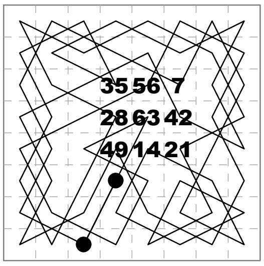



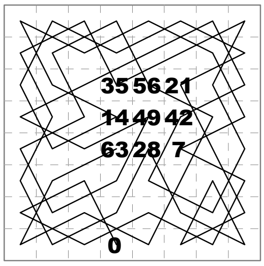




Four tours by Dawson and Hall with APs with c.d. = 7. Beginning at 1 the series has 10 terms ending at 64 and the first shows 9 of the terms in a square. The reverse tour has the same property but with 64 as the isolated term. The other tours show the 9 multiples of 7 in 3×3 array.



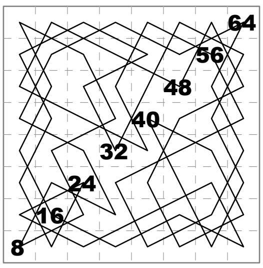



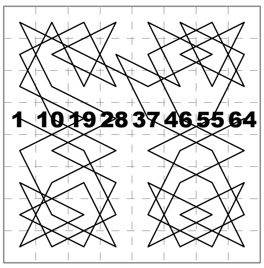




The first two tours above are by Dawson (*CIRM* 1935). The others are by S. H. Hall (*PFCS* 1936 vol.2 #16 Feb 1936 p.172 ¶2178 and *FCR* vol.3 #13 Aug 1938 p.140 ¶3249) one being closed and the other being numbered from 0 to 63.

The next two tours (Dawson *CIRM* 1935) show APs with c.d. = 8 and 9. The first has the multiples of 8 in order of magnitude along the diagonal (composed without knowledge of the earlier work by the Rajah of Mysore 1871, shown above p.417). The second tour shows the AP along the fifth rank. The 9-move segments of this tour are alternately in the lower and upper ranks of the board.

Two tours with the eighteen odd primes in a rectangle (Dawson open, Hall closed, *PFCS* vol.2 #16 Feb 1936 p.172 ¶2179).

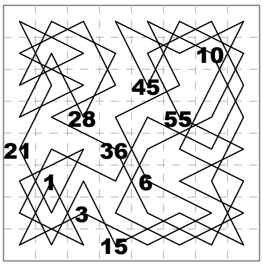


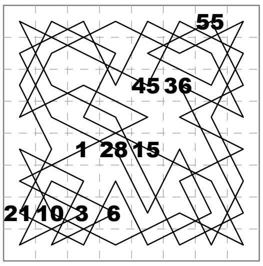



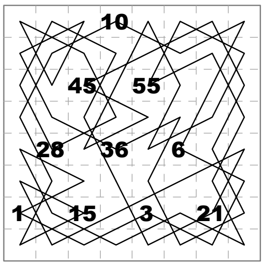

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 50 | 39 | 14 | 33 | 52 | **61** | 12 | 63 |
| 15 | 36 | 51 | 38 | **13** | 64 | **53** | 60 |
| 40 | 49 | 34 | **17** | 32 | **59** | 62 | **11** |
| 35 | 16 | **37** | 48 | **29** | 24 | **31** | 54 |
| 4 | **41** | 18 | **23** | 58 | **47** | 10 | 25 |
| **1** | 20 | **3** | 28 | **7** | 30 | 55 | 46 |
| 42 | **5** | 22 | **19** | 44 | 57 | 26 | 9 |
| 21 | 2 | **43** | 6 | 27 | 8 | 45 | 56 |

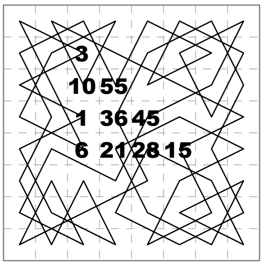
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 50 | 15 | 62 | 27 | 60 | **17** | 42 | 39 |
| 63 | 28 | 49 | 16 | **41** | 38 | **59** | 18 |
| 14 | 51 | 26 | **61** | 48 | **19** | 40 | **43** |
| 25 | 64 | **29** | 12 | **37** | 44 | **31** | 58 |
| 52 | **13** | 24 | **7** | 30 | **47** | 20 | 45 |
| **1** | 8 | **3** | 36 | **11** | 34 | 57 | 32 |
| 4 | **53** | 10 | **23** | 6 | 55 | 46 | 21 |
| 9 | 2 | **5** | 54 | 35 | 22 | 33 | 56 |

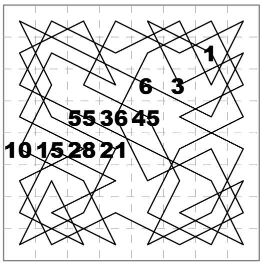
Six closed tours by T. R. Dawson in *CIRM* (1935) with the ten triangular numbers 3, 6, 10, 15, 21, 28, 36, 45, 55 in triangular formations. I also have a note of three in *British Chess Magazine*: (vol.60 1940) which are probably from this set.

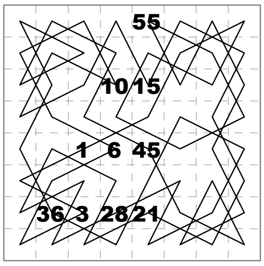


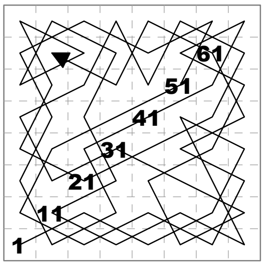



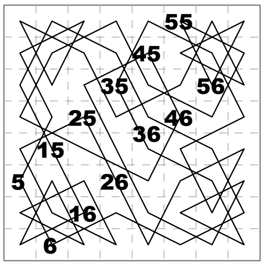



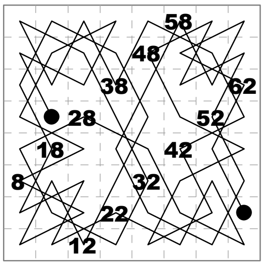



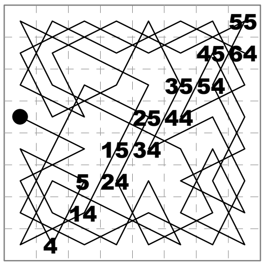




E. W. Bennett in *FCR* 1949-52 constructed four tours showing arithmetic progressions with common difference 10 arranged along diagonals.



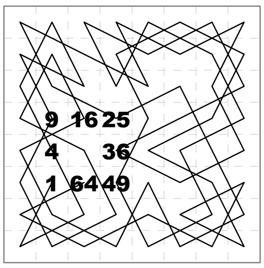

This returns to the original theme of Sachsen-Gotha who showed multiples of 10.

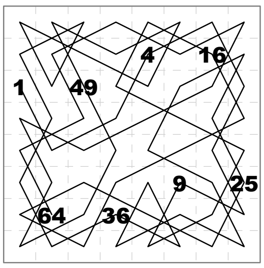
---

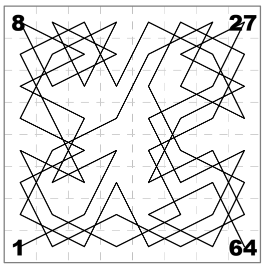
# My Own Work on Figured Tours

My interest in figured tours began 30 years after Dawson's work through the influence of Anthony Dickins. Alongside his solution of the Carpenter problem, I gave a symmetric tour with the multiples of eight along a diagonal (first diagram below). This was in *Chessics* #5 Jul 1978 p.8.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **8** | 5 | 26 | 15 | 28 | 11 | 52 | 13 |
| 25 | **16** | 9 | 6 | 51 | 14 | 29 | 34 |
| 4 | 7 | **24** | 27 | 10 | 33 | 12 | 53 |
| 17 | 22 | 63 | **32** | 55 | 50 | 35 | 30 |
| 62 | 3 | 18 | 23 | **64** | 31 | 54 | 49 |
| 21 | 44 | 1 | 42 | 59 | **56** | 39 | 36 |
| 2 | 61 | 46 | 19 | 38 | 41 | **48** | 57 |
| 45 | 20 | 43 | 60 | 47 | 58 | 37 | **40** |



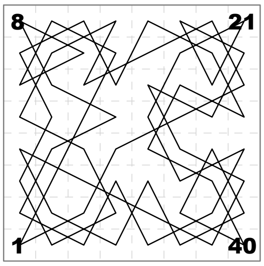




My next ventures into the subject were to look at the Carpenter problem for wazir tours (see ⌘ 2) and for the knight on the 6×6 board in *Chessics* 1985 (see ⌘ 5). In the same issue (*Chessics* #22 1985 p.61) I presented two open 8×8 knight tours, showing squares in closed Wazir and Giraffe circuits and another with cubes in the corners:

The next year in ‘Further Notes on the Knight's Tour’ (*Chessics* #25 Spring 1986 p.106-7) there were three new figured tours. (a) Closed tour with squares in a symmetric knight chain with axis of symmetry an axis of the board. (b) Closed tour of squares and diamonds with so-called ‘octagonal’ numbers $n\cdot(3\cdot n - 2) = 1, 8, 21, 40$ in the corners. (c) Closed tour with the squares in a knight path delineating a tour of the edges of a cube. This tour within a tour is one of my favourite results.

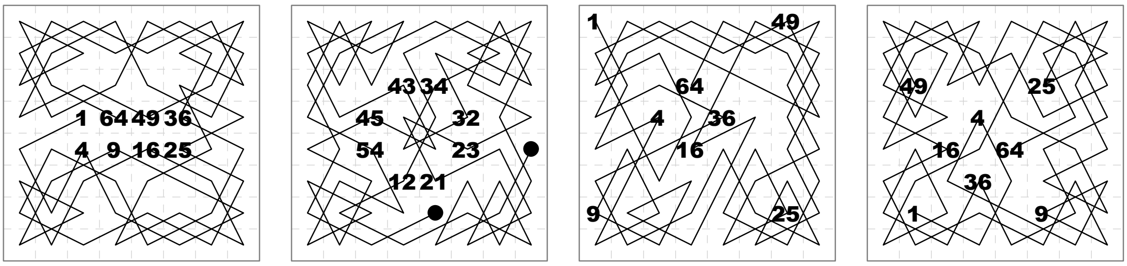
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 14 | 19 | 54 | 31 | 12 | 33 | 52 | 47 |
| 55 | 30 | 13 | 18 | 53 | 48 | 11 | 34 |
| 20 | 15 | 28 | **49** | 32 | 17 | 46 | 51 |
| 29 | 56 | 21 | **16** | **1** | 50 | 35 | 10 |
| 22 | 27 | **64** | 57 | **36** | **9** | 42 | 45 |
| 61 | 58 | **25** | **4** | 43 | 2 | 39 | 8 |
| 26 | 23 | 60 | 63 | 6 | 37 | 44 | 41 |
| 59 | 62 | 5 | 24 | 3 | 40 | 7 | 38 |




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 20 | 47 | 38 | 5 | 10 | 45 | 40 |
| 48 | 51 | **4** | 19 | 46 | 39 | 8 | 11 |
| 21 | 2 | 37 | 50 | **9** | 6 | 41 | 44 |
| 52 | **49** | 32 | **1** | 18 | 43 | 12 | 7 |
| 63 | 22 | 53 | **36** | 31 | **16** | 59 | 42 |
| 54 | 33 | **64** | 17 | 60 | 13 | 30 | 27 |
| 23 | 62 | 35 | 56 | **25** | 28 | 15 | 58 |
| 34 | 55 | 24 | 61 | 14 | 57 | 26 | 29 |

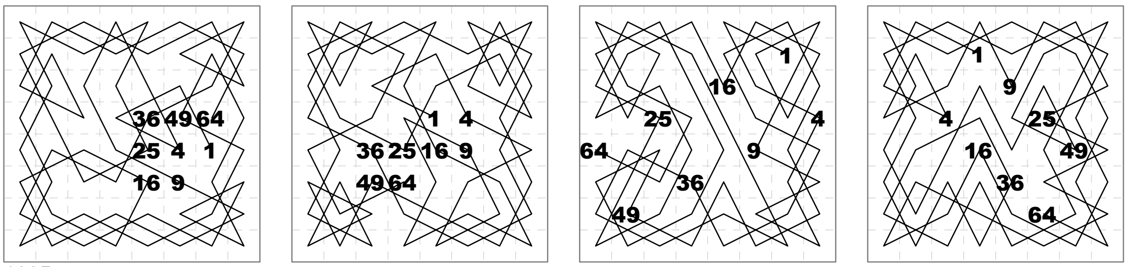
Some more new results appeared in my article on ‘Figured Tours’ in *Mathematical Spectrum* (vol.25 1992/3 #1 p.16-20): Symmetric open tours 7×7 and 9×9 with arithmetical progressions with c.d. 8 and 10 (analogous to the 5×5 with c.d. 6 shown by Euler). See ⌘ 5.

Three assorted 8×8 examples. (a) square numbers in a wazir rectangle, (b) pairs of numbers aa, bb and ab, ba on adjacent cells (the central numbers, not shown, are 11, 22, 33, 44). (c) and (d) square numbers in a closed zebra path (d) from my *Figured Tours* booklet.




For the frontispiece of *The Games and Puzzles Journal* vol.2, #13 I constructed a Figured tour on a 12×12 board (see later) with the Fibonacci numbers forming an oval (representing an Easter egg). The same was used for the cover of my booklet *Figured Tours: A Mathematical Recreation* 1997. This was a 22-page A4 booklet containing more than 225 tours, including all 100 Dawson tours.

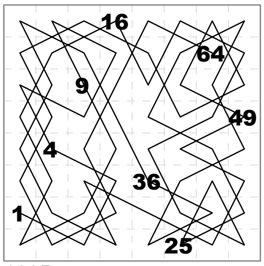
Original tours from the booklet show the square numbers in: (a) closed wazir path in open knight tour. (b) symmetric open wazir path in closed knight tour, (c) and (d) symmetric open knight paths.



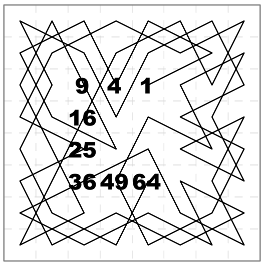

1995 1992 1992 1995

Despite these efforts to raise further interest in this subject I seem to have been the only composer active in this field recently.

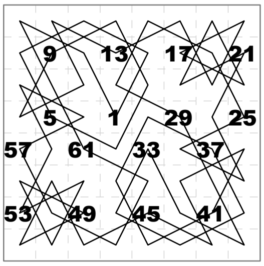
Another seven tours from the booklet: (e) square numbers in an 8-queens unguard position (after Dawson), (f) square numbers in a C-shaped wazir path, (g) a symmetric tour with an AP of CD 4 in a symmetric path of dabbaba and fers moves, (h) the octuples in a knotted octagon. (i) the powers of 2 in a wheel around 1, (j) and (k) the squares and cubes in symmetric open paths.



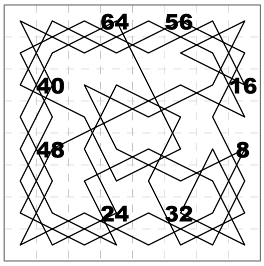
1995



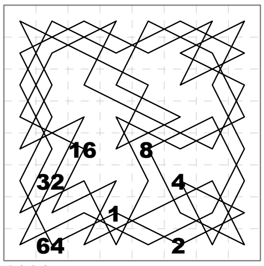
1995



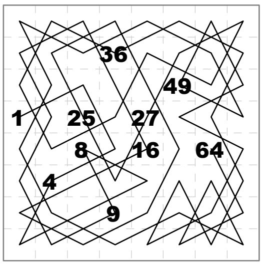
1992



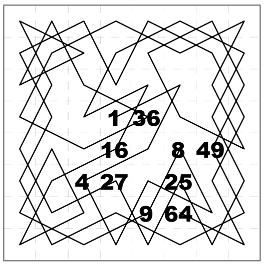
1995



1992



1992



1992

The following three tours are constructed so that a Queen guards all 17 odd primes; in other words the cells containing the odd prime numbers are all on the rank, file and diagonals through one particular cell. The first two solutions appeared on Mike Keith's fascinating "World of Words & Numbers" website, where it was stated that the problem was proposed by G. L. Honaker Jr. in 1998. (A) by Mike Keith is an open tour showing all 18 primes (including 2) guarded by the queen. (B) by someone identified as "gscgz" is a closed tour showing the same. In my version (C, 11 Oct 2000) the queen, stands on square 1, 'observes' all 17 odd primes and 'ignores' all 14 odd composite numbers.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 37 | 24 | 45 | 4 | 39 | 22 | 47 | 62 |
| 44 | 5 | 38 | 23 | 46 | 61 | 40 | 21 |
| 25 | 36 | 43 | 60 | 3 | 20 | 63 | 48 |
| 6 | 59 | 26 | 35 | 64 | 41 | 2 | 19 |
| 27 | 30 | 57 | 42 | 1 | 34 | 49 | 12 |
| 58 | 7 | 54 | 29 | 52 | 13 | 18 | 15 |
| 31 | 28 | 9 | 56 | 33 | 16 | 11 | 50 |
| 8 | 55 | 32 | 53 | 10 | 51 | 14 | 17 |

Keith 1998

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19 | 58 | 33 | 6 | 21 | 16 | 13 | 8 |
| 32 | 5 | 20 | 17 | 34 | 7 | 22 | 15 |
| 57 | 18 | 59 | 4 | 29 | 14 | 9 | 12 |
| 42 | 31 | 56 | 35 | 62 | 11 | 28 | 23 |
| 55 | 60 | 43 | 30 | 3 | 24 | 63 | 10 |
| 44 | 41 | 46 | 61 | 36 | 27 | 50 | 25 |
| 47 | 54 | 39 | 2 | 49 | 52 | 37 | 64 |
| 40 | 45 | 48 | 53 | 38 | 1 | 26 | 51 |

GSCGZ 1999

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15 | 12 | 19 | 64 | 23 | 10 | 7 | 4 |
| 20 | 63 | 14 | 11 | 18 | 5 | 24 | 9 |
| 13 | 16 | 61 | 22 | 1 | 8 | 3 | 6 |
| 62 | 21 | 50 | 17 | 60 | 37 | 30 | 25 |
| 51 | 46 | 53 | 42 | 29 | 2 | 59 | 36 |
| 54 | 43 | 40 | 49 | 38 | 33 | 26 | 31 |
| 47 | 52 | 45 | 56 | 41 | 28 | 35 | 58 |
| 44 | 55 | 48 | 39 | 34 | 57 | 32 | 27 |

Jelliss 2000

My most recent figured tour results were published in my *Jeepyjay Diary* online. First an asymmetric closed tour (26 Nov 2015) numbered 0 to 63 with the Metasquare Numbers n·(n+1) in cyclic order along a diagonal using n = 0 = 0·1 to make up a set of eight.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | 46 | 61 | 26 | 11 | 22 | 5 | **20** |
| 60 | 27 | 10 | 45 | 62 | 19 | **12** | 23 |
| 47 | 8 | 1 | 18 | 25 | **6** | 21 | 4 |
| 28 | 59 | 44 | 7 | **2** | 63 | 24 | 13 |
| 43 | 48 | 39 | **0** | 17 | 52 | 3 | 34 |
| 58 | 29 | **56** | 51 | 40 | 35 | 14 | 53 |
| 49 | **42** | 31 | 38 | 55 | 16 | 33 | 36 |
| **30** | 57 | 50 | 41 | 32 | 37 | 54 | 15 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 18 | 5 | 44 | 25 | 16 | 33 | 42 | **31** |
| 45 | **28** | 17 | 6 | 43 | 30 | **23** | 34 |
| 4 | 19 | 26 | 29 | 24 | **15** | 32 | 41 |
| 27 | 46 | 53 | 20 | **7** | 40 | 35 | 22 |
| 54 | 3 | 8 | **39** | 52 | 21 | 14 | 59 |
| 9 | 64 | **47** | 56 | 61 | 58 | 51 | 36 |
| 2 | **55** | 62 | 11 | 38 | 49 | **60** | 13 |
| **63** | 10 | 1 | 48 | 57 | 12 | 37 | 50 |

Second a symmetric tour (9 Dec 2016) with the Legendre Numbers (4^k)·(8·n + 7) in the diagonals. These are numbers that are not the sum of three squares.

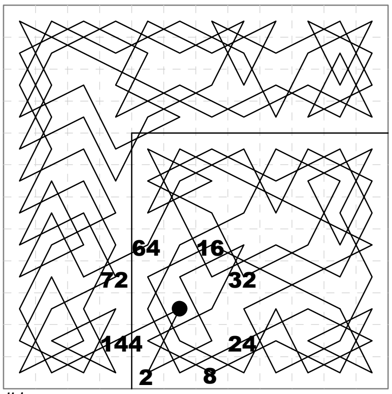
------------------------------------------------------------------------------------------------------------------------------

## Figured 12×12 Tours

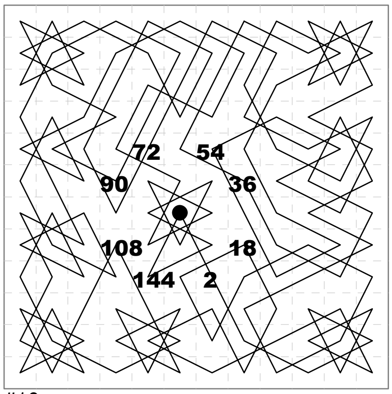
The **Rajah of Mysore, Krishnaraj Wodeyar** (1794-1868) whose work on tours was collected in the Harikrishna ms 1871, printed in Indian script in 1900, and reproduced with English commentary by S. R. Iyer in *Indian Chess* 1982, appears to have originated the idea of figured tours with numbers in a knight wheel. He composed two figured tours on the 12×12 board and five on the 8×8 board.

#1 is an 8×8 with gnomon, numbered from f3, has the numbers 2, 8, 16, 24, 32, 64, 72, 144 in a circle round the initial cell. These apart from 2 are multiples of 8, but the other 11 multiples do not form any particular pattern. The text says that the nine marked cells add to 363 but the significance of this number is not explained.

#10 numbered from f6 has the numbers 2, 18, 36, 54, 72, 90, 108, 144 in a circle round the initial cell. These apart from 2 are all multiples of 18 (the 7th multiple 126 is at b9).

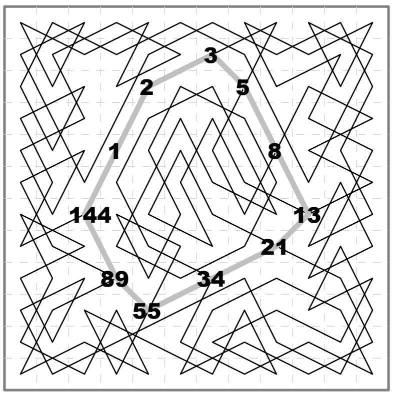


#1



#10

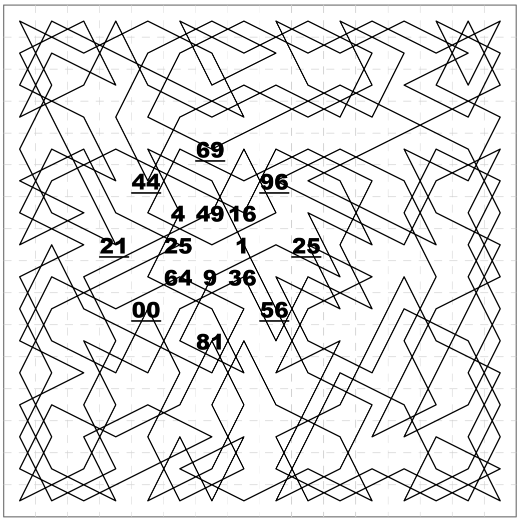
For the frontispiece of the first issue in the second series of *The Games and Puzzles Journal* (v.2 #13 May 1996 p.201) I constructed this Figured tour on a 12×12 board with the Fibonacci numbers forming an oval (representing an Easter egg). The same was used for the cover of my booklet *Figured Tours: A Mathematical Recreation* 1997. This was a 22-page A4 booklet containing more than 225 tours, including all 100 Dawson tours showing square numbers in knight chains. It was advertised in *The Games and Puzzles Journal* v.2 #15 Dec 1997 p.252.




Each number in the Finonacci sequence is the sum of the two preceding numbers, beginning here with 1 and 2. All moves are either internal or external to the oval, i.e. there are no knight's move lines crossing the convex polygon shown by the heavier lines. I believe one other oval arrangement is possible, but it may not be usable in a tour. (The links 3-5, 13-21, 55-89 connecting odd numbers cannot of course be knight moves.)

==========================================================================

# Figured 16×16 Tour




The back page of my booklet on *Figured Tours* (1979) showed this 16×16 tour with the square numbers forming a tour of a hypercube (analogous to my 8×8 tour with the squares delineating a cube). Add 100 or 200 to underlined numbers.

Figured tours can also be found in the monographs on Walker tours (⌘ 2), Shaped & Holey tours (⌘ 3), Oblong boards (⌘ 4), Odd & Oddly Even batds (⌘ 5) and among the Augmented Knight tours (⌘ 10). Here we have dealt only with knight tours on evenly even boards.

# Alternative Boards

## Bent Boards

By bending boards round so opposite edges come together pieces acquire extra move-patterns due to being able to pass across the edges. A **circular** board m×n is an array of cells consisting of m concentric circles of n cells each. Thus the ranks circle about the centre point and the files are radial. Various tours have historically been constructed on 8×8 and 4×16 circular boards, and the 4×16 board in particular has been used for a distinctive variant of chess. A **cylinder** board is a rectangular board in which the left and right edges are considered to be contiguous as if the paper were bent round to form a tube. This is also described as a **vertical cylinder**. If the top and bottom edges are instead taken to be contiguous we have a **horizontal cylinder**. There is no real difference between cylinder and circular boards as regards the way moves connect the cells. They are just differently presented to view. If both pairs of edges are identified the board becomes a **torus**.

Boards with one or both edge-connections given a twist can also be considered, resulting in boards of Moebius Strip, Klein Bottle and other designs. Spherical boards have also been played with, where edges reduce to a point, though it is not always clear how pieces move when they pass across the poles. These boards however are beyond our current remit. See for example G. Cairns *Mathematics Magazine* (vol.70 #3 Jun 2002) ‘Pillow Chess’. This is about a type of spherical chess and the bibliography has 81 references. [online]

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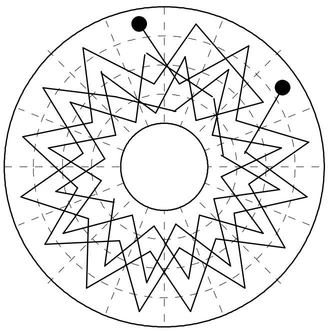
### Circular Board

The earliest example of a tour on a circular chess board is undoubtedly that given by **Richard Twiss** in volume 2 of his *Chess* (1789).



| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 32 | 51 | 46 | | 17 | 2 | 47 | 50 |
| 52 | 45 | 16 | 31 | | 48 | 33 | 18 | 3 |
| 15 | 30 | 53 | 44 | | 19 | 4 | 49 | 64 |
| 54 | 43 | 14 | 29 | | 34 | 63 | 20 | 5 |
| 13 | 28 | 55 | 42 | | 21 | 6 | 35 | 62 |
| 56 | 41 | 12 | 27 | | 36 | 61 | 22 | 7 |
| 11 | 26 | 57 | 40 | | 23 | 8 | 37 | 60 |
| 58 | 39 | 10 | 25 | | 38 | 59 | 24 | 9 |

He describes a manuscript from the Cotton collection in the King's Library in which is a drawing of a round Chess-board (in effect a 4×16 rectangle bent round so the ends meet). He writes: “The figures on this board (in the plate) show the march of the Knight in order to cover the sixty-four squares in as many moves. This I found after four or five hours trial on a slate at different times; it probably has never been done before, and will be found much more regular than any of the like marches on the square board.” [details from Ken Whyld] Twiss shows the tour in numerical form on a circular diagram. It is shown above in graphical form, which makes the regularity much clearer. Alongside we show the same tour in numerical form in a conventional representation.

Articles by John D Beasley in *Variant Chess* on ‘Circular Chess in Lincoln’ (#31 p.33-34, 48 and #32 p.55) give the Twiss tour as above and two new tours, and he notes that “on a circular board we can achieve a symmetry impossible on an ordinary rectangular board”.

The first tour is formed by a systematic forward and across repeated movement. The pattern has two ‘centres’ midway between 22 and 43 and between 59 and 6. Working outward from these centres the opposite numbers are complements adding to 65. The second tour is a sort of zigzag step-sidestep tour with 1 and 64 diametrically opposite. The centres are between 20 and 45 and 41 and 24 (crossed by the middle move 32-33).

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 34 | 45 | 24 | 3 | 56 | 35 | 14 | 25 | | 1 | 18 | 43 | 60 | 9 | 26 | 51 | 34 |
| 13 | 2 | 55 | 44 | 15 | 4 | 57 | 46 | | 52 | 35 | 10 | 27 | 42 | 59 | 8 | 17 |
| 54 | 33 | 12 | 23 | 36 | 47 | 26 | 5 | | 19 | 2 | 61 | 44 | 25 | 16 | 33 | 50 |
| 1 | **22** | **43** | 64 | 27 | 16 | 37 | 58 | | 36 | 53 | 28 | 11 | 58 | **41** | **24** | 7 |
| 42 | 53 | 32 | 11 | 48 | **59** | **6** | 17 | | 3 | **20** | **45** | 62 | 15 | 32 | 49 | 40 |
| 23 | 10 | 63 | 52 | 7 | 28 | 49 | 38 | | 54 | 37 | 12 | 29 | 48 | 57 | 6 | 23 |
| 62 | 41 | 20 | 31 | 60 | 39 | 18 | 29 | | 21 | 4 | 63 | 46 | 31 | 14 | 39 | 56 |
| 9 | 30 | 51 | 40 | 19 | 8 | 61 | 50 | | 38 | 55 | 30 | 13 | 64 | 47 | 22 | 5 |

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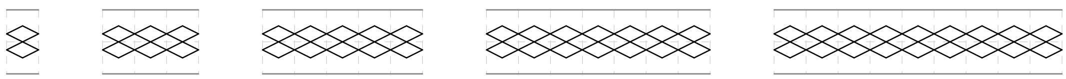
## Cylindrical Boards

Knight's tours on cylinder boards were considered by S. Vatriquant in *L'Echiquier* (Sep 1929 p.414-415), where he gave schemes for tours on boards of 2, 3 and 4 ranks.

The knight's moves on a strip 2×n form a braid of four strands. However if we join the two ends of the board to make a cylindrical strip the number of strands reduces to one or two depending on whether the board is of odd or even length, so in the odd cases we get closed tours, surprisingly including the case n = 1. To visualise the moves it is helpful to draw the squares on a larger scale.



| | |
|---|---|
| 1 | |
| 2 | |

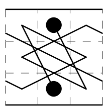
| | | |
|---|---|---|
| 1 | 3 | 5 |
| 4 | 6 | 2 |

| | | | | |
|---|---|---|---|---|
| 1 | 9 | 7 | 5 | 3 |
| 6 | 4 | 2 | 10 | 8 |

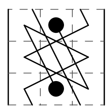
| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 5 | 9 | 13 | 3 | 7 | 11 |
| 8 | 12 | 2 | 6 | 10 | 14 | 4 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 15 | 11 | 7 | 3 | 17 | 13 | 9 | 5 |
| 10 | 6 | 2 | 16 | 12 | 8 | 4 | 18 | 14 |

The 3×3 diagonal magic square can be interpreted as an open knight tour on a cylinder board, in either orientation, as illustrated.

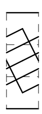


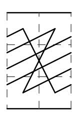
| | | |
|---|---|---|
| 2 | 9 | 6 |
| 7 | 5 | 3 |
| 4 | 1 | 8 |

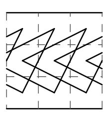


However a closed tour (not magic) is also possible, since where the edges are joined the chequering is violated and the knight can move to a cell of the same colour. The 3×3 tour by Vatriquant can also be toured, with the cells visited in the same sequence, by taking the board to be a horizontal cylinder.

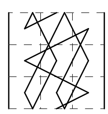
On 3×n cylinders the tours can be of any length. Vatriquant used a repeated cycling pattern of three moves that will suffice to tour a 3×n board of any length.

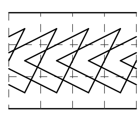




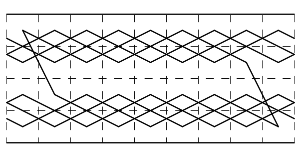


| | | |
|---|---|---|
| 1 | 4 | 7 |
| 8 | 2 | 5 |
| 3 | 6 | 9 |

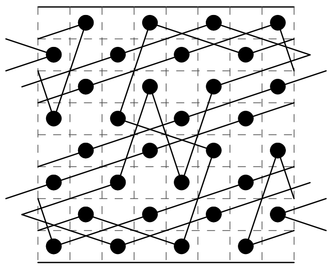


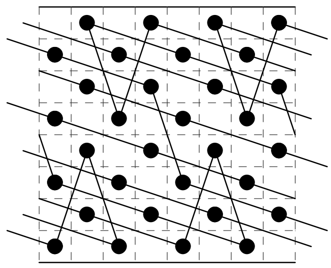


Here is a 4×9 cylinder with a closed knight tour by S. Vatriquant. Analogous tours are possible on any cylinder boards 4×n with n odd, including 4×1.



Camel tours on the 8×8 cylinder were considered by Frans Hansson (*Problemist Fairy Chess Supplement* Apr and Jun 1933, ¶714). “On the 8×8 cylinder, ah files joined, trace a closed diametrally symmetrical camel tour in 32 moves containing two 7-unit lines, one with endpoint at a1, the other with endpoint at h8.” D. H. Hersom found the second solution shown.





When numbered 1 to 32 from d6 to e3 these tours have diametrally opposite cells adding to the constant value 33 (not a constant difference). This is what we have called ‘negative symmetry’, though only the first tour has a Bergholtian crossover at the ‘centre’. Of course on a cylinder board there is a centre line but no actual centre point. The Hersom tour crosses the centre line at two ‘diametrally opposite’ points of the cylinder.

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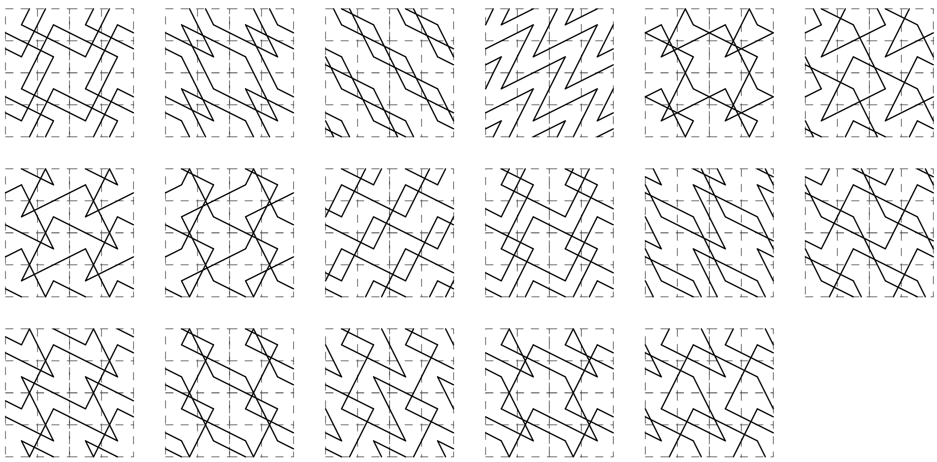
## Toral Boards

Boards with both pairs of opposite sides joined form a **torus**. On a 2×2 board one would naturally suppose that the only moves possible would be wazir and fers steps. However, Dr C. M. B. Tylor (*Chessics* #14, 1982, p12, ¶500) posed the problem: “How many geometrically distinct knight tours are possible on a 2×2 torus?”

There are eight possible knight moves from a1, four lead to a2 and four to b1; they are equivalent to wazir moves, but follow different paths. Two of the a1-a2 moves pass round the board cylindrically, while the other two cross both pairs of joined edges. Dr Tylor found that there are 17 different knight tours on the 2×2 torus, and that there are 8 different types of symmetry involved.

If the boards are expanded to form a lattice they come into 7 of the 17 possible two-dimensional space groups (i.e. wall-paper patterns). The following are diagrams of the tours in this expanded form; any four squares within the pattern form a representation of the tour.



The use of Toral Boards to generate magic tours by the step-sidestep method has been explained earlier in the Theory (p.35), with many examples among the Augmented Knights and Big Beasts.

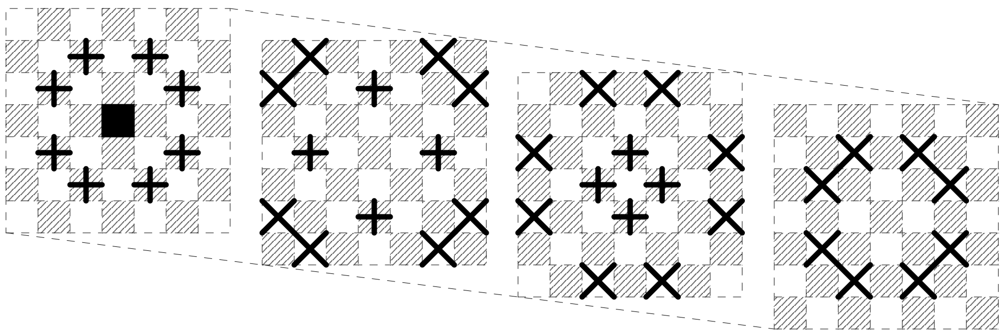
# Space Chess

## Space Pieces

By Space Chess is meant any form of chess that uses a three-dimensional board <u>and in which the rules for the pieces, other than the pawns, are the same in all lines and planes of the board</u>. This latter condition is necessary to exclude games in which the moves between the boards are by 'lift'.

Three-dimensional 'boards' are shown by means of a set of 2-dimensional boards representing successive slices through the 3-dimensional region, the labels of the boards, A, B, C, ... acting as the third coordinate. A move in three dimensions is represented by three coordinates ($r$, $s$, $t$), indicating that the move is equivalent to $r$ (wazir) steps right, $s$ steps forward and $t$ steps up. (Or respectively left, backward and down if the numbers are negative.) We express the 'pattern' of the move by the corresponding positive or zero values {r, s, t}. The length of an {r, s, t} move is $\sqrt{(r^2 + s^2 + t^2)}$.

If all three numbers are different and none is zero an {r, s, t}-leaper can move in 48 different directions! The simplest leaper of this type is the {1, 2, 3}-leaper, or √14-leaper (called a **Hippogriff** in Kogbetliantz's game). If one pair of the coordinates is taken to specify a move on one of the layers of the board, then the third coordinate represents a move up or down. Thus a {1, 2, 3}-leaper either moves vertically one layer and makes a {2, 3} zebra move or vertically 2 and makes a {1, 3} camel move or vertically 3 for a {1, 2} knight move, as illustrated by the marks × in the figure. Since this piece has an even number of odd coordinates it is confined to cells of one shade in the chequering.



Fortunately when there is a zero coordinate or two coordinates are equal the number of moves reduces considerably. Any 2-D move can be made in 3-D space in each of the three coordinate planes passing through the cell initially occupied by the moving piece. In this way each 2-D (or 1-D) piece defines a corresponding Space Piece.

Thus the **Space Rook** makes moves of type {0, 0, n} in 6 directions (up-down, left-right, and to-fro), where n can take any value except zero, the **Space Bishop** makes moves of type {0, n, n} in 12 directions and the **Space Knight** moves {0, 1, 2} in 24 directions (shown by the marks + in the figure) that is 8 in each of the three planes through its cell. Moves with all three coordinates non-zero are 'true' 3-dimensional moves, and pieces making them are 'essentially' 3-D pieces. The simplest is the **Sprite**, which is the {1, 1, 1}-leaper, or √3-leaper. The corresponding rider, the {n, n, n}-mover, or √3-rider, is called the **Unicorn** in Maack's game (but 'Fool' in Kogbetliantz's). The rook moves through the faces of the cubic cells, the bishop through the edges of the cubes, while the unicorn moves through the corners. Thus a unicorn has a choice of 8 directions of movement. A rook has access to all the cells of the board, in a series of moves, but it takes two bishops to patrol the whole space, and four unicorns.

The king in two-dimensions can be defined either as a wazir + fers or as a piece that moves to every 'neighbouring' cell (by which we mean any cell that has a boundary point in common with the initial cell). <u>In three dimensions these two definitions are not equivalent</u>. From the definition of space piece given above it follows that a piece that has the usual king moves in any plane is properly called a **Space King**. A piece that moves to every neighbouring cell in space is a wazir + fers + sprite which I call a **Cubi-King**. The royal pieces in the Maack and Kogbetliantz games are of this type.

The cubi-king moves to all 26 outer cells of the 3×3×3 cube around it. The choice of the cubi-king rather than the space king as the 3-D monarch is supported by the fact that one space king could stalemate another on a cubical board (e.g. on Aa1 and Bb2) which may be considered an undesirable property for the royal piece, or at least not analogous to the 2-D case. The **Space Queen** is of course rook + bishop. Its more powerful cousin the **Cubi-Queen** runs in the directions that the Cubi-King walks, i.e. rook + bishop + unicorn.

The 3D moves with coordinates non-zero and less than 3 are {1,1,1}, {1,1,2}, {1,2,2}, {2,2,2}. The first and last are unicorn moves. The {1,1,2} move is of length √6, and I call the corresponding rider a **Sexton**, it is confned to cells of one colour and can reach all the cells of that colour. The {1,2,2} move is of length √9 = 3. Thus the **Threeleaper** becomes of special interest in space! Besides its 1D move {0,0,3} it has this 3D move. Unlike the threeleaper in two dimensions, which is confined to 1 cell in every 9, in three dimensions it can get to every cell of the board.

In space we also encounter two other double-pattern fixed-distance leapers before we reach the familiar Fiveleaper. These are the √17 leaper or **Space Giraffe** which moves {0,1,4} and {2,2,3} and can reach any cell, and the 3√2 leaper or **Space Tripper** which moves {0,3,3} and {1,1,4} and is confined to one colour.

[The above notes are from my article ‘Space Chess’ in *Variant Chess* (#23 Spring 1997, p.52-3)].

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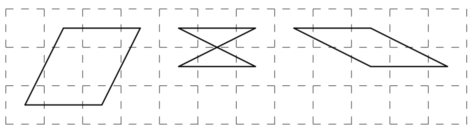
## Space Knight Tours

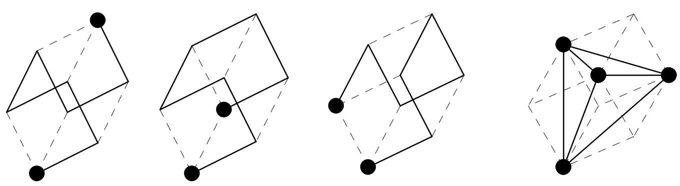
We now look at knight tours in space. It is easily possible to construct an open knight tour on any board p×q×r where the p×q board has a closed tour. Start the closed tour on the first board at any cell, then jump from the end cell up to the next level and take the cell reached as the start point of the same closed tour, and so on, and on. In the case of a p×q board that will not admit a closed tour, if the same tour is to be used on every level then either the end cells must be two steps apart {0,2} or can be made two steps apart by a rotation or reflection (see the 2×3×4 and Methuselah tours below).

To join two closed tours on successive layers together to make a closed tour we need to delete one move in each and cross-connect. There are three formations that allow this. All these knight-move pairs are of quite common occurrence in tours, especially the crossing pair, so almost any tour can be used in this way; and any two different tours can usually be linked up in these formations.



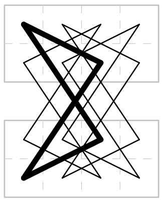
Once a two layer closed tour has been formed in this way other layers can be added similarly, making a pile of boards to any size r. It may be noted that the centres of all three tetragons are at the mid-point of a cell edge. So this method of linking will only preserve symmetry in tours on odd by even boards. For examples see the sections below on 2×3×10 and 2×5×6 boards.

**2×2×2 = 8 cells** T. R. Dawson (*Chess Amateur* Jun 1926) considered moves of rook and bishop (i.e. wazir and fers) in a 2×2×2 board, noting that the net of rook moves form the edges of a cube and that there are 18 oriented tours, 3 geometrically distinct, while the net of bishop moves on one colour form the edges of a tetrahedron and that there are 6 oriented tours, but only 1 geometrically distinct. He also noted that the bishop can describe a triangle in three dimensions.



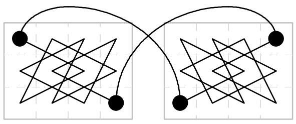
## KNIGHT'S TOUR NOTES

**2×2×4 = 16 cells.** The knight's moves on this board form a pseudotour consisting of four circuits of four moves, so no tour is possible. In fact on any board 2×2×n with n > 2 the moves from Aa1 and Ba2 go to Ac2 and Bc1, forming a circuit, so no tour is possible on any of these boards.



**2×3×3 = 18 cells**. No knight move is possible from the two cells of the centre column, so no tour is possible. However, tours of the remaining 16 cells can easily be formed.

**2×3×4 = 24 cells.** The corner-to corner 3×4 tour found by Euler (1759) can be combined with a reflected copy of itself on a second board to give a closed 2×3×4 tour, as illustrated below.



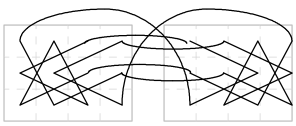
A

| | | | |
|---|---|---|---|
| 1 | 4 | 7 | 10 |
| 8 | 11 | 2 | 5 |
| 3 | 6 | 9 | 12 |

B

| | | | |
|---|---|---|---|
| 22 | 19 | 16 | 13 |
| 17 | 14 | 23 | 20 |
| 24 | 21 | 18 | 15 |

This seems obvious. However an article in the *Mathematical Gazette*, 1944 by N. M. Gibbins stated that the minimal board for a tour was 3×3×4, a surprising error, (perhaps a misprint?). It seems that this error was first pointed out by Awani Kumar in a Note on ‘Magic Knight's Tours for chess in three dimensions’ in the same journal (Mar 2008). However the four tours given there do not include the above simple case. Fig 1 is an irregular open tour while Figs 3-4 are partially magic, adding to 50 in the four-cell lines. I show Fig. 3 for comparison.



A

| | | | |
|---|---|---|---|
| 1 | 18 | 21 | 10 |
| 20 | 11 | 16 | 3 |
| 17 | 2 | 19 | 12 |

B

| | | | |
|---|---|---|---|
| 22 | 9 | 6 | 13 |
| 15 | 4 | 23 | 8 |
| 24 | 7 | 14 | 5 |

**3×3×3 = 27 cells.** There is no knight move from the centre cell so no knight tour is possible. However tours of the remaining 26 cells are easily found. Wazir tours might be of interest.

**2×4×4 = 32 cells**. In his 2008 *Maths Gazette* article Awani Kumar gave 16 figures showing tours on this board that have partially magic properties. The ranks and files of the 4×4 boards all add to 66, and in some cases half of the two-cell columns add to 33, as in this example (his Fig 6):

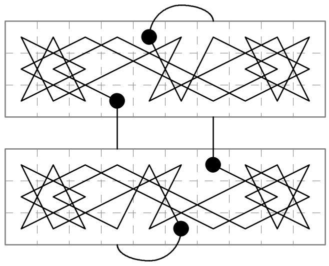
A

| | | | |
|---|---|---|---|
| 1 | 14 | 23 | 28 |
| 24 | 27 | 2 | 13 |
| 15 | 22 | 25 | 4 |
| 26 | 3 | 16 | 21 |

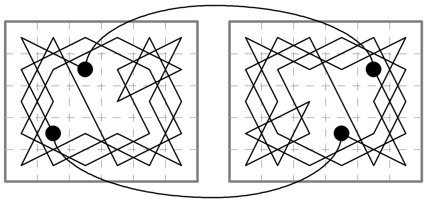
B

| | | | |
|---|---|---|---|
| 10 | 19 | 32 | 5 |
| 31 | 6 | 9 | 20 |
| 8 | 11 | 18 | 29 |
| 17 | 30 | 7 | 12 |

**2×3×10 = 60 cells**. To make a closed tour in the two-layer case (by joining closed tours on the two layers) it is necessary that the end cell on the upper board be two steps from the start cell on the bottom board. Here is a simple example (Jelliss September 2014) using the 3×10 board which (along with the 5×6) is the smallest rectangle with a closed tour. This tour is symmetric. (It uses a tour by Bergholt 1918.)



**2×5×6 = 60 cells**. The same principle as illustrated for the 3×10 board above can be used on the 5×6 board, but since no tours on that board have a symmetrically arranged pair, the symmetry in this case is achieved by rotating the tour used (which is due to Euler 1759).



**4×4×4 = 64 cells.** A. T. Vandermonde (1771) was the first to construct a three-dimensional knight's tour, using a board 4×4×4. As for his 8×8 tour in the same article, he constructed this closed tour by joining up four circuits. (Some sources mistakenly state that Vandermonde constructed an 8×8×8 tour.) He presented his tour in the form of a series of spatial coordinates (x,y,z) expressed in a fraction style of notation, but translated into numbered cells it becomes:

| A | | | | B | | | | C | | | | D | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 63 | 12 | 25 | 34 | 26 | 41 | 54 | 13 | 5 | 62 | 33 | 18 | 42 | 17 | 4 | 55 |
| 28 | 35 | 64 | 15 | 59 | 16 | 29 | 46 | 38 | 21 | 8 | 51 | 7 | 56 | 43 | 20 |
| 39 | 24 | 11 | 52 | 6 | 53 | 40 | 19 | 27 | 48 | 61 | 14 | 60 | 3 | 32 | 47 |
| 10 | 49 | 36 | 23 | 37 | 30 | 1 | 50 | 58 | 9 | 22 | 45 | 31 | 44 | 57 | 2 |

Two 4×4×4 closed tours are given in Prof Hermann Schubert's *Mathematische Mussestunden* 1st ed 1898 p.220 (also in 3rd ed 1909 p.34-35; but only one is retained in the 4th ed 1924, edited by Prof F. Fitting p.244. The first is also quoted in Ahrens 1901 p.198). The first is given as the solution of a cryptotour puzzle.

| A | | | | B | | | | C | | | | D | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 7 | 22 | 17 | 27 | 62 | 15 | 2 | 42 | 37 | 56 | 51 | 57 | 30 | 47 | 36 |
| 21 | 18 | 9 | 6 | 14 | 1 | 26 | 63 | 55 | 52 | 43 | 40 | 48 | 33 | 58 | 31 |
| 8 | 11 | 20 | 23 | 61 | 28 | 3 | 16 | 38 | 41 | 50 | 53 | 29 | 60 | 35 | 46 |
| 19 | 24 | 5 | 12 | 4 | 13 | 64 | 25 | 49 | 54 | 39 | 44 | 34 | 45 | 32 | 59 |

| A | | | | B | | | | C | | | | D | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 24 | 39 | 58 | 9 | 35 | 44 | 3 | 14 | 22 | 29 | 54 | 59 | 49 | 34 | 15 | 2 |
| 19 | 28 | 51 | 62 | 50 | 31 | 18 | 1 | 25 | 40 | 57 | 10 | 36 | 43 | 6 | 13 |
| 38 | 45 | 4 | 11 | 23 | 42 | 55 | 8 | 48 | 33 | 16 | 63 | 21 | 30 | 53 | 60 |
| 47 | 32 | 17 | 64 | 20 | 27 | 52 | 61 | 37 | 46 | 5 | 12 | 26 | 41 | 56 | 7 |

Here is another closed tour by T. W. Marlow (*Chessics* 1987 p.162).

| A | | | | B | | | | C | | | | D | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 8 | 3 | 10 | 24 | 43 | 26 | 19 | 63 | 32 | 55 | 52 | 44 | 61 | 38 | 31 |
| 16 | 11 | 6 | 13 | 27 | 20 | 15 | 30 | 56 | 53 | 48 | 33 | 37 | 50 | 45 | 60 |
| 7 | 2 | 9 | 4 | 64 | 25 | 18 | 41 | 23 | 42 | 57 | 54 | 62 | 39 | 36 | 51 |
| 27 | 5 | 12 | 29 | 17 | 28 | 21 | 14 | 58 | 49 | 34 | 47 | 35 | 46 | 59 | 40 |

Günter Stertenbrink (2003) appears to have been the first to realise that a 4×4×4 knight's tour could be magic! That this is possible on a board with such a short side is surprising to anyone who has studied 8×8 magic tours. The space diagonals (but not the plane diagonals) also add to the magic constant, 130. I'm not sure where it was first published. Here is the tour as shown by John Beasley in *Variant Chess* 'A remarkable magic knight's tour in three dimensions' (#55 Sep 2007, p.8) and 'Latin Cube Magic Tours' (#56 February 2008, p.32).

A

| | | | |
|---|---|---|---|
| 53 | 16 | 43 | 18 |
| 42 | 19 | 56 | 13 |
| 15 | 54 | 17 | 44 |
| 20 | 41 | 14 | 55 |

B

| | | | |
|---|---|---|---|
| 2 | 59 | 32 | 37 |
| 29 | 40 | 3 | 58 |
| 60 | 1 | 38 | 31 |
| 39 | 30 | 57 | 4 |

C

| | | | |
|---|---|---|---|
| 47 | 22 | 49 | 12 |
| 52 | 9 | 46 | 23 |
| 21 | 48 | 11 | 50 |
| 10 | 51 | 24 | 45 |

D

| | | | |
|---|---|---|---|
| 28 | 33 | 6 | 63 |
| 7 | 62 | 25 | 36 |
| 34 | 27 | 64 | 5 |
| 61 | 8 | 35 | 26 |

I understand over 70 such tours were constructed between 2006 and 2009 by Awani Kumar, Günter Stertenbrink and Francis Gaspalou. Here is an open tour by Awani Kumar published in 'Magic Knight's tours in higher dimensions' on the Arxiv website: See also 'Magic knight's tours for chess in three dimensions' *Mathematical Gazette* Vol.92, #523; p.111-114 (2008) and 'Construction of Magic Knight's Towers' *Mathematical Spectrum* Vol. 42, #1 p.20-25 (2009).

A

| | | | |
|---|---|---|---|
| 19 | 46 | 63 | 2 |
| 48 | 17 | 4 | 61 |
| 29 | 52 | 33 | 16 |
| 34 | 15 | 30 | 51 |

B

| | | | |
|---|---|---|---|
| 58 | 7 | 22 | 43 |
| 5 | 60 | 41 | 24 |
| 40 | 9 | 28 | 53 |
| 27 | 54 | 39 | 10 |

C

| | | | |
|---|---|---|---|
| 47 | 18 | 3 | 62 |
| 20 | 45 | 64 | 1 |
| 49 | 32 | 13 | 36 |
| 14 | 35 | 50 | 31 |

D

| | | | |
|---|---|---|---|
| 6 | 59 | 42 | 23 |
| 57 | 8 | 21 | 44 |
| 12 | 37 | 56 | 25 |
| 55 | 26 | 11 | 38 |

**3×4×6 = 72 cells**. A tour on this board is given in Schubert's *Mathematische Mussestunden* (1st ed 1898 p.222, 3rd ed 1909 p.36, 4th ed 1924 p.245). It is shown there as four boards 3×6, but for reasons of space and consistency I show it here as three boards 4×6.

A

| | | | | | |
|---|---|---|---|---|---|
| 61 | 14 | 59 | 10 | 51 | 6 |
| 58 | 11 | 62 | 7 | 54 | 9 |
| 3 | 60 | 13 | 56 | 5 | 52 |
| 12 | 57 | 4 | 53 | 8 | 55 |

B

| | | | | | |
|---|---|---|---|---|---|
| 2 | 25 | 72 | 15 | 66 | 19 |
| 71 | 22 | 69 | 18 | 63 | 16 |
| 26 | 1 | 24 | 65 | 20 | 67 |
| 23 | 70 | 21 | 68 | 17 | 64 |

C

| | | | | | |
|---|---|---|---|---|---|
| 27 | 44 | 31 | 48 | 35 | 50 |
| 30 | 41 | 28 | 45 | 38 | 47 |
| 43 | 32 | 39 | 36 | 49 | 34 |
| 40 | 29 | 42 | 33 | 46 | 37 |

**5×5×5 = 125 cells.** The 5×5×5 cube is the board used for the Maack version of space chess. P. C. Taylor (*Fairy Chess Review*, 1939, vol.4, no.3 and 1940, vol.4, no.4, problem 3928) constructed a figured knight's tour in a 5×5×5 cube under the conditions that (a) the cube numbers (1, 8, 27, 64, 125) are from Cc1 to Cc5, (b) the square numbers are all on the C-plane in a knight's chain and (c) the maximum number of cells are filled before the E-plane is entered. The chain of squares evidently represents a D for Dawson. Add 100 to underlined numbers:

A

| | | | | |
|---|---|---|---|---|
| 17 | 52 | 57 | 46 | 11 |
| 58 | 45 | 10 | 51 | 56 |
| 53 | 26 | 41 | 12 | 3 |
| 22 | 59 | 44 | 55 | 6 |
| 43 | 54 | 23 | 30 | 13 |

B

| | | | | |
|---|---|---|---|---|
| 34 | 47 | 40 | 15 | 32 |
| 39 | 20 | 33 | 48 | 5 |
| 28 | 35 | 94 | 31 | 14 |
| 19 | 38 | 21 | 76 | 95 |
| 24 | 29 | 42 | 37 | 2 |

C

| | | | | |
|---|---|---|---|---|
| 81 | 16 | $\underline{25}$ | 90 | 63 |
| 18 | 61 | 64 | 9 | 50 |
| 25 | $\underline{00}$ | 27 | 62 | 93 |
| 60 | 65 | 8 | 49 | 4 |
| $\underline{21}$ | 36 | 1 | 66 | 7 |

D

| | | | | |
|---|---|---|---|---|
| 84 | 89 | 80 | 71 | 92 |
| 79 | 72 | 83 | 88 | 69 |
| 82 | 85 | 70 | 91 | $\underline{12}$ |
| 73 | 78 | 87 | 68 | 75 |
| 86 | 67 | 74 | 77 | 96 |

E

| | | | | |
|---|---|---|---|---|
| $\underline{07}$ | $\underline{18}$ | $\underline{03}$ | $\underline{24}$ | $\underline{13}$ |
| $\underline{02}$ | $\underline{23}$ | $\underline{06}$ | $\underline{17}$ | $\underline{04}$ |
| $\underline{19}$ | $\underline{08}$ | 99 | $\underline{14}$ | 97 |
| $\underline{22}$ | $\underline{01}$ | $\underline{10}$ | $\underline{05}$ | $\underline{16}$ |
| $\underline{09}$ | $\underline{20}$ | $\underline{15}$ | 98 | $\underline{11}$ |

**2×8×8 = 128 cells**. There are numerous chess variants that use two boards, but where transfer between the boards is permitted this is usually to the corresponding cell on the other board. (The most popular is probably Alice Chess by V. R. Parton where the second board is the land "Through the Looking Glass" rather than on another level. One that is specifically described as a 3D game is Flying Chess by David Eltis 1984 but the vertical moves are straight up and down. See the *Encyclopedia of Chess Variants* by David Pritchard and John Beasley for more details.)

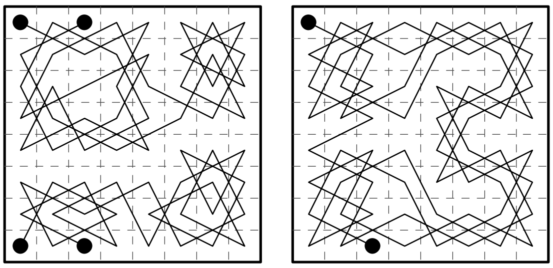
Tours formed on the same principle as the 2×3×10 and 2×5×6 tours above can only produce asymmeric tours on the 2×8×8 board. It is a matter of finding a tour with a pair of knight moves separated by a two step shift, or that can be placed in this relation by rotation or reflection (assuming the same tour is used on both levels).

By repeated linking of tours, closed tours of boards of any size n×8×8 can be constructed. These can moreover be spatially symmetric in the cases when n is odd; for example a symmetric middle tour with asymmetric tours above and below, oppositely oriented, and linked to the middle tour in corresponding positions.

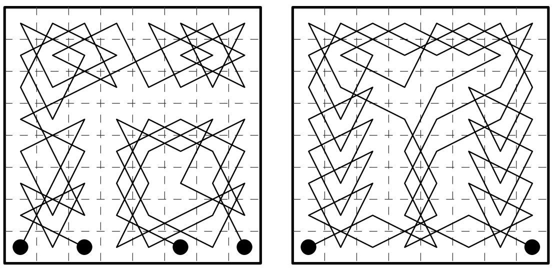
**Cuboid Chess Board = 312 cells**. Cuboid Chess is a 3D game that I described in the *Variant Chess* article mentioned above. It uses a board 6×6×6 with 4×4 extensions added to the middle of each of the six faces. However I have not attempted any tour of it so far.

**8×8×8 = 512 cells.** I understand there is a tour 8×8×8 by J. Stewart in the *Journal of Recreational Mathematics* (vol.4, no.1, 1971) consisting of 8 planar tours stacked so the ends can be connected by knight moves in vertical planes, but I have not seen this article. In the note above on the 2×8×8 case I indicate how tours can be used to make a closed tour of any size 8×8×n.

Here is another scheme for a closed tour on an 8×8×n board of any size. The tour on the right is Beverley's magic tour and any other magic tour of the 27 type can be substituted. This tour is for the upper and lower levels. The other diagram serves for all the intermediate floors. It shows the 8×8 board covered by two separate tours of 27 and 37 cells. After traversing the ground level tour the knight moves up to the next floor and traverses the 27-cell tour, then up again to tour another 27-cell tour, and so on until it reaches the roof. Here it traces out the Beverley tour in reverse. It can then descend the back staircase, touring the 37-cell part-tours on each floor on the way down, reaching the start again for a 3D closed tour. Any of the intermediate floors can also be reflected front to back.

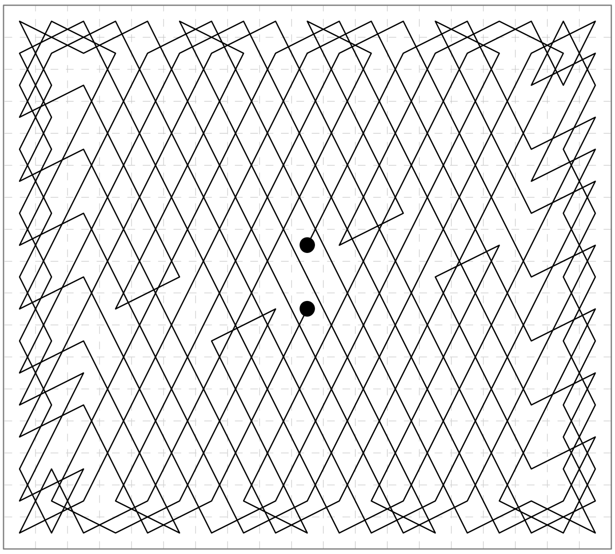


Here is an alternative scheme of this type with the staircases on right and left and using any corner-to-corner tour at top and bottom. The corner-to-corner tour shown is from Käfer (1842). The two odd-numbered part-tours on the intermediate floors are of 25 and 39 cells this time, and can be reflected left to right.

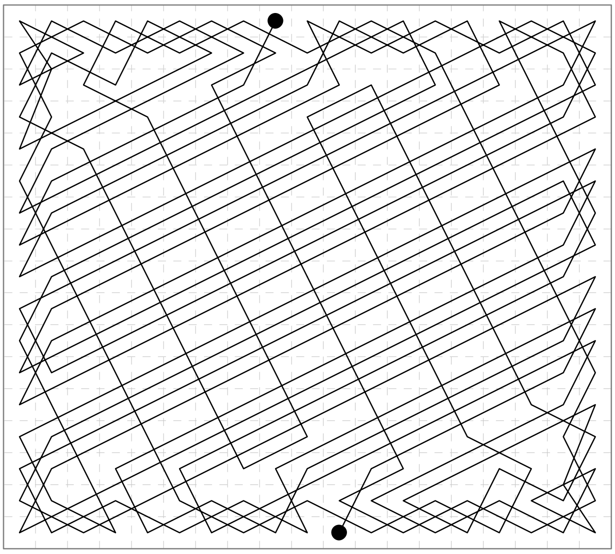


The sub-tours have to be odd of course snce their ends have to be on the same colour cells when chequered. These schemes were devised by me while writing this article.

**3×17×19 = 969 cells.** The diagrams below (Jelliss 1986) show 17×19 = 323-cell tours based on nightrider patterns 2 and 5 in translational symmetry (see ⌘ 1). Three copies of the first tour will join by knight-move ramps between the floors, to form a 3×17×19 = 969-cell ‘Methuselah Tour’, representing the alleged age to which the patriarch lived.

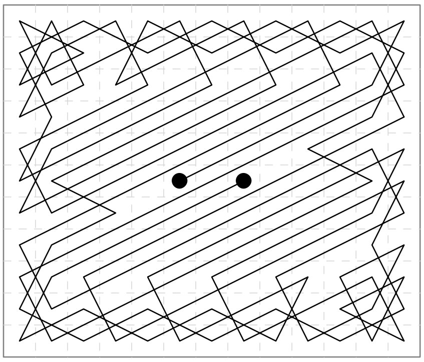
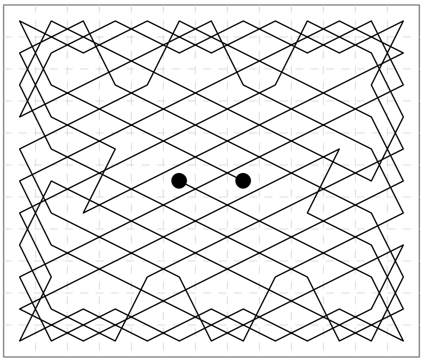
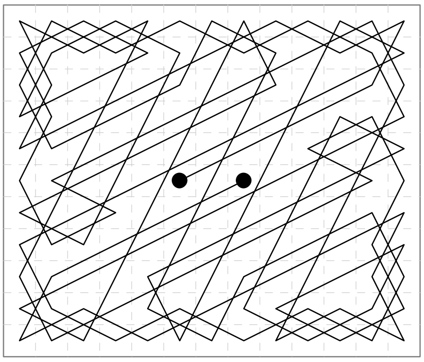
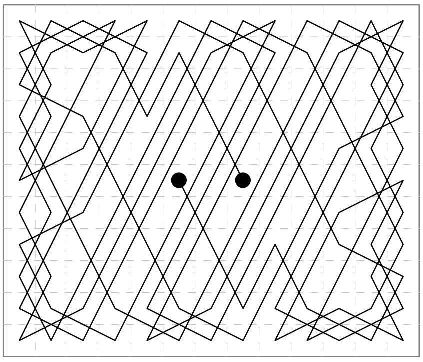
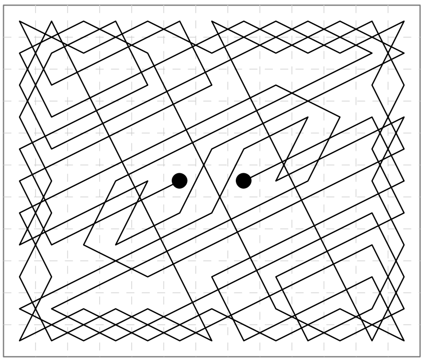
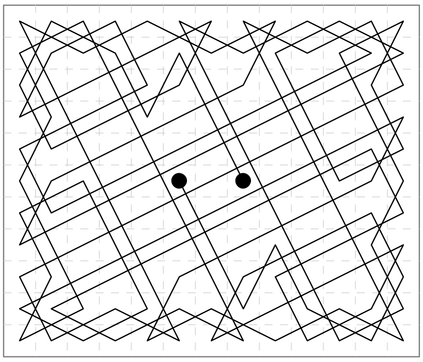
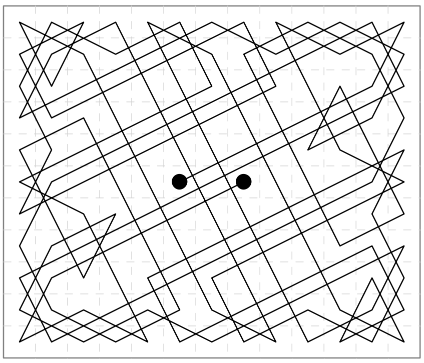
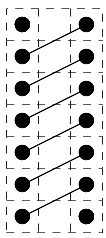


Three copies of the second tour, with the middle copy reflected, will also form a Methuselah tour, or alternatively two will form a 2×17×19 = 646 cell closed tour.



**7×11×13 = 1001 cells.** Board 11×13. These 143-cell tours (G. P. Jelliss, New Year's Greeting card 1986-7 and *Chessics* 1987 p.162) join together in three dimensions to form a 1000-move knight's tour of a 7×11×13 = 1001-cell board, with the title ‘The Sheherazade Tour’. This was dedicated to Anthony Dickins, who kindly commented: “It was a brilliant idea to make it a 1001 tour and thus simultaneously give it the right to a most apt and fitting title. For Sheherazade used to play chess with her Shah as well as tell him stories - and the Shah (Caliph) was the one in Baghdad at the time of the first great explosion of Fairy Chess under the Moslems.”

Each "floor" 11×13 is symmetrically toured from f6 to h6 before the knight moves up the central "staircase" to the next level. These 7 tours are based on 7 of the 8 possible area-coverings by nightrider lines (see ⌘ 1) and can be taken in any order. The 8th pattern (type 34) cannot be placed symmetrically on an odd-sided board.

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## Higher Dimensions

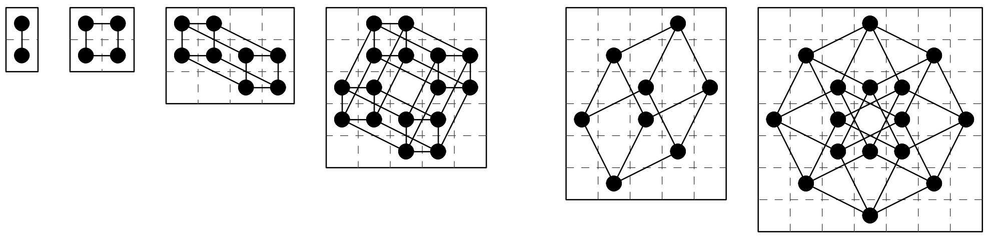
J. A. Lewis (*Fairy Chess Review* vol.5 1944 Feb p.74, Apr p.85-86 ¶5818). In a 4-dimensonal hypermodel 3×3×3×2 put black knights at IBb2 and IIBb2 (shown as holes below). A white knight then plays a closed tour of the remaining 52 hypercells.

| | A | | | B | | | C | | |
|---|---|---|---|---|---|---|---|---|---|
| | 17 | 12 | 15 | 42 | 37 | 40 | 49 | 46 | 51 |
| | 14 | 1 | 18 | 39 | | 43 | 52 | 19 | 48 |
| I | 11 | 16 | 13 | 44 | 41 | 38 | 47 | 50 | 45 |

| | A | | | B | | | C | | |
|---|---|---|---|---|---|---|---|---|---|
| | 6 | 9 | 4 | 35 | 32 | 29 | 28 | 23 | 26 |
| | 3 | 20 | 7 | 30 | | 34 | 25 | 2 | 21 |
| II | 8 | 5 | 10 | 33 | 36 | 31 | 22 | 27 | 24 |

Awani Kumar in the Arxiv article referred to above constructed a magic knight's tour in a four-dimensional 4×4×4×4 board, and also on a 5-dimensional board 4×4×4×4×4. See Figures 9 and 10 in that article for diagrams.

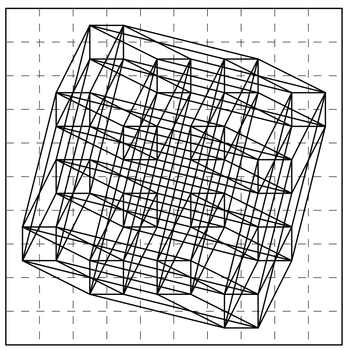
**Representation of Hypercubes** If we represent each of the 2^k binary numbers of k digits by a point and join two points by a line when their numbers differ only by one digit in one position, then the resulting diagram is a net called a **k-cube**. Equilateral diagrams of the 1, 2, 3 and 4-dimensional cases are shown below, on left and right, in which the links are shown as wazir or knight moves.



To draw similar equilateral diagrams for the 5D and 6D cases requires use of a root-65 leaper, that is a {1,8}+{4,7} leaper, but the diagrams take up a lot of room (31×24 board and 39×25 board respectively) and do not show the structure very clearly. A 5-leaper or root-50 leaper does not suffice since there is overlap of the orthogonal {0,5} leaps or of the diagonal {5,5} leaps in places.

Clearer diagrams can be achieved by allowing the edges to be of different lengths and minimising the size of the board on which the diagram is to be drawn. The middle illustrations above show the 3D case fitted onto a 3×4 board, and a 4D cube fitted onto a 5×5 board, 5D cube can be fitted onto a 6×9 board. It is clear from these how each successive dimension is reached by duplicating the diagram from the previous case and joining all pairs of corresponding points of the two diagrams by parallel lines. The 4D net is evidently equivalent to the net of wazir moves on a 4×4 torus board.

A 6D cube can be fitted onto a 10×10 board, using the moves of a {0,1}+{1,2}+{1,4} leaper. An enlarged diagram of this type was used as the cover illustration on *Chessics* #14, 1982, with the caption: ‘The chessboard as a six-dimensional hypercube’.



See also the note on the Hyperwazir in the rook-move section (p.56).

# Honeycomb Boards

As is well known, there are just two ways of covering a plane area with regular polygonal tiles all of the same shape, size and orientation, namely by squares or hexagons. Equilateral triangles can also tile an area, but they must occur in two orientations. Without the restrictions on size, regularity and orientation, or on all the tiles being the same shape, there are of course many other types of coverings. The study of these is a subject in itself, known as ‘tessellation’. Here we confine our attention to the hexagon pattern, which occurs in nature in the bees' honeycomb. A playing area covered with hexagons is also termed a hex-board.

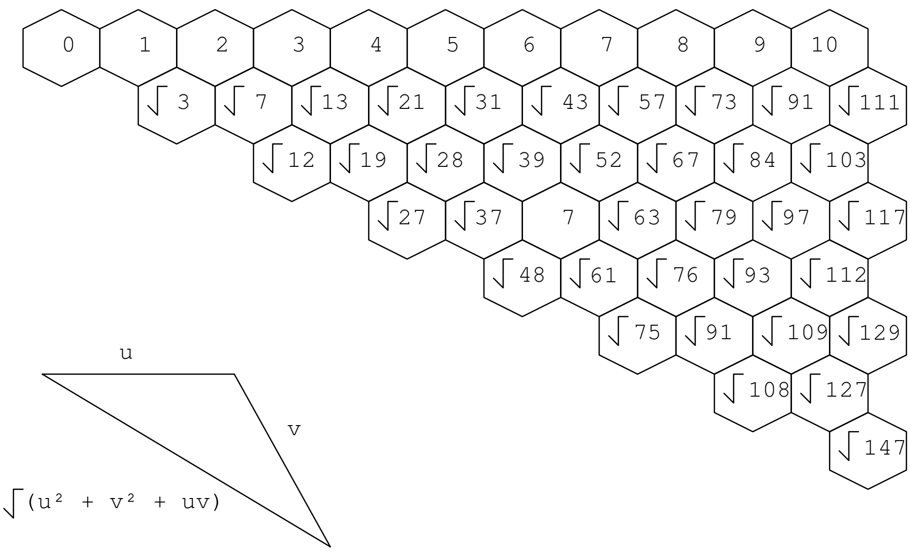
It may be pointed out that it is difficult to draw regular hexagons in most computer drawing programs. The diagrams that follow have been produced by drawing irregular hexagons, by joining dots in a square array, and resizing them to give an approximation to regularity.

## Honeycomb Chess Pieces

Many forms of variant chess have been developed that make use of boards of hexagon shaped cells arranged in the honeycomb pattern, usually termed **hexagonal chess**. As on boards of squares moves can be described by two coordinates, but with 120° instead of 90° between the axes. It turns out that, as on square-cell boards, the **wazir**, moving through the edges of the cells, is the {0,1}-mover, the **fers**, passing through the corners of the hexagons, along the edge common to two adjacent hexagons, is the {1,1}-mover. The corresponding long-range riders in these directions are **rook** and **bishop**. The **knight**, moving to the nearest cells not reachable by a **queen** (rook plus bishop) from the same cell, is the {1,2}-mover.

As on the squares the honeycomb bishops are confined to a part of the board only, but there are three types instead of two. Their domains on the board are usually distinguished by tri-coloured ‘chequering’ of the honeycomb. The piece moving two cells along a rook line, called a **dabbaba**, defines a different ‘chequering’ of the board, requiring four colours. Intuitively this seems rather strange on a board with structure based on threes and sixes.

Taking as unit the centre-to-centre distance of hexagons with a common side it is possible to calculate the length of a move {u,v} from centre to centre of any cells on the honeycomb by using the modified Pythagorean formula $\sqrt{(u^2 + v^2 + u \cdot v)}$. From this it follows that, apart from moves along rook lines, the numerical values of the lengths are different from those for the corresponding square-board pieces with the same move coordinates. For example, while the wazir move is still of length 1, the fers is now a $\sqrt{3}$ mover (instead of $\sqrt{2}$) and the knight is a $\sqrt{7}$ mover (instead of $\sqrt{5}$).



As on squared boards it is possible to investigate the properties of longer **leapers**. The lengths of leaps work out as shown in this chart .[Source: *Chessics* #7, 1979] The first double-pattern fixed distance leaper on the honeycomb is the **7-leaper** (analogous to the 5-leaper on the chessboard). Its two components are {0,7} and {3,5}. The next case, also shown in the above diagram is the **√91-leaper** whose components are {1,9} and {5,6}.

---

## Honeycomb Board Shapes

The number of cells in a hexagon-shaped board of honeycomb pattern with n cells in a hex-rook path along each side (or $2\cdot n - 1$ cells in the rook move through the centre) is $3\cdot n\cdot(n-1) + 1$, as can be visualised by removing the centre cell and cutting the remainder up into three parallelograms, each of n by (n–1) cells. This formula can also be expressed as $n^3 - (n-1)^3$ since the board can also be visualised as three faces of a cube formed of near-spherical cells, the central hexagon being a corner of the cube. The sequence runs: 1, 7, 19, 37, 61, 91, 127, 169, 217, 271, 331, 397, 469, 547, 631, 721, 817, 919, 1027, ... There are also less familiar hexagon-shaped boards where the edge cells are in bishop lines. The number of cells works out at $9n^2 - 15n + 7$. The sequence runs: 1, 13, 43, 91, 157, 241, 343, 463, 601, 757, 931, 1123, ....

It may also be noted here that, unlike boards of squares, honeycomb boards can occur in two orientations on the page, namely with sides of the hexagonal cells vertical or horizontal. We use both orientations, but tend to orient axially symmetric tours so that the axis is vertical, which makes the symmetry more readily apparent to the human eye.

Other shapes of honeycomb board can also be used: Triangular boards, where the number of cells is $n\cdot(n+1)/2$ when the sides are rookwise (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, ...), but $3\cdot n\cdot(n+1)/2 + 1$ when they are bishopwise (1, 4, 10, 19, 31, 46, 64, 85, 109, ...). Lozenge-shaped boards, where the number is of course $n^2$ when the sides are rookwise, since it is merely a distorted square (1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...), but $n^2 + 2\cdot(n-1)^2$ when the side is bishopwise (1, 6, 17, 34, 57, 86, 121, ...). It is also possible to arrange the hexagons in shapes with knight-wise edges.

**H. E. de Vasa** invented a form of chess on a lozenge-shaped honeycomb board. Some tours of these boards by the hexagonal equivalent of knights were mentioned by him in late issues of *Fairy Chess Review* (¶9941, 1954 and ¶10056, 1955). These asked for a knight tour of a 36-cell lozenge (and it was mentioned that the smallest board on which a tour is possible is the 25-cell), and knight tours on an 81-cell lozenge showing the square numbers on (a) the short and (b) the long diagonal, in order of magnitude. The latter are shown here in numerical form.

12 22 38 19 28 48 56 62 **81**
20 27 11 57 37 80 60 **64** 35
18 13 21 26 63 58 **49** 55 61
23 39 17 29 47 **36** 79 59 50
30 10 14 07 **25** 65 5 34 54
08 24 02 **16** 33 06 74 66 44
40 31 **09** 46 03 78 68 51 71
15 **04** 77 73 42 70 53 75 67
**01** 41 32 05 76 72 43 69 52

**01** 21 13 11 19 32 23 29 74
10 **04** 02 22 30 26 15 33 76
06 18 **09** 14 44 75 31 24 46
20 12 05 **16** 08 77 28 73 34
03 07 19 27 **25** 45 59 78 63
52 38 40 43 54 **36** 66 47 72
42 55 51 58 79 62 **49** 35 69
57 53 37 67 60 70 80 **64** 48
39 41 56 50 65 68 61 71 **81**

No more work on the subject of honeycomb board tours is known to me until my own which was published in *Chessics* #7 (1979). This was initially on the 91-cell hexagonal board (6 cells to each side) proposed by **Wladislaw Glinski** that was popular at the time, and I have noted that I sent my first results to the inventor on 10 Jan 1974 but whether he included a tour in any of his works I'm not sure. See the 91-cell section below for diagrams.

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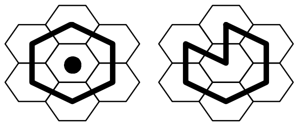
## 7 cells

On the 7-cell hexagon board there is one centreless closed wazir tour, hexasymmetric, and one geometrically distinct closed wazir tour, axially symmetric.



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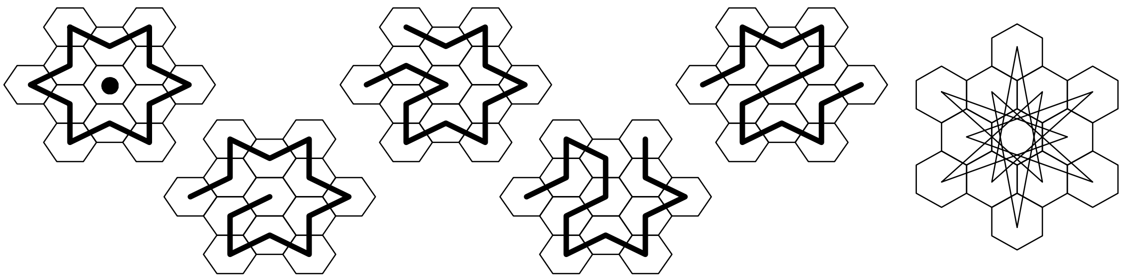
## 13 cells

On the 13-cell hexagon board there is one centreless hexasymmetric closed wazir tour, and four geometrically distinct open wazir tours. Paths through the outer six cells are fixed, forming the star-shaped centreless circuit. The four open tours either have an end at the centre or pass through it at angles of 60°, 120° or 180°; this last is centrosymmetric.

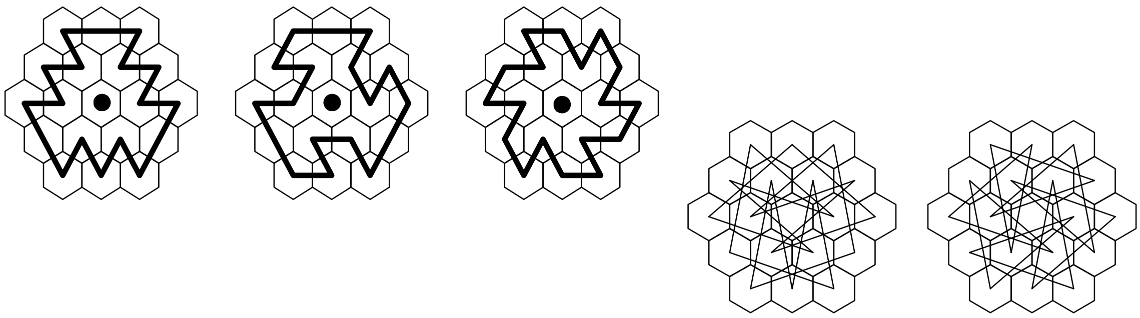


On this board there is a uniquely defined 12-move centreless knight tour, forming a twelve-pointed star (with six long points and six short points). This is the maximum type of symmetry on a board of hexagons, with 6 axes and 1/6 cycle rotation (analogous to octonary on a board of squares, with 4 axes and 1/4 cycle rotation).

---

## 19 cells

On the 19-cell hexagon board I have found 42 centreless wazir tours of which 12 are symmetric. Shown here are three unique cases: one with a triple axis, one unchanged by rotation of 60° (the 'cogwheel' pattern), and one unchanged by 120° rotation. On this board I also find 107 geometrically different closed wazir tours, none symmetric. They consist of 49 with a 60° angle at the centre, 40 with a 120° angle and 18 with a 180° angle (i.e. a straight path).

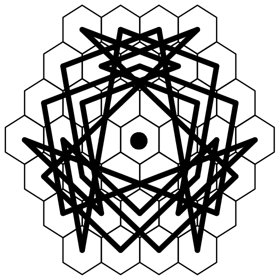


On the 19-cell board I find 9 knight tours with a single axis, 1 with three axes (and 120° rotation), 5 with simple 180° rotational symmetry, and 1 with 60° rotary symmetry. The two special cases are shown here. The enumeration is tricky, since the human eye finds it difficult to recognise rotations or reflections of these tours as being the same.

---

## 37 cells

Axial symmetry is still possible on the size 4 (37-cell) hexagon, in a centreless tour, since there are only two cells on the bishop-move axis. I show a triaxial tour (with 120° rotation). A tour with 60° rotational symmetry is not possible on this board, since the moves between the inner edge cells form two hexagons, which are closed circuits.



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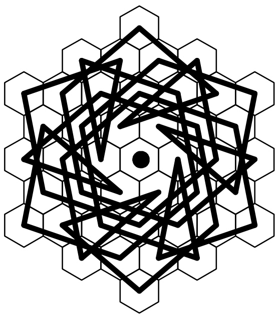
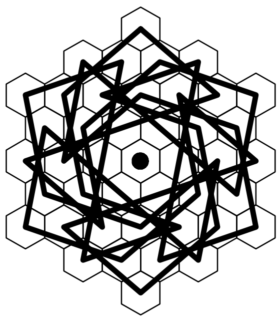
## 43 cells

Two tours with 60° rotational symmetry are possible on the 43-cell board.





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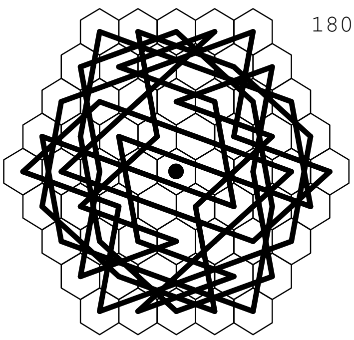
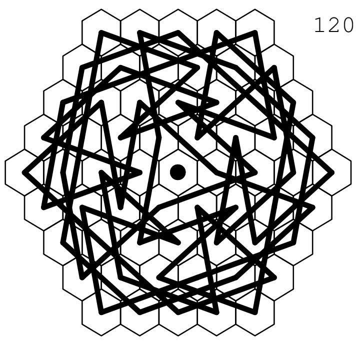
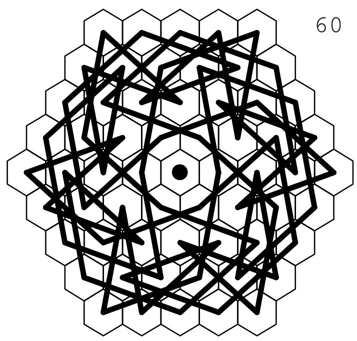
## 61 cells

Axial symmetry is not possible on hexagon boards of more than 37 cells, but rotary symmetry is possible. In centreless tours; this can be of three degrees: bi-, tri- or hexa- in which the smallest rotations that leave the pattern unchanged are 180°, 120° and 60° respectively. Here are examples on the size 5 board (61 cells).



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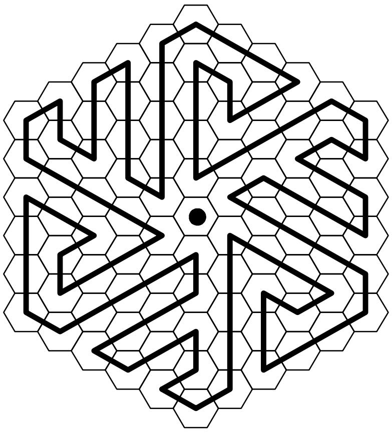
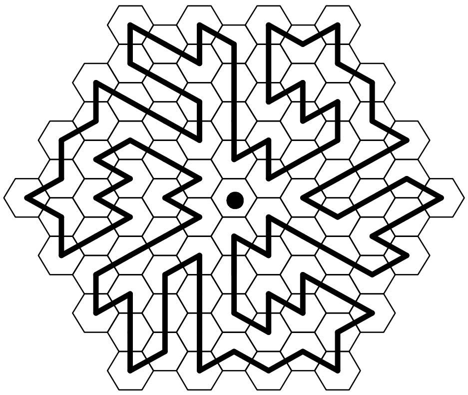
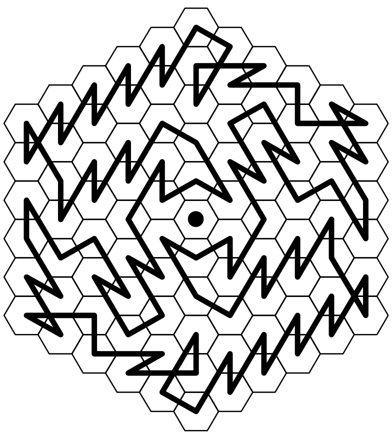
## 91 cells

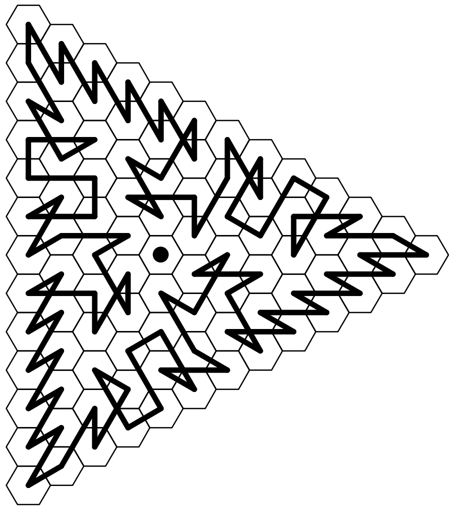
There are three different designs of 91-cell boards formed of hexagons. Here are two centreless wazir tours of the lateral and diagonal sided hexagons.



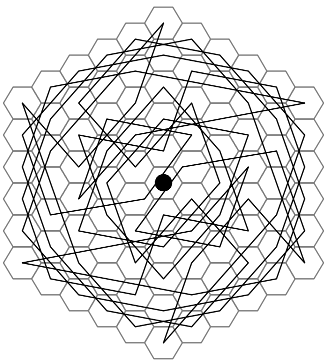


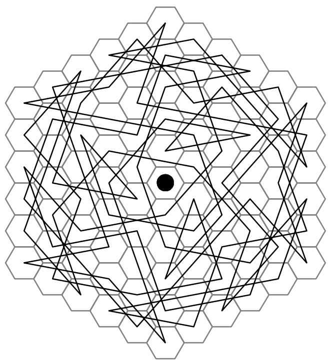
Here are symmetric centreless tours by a hex-king on the 91-cell hexagon and triangle boards. The triangular tour, with 120° symmetry has alternating wazir and fers moves, this is possible since 90/3 = 30, an even number. The hexagonal tour, with 180° symmetry cannot quite alternate, since 90/2 = 45 is an odd number.

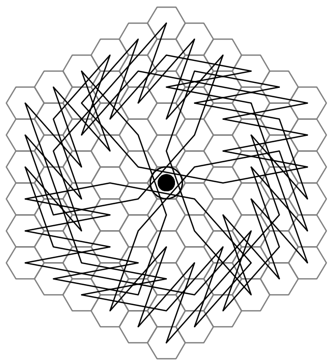




Here are three knight tours on the Glinski 91-cell board showing bi- tri- and hexa- symmetry,



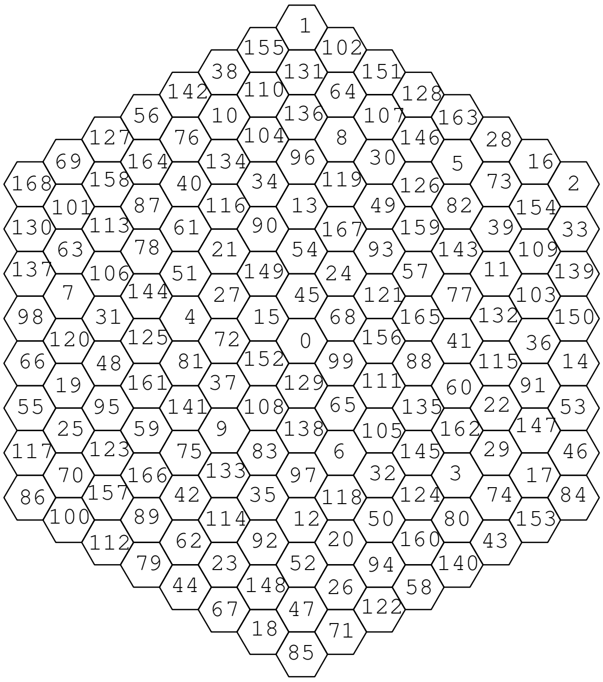




omitting the centre cell (*Chessics* #7 1979 p.5).

## 169 cells

The size 8 board (169 cells) is the smallest hexagonal board on which the 7-leaper can make a complete tour. The tour as a whole is asymmetric, but by omitting the centre cell (numbered 0) and making the last leap from 168 to 1 the resulting centreless tour is bi-symmetric. We show the tour in numerical form since it makes a lot of use of the rookwise 7-leap which makes it difficult to show clearly in graphical form (Jelliss *Chessics* #11 1981 p.1).



# Cryptotour Solutions

**PUZZLE 1** (p.5): Some Solutions to the al Buni Lettered Magic Squares Question.

| | | | |
|---|---|---|---|
| **A** | **N** | **K** | **H** |
| G | L | M | B |
| P | C | F | I |
| J | E | D | O |

| | | | |
|---|---|---|---|
| 1 | 14 | 11 | 8 |
| 7 | 12 | 13 | 2 |
| 16 | 3 | 6 | 9 |
| 10 | 5 | 4 | 15 |

| | | | |
|---|---|---|---|
| P | K | F | A |
| E | B | O | L |
| **I** | **N** | **C** | **H** |
| D | G | J | M |

| | | | |
|---|---|---|---|
| 16 | 11 | 6 | 1 |
| 5 | 2 | 15 | 12 |
| 9 | 14 | 3 | 8 |
| 4 | 7 | 10 | 13 |

| | | | |
|---|---|---|---|
| **F** | **I** | **D** | **O** |
| L | G | N | A |
| M | B | K | H |
| C | P | E | J |

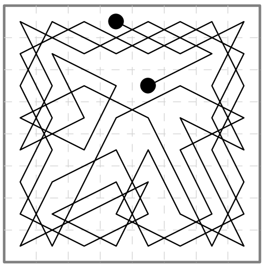
| | | | |
|---|---|---|---|
| 6 | 9 | 4 | 15 |
| 12 | 7 | 14 | 1 |
| 13 | 2 | 11 | 8 |
| 3 | 16 | 5 | 10 |

By reflecting the second the word is altered to LOBE in the second row.

# KNIGHT'S TOUR NOTES

**PUZZLE 2** (p.6): **Solution to the C. J. Morse Revolver Practice Puzzle**. To construct an alphabetical tour (lettered ...A...Z&A...Z...) showing the word REVOLVER along the back rank, forming the same pattern as the chess pieces. My solution is a K...T tour. The letters on the opposite rank don't show the same pattern however.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| P | Z | A | K | R | A | Z | M |
| B | L | Q | & | & | L | S | B |
| Y | O | H | K | T | H | N | Y |
| I | C | M | G | G | P | C | I |
| N | X | J | T | T | J | X | O |
| D | S | F | M | P | F | Q | D |
| W | N | Q | U | U | S | K | W |
| **R** | **E** | **V** | **O** | **L** | **V** | **E** | **R** |



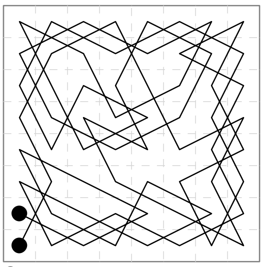
**PUZZLE 3** (p.10): **Staunton Cryptotour (I).** The tour is the Jaenisch magic tour (00a). The verse reads: “The man that hath no love of chess / Is, truth to say, a sorry wight, / Disloyal to his King and Queen, / A faithless and ungallant Knight. / He hateth our good mother church, / And sneereth at the Bishop's lawn, / May ill luck force him soon to place / His Castles and estates in Pawn!”

**PUZZLE 4** (p.11): **Staunton Cryptotour (V).** “The knight moves from one corner to the opposite on a rectangle of six squares.” (Diagram below). Murray (1942) cites Staunton's *Chess Player's Handbook*: “Mathematical definition of the move of the knight: the knight's line of motion and attack is along the diagonals of parallelograms, 3 by 2, in every direction, to the opposite square” and notes that this formulation “though often repeated, e.g. by W. H. Cozens, is deplorably unmathematical.” It is now more customary to describe the knight as a {1,2} mover.

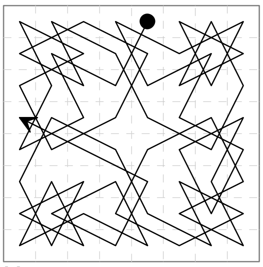
**PUZZLE 5** (p.12): **Rowland Cryptotour.** The tour is a magic one (12o) by Jaenisch. The verse reads: “Our life is but a pilgrimage, / From birth unto the bier, / We journey onward as the Knight, / Our pathway never clear. / O'er checkered course of sixty-four, / Through ways that lead astray, / Now bright, then dark, we travel on, / Till darkness veils our day.” (The division of the letters to give repeated entries is intended to “lead astray”).

**PUZZLE 6** (p.12): **Murray Cryptotour**. The verse by Tomlinson is: “The wise man thinks before he speaks / And words of wisdom from him fall. / The fool speaks first then haply thinks / Or he may never think at all. / So in our royal game of Chess / The rule of life is still the same. / Folly frets out its thoughtless hour / And wisdom plays a cautious game.” What the three maxima are I'm not sure, one is probably the 34 slants. (Diagram below).

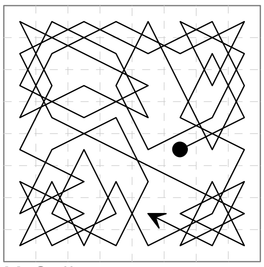
**PUZZLE 7** (p.13): **McGuffey Cryptotour.** “When I take the humour of a thing once, I am like your tailor's needle — I go through”. (Diagram below).



Staunton



Murray



McGuffey