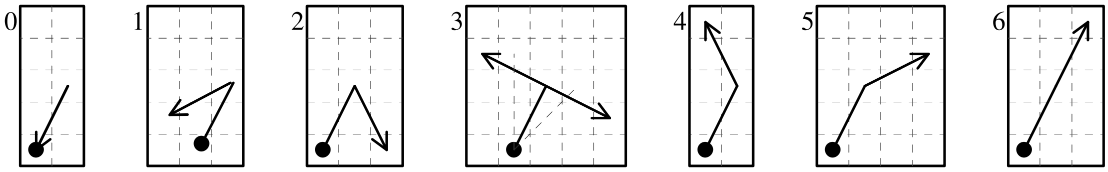
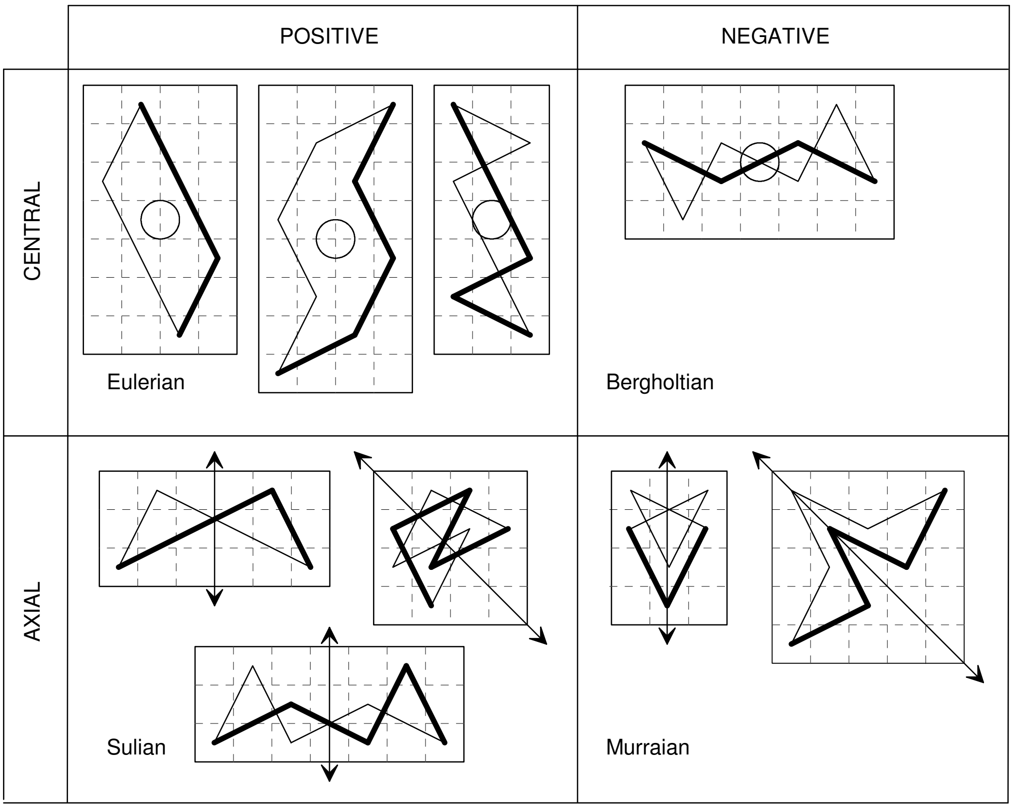
⌘ 1

# Theory of Moves




by G. P. Jelliss

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| | | | | | | |
|---|---|---|---|---|---|---|
| **9** | 8 | 7 | 6 | 7 | 8 | **9** |
| 8 | **5** | 4 | 3 | 4 | **5** | 8 |
| 7 | 4 | **2** | 1 | **2** | 4 | 7 |
| 6 | 3 | 1 | **0** | 1 | 3 | 6 |
| 7 | 4 | **2** | 1 | **2** | 4 | 7 |
| 8 | **5** | 4 | 3 | 4 | **5** | 8 |
| **9** | 8 | 7 | 6 | 7 | 8 | **9** |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **9** | 8 | 7 | 6 | 6 | 7 | 8 | **9** |
| 8 | **5** | 4 | 3 | 3 | 4 | **5** | 8 |
| 7 | 4 | **2** | 1 | 1 | **2** | 4 | 7 |
| 6 | 3 | 1 | **0** | **0** | 1 | 3 | 6 |
| 6 | 3 | 1 | **0** | **0** | 1 | 3 | 6 |
| 7 | 4 | **2** | 1 | 1 | **2** | 4 | 7 |
| 8 | **5** | 4 | 3 | 3 | 4 | **5** | 8 |
| **9** | 8 | 7 | 6 | 6 | 7 | 8 | **9** |

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**Title Page Illustrations:**
The angles of the double moves of a knight.
Symmetries in Knight Circuits
Centrifugal cell coding.

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# Contents



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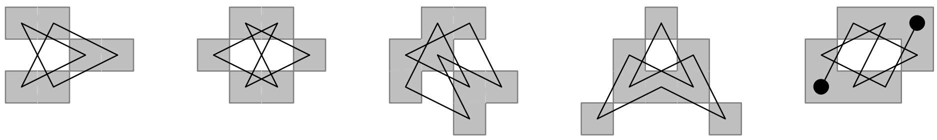
http://www.mayhematics.com/
Knight's Tour Notes, Volume 1, Theory of Moves.

# Movement

## Boards

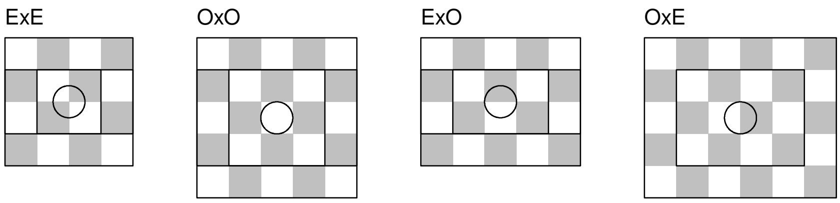
For the purposes of this study a **board** is essentially an array of **cells** with common edges or faces forming a **connected** whole, not separable into two or more parts. Another way of expressing this connectedness is that it must be possible for a token to get from any cell to any other in a series of moves between adjacent cells. This convention rules out for example arrays of cells where some are supposedly ‘attached’ only at a point. Here are some examples showing knight paths on **loosely connected** arrays of six or eight small square cells. These do not count as boards.



By a **chessboard** we mean any array of cells in the form of small squares all of the same size connected edge to edge and lying in a lattice formed of two sets of parallel lines crossing each other. In Alternative Worlds (⌘ 11) we also study pieces and moves on **hexboards** formed by three sets of parallels, and **spaceboards** formed by sets of parallel planes in three dimensions.

In diagrams we show the square cells with their sides parallel to the edges of the page. The cells form lines running up and down the page called **files** and lines running across the page called **ranks**. A **rectangular** board formed by r ranks and s files is described as an ‘r by s’ board, abbreviated r×s. In a complete rectangle r×s the number of cells is r·s where we use a raised dot to denote multiplication. The boards we consider may also be **shaped** (having cells missing round the edges) or **holey** (having cells missing internally, but any such board has a unique containing rectangle.

Some important characteristics of a rectangular r×s board depend on the **parity** of its sides, that is whether r and s are odd or even. The parity of the sides determines the geometry of the centre of the board. The centre of an even×even (EE) board (2×2, 2×4, 4×4, ...) is at the corner of a cell, i.e. where four cells meet. An odd×odd (OO) board (3×3, 3×5, 5×5, ...) has a central cell. An even×odd (EO) board (2×3, 2×5, 4×5, ...) or odd×even (OE) board (3×4, 3×6, 5×6, ...) has its centre at the mid-edge of a cell. As these diagrams of the simplest cases illustrate.




The cells in the boards we study will generally not be coloured unless it is helpful to prove a point. Any rectangular board can be **chequered**, with the cells coloured alternately light and dark, and there is a convention that the top left cell is taken to be light. In the case of **odd** rectangles (with r·s odd) this means the light cells are in the majority. On **even** rectangles (with r·s even) there are always the same number of light and dark cells. In fact an even board can be divided into pairs (**dominos**) of adjacent light and dark cells.

If we number the files and ranks from a corner of the rectangular area then the position of a cell on the board is specified by a pair of **coordinates** (x,y) specifying the file and rank in which it occurs. Chessplayers replace the *x* coordinate by letters and take (1,1) = a1 to be the bottom left cell. Algebraists tend to take (1,1) to be the top left cell (as in matrix algebra). Geometers sometimes prefer to take the bottom left cell to be (0,0). These varied conventions can lead to some confusion.

For consistency of appearance our diagrams, drawn using Lotus WordPro, all use 1/6 inch cells which allow four 8×8 boards across the A4 page between inch margins.

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## Moves

A **move** takes a token from point (x,y) to point (x',y') and is represented by the ordered pair of signed numbers (x'–x, y'–y) = (*r*,*s*), using italics to indicate signed numbers. However the **pattern** of the move is represented by an unordered pair of unsigned numbers {r,s}. Thus a move of pattern {r,s} takes a piece from (x,y) to any of the up to eight cells (x±r, y±s) and (x±s, y±r) that are available within the confines of the board. The number of directions of move possible will be less than eight if one coordinate is zero or if the two coordinates are equal.

If r = 0 and s = 0 we have the somewhat paradoxical concept of a **null** move.

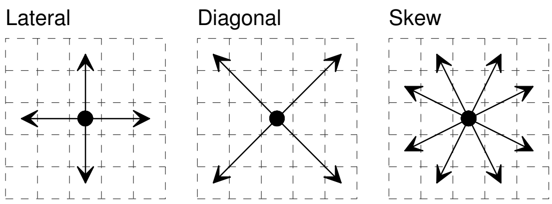
If r = 0 or s = 0, not both, the move is **lateral**; through the sides of the squares.

If r = s, not both zero, the move is **diagonal**; through the corners of the squares.

If r and s are unequal and neither is zero the move is termed **skew**. In the case of a skew move all eight moves from a given cell will be distinct, making a **wheel** formation (on a large enough board).

If r < s then the four moves (±r, ±s) are termed **vertical** while (±s, ±r) are **horizontal**.

For diagonal or lateral moves the number of directions reduces to four.




Depending on the convention in use the **principal** diagonal of cells with equal coordinates (m,m) may be the one from top left to bottom right (as in matrix algebra) or bottom left to top right (as in chess, algebraic geometry and chart plotting).

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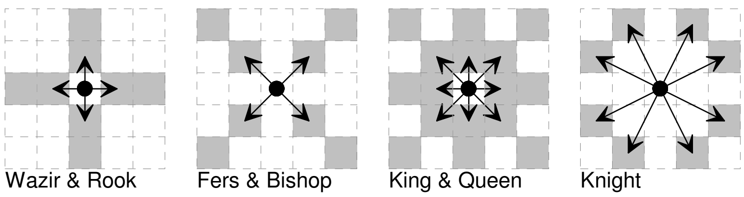
## Pieces

A **piece** is a token that can occupy a cell and whose distinctive design or symbol indicates the moves that it is able to make to other specified cells. The simplest possible piece is the **wazir** which makes a single step through an edge of the cell on which it stands to the next cell. The wazir move is a {0,1} move, signifying a move of one rank without change of file, or of one file without change of rank. A similar but more powerful piece is the **rook** which can make any number of wazir steps in a single move in a straight line. This is a move {0,n}.

Similarly the **fers** is a piece that moves through the corner of a cell to the next cell. This is a move {1,1} indicating a change of both rank and file. Analogous to the rook, the **bishop** can make any number of fers steps in a single move in a straight line. This is a move {n,n}.

The wazir and fers are not pieces used in modern orthodox chess, but a piece capable of either of these moves is the **king**. Similarly a piece combining powers of rook and bishop is the **queen**.

We now come to the **knight**, which is a piece capable of reaching the nearest cells that cannot be reached by a queen on the same cell. It is a {1,2} mover. That is it moves one file and two ranks, or two files and one rank.




More generally, given any pattern or set of patterns of moves then a **piece** can be defined, able to make just those moves from whichever cell of the board it is on.

A piece that can move regardless of the presence or absence of pieces on any intervening cells is called a **leaper**, in contrast to other pieces like **riders** and **hoppers** that require absence or presence of other pieces for their moves to be possible. There are many more families of pieces employed in chess and its variants, but leapers are sufficient to our requirements here.

Because of their use in varieties of chess, many of the simpler **single-pattern leapers** have acquired colourful individual names, some of them dating back to mediaeval times:

| | | | | |
|---|---|---|---|---|
| {0,0} **dummy** | | | | |
| {0,1} **wazir** | {1,1} **fers** | | | |
| {0,2} **dabbaba** | {1,2} **knight** | {2,2} **alfil** | | |
| {0,3} **3-leaper** | {1,3} **camel** | {2,3} **zebra** | {3,3} **tripper** | |
| {0,4} **4-leaper** | {1,4} **giraffe** | {2,4} **lancer** | {3,4} **antelope** | {4,4} **commuter** |

Pieces are sometimes described in terms of the geometrical length of their moves. The length of an {r,s} move, measured in a straight line from centre to centre of the squares involved is, by the theorem of Pythagoras, $\sqrt{(r^2 + s^2)}$. Thus the knight may be described as a ‘root-five mover’.

There are 36 different single-pattern pieces on the 8×8 board. More generally, on an m×m board there are $m\cdot(m+1)/2$ patterns of move possible, and on an m×n board, with $m \leq n$, we must add a further $m\cdot(n-m)$ patterns. The total is $m\cdot n - (m^2 - m)/2$ moves, ranging from (0,0) to (m–1,n–1).

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## Composite Pieces

A **composite** piece is one that can move either as a piece of type A or as one of type B, at will; it is denoted A+B. Here are some (traditional and suggested) names for double-pattern leapers:

| | | |
|---|---|---|
| {0,1} + {1,1} **king** | {0,1} + {0,2} **wazaba** | {0,1} + {1,2} **emperor** |
| {0,1} + {2,2} **weevil** | {1,1} + {0,2} **duke** | {1,1} + {1,2} **prince** |
| {1,1} + {2,2} **ferfil** | {0,2} + {1,2} **templar** | {0,2} + {2,2} **alibaba** |
| {1,2} + {2,2} **hospitaller** | {1,2} + {1,3} **gnu** | {1,2} + {2,3} **okapi** |
| {1,3} + {2,3} **bison** | {0,5} + {3,4} **5-leaper** | {5,5} + {1,7} **root-50-leaper** |

For the purposes of our study of tours, where such tactics as pins and batteries do not come into consideration, there is little difference between riders and compound pieces. For instance the wazaba can be regarded as a 1-2-rook, restricted to steps of one or two cells, and the ferfil a 1-2-bishop.

The total number of pieces possible on the 8×8 board, single or multiple, and including the dummy, is the number of ways of making a selection of moves from the 35 non-null moves available. This total is 2^35, since each move can either be selected or not, giving two choices in each of 35 cases. This works out to 34,359,738,368. It is thus unlikely they will ever all be individually named!

As longer moves are considered, cases arise in which different patterns of move have the same geometric length. The first two cases of this define the **fiveleaper** and the **root-fifty leaper** in the list above. T. R. Dawson gave a list of two-pattern (and larger) fixed distance leapers on larger boards in *Chess Amateur* Aug 1925. His results were no doubt found by use of the following general result (R. D. Carmichael *Diophantine Analysis* 1915 p.25): the equation $x^2 + y^2 = u^2 + v^2$ is satisfied by any numbers of the forms $x = m\cdot p + n\cdot q$, $y = m\cdot q - n\cdot p$, $u = m\cdot p - n\cdot q$, $v = m\cdot q + n\cdot p$. For example m = 4, n= 3, p = 2, q = 1 gives $11^2 + 2^2 = 5^2 + 10^2$.

The following are all the double-pattern fixed distance leapers and their lengths up to length 17. {3,4}+{0,5} = 5L, {5,5}+{1,7} = $\sqrt{50}$L (~7.07), {4,7}+{1,8} = $\sqrt{65}$L (~8.06), {6,7}+{2,9} = $\sqrt{85}$L (~ 9.22), {6,8}+{0,10} = 10L {5,10}+{2,11} = $\sqrt{125}$L = $5\sqrt{5}$L (~11.18), {7,9}+{3,11} = $\sqrt{130}$L (~11.40) {8,9}+{1,12} = $\sqrt{145}$L (~12.04), {5,12}+{0,13} = 13L, {7,11}+{1,13} = $\sqrt{170}$L (~13.04), {8,11}+{4,13} = $\sqrt{185}$L (~13.60), {10,10}+{2,14} = $\sqrt{200}$L = $2\sqrt{50}$L (~14.14), {6,13}+{3,14} = $\sqrt{205}$L (~14.32), {10,11}+{5,14} = $\sqrt{221}$L (~14.87), {9,12}+{0,15} = 15L, {9,13}+{5,15} = $\sqrt{250}$L = $5\sqrt{10}$L (~15.81), {8,14}+{2,16} = $\sqrt{260}$L = $2\sqrt{65}$L (~16.12), {11,12}+{3,16} = $\sqrt{265}$L (~16.28), {8,15}+{0,17} = 17L.

Note: multiplying the length of any single-pattern leaper by 5 produces a double-pattern leaper.

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## Freedom and Multiplicity

The **freedom** (of movement) of a piece can be measured by the fraction of the board to which it has access. Conversely the **multiplicity** of a piece is the number of pieces of that type needed to patrol all the cells of the board. Obviously a requirement (though not a sufficient condition) for a piece to be able to make a tour of a given board is that it must be **free** to reach any cell from any other by a series of moves. Its freedom and its multiplicity are both 1. For the {r,s} leaper to be free on the 2s×2s or any larger board we must have **highest common factor** hcf (r,s) = 1 and r+s odd. [Sources: Jelliss, *Chessics*, #2 1976, #24 1985; Knuth, 'Leaper Tours', *Mathematical Gazette*, 1994.]

On the 8×8 chessboard there are five single-pattern **free leapers** namely: the **wazir** {0,1}, **knight** {1,2}, **zebra** {2,3}, **giraffe** {1,4} and **antelope** {3,4}. On larger boards the series continues: {2,5}, {4,5}, {1,6}, {5,6}, {2,7}, {4,7}, {6,7}, {1,8}, {3,8}, {5,8}, {7,8}, {2,9}, {4,9}, {8,9} and so on. All leapers of the families {z, z+1}, {1, 2·z} and {2, 2·z + 1} are free.

For the {r,s} leaper to be **half-free** (that is with access to half the squares of the board; the white or black squares if the board is chequered) we must have hcf(r,s) = 1 and r+s even. The half-free leapers, with freedom 0.5 and multiplicity 2, are: **fers** {1,1}, **camel** {1,3}, {1,5}, {3,5}, {1,7}, {3,7}, {5,7}, {1,9}, {5,9}, {7,9} and so on. All leapers {1, 2·z + 1} are half-free.

The freedom of an {r,s} leaper, with $r \leq s$, on a 2s×2s board is $1/k^2$ or $1/(2\cdot k^2)$ according as (r+s)/k is odd or even, where k = hcf (r,s). If hcf(r,s) = k then the {r,s} mover is confined to squares with coordinates $(x \pm u\cdot k,\ y \pm v\cdot k)$ relative to its initial square (x,y) and cannot, for example, get to those squares $(x\pm a,\ y\pm b)$ with a and b between 0 and k. If (r+s)/k is even the piece is confined to those squares $(x \pm u\cdot k,\ y \pm v\cdot k)$ with u+v even, which is half of them.

Other leapers with coordinates up to 9 are: (a) with odd multiplicity; 9: {0,3}, {3,6}, {6,9}; 25: {0,5}; 49 {0,7}; 81 {0,9}; these, like free leapers always move to squares of a different colour: (b) with even multiplicity; 4: {0,2}, {2,4}, {4,6}, {2,8}, {6,8}; 8: {2,2}, {2,6}; 16: {0,4}, {4,8}; 18: {3,3}, {3,9}; 32: {4,4}; 36: {0,6}; 50: {5,5}; 64: {0,8}; 72: {6,6}; 98: {7,7}, 128: {8,8}, 162: {9,9}; these are all on squares of one colour.

Any combination of a piece with a free piece is obviously free. It can however happen that a multi-pattern piece is free even though none of its components (single-pattern components or combinations of them) are free. A combination that has greater freedom than any of its components I call an **amphibian**, since it is able to move in two or more distinct realms. The simplest cases are {1,1}+{0,3}, {0,2}+{0,3} and {2,2}+{0,3}. A combination of pieces with multiplicities p and q produces a piece with multiplicity hcf(p,q). Thus for an amphibian p must not be a multiple of q. Any component with odd multiplicity combined with a component of even multiplicity is free.

**PUZZLE** 1: What is the smallest triple-pattern amphibian? Solutions at end.

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## Mobility

The **mobility** of a piece on a given board can be measured by the average number of moves it can make when placed at random on the board. This can be calculated by totalling the moves available to it at all the cells of the board and dividing by the number of cells. For comparison the mobilities of various leapers on the 8×8 board are: tripper 1.5625, alfil 2.25, antelope and threeleaper 2.5, dabbaba 3, fers 3.0625, wazir and giraffe 3.5, zebra 3.75, camel 4.375, knight 5.25. The knight is in fact the most mobile single-pattern leaper on any rectangular board larger than 4×4. This goes a long way to explaining why the knight is specially interesting. Its mobility singles it out for particular study.

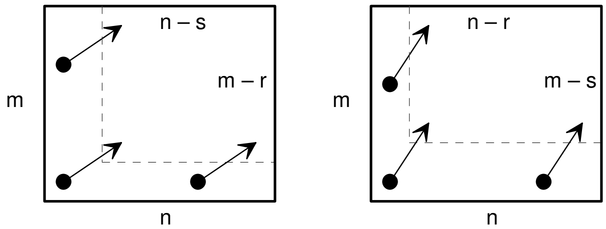
Another measure of manoeuvrability is the minimum number of moves needed to take a piece to an adjacent cell. This is 1 for wazir, 3 for knight, 5 for zebra and giraffe, 7 for antelope.

The number of moves available to a skew leaper reaches its maximum value of eight only in the central regions of a sufficiently large board. The board may be divided into regions according to the number of moves available to the leaper on cells in those regions. Each of these **mobility patterns** can be seen as a pattern of intersections of the eight rectangles described in the following theorem. The number of moves at a cell is the number of these rectangles in which the cell is contained.

KNIGHT'S TOUR NOTES

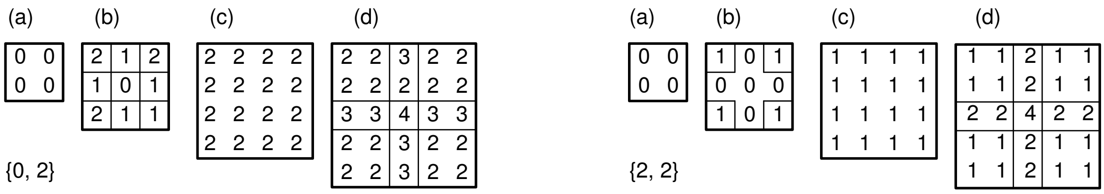
THEOREM: <u>For an {r,s} skew leaper on a square board n×n the mobility is $8\cdot(n-r)\cdot(n-s)/n^2$ and on a rectangular board m×n the mobility is $[4\cdot(m-r)\cdot(n-s) + 4\cdot(m-s)\cdot(n-r)]/(m\cdot n)$.</u>

*Proof*: On the square board we consider all its moves in one of the eight directions. The destinations of these moves must lie within a rectangle of (n–r)×(n–s) cells. Similarly on the rctangular board moves in the four horizontal directions lead to one rectangle, and moves in the 4 vertical directions to another rectangle.




For lateral and diagonal leapers the mobilities are half the above values. Historical note: Edouard Lucas (*Recréations Mathématiques* vol.4, 1894, p.130) gives this result, without proof, in the form: $2\cdot\{(2\cdot m - r - s)\cdot(2\cdot n - r - s) - (r - s)^2\}/(m\cdot n)$.

For lateral {0,s} or diagonal {s,s} leapers on a square board n×n only four mobility patterns can occur in each case, according as (a) n ≤ s, (b) s < n < 2·s, (c) n = 2·s, (d) n > 2·s, as illustrated below. Case (b) does not occur when s = 1.



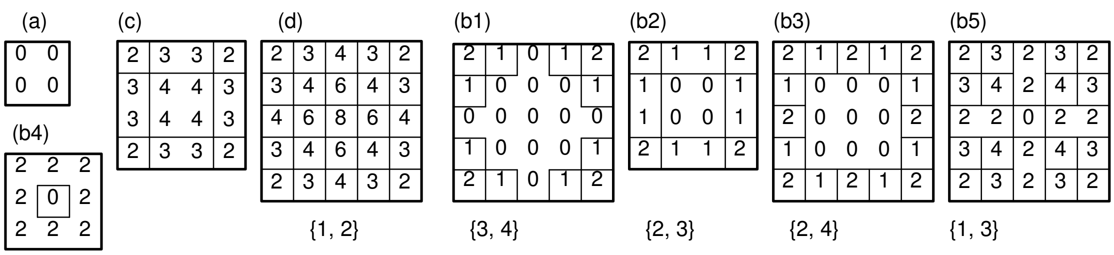

In the case of {r,s} skew leapers, r ≤ s, eight different mobility patterns can occur on square boards according to the size of the board n×n and the relative proportions of the numbers r, s and n.

The case (b) s < n < 2·s divides into five separate cases according to the value of r: (b1) s < n < 2·r, (b2) n = 2·r, (b3) 2·r < n < r+s, (b4) n = r+s, (b5) r+s < n < 2·s.

Cases (a), (b4), (c), (d) occur for all skew leapers, for example the knight.

The mobility pattern of the knight on the 8×8 board is of type (d) and differs from its pattern on the 5×5 board only in that some of the regions become larger; e.g. the 8-move area consists of the central 4×4 region of the board, and the 6-move areas are rectangles 1×4 adjacent to the central area.

The other four cases only occur under special conditions: (b1) s < 2·r – 1, (b2) s < 2·r, (b3) r+1 < s ≤ 2·r, (b5) r+1 < s.




The mobility pattern of an {r,s} skew leaper on a square board of edge r+s has two moves available at every cell except the central ones, this means the moves must form closed circuits. The {1, 2·k}-movers form a single circuit. [Source: *Chessics*, #2 1976 p.2 and #24 1985 p.94.]

**PUZZLE** 2: What is the smallest board on which an {r,s} leaper can move from every cell?

**PUZZLE** 3: On what board does the mobility of a {0,s} leaper equal 3?

**PUZZLE** 4: Which is the smallest skew leaper that exhibits all eight mobility patterns?

# Journeys

Any sequence of moves of a piece constitutes a **journey**. In general it may traverse the same links or cells more than once. The resulting change in position of the moved man is always equivalent to a single leap. We thus speak of a (p,q)-journey if the final cell reached is a (p,q) move from the initial cell. A journey may be **open** (ending at a different node from that where it started) or **closed** (ending at the initial node). A closed journey (ending at the initial cell) may be considered a (0,0)-journey.

**Reordering Journeys**

THEOREM (Reordering) On a sufficiently large board a series of moves will reach the same destination, from a given initial cell, regardless of the order in which the moves are made. *Proof*: Any move in the journey takes the piece a given distance horizontally and a given distance vertically (these distances may be positive, negative or zero). The sums of the horizontal and of the vertical moves give the coordinates of the end point relative to the start point. A sum of distances in a line is independent of the order in which they are added.

**Counting Journeys**

THEOREM The number of ways of ordering a journey of j = a + b + c + d + a '+ b '+ c '+ d ' moves in the eight directions of an {r,s} skew-mover (a and a ' being the number of moves in the forward and reverse direction of one line, and so on) is j!/(a!·b!·c!·d!·a '!·b '!·c '!·d '!). *Proof*: This elegant formula is well-known algebra (e.g. Crystal *Textbook of Algebra* 1900, vol.2, p.5).

**The Journey Equations**

By adding the horizontal and vertical components we get the journey equations:

(1a) $p = [(a - a') + (d - d')]\cdot r + [(b - b') + (c - c')]\cdot s$

(1b) $q = [(a - a') + (d' - d)]\cdot s + [(b - b') + (c' - c)]\cdot r$.

Substituting $A = a - a'$, $B = b - b'$, $C = c - c'$, $D = d - d'$ the equations take the simpler form:

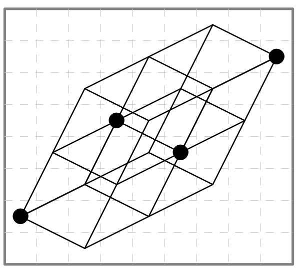
(2a) $p = (A + D)\cdot r + (B + C)\cdot s$

(2b) $q = (A - D)\cdot s + (B - C)\cdot r$.

In the above equations italic letters indicate signed numbers.

**Journey Patterns**

The interesting patterns formed by the shortest journeys of a given leaper between two given cells were pointed out to me by Dr C. M. B. Tylor [*Chessics* vol.2 #19 p.30 1984] but I have since found similar work in *L'Echiquier* 1926-7 from which the first two puzzles below are taken. It is an intriguing recreation to try constructing all the shortest knight journeys between two arbitrarily chosen cells to see the pattern that appears.

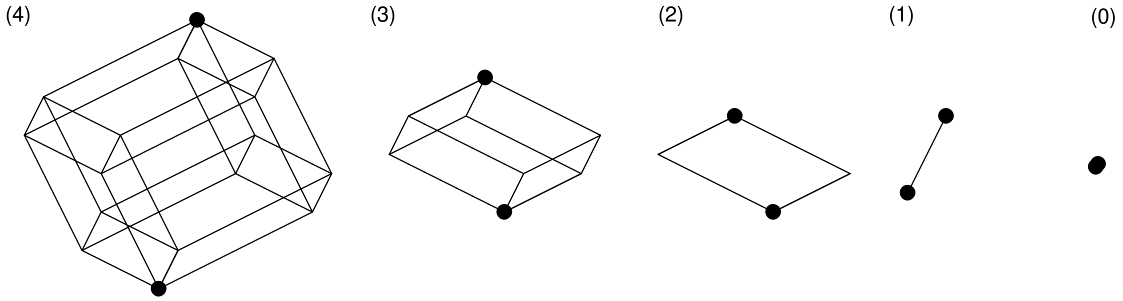


The above diagram illustrates the 5!/(2!·2!) = 30 shortest knight journeys a2 to i7, an (8,5)-journey of 5 moves.

**Reduced Journeys**

In a journey involving moves in opposite directions we can re-order the moves to bring the opposing moves together to cancel each other out. The journey of j moves is reduced by this method to a journey of i = $|A| + |B| + |C| + |D|$ moves in four or fewer of the eight directions. A series of moves all in the same direction is a **ride**. Thus by reordering we reduce any journey to 4 or fewer rides in different lines.

The four rides can be taken in 4! = 24 different orders, forming a ‘hyper-parallelogram’ pattern of the type illustrated below. The diagrams simplify considerably if fewer than 4 directions are used, becoming a parallelepiped (3! = 6) when three are used, parallelogram (2! = 2) when two are used, line (1! = 1) when only one is used, and point (0! = 1) when all 4 rides are null.




**Closed Journeys**

In the case of a closed journey $p = q = 0$, so the journey equations reduce to:

$$0 = (A + D)\cdot r + (B + C)\cdot s \text{ and } 0 = (A - D)\cdot s + (B - C)\cdot r,$$

from which, since s is not 0, we get (when $B - C$ is nonzero)

$$r/s = -(B + C)/(A + D) = -(A - D)/(B - C)$$

and multiplying through by $-(A + D)(B - C)$ gives us

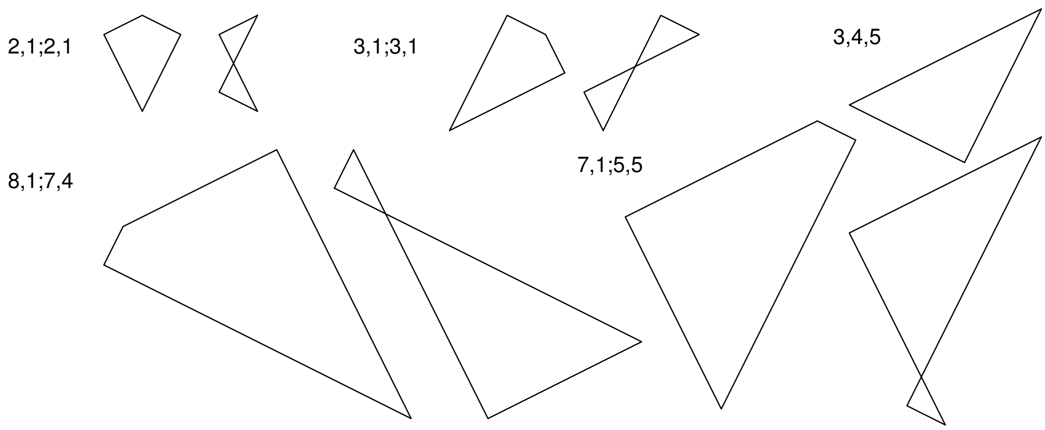
$$(B + C)\cdot(B - C) = (A + D)\cdot(A - D)$$

which is the same as $B^2 - C^2 = A^2 - D^2$. We can now substitute the absolute values since the squares of negative numbers are positive, and so finally

$$|A|^2 + |C|^2 = |B|^2 + |D|^2$$

When $B = C$ we also have (since q = 0) $A = D$ and this equation remains true.

Geometrically this relation means that the four rides (when none is zero) form two right angled triangles with a common longest side, $A$ being perpendicular to $C$ and $B$ to $D$. This is obviously possible whenever $A = B$ and $C = D$ or when $A = D$ and $B = C$ in which cases the quadrilateral is either a ‘kite’ when arranged so that no two lines cross, or a ‘bow’ when they cross. But there are also skew cases such as ($A$,$C$) = (5,5), ($B$,$D$) = (1,7). The cases with perimeters up to 30 units are: 7,1,5,5 (18); 8,1,7,4 (20); 9,2,7,6 (24); 11,2,10,5 (28); 11,3,9,7 (30); 12,1,9,8 (30).



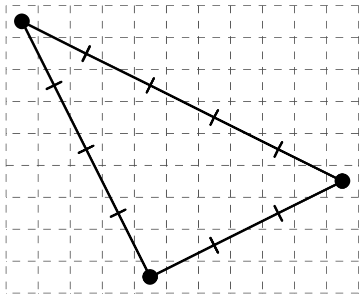
**Triangular Journeys**

Since wazir moves can only form right angles with each other no triangle formed of a series of wazir moves is possible. The knight and other skew leapers however can describe triangles, though they require larger boards than the standard chessboard.

When one of the rides reduces to zero the three rides form a right-angled triangle. For an {r,s}-mover with $r < s$ the sides of this triangle must be in the ratios

$s^2 - r^2 : 2{\cdot}r{\cdot}s : r^2 + s^2$.

In the case of the knight the smallest triangle that can be formed by three successive lines of knight moves is the famous 3:4:5 right-angled triangle. This property was known to T. R. Dawson in *Chess Amateur* and he has a diagram of such a triangle formed of 3, 4 and 5 complete knight moves.

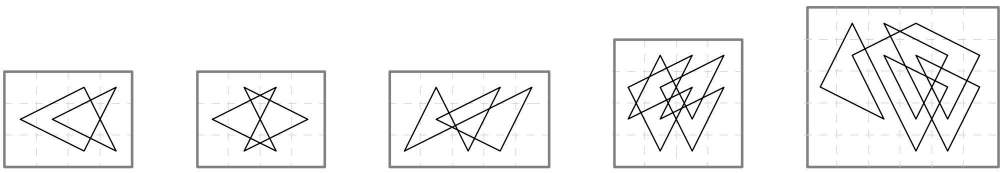


Since a knight move has length $\sqrt{5}$ (taking the sides of the squares of the board as unit), this triangle has area ½ base × height = $\frac{1}{2}(4\sqrt{5})(3\sqrt{5})$ = 30 square units. To make this complete triangle the knight requires a 9×11 board, somewhat larger than its usual domain.

For an {r,s} skew leaper generally a triangle will have angles {$\alpha$, $\beta$, 90} forming a right-angled triangle with sides in the ratios $(s^2 - r^2) : 2{\cdot}r{\cdot}s : (s^2 + r^2)$ . In the case of the Zebra {2,3} the triangle is of the 5:12:13 type and requires a board 37 by 40. For the Giraffe {1,4} the triangle is of the 15:8:17 type and requires a board 33 by 69. For the Antelope {3,4} the triangle is another of the long thin type, 7:24:25, needing a board 97 by 101, surprisingly large.

**Compact Journeys**

H. J. R. Murray (1942) considered the problem of arranging the moves in ‘irreducible chains’ so as to pack them within the smallest possible board. The following are his results for the 2,1;2,1 and 3,1;3,1 and 5,4,3 cases, and my own best result for the 7,1;5,5 case.



2,1;2,1 2,1;2,1 3,1;3,1 3,4,5 7,1;5,5

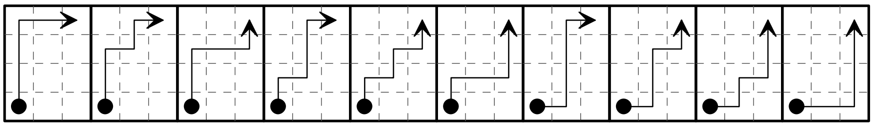
**PUZZLE** 5: How many 3-move knight routes from c3 to d3 on a 5×6 board?
**PUZZLE** 6: How many 4-move routes from a5 to g3 on a 7×7 board?
**PUZZLE** 7: How many 4-move paths to make a (0,6) journey d1 to d7 on 7×7 board?
**PUZZLE** 8: Fit the 8,1;7,4 and 9,2;7,6 cases onto a smallest possible board.
(Solutions p.738)
**EXERCISE**: Draw four-move (0,10)-journeys by zebra and giraffe for comparison.

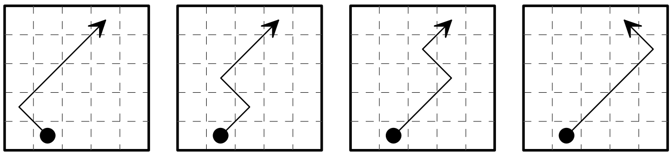
## The Shortest Path Problem

The general problem is: How few moves are needed for an {r,s}-mover to make a {p,q} journey? In answering this problem we assume, for now, that the board edges are far enough away so as not to get in the way of any route we may try. Later, the problem of fitting the path onto the smallest possible board will be considered. If a smaller board than the minimum is specified then either the number of moves necessary will increase or the journey will become impossible.

### Lateral and Diagonal Movers

For the wazir {0,1} the answer is easy: i = p+q, with p moves in one direction and q in the perpendicular direction. Shown are the (p+q)!/(p!·q!) = 10 routes of a wazir to make a {2,3} journey of fewest moves, p+q = 5. The more general {0,s}-mover requires i = (p+q)/s moves.



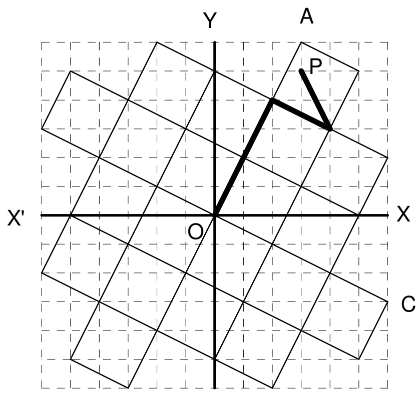


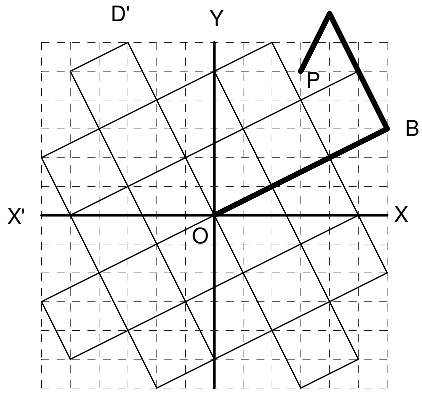
For the fers {1,1} we must have p+q even, since it is confined to squares of one colour. Thus, assuming p ≤ q the minimum number of moves required is, i = q, composed of (q–p)/2 in one direction and p+(q–p)/2 = (p+q)/2 at right angles. Illustrated are the q!/[(p+q)/2]!·[(q–p)/2]! = 4 routes of a fers in making a {2,4} journey. The longer {r,r}-mover requires i = q/r moves.

This leaves us with the more general case of skew movers. Any {p,q} journey by an {r,s} mover is geometrically similar to (that is a magnified version of ) a {p/k,q/k} journey by an {r/k,s/k} mover, where k = hcf (r,s). (The journey equations show that any factor of r and s is also a factor of p and q). We are thus left with the case hcf (r,s) = 1 to solve. The knight is a special case of this.

### Shortest Knight Paths.

The following notes are from H. J. R. Murray (1942): “Any two points on the lattice can always be connected by a standard chain containing three vectors at most, in two ways: $m{\cdot}A + n{\cdot}C \pm Y$ and $p{\cdot}B + q{\cdot}D \pm Z$, where *m*, *n*, *p* and *q* may have any integral values positive or negative, or be zero, and Y is either B or D and Z is either A or C. Consider the position of a point of the lattice with respect to the network of parallels to OA and OC.” [in the figure below]



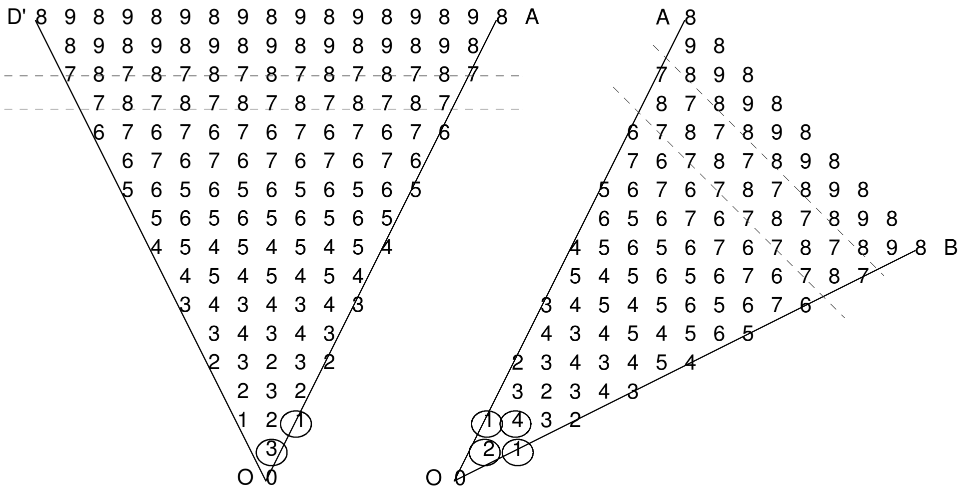




“Every point lies either on a corner of the squares formed by the intersections of paralels to OA and OC, or within one of these squares. There are four points inside every square, and each of these points links with a corner of the square by a vector which is one of B, B ', D and D '.

Thus, a chain O to P is 2·A + C + D ', i.e. 2·A + C – D. A similar relationship of points to the network of parallels to OB and OD holds, and a chain O to P is 3·B – 2·D – A." This property is unique to the knight, but these routes are not necessarily the shortest in number of moves.

Murray continues: "Jaenisch (1862, vol. 1 pp. 188-237) who first used the term 'minimum chain', dealt at length with them, deciding finally that there was no general formula for them." ... "Although Jaenisch decided that there was no formula giving the minimum length of the chain O to P on the lattice, we have only to plot the lengths of the minimum chain O to P for a number of positions of P to see that they must be governed by some law. The field surrounding O is divided into sectors by the vector directions OA, OB, etc, and the minimum chain O to P is conditioned by the bounding lines of the sector in which P is situated. In [the figure below] we give the lengths of the minimum chains from O to P in the sectors bounded by OA and OD ' and by OA and OB."

Murray`s rule, slightly simplified, is that: For points lying between OA and OD ' and on the lines $y = 2 \cdot n - 1$ and $y = 2 \cdot n$, and for points lying between the directions OA and OB and on or between the diagonals $x + y = 3 \cdot n + 1$ and $x + y = 3 \cdot n - 2$ the number of links in the minimum chain from O is n or n+1, the even number of these being on the cells of the same colour as O. The only exceptions to the rule are the cells (0,1), (1,1), (1,2), (2,1), (2,2) shown circled in the figure.




Example of restriction to a board with fixed edges: The knight takes two moves minimum to make a diagonal step (e.g. a2-c3-b1) but in the corner of most rectangular boards requires four moves to get from a1 to b2 (e.g. a1-b3-d2-c4-b2) and on the 3×3 board cannot make the journey at all.

**Shortest Leaper Paths**

The case of a closed path, when {p,q} = {0,0} was considered in the last section. The shortest path is the null path, which means the piece just remains where it is. If it has to move a positive distance then the shortest journey is a switchback (one move out and back again). If it must not retrace any section of its path the shortest journey is a circuit of four moves. If we must not make two moves in opposite directions the shortest path is of six moves: four rides of lengths 1, 1, 2, 2 in some sequence, forming a kite or bow shape. If we require a journey in the minimum number of rides without swtchbacks, the answer is the triangular circuit. The shortest closed journey in four rides of different lengths is the 8,1;7,4 case considered in Puzzle 8 above.

**Houston's Problem**

After the closed circuit the next case to consider is a path to an adjacent cell, {p,q} = {0,1}. This question was posed by A. I. Houston to the Fairy Chess Correspondence Circle in the 1970s.

Consider the journey equations with p = 0, q = 1 (a single wazir step in direction 1).

(1) $0 = (A + D) \cdot r + (B + C) \cdot s$ and (2) $1 = (B - C) \cdot r + (A - D) \cdot s$

For equation (1) to be true we must have, for some odd $m$:

(3) $A + D = m\cdot \mathrm{s}$ and $B + C = -m\cdot \mathrm{r}$

Note: $m$ is odd, since, substituting the values of $D$ and $C$ from (3) into (2) gives

$1 = (2\cdot A - m\cdot \mathrm{s})\cdot \mathrm{s} + (2\cdot B + m\cdot \mathrm{r})\cdot \mathrm{r}$

and if $m$ is even the right-hand side is divisible by 2 and cannot equal 1.

Equation (2) is of the form $1 = X\cdot \mathrm{r} + Y\cdot \mathrm{s}$. If we can find any one solution $(x,y)$ of this then the other solutions are given by $X = x + n\cdot \mathrm{s}$ and $Y = y - n\cdot \mathrm{r}$. We therefore require

(4) $B - C = x + n\cdot \mathrm{s}$ and $A - D = y - n\cdot \mathrm{r}$

Equations (3) and (4) provide us with expressions for $A$, $B$, $C$, $D$ in terms of $x$, $y$, $m$, $n$, r, s:

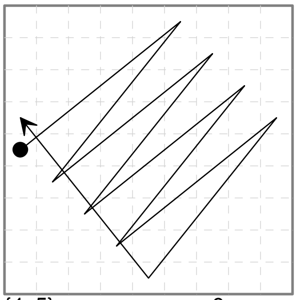
(5) $A = (y - n\cdot \mathrm{r} + m\cdot \mathrm{s})/2$, $B = (x - m\cdot \mathrm{r} + n\cdot \mathrm{s})/2$, $C = -(x + m\cdot \mathrm{r} + n\cdot \mathrm{s})/2$, $D = (-y + n\cdot \mathrm{r} + m\cdot \mathrm{s})/2$

We must choose $m$ and $n$ so that these expressions represent whole numbers.

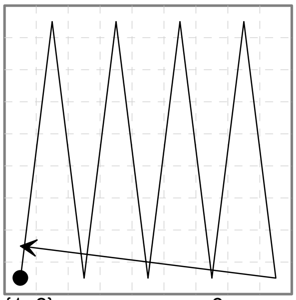
The problem in the above form was posed by A. I. Houston and he solved it for **{r,s} = {z,z+1}**, a case that includes knight, zebra and antelope. We see that (z+1) – z = 1, so $x = -1$ and $y = 1$. Trying $m = 1$ and $n = 1$ we calculate from (5): $A = 1$, $B = 0$, $C = -\mathrm{z}$, $D = \mathrm{z}$, so $\underline{i = 2\cdot \mathrm{z} + 1}$

Another family of cases is **{r,s} = {1,2·z}**, which includes knight and giraffe. Here we can use $1 + 0\cdot(2\cdot \mathrm{z}) = 1$, which gives $x = 1$, $y = 0$. Taking $m = 1$ and $n = 0$ we find from (5): $A = \mathrm{z}$, $B = 0$, $C = -1$, $D = \mathrm{z}$ and so again $\underline{i = 2\cdot \mathrm{z} + 1}$.

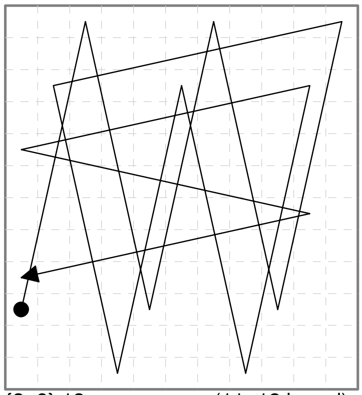
These two cases, illustrated below, conform to the rule i = r+s. In general however r+s ≤ i



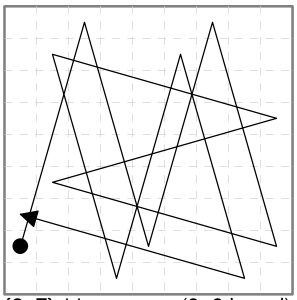
{4, 5}-mover 9 moves



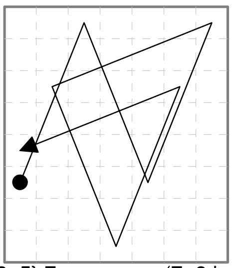
{1, 8}-mover 9 moves



{2, 9} 13 moves (11×12 board)



{2, 7} 11 moves (9×9 board)



{2, 5} 7 moves (7×8 board)

The first cases not included in the above families are {2,5}, {2,7}, {2,9}, ... which are of the form **{r,s} = {2, 2·z + 1}**. A general solution for this family proves a little more difficult. Noting that $(2\cdot \mathrm{z} + 1) - 2\cdot \mathrm{z} = 1$ we have $x = -\mathrm{z}$, $y = 1$. Taking $m = 1$ and $n = 0$ we find from (5): $A = \mathrm{z} + 1$, $B = -(\mathrm{z} + 2)/2$, $C = (\mathrm{z} - 2)/2$, $D = \mathrm{z}$. This only gives whole numbers when z is even. For $\mathrm{z} = 2\cdot \mathrm{k}$, so that $\mathrm{s} = 4\cdot \mathrm{k} + 1$, we have $A = 2\cdot \mathrm{k} + 1$, $B = -(\mathrm{k} + 1)$, $C = \mathrm{k} - 1$, $D = 2\cdot \mathrm{k}$, so $\underline{i = 6\cdot \mathrm{k}+1}$.

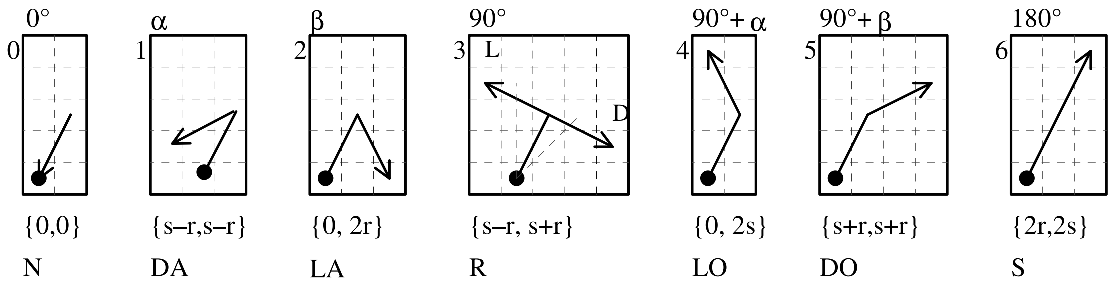
This solves {2,5} in 7 moves and {2,9} in 13 moves. Choosing $n = 1$ instead we find: $A = \mathrm{z}$, $B = (\mathrm{z} - 1)/2$, $C = -(\mathrm{z}+3)/2$, $D = \mathrm{z} + 1$. This only gives whole number values when z is odd. For $\mathrm{z} = 2\cdot \mathrm{k} - 1$, that is $\mathrm{s} = 4\cdot \mathrm{k} - 1$, we have $A = 2\cdot \mathrm{k} - 1$, $B = \mathrm{k} - 1$, $C = -(\mathrm{k} + 1)$, $D = 2\cdot \mathrm{k}$, so $\underline{i = 6\cdot \mathrm{k} - 1}$.

This solves {2,7} in 11 moves.

**PUZZLE** 9: The first cases not covered by the above general rules are the {3,8} and {4,7} movers. Solve these cases and fit the journeys onto the smallest possible boards.

# Two-Move Journeys: Angles

Two-move journeys of a skew mover {r,s} are of seven types when classified in terms of the angle between the two lines. We will number the angles 0 to 6 so that, in the case of the knight, the number measures the angle to the nearest multiple of 30°.




Angles less than a right angle (0, 1, 2) are termed **acute** while those greater than a right angle (4, 5, 6) are **obtuse**. The angles 1 and 5 are termed **diagonal** and the angles 2 and 4 **lateral**, because of the directions of their bisectors and resultants. Thus the six angles can also be described as Null, Diagonal Acute, Lateral Acute, Right, Lateral Obtuse, Diagonal Obtuse and Straight.

The two-move journeys can also be classified in terms of the resultant (p,q) move from beginning to end of the journey. These take the values: 0 {0,0}, 1 {s–r,s–r}, 2 {0,2·r}, 3 {s–r,s+r}, 4 {0,2·s}, 5 {s+r,s+r}, 6 {2·r,2·s}. In the case of the knight the resultant moves, connecting the initial and final cells of the two moves, range over the values {0,0}, {1,1}, {0,2}, {1,3}, {0,4}, {3,3}, {2,4}.

Unlike the other angles, there are always two choices for the right-angle move. When necessary we distinguish them as 3D and 3L (or just D and L) according to whether the second move crosses a diagonal or lateral from the starting point of the first move. [These letters could also be taken to stand for *dextro* and *laevo*, i.e. right and left, or clockwise and anticlockwise, but this may be misleading, since whether an L or D move is to the left or right depends on the direction of the first part of the move. Either type can occur in either direction.] The reverse journeys of 0, 1, 2, 4, 5, 6 have the same code, but each of the right-angle moves is the reverse of the other.

The two acute angles are $\alpha = 2\tan^{-1}[(s-r)/(s+r)]$ and $\beta = 2\tan^{-1}(r/s)$, related by $\alpha + \beta = 90°$. In the case of the knight it is obvious that these angles are different since the tangent of half the smaller angle is 1/3 and of the larger 1/2. In fact there is no leaper for which the angles are equal, since this would require $r/s = (s-r)/(s+r)$ that is $r\cdot(s+r) = s\cdot(s-r)$ or $r\cdot s + r^2 = s^2 - s\cdot r$ whence $(r+s)^2 = r^2 + 2\cdot r\cdot s + s^2 = 2\cdot s^2$ which implies $(r+s)/s = \sqrt{2}$, but the square root of two cannot be expressed as a ratio of whole numbers.

The angle α is smallest if $(r + s)/s > \sqrt{2}$ (as it is for the knight 1.5 > 1.414...), but β is smallest if $(r + s)/s < \sqrt{2}$. The angles α and β are the acute angles of the right-angled triangle with sides in the proportions $(s^2 - r^2) : 2\cdot r\cdot s : (s^2 + r^2)$. In the case of the knight this is the 3 : 4 : 5 triangle.

Murray (1942) gives the values of these angles for the knight in degrees, minutes and seconds as follows: α = 36° 52' 11.4" and β = 53° 7' 48.6".

The shape formed by a sequence of moves can be specified by naming the angles between the pairs of successive moves. Thus a straight rider line would be a series of sixes.

Shapes formed by lines of free-leaper moves must have the angles α, 90–α, 90, 90+α, 180–α, 180 that are the only ones possible between free-leaper moves, α being the most acute angle. In a triangle the three angles add to 180° so the only set of angles possible in a free-leaper triangle is {α, β, 90}, forming an $(s^2 - r^2) : 2\cdot r\cdot s : (s^2 + r^2)$ right-angled triangle, as noted earlier.

**PUZZLE** 10: For which leaper on the 8×8 board are α and β most nearly equal?

**PUZZLE** 11: What leapers other than {k·r,k·s} can make the same angles as {r,s}?

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## Schuh's Theorem

Some general propositions concerning the numbers of the different angles in closed knight's tours were proved by Fred Schuh (1943) which can be generalised to closed journeys by free-leapers of all types. The number of angles in a closed journey, including the angle where the last and first moves meet, may be denoted by N(1, 2, 3, 4, 5, 6) = N(1) + N(2) + N(3) + N(4) + N(5) + N(6) which equals the total number of moves and is thus even. The move pairs with angles 1, 3 and 5 move the leaper an odd number of ranks and files, but N(1, 3, 5) must be even in a closed journey. From the previous two results we must have N(2, 4, 6) even also in a closed path.

Move-pairs with angles 1, 2, 4 and 5 rotate the leaper's move by angles $\pm\alpha$, $\pm(90°-\alpha)$, $\pm(90°+\alpha)$ and $\pm(180°-\alpha)$ clockwise or anticlockwise, and $\alpha$ and 90° are incommensurable (i.e. no whole number multiple of $\alpha$ is a whole number multiple of 90°). The total angle moved through must be a multiple of 360°, since the angle between the last and first moves reorientates the piece to face in the same direction as it started out. Therefore N(1, 2, 4, 5) is even in a closed path, because positive and negative alphas must balance out. Combining results tells us that N(3, 6) is even. Saying 'the sum of two quantities is even' can be expressed as'two quantities are of the same parity', that is, if one is odd the other is odd, and if one is even the other is even. Thus the above results can be summarised by saying that in a closed path the numbers of right (3) and straight (6), diagonal (1 or 5) and lateral (2 or 4) angles are always of the same parity. In a symmetric path they are obviously all even.

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## Three-Move Journeys

Three-move journeys without switchbacks, considered directionally, are of 7×7 = 49 types, since at the second and third moves there is in each case a choice of 7 directions for the next move. We can indicate them by ordered pairs of the angle-codes for two-move journeys. There are 7 three-move paths that are of the same type when traversed in the opposite direction (11, 22, DL, LD, 44, 55, 66). There are 20 not using right angles that are of type uv reversing to vu (12/21, 14/41, 15/51, 16/61, 24/42, 25/52, 26/62, 45/54, 46/64, 56/65). Then there are 20 using one right angle, the reverse of uD being Lu and of uL being Du (u not D or L). There remain the pair of cases DD and LL, each the reverse of the other. The rule is: <u>to code the reverse journey reverse the sequence and replace D by L and L by D</u>. Geometrically there are 7(7+1)/2 = 28 different 3-move paths: 11, 12/21, 1D/L1, 1L/D1, 14/41, 15/51, 16/61, 22, 2D/L2, 2L/D2, 24/42, 25/52, 26/62, LL/DD, DL, LD, L4/4D, 4L/D4, L5/5D, 5L/D5, L6/6D, 6L/D6, 44, 45/54, 46/64, 55, 56/65, 66.

However, there are only 12 types of triple leap when classified in terms of equivalent {p,q} move: 11, 16, $\Rightarrow \{|2\cdot r - s|, |2\cdot s - r|\}$, 12, 1D, 2L $\Rightarrow \{r, |2\cdot r - s|\}$, 1L, 14, L4 $\Rightarrow \{s, |2\cdot s - r|\}$, 15, 24, DD $\Rightarrow \{r,s\}$, 22, 26 $\Rightarrow \{3\cdot r, s\}$, 2D, 25, D5 $\Rightarrow \{2\cdot r + s, r\}$, DL, L6 $\Rightarrow \{|2\cdot s - r|, 2\cdot r + s\}$, LD, D6 $\Rightarrow \{2\cdot s + r, |2\cdot r - s|\}$, 34, L5, 45 $\Rightarrow \{s, 2\cdot s + r\}$, 44, 46 $\Rightarrow \{r, 3\cdot s\}$, 55, 56 $\Rightarrow \{2\cdot r + s, 2\cdot s + r\}$, 66 $\Rightarrow \{3\cdot r, 3\cdot s\}$.

For the particular cases of knight and camel the number is less. For the knight {1,2} we have $\{s, 3\cdot r\} = \{2,3\} = \{r, 2\cdot s - r\}$. For the camel {1,3} we have $\{s - 2\cdot r, 2\cdot s - r\} = \{1,5\} = \{r, 2\cdot r + s\}$.

These results provide another proof that a triangle of three moves by an {r,s} skew leaper is impossible. Neither r nor s is zero so the only case where {p,q} might be {0, 0} is where subtractions occur in both coordinates. The only case is $\{|2\cdot r - s|, |2\cdot s - r|\}$ which requires 2r = s and 2s = r which implies r = s = 0.

**PUZZLE** 12: Why are DL and LD of the same length but in different directions? Related to this question is this knight circuit puzzle, described as a pretty paradox, by W. H. Cozens: Draw two closed knight-paths, identical in shape and size, but one requiring a 9×9 board and the other 11×11. (Solutions p.739)

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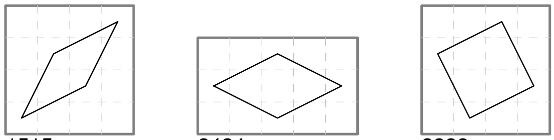
## Four and Six-Move Circuits

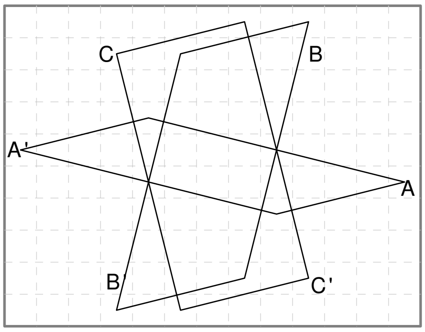
We can combine two 2-move journeys to form a 4-move circuit in three ways (1515, 2424, 3333). In coding a closed path of 2·k moves all 2·k angles are stated, the cyclic sequence being broken to give the smallest multi-digit number. Note that in a convex circuit the angle codes must add to 6·k.

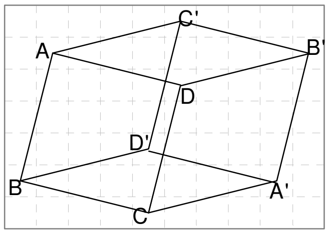


1515 2424 3333

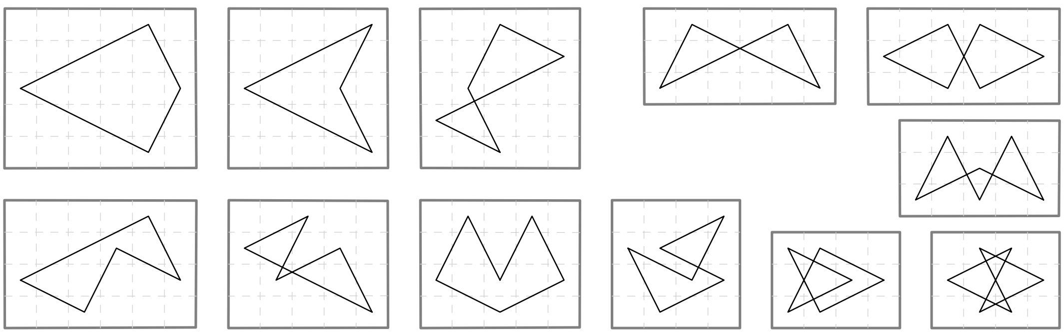
In terms of sequences of angle codes four-move journeys without switchbacks number $7^3 = 343$. This count includes 6 entries for the three four-move circuits (151, 515; 242, 424; DDD, LLL) and 6 entries for three skew-symmetric paths, having an axis of symmetry in an {r,s} direction and formed of right angles and straights (DLD/LDL; D6D/L6L; 6D6/6L6). The remaining 331 codes represent 150 asymmetric, counted twice, and 31 symmetric open paths. These symmetric paths of course give the same code when traversed in either direction.

By combining two three-move paths between the same cells we can form six-move circuits. The following two diagrams (using Giraffe {1,4} leaps as an example) show the 13 different circuits that are always possible with any {r,s} skew leaper. The ‘open book’ pattern gives 9 cases (AA', AB, AB', AC, AC', BB', BC, BC', CC') while the ‘cube’ pattern gives 4 cases (ABC, ABD, ACD, BCD, where for example ABD denotes the hexagon ABD'A'B'D').




In the case of knight {1,2} and camel {1,3} however, there are 12 further circuits possible in addition to the above 13, making 25 in all. [A note in *Fairy Chess Review* (Nov. 1949, p.68) states this total was reported “long ago in *Chess Amateur*”, but I have not been able to trace this reference.]



The reason for the extra circuits being possible by the knight is that certain of the three-move paths in which the end-points are separated by $|2\cdot r - s|$ ranks or files in the general case are on the same rank or file, since for the knight $2\cdot r = s$. Similarly for the camel end-points lie on a diagonal.

The above results account to a considerable extent for the greater touring ability of the knight as compared to other leapers and single it out for study, and the camel has similar attributes.

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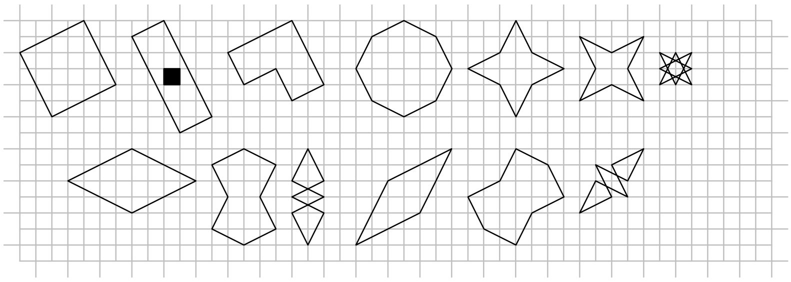
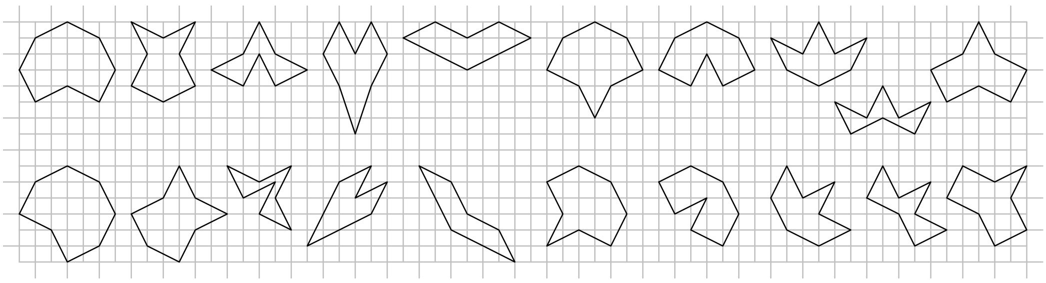
## Eight-Move Symmetric Circuits

The symmetric 8-move circuits were enumerated by T. R. Dawson finding 106 and he illustrated 100 of them in his series of knight`s tours with square numbers in symmetric closed knight chains. (See the Figured Tours in ⌘ 11). The following diagrams are the results of a check on this enumeraton using the angle-coding method. The enumeration is quite tricky, there being several special cases easily overlooked.

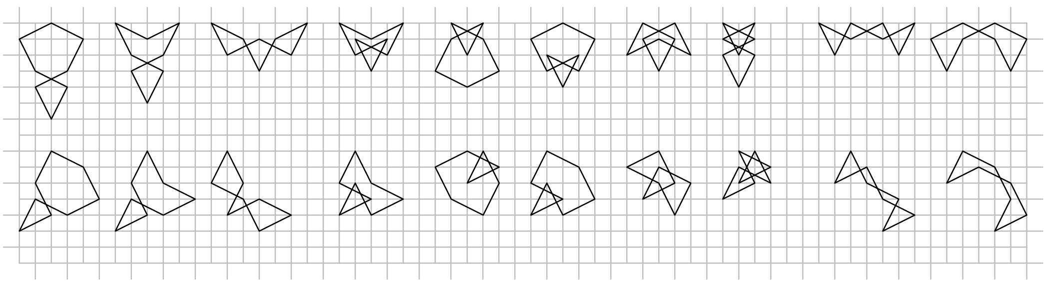
In particular there are three showing skew symmetry (1 with octonary symmetry, 1 with biaxial symmetry and 1 with a single axis). By joining together four equal two-move paths we can form 4 octonary circuits. By joining together two equal symmetric four-move paths we can form a further 6 circuits with biaxial symmetry (3 with lateral axes and 3 with diagonal axes). These occur in modally related pairs, where diagonal angles become lateral and vice versa.



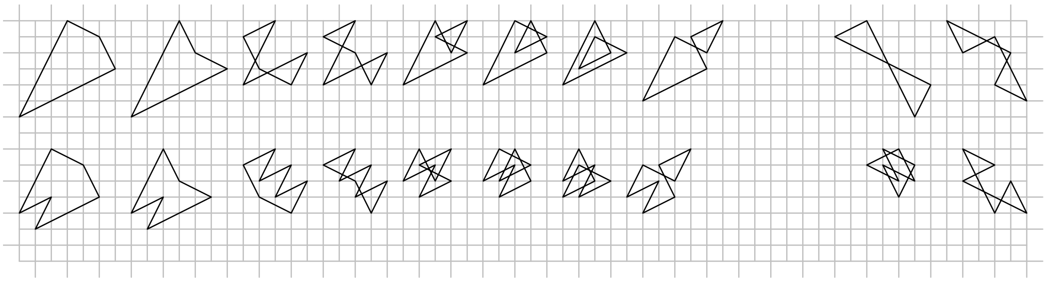
By joining together two different symmetric four-move paths we can (for the general {r,s} skew leaper) form 36 symmetric circuits with a single axis of symmetry, in modally related pairs.



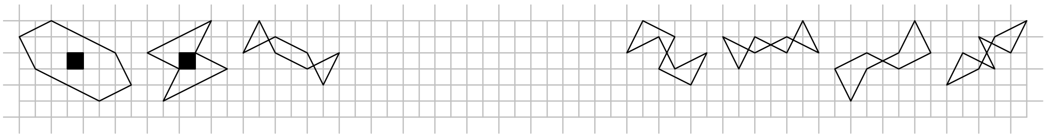
Ten pairs without intersection (above) and eight pairs with intersection. Plus four special axial cases formed by reflecting an asymmetric four-move path in the perpendicular bisector of the line joining its end-points (2 with lateral axis and 2 with diagonal axis) shown on the right here.



In the special cases of knight and camel we have to add a further 16 circuits formed by the first method, and 4 formed by the second method, all with diagonal axis for the knight (shown below) but lateral axis for the camel (not shown). These connections are not possible in the general case since the end-points of the two component paths are at different distances apart in the first method, and are not in the same lateral or diagonal in the second method.

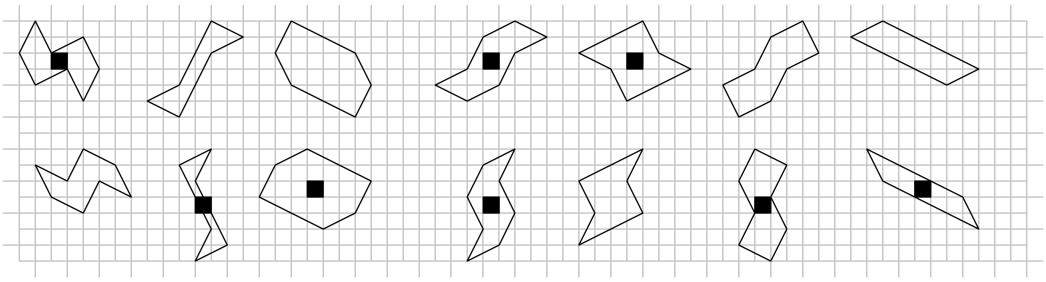


We now come to the cases with pure rotational symmetry. By joining an asymmetric four-move path of an {r,s} skew leaper with a 180° rotated copy of itself we can form a symmetric circuit with pure rotatory symmetry in 29 cases, classified into 3 self-modal forms and 13 modally related pairs.

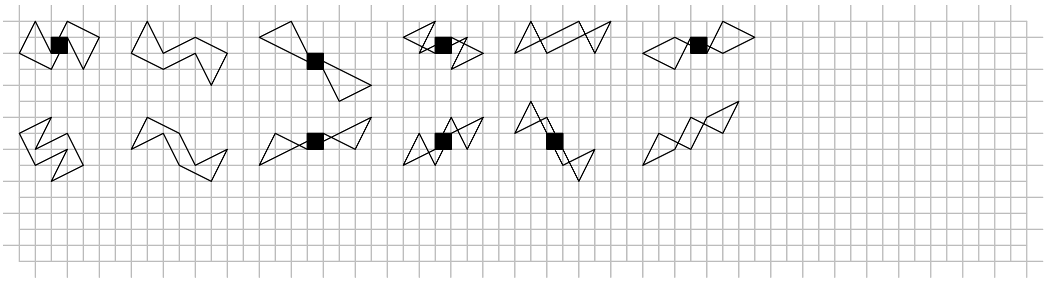


It is very easy to miss four further circuits with pure rotational symmetry that are possible in the case of knight and camel (on the right above). These have Bergholtian symmetry, in which the path crosses at the centre. These cannot be split into two equal four-move paths, but can be analysed into two equal three-move paths whose ends in the case of the knight are separated by {0,1} or {0,2} moves thus permitting them to be joined to the X of knight moves (in the case of the camel moves the ends of the X are separated by {1,1} and {2,2} moves). Alternatively they can be analysed into two three-move paths with rotary symmetry which cross at their mid-points and whose ends can be joined.

In the case of the knight 8 of the 29 circuits self-intersect, but this is not necessarily the case for other leapers: while the circuit 2356 is non-intersecting for the knight or zebra, its modal twin 1346 intersects, but for the giraffe, the first is intersecting and the second not. Seven pairs



and six pairs



The black squares in the above diagrams mark those circuits that can be centred on an even board. They often appear as the linkage polygons in forming a centrosymmetric tours by Simple Linking of circuits.

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# Touring Tests

A net is said to be **tourable** if it has a tour, open or closed. No simple criterion, such as that given by Euler for unicursal routes, has been found to determine whether a net is tourable or not, but some simple tests are possible which help toward the soluton of the question in many cases. The following is a summary of rules that are useful in constructing tours. Most of the rules are obvious, but the results of their cumulative application may be less so.

**1. Forced Moves.** A net with a node of degree 1 has no closed tour but may have an open tour with that node as an end. A net with two nodes of degree 1 if tourable must have those nodes as ends of the tour. A net with more than two nodes of degree 1 has no tour. In a net with nodes of degree 2, the paths through those nodes are forced in a closed tour, or in an open tour whose ends are known. This is the case for the corner cells in any rectangular knight tours for example.

**2. Redundancy.** Forced moves may have a knock-on effect. If links from two nodes of degree 2 meet at another node, then the path through that node, in any closed tour of the net, is determined. Any other links at that node therefore cannot be used in any tour and may be deleted, since they are redundant. Further tests can then be applied to the reduced net. If the net becomes disconnected into two or more nets no tour is possible. This process may be iterative, since the reduced net may contain nodes of degree 2, and these may converge to determine further passage nodes, and further redundant links. The same principle applies in open tours if the ends of the tour have already been established.

**3. Short Circuits.** If the forced moves form a circuit that is not a complete tour then no closed tour is possible, because of this short circuit. However, an open tour may be possible with one end on a node in this circuit. If the passages form two circuits with no node in common, then an open tour if possible must begin in one circuit and end in the other. If passages form three or more separate circuits then no tour, open or closed, is possible.

**4. Snags.** If three or more links from nodes of degree 2 meet at one node we call this node a **snag**. If a snag exists then a closed tour is not possible, but an open tour may be possible with one of the adjacent nodes of degree 2 as an end. If two snags exist an open tour, if possible, must have its ends on nodes of degree 2 adjacent to each of these snags. If a double snag exists (four links from nodes of degree 2 meeting at one node) then an open tour, if possible, must have its ends on two of the adjacent nodes of degree 2. If more than two snags exist, or a more than double snag (five links from nodes of degree 2 meeting at one node) then no open tour is possible. (See p.684 for examples).

**5. Equipartition.** A bipartite net is one whose nodes fall into two sets A and B, with no A-A or B-B links. A closed tour of a bipartite net is impossible unless the two sets have the same number of elements, since each move is A-B or B-A and so the tour visits As and Bs alternately. For an open tour the numbers of As and Bs must not differ by more than one. If one set is in the majority then the ends of an open tour must be on these cells. The preceding rules then apply to the minority cells.

**6. Enclosure.** A set of nodes A may be said to enclose a set of nodes B if the Bs link only to As. If the number of As is equal to the number of Bs then a closed tour is impossible unless they comprise all the nodes. This is proved by noting that in any path the Bs must occur in alternating sequences like ABA, ABABA, ABABABA which always contain more As than Bs, unless the last A coincides with the first, in which case we have a short circuit, except when the As and Bs comprise all the nodes. If there is one more A than B then in a tour the As and Bs form one section of the tour. A striking example is provided by the Giraffe tour on the 9×10 board (see ⌘ 10).

**7. Bottlenecks.** If k nodes of a net with a closed tour are deleted (and all links connected to them) then the resulting net has not more than k components. Consequently, if deletion of k nodes gives more than k components no tour is possible, and the k nodes form a **bottleneck** [J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan 1977, p.53–4. Theorem 4.2].

**8. Completeness.** Obviously, any complete net of more than two cells has a closed tour. The following is a stronger result for nets that approach completeness. A net with $N > 3$ nodes and no node of degree less than $N/2$ has a closed tour [G. A. Dirac, The structure of k-chromatic graphs, *Fund. Math.* 40, p.50, 1953].

[This note is based on my article ‘Generalised Knights and Hamiltonian Tours’ *Journal of Recreational Mathematics*, vol 27, #3, p.191-200, 1995.]

# Symmetry

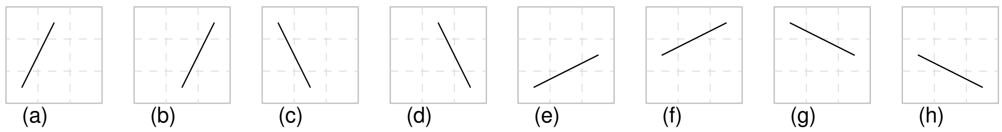
## What are we Counting?

The counting of anything requires first of all that we have a clear idea of what it is we are counting. This seems obvious enough, but there is often considerable confusion arising from carelessness on this basic point. It seems preferable where possible to work with small numbers and accordingly to regard a **pattern**, such as a chess-piece tour, as a geometrical object that can be rotated and reflected and seen in various different orientations. In counting tours we consider only **geometrically distinct** tours (also termed *generic* tours).

This idea was expressed graphically by H. J. R. Murray (1942) in this way: “If we trace a tour in ink on a piece of transparent paper, turn the paper over and ink in the tour as it appears through the paper, and then carefully cut out the diagram, it is impossible for anyone to say with certainty how the tour was oriented from which it was traced”.

In counting patterns a clear distinction must be made between the pattern considered as a geometrical object, and a **diagram** of the pattern by which it is represented on the page. Assuming that a square diagram is printed with its sides parallel to the page edges, one pattern may have eight different appearances according to its **orientation**.

These diagrams show that a simple pattern, formed by a single skew move on a 3×3 board, can have eight different appearances:




The pairs are related by reflections and rotations as follows: a-b, c-d, e-f, g-h **180° rotation** (**half-turn**). a-c, b-d, e-g, f-h, **reflection** in horizontal. a-d, b-c, e-h, f-g, reflection in vertical. a-e, b-f, c-h, d-g, reflection in upward diagonal. a-f, b-e, c-g, d-h, reflection in downward diagonal. a-g, b-h, c-f, d-e, **90° rotation** (**quarter-turn**) clockwise. a-h, b-g, c-e, d-f, 90° rotation anticlockwise.

Anyone familiar with using a drawing program will know that a half-turn can be accomplished by combining reflections in the vertical and horizontal. In fact combination of any two of the operations is equivalent to another or to no change. In mathematical terms they form a group algebra.

If we are considering an oblong pattern instead of a square, we usually follow the convention of having the long side horizontal, so only the first four cases (a) to (d) apply.

## Terminology

The terms used to describe symmetry vary widely, even among mathematicians. The terms adopted here, some of which I have coined myself, are shown in bold, and alternative terms in italic. In the case of a **symmetric** pattern some of the diagrams produced by rotation or reflection of the pattern will look the same as the original diagram. If all eight diagrams are the same I call the symmetry **octonary** since lines through the centre will divide such a pattern into eight congruent components. (This symmetry is traditionally more often called ‘*square*’ or ‘*perfect square*’ symmetry, even if the pattern bears little resemblance to a square.) If the diagrams occur in two sets of four alike the symmetry is **quaternary** and lines can be drawn through the centre to divide the pattern into four congruent components. If the diagrams occur in pairs alike the symmetry is **binary** and the pattern can be divided into two congruent components. If all eight diagrams are different, as in the example above, the pattern might be called **unary**, since it is formed of one indivisible component, but the more usual term is **asymmetric** (or *non-symmetric*)

Instead of measuring the amount of symmetry in a pattern by counting the number c of congruent components we could use the **degree** of symmetry which is defined by $d = \log_2 c$ (i.e. c = 2^d, that is 2 to the power d). Patterns with degrees of symmetry 0, 1, 2 and 3 may then be referred to as *nully*, *singly*, *doubly* and *triply* symmetric, corresponding to unary, binary, quaternary and octonary.

A terminology of this type for tours was favoured by Sharp (1925) but has never become widely used. In cases where c is not an exact power of 2, d becomes fractional. For example when c = 3, which is possible in tours on honeycomb boards (see ⌘ 11), then d = 1.585 approximately.

**Reflections**

A pattern that is unaltered by reflection in a line is said to have **reflective** symmetry, and the line is called an **axis** of symmetry (plural **axes**). A pattern with one axis is said to have **axial** symmetry, and a pattern with two axes is **biaxial** though there is sometimes ambiguity as to whether 'axial' means one or at least one but maybe more (octonary symmetry has four axes.). Murray, following Bergholt (1918), used the term *direct* for axial symmetry, which leads to the longer phrases *direct binary* and *direct quaternary* for axial and biaxial symmetries. However 'direct' is used by Coxeter (1969) to describe any transformation that does not alter the direction of description of curves (i.e. rotations and translations, as opposed to reflections) which is the opposite of Bergholt's sense. An alternative term for axial with one axis is *monaxial* (Sharp 1925).

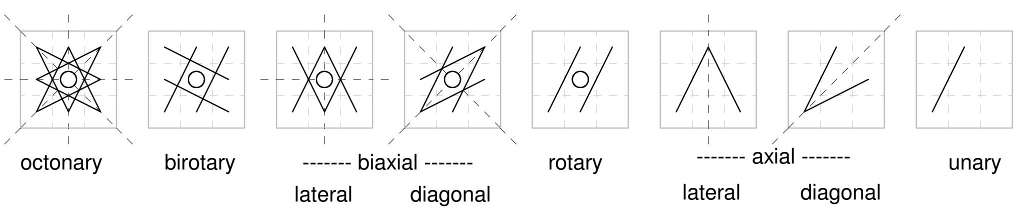
In axial symmetries on chess boards the axes may be **lateral** (parallel to the edges of the board) or **diagonal**. (Murray favoured the term *orthogonal* in place of lateral, but this now has the established mathematical sense of referring to two lines crossing at right angles.) Patterns of leaper moves, considered apart from the board on which they are drawn, can have **skew** axes. For example a square of four successive knight moves. This can be termed **intrinsic** symmetry, but the board itself does not share this symmetry, so the diagram as a whole does not have this symmetry. Axial symmetry is sometimes called *bilateral* (e.g. Weyl 1952) though this may exclude the diagonal case.

**Rotations**

A pattern that is unaltered by rotation about a point (other than by a multiple of 360°) is said to have **rotational** symmetry, and the point is called a **centre** of symmetry. If such a point exists it is unique. A pattern of this type is also said to exhibit **central** symmetry. For brevity I use the term **rotary** for a pattern unaltered only by rotation through 180°, and **birotary** for a pattern unaltered by rotation of 90° (and hence also 180°, or 270° which is just 90° in the opposite direction). No other angle rotation symmetries are possible on chessboards but do become possible on honeycomb type boards (see ⌘ 11). Again there may be some ambiguity as to whether rotary excludes 90° rotation or existence of an axis. Murray and Bergholt used the longer phrases *oblique binary* and *oblique quaternary* for these symmetries, where *oblique* means having rotational symmetry but no axes of symmetry. However 'oblique' has a long-established dictionary sense of 'non-orthogonal' (i.e. diagonal or skew). Alternative terms for rotary are *monorotary* (Sharp 1925), *diametral* (Jaenisch 1862) referring to the fact that corresponding cells are diametrically opposite, i.e. on the same line through the centre of symmetry. French writers (e.g. Feisthamel 1880) use the term *angulaire*. Birotary symmetry might also more graphically be called '*swastika*' or '*windmill*' symmetry.

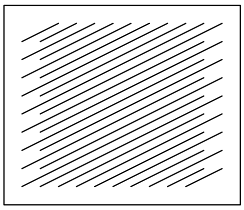
**Combined Symmetry**

A pattern can have both an axis and a centre of symmetry; in fact any pattern that has two axes is necessarily symmetric by rotation about the point of intersection of the axes, the minimum angle of rotation that results in superposition being twice that between the two axes. In the case of patterns of skew leaper moves on square boards eight types of symmetry can be distinguished, in terms of reflections and rotations, as illustrated here by simple patterns of knight moves on the 3×3 board.



## Translational Symmetry

Tours are usually constructed within a circumscribed area. However, move patterns can also be constructed that can be repeated at regular intervals so as to cover an area of any size. This type of design is said to exhibit **translational** symmetry of the type seen in wallpaper patterns. The simplest translatory knight-move patterns are those consisting solely of straight lines. A study I made in 1985 found eight possible patterns of this type, as shown below.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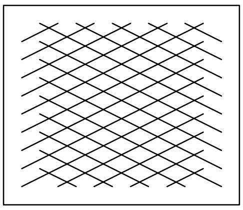
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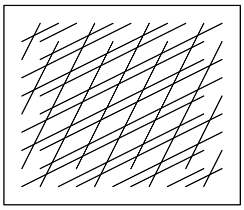
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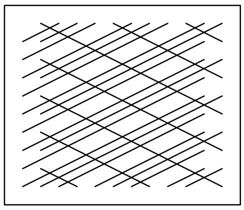
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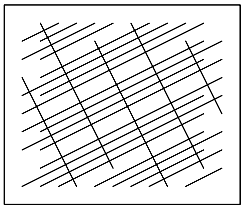
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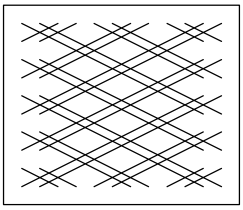
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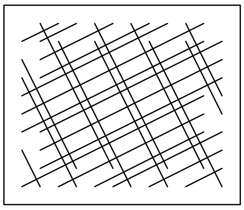
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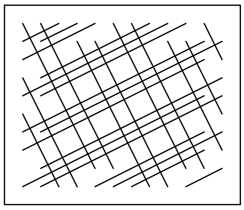

45

Pattern 1 is the basic pattern formed of one set of close-packed parallels all in the same direction. Patterns 2, 3, 4 and 5 are formed from pattern 1 by rotating every second, third, fourth or fifth line through a suitable angle. Patterns 34, 35 and 45 are similarly formed from pattern 1 by rotating every 3rd and 4th, 3rd and 5th, or 4th and 5th pairs of lines. These, and more complex patterns involving non-straight arrangements of moves, can be used to fill areas in large tours.

# Aesthetics of Symmetry

An aesthetic view of symmetry is expressed by Murray (1942): “The graphical diagram of a tour takes the form of a geometrical pattern, and the pattern becomes of special interest when the figure also satisfies the principles of artistic design, that is to say, when the diagram exhibits an orderly or symmetrical repetition of detail. The more evident are these artistic features, the greater is the pleasure afforded by the tour.”

It can however also be argued to the contrary, as has been done in other fields of art such as chess problem composition and in music, that repetition and formal structure are indicative of shortage of original content, or of a mechanistic rather than a naturalistic approach, and conducive to boredom. My own experience is that the pleasure given by a tour comes from the appreciation of the uniqueness of the properties fulfilled by the tour, and symmetric properties are but one type of property, but nevertheless an important type.

In terms of the historical development of the subject the earliest work done on knight's tours, which dates back to the ninth century (circa 850) was largely concerned simply with the problem of how to construct a tour on the chessboard or the half-chessboard without regard to any symmetry. Although mathematicians like Euler and Vandermonde were aware of rotational symmetry it seems that symmetry as understood among the general populace meant axial or reflective symmetry. Rotational symmetry was scarcely regarded as symmetric at all. This is probably because the human eye has developed through natural selection to instinctively react to the axial symmetry seen in nature, particularly in the bodies of animals and humans when seen head-on. The axial symmetry in a diagram is more readily noticed if it is oriented with the axis vertical than when it is horizontal. There is (or has been) a tendency for axial symmetry to be regarded as ‘true’ symmetry and rotational symmetry as ‘pseudosymmetry’ (a term used by T. R. Dawson in the 1940s, though in crystallography it has a different meaning).

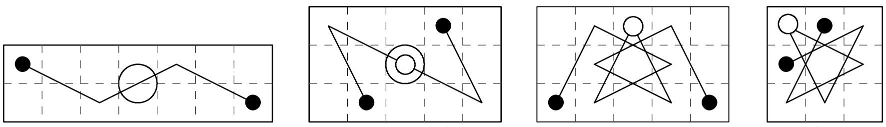
For example, when Ernest Bergholt published some knight's tours with rotary symmetry in *Queen* in 1915 (confusingly calling it ‘bilateral’) he wrote in the following issue (1 January 1916):

“Mr Henry E. Dudeney, who is a great authority on puzzles, but by no means an authority on the meaning and use of words ... writes: ‘I cannot agree with you that your bilateral symmetry is symmetrical at all. Symmetry is a matter for the eye; this is a sort of intellectual symmetry.’ It would appear that Mr Dudeney's eye is differently constituted from the eye of the ordinary person. To me, indeed, these tours convey a most pleasingly symmetrical effect, and I have little doubt that the vast majority of my readers will take equal delight in contemplating them. To Mr Dudeney, Hogarth's well-known line of beauty would be quite devoid of symmetry; he would admit that a capital X might be drawn symmetrically, but not a capital S. Such are the eccentricities of genius.”

The reference is to the painter William Hogarth's book *The Analysis of Beauty* (1753) in which he argued that the eye should be led along a serpentine ‘line of beauty’, though how serpentine it could be was not clear.

## Symmetry in Open Paths

Open knight paths, having two loose ends that must correspond, can only show binary symmetries, or asymmetry. For rotational symmetry the mid-point of the path must be at the centre point of the board. Two cases can be distinguished: (a) The centre of the board is the mid-point of the edge of a cell, and the mid-point of the middle move of the path. The path therefore has an odd number of moves (even number of cells) and uses a board of odd×even dimensions. (b) The centre of the board is the centre of a cell, and the two middle moves of the path meet there, making a two-move straight line. The path has an even number of moves (odd number of cells) and uses a board odd×odd.



(a) move-central (b) cell-central (c) lateral (d) diagonal

For axial symmetry the mid-point of the path must lie on the axis, i.e. on a lateral or diagonal line through the board centre. Since a knight move is always skew it cannot cross a lateral or diagonal axis at right angles, so a knight path in axial symmetry must have an even number of moves (odd number of cells), the mid-point of the path being the centre of a cell on the axis. There are two cases: (c) lateral axis, board odd×any, (d) diagonal axis, board square. Only case (d) can occur on a board even×even (but not as a complete tour).

If the cells in an open path are numbered 1 to C then the numbers in the end cells add to C+1, and in the case of a symmetric open path the numbers in any pair of cells related by the rotation or reflection also add to the constant value C+1. To be more explicit, these are cells x moves from one end, numbered x+1, and x moves from the other end, numbered C–x. When the number of cells C is odd (or the number of moves C–1 is even) there is a middle cell in the path, and the number on it is (C+1)/2, which is the average of all the numbers 1 to C.

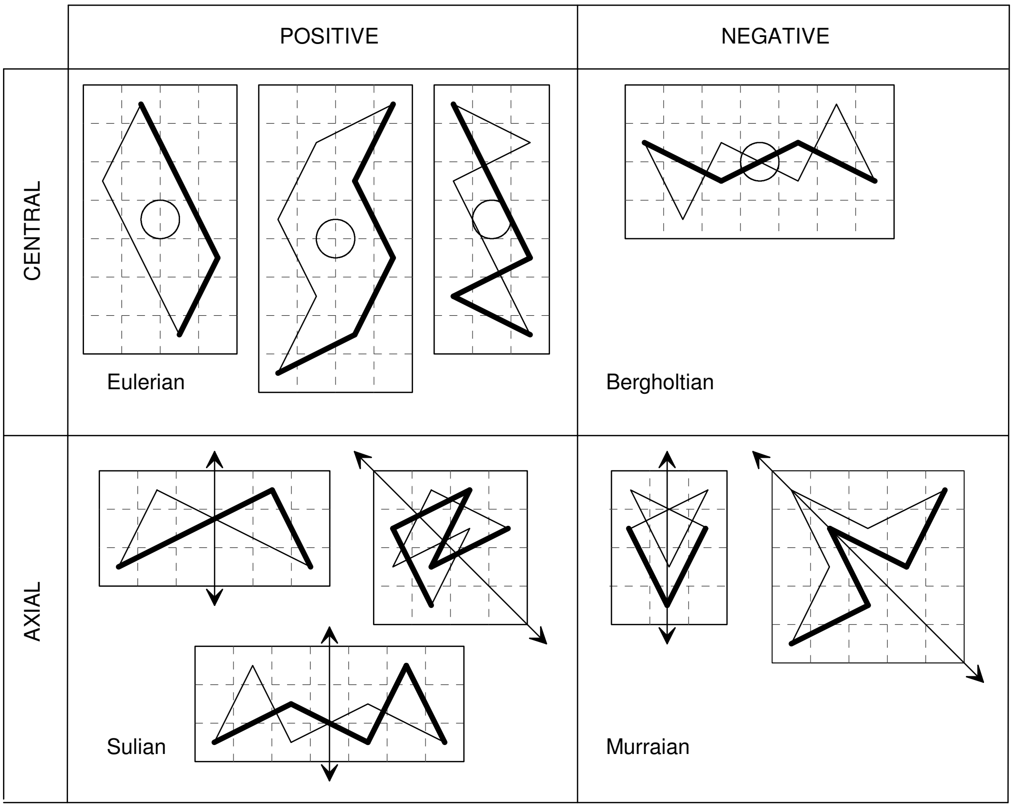
The above observations remain true for any {r,s} free-leaper except for the wazir {0,1} where a straight move along a rank or file exhibits biaxial symmetry. Half-free leapers like the camel {1,3} can pass symmetrically over the centre point of an even board, where four cells meet. Pieces whose move cordinates have a common factor, like the lancer {2,4}, can pass symmetrically over a cell, like the centre point of an odd board.

## Symmetry in Closed Paths

All these types of symmetry can also occur in patterns formed by closed knight paths. If A and A' are a pair of corresponding cells in a closed path with binary symmetry then the closed path consists of two joined open paths A—A'. For binary symmetry these two paths must either be congruent, one being the rotation or reflection of the other (we call this **positive** symmetry), or each must itself be symmetric, with the same type of symmetry (we call this **negative** symmetry). If B and B' are another pair of corresponding cells then in the positive case the points occur in the cyclic sequence ABAB...

along the path, but in the negative case in the sequence ABBA... (or AABB... which is the same cyclically). The symmetries can be central or axial.

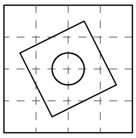
We thus have four types of binary symmetry in closed paths. Namely: **Eulerian** (positive central) which circles round the centre without passing through it. **Bergholtian** (negative central) which passes twice through the centre point. **Sulian** (positive axial) which has no cells on the axis. **Murraian** (negative axial) which has two cells on the axis. These symmetries are so-called since examples were shown by Euler, Bergholt, As-Suli and Murray.



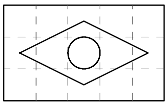

If we number the cells of a closed path 1, 2, ..., C (beginning at any cell and proceeding in either of the two directions) and the cell corresponding to cell u under binary symmetry is the cell numbered u' then the cell corresponding to u+1 will be either u'+1 or u'–1. These correspond to the cases of positive and negative symmetry defined geometrically above. In the positive case numbers in corresponding cells have constant difference u'–u, while in the negative case they have constant sum u'+u. In the positive case we have |u'–u| = C/2 throughout. In the negative case, when the move 1 to C passes through the centre we have u'+u = C+1, but if the origin of numbering is differently placed there will be two different values for the constant u'+u along different sections of the path.

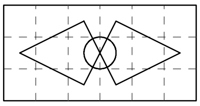
The above symmetry classes can be further subclassified according to whether the number of moves in the connecting paths A—A' are even or odd, or equivalently, whether the centre is on a cell corner, centre or edge or whether the axis is through cell centres or edges. Eulerian and Sulian paths can be even or odd, but Bergholtian must be odd and Murraian must be even.

In cases of quaternary and octonary symmetry the paths A—A' (where A' is the cell related to A by 180° rotation) will be congruent and formed of two congruent parts. Quaternary paths with lateral axes can be either Eulerian or Bergholtian, but all those with diagonal axes or central can only be Eulerian and square. The simplest examples are shown:

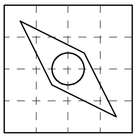


quaternary central

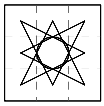




quaternary lateral, positive and negative



quaternary diagonal



octonary

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## Symmetry in Rectangular Knight Tours

Above we make a study of symmetry in leaper paths in general. Here we look more specifically at the symmetries possible in knight tours. Since each move of a knight on a chequered board is to a cell of the other colour, for a closed knight tour the board must contain an even number of cells, so at least one of the sides must be even.

THEOREM (Open Tour): The only symmetry possible in a rectangular open knight's tour is 180° rotative symmetry, and at least one side of the board must be odd. *Proof*: (a) For reflective symmetry of an open path with an odd number of moves the middle move would have to cross the lateral or diagonal axis of symmetry at right angles, which is impossible for a knight's move. (b) For reflective symmetry of an open path with an even number of moves the mid-point of the path must be the centre of a cell on the axis of symmetry. The path cannot enter any other cell on the axis since the other half of the path would by symmetry also enter the same cell, thus making a closed path. For the path to be a tour all cells on the axis must be entered; thus there can only be one cell on the axis, so the board must be 1×1 for diagonal axis or 1×odd for lateral axis, but knights cannot move on such boards. (c) For rotative symmetry of an open path the mid-point of the tour must be either the end-point or the mid-point of a knight's move, i.e. a cell centre (even number of moves) or the mid-point of the side of a cell (odd number of moves). Thus the board cannot be even×even, since at the centre of such a board the corners of four cells meet, so the board must have at least one side odd. (d) The rotative symmetry canot be 90° rotatory since that would require four end-points.

THEOREM (Axial Tour): A rectangular closed knight's tour with reflective symmetry requires a board with one side odd and the other singly-even, and cannot show quaternary symmetry (i.e. can only have one axis of symmetry). *Proof:* (a) If A and A' are corresponding cells, not on an axis of symmetry, in a tour with reflective symmetry, then it consists of two paths A—A'. (b) The two paths cannot be themselves symmetric, with their mid-points on the axis, since there would then only be two cells on the axis and the board would be 2×n on which no knight's tours are possible. In other words the symmetry must be Sulian (no cells on the axis) and not Murraian. (c) The two paths A—A' must be reflections of each other in the axis and there must be no cells on the axis. The axes are therefore lateral, not diagonal, and the cells A and A', being equidistant from the axis and on opposite sides of it, must be of opposite colour when chequered. (d) The paths A—A', connecting cells of opposite colour, are therefore of an odd number of moves. The tour therefore occupies twice an odd number of cells. The board must therefore be odd by singly even, i.e. $(2{\cdot}m + 1)$ by $(4{\cdot}n + 2)$. The axis is the bisector of the even side. (e) It follows that biaxial symmetry is impossible in a rectangular knight's tour since the side parallel to an axis must be odd and the side perpendicular to it even, and if this is true for one axis it cannot be true for the perpendicular axis. In the case of Sulian tours on rectangular boards one side must be odd and the other singly even.

THEOREM (Rotary Tour): A symmetric rectangular closed tour on a board even×even can only be Eulerian, while on a board with one side a multiple of four and the other odd it can only be Bergholtian. On a board singly-even×odd all three types, Sulian, Eulerian and Bergholtian may be possible. *Proof*: (a) As proved above direct symmetry requires singly-even by odd and so is impossible on the first two types of board mentioned. (b) Bergholtian symmetry requires a board odd×even since the centre point must be the mid-point of a knight's move and therefore the mid-point of the side of a cell, so on the even×even board only Eulerian symmetry remains (it can be binary or quaternary, i.e. 180° or 90° type). In Eulerian symmetry on an odd×even board the corresponding cells A and A' (on a line bisected by the centre of the board) are of opposite colour, so the number of moves in A—A' is odd and the number of moves in the tour is twice this (singly even). Therefore if the even side is not singly even (thus a multiple of 4) only Bergholtian symmetry remains feasible.

THEOREM (Birotary Tour): A rectangular closed knight's tour with birotary symmetry requires a square board with singly even side. *Proof*: Such a tour consists of four equal paths, the board must be square, for the 90° rotations to leave it invariant, and even-sided, for closure. On an even-sided square the cell a 90° rotation away is of opposite colour to the original cell, and so the path joining them is of an odd number of moves. The whole tour is thus 4 times an odd number of moves, i.e. it contains 2 as a factor only twice. The side must be $4 \cdot n + 2$ (i.e. 6, 10, 14, 18 ...).

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## Symmetry in Knight's Tours on Square Boards

On square boards the range of overall symmetries possible is limited, compared with more general rectangular or shaped boards. On square boards of odd side $(2 \cdot k - 1)$ the number of cells, $(2 \cdot k - 1)^2 = 4 \cdot k^2 - 4 \cdot k + 1$, is an odd number so tours can only be open, and the only form of symmetry possible is centrosymmetry, that is invariance to 180° rotation, with the two middle moves forming a straight line through the centre cell. If the board is chequered, then a tour must start and finish in a cell of the majority colour, that is the colour of the corner cells and centre cell.

On square boards without holes we can conclude that it is impossible to construct a knight's tour that has exact axial symmetry, in which each move in the left half of the board has a corresponding move in the right half. This was well known for the 8×8 board but a formal proof was not given until Jaenisch (1862). It follows that quaternary symmetry of reflective type is even more impossible.

Quaternary symmetry of rotative type, unchanged by 90° rotation, is possible on square boards but only on those with side twice an odd number $(4 \cdot h + 2)$ that is 6×6, 10×10, 14×14 etc. On boards of side $4 \cdot h$ (that is 8×8, 12×12 and so on) the only possible form of exact overall symmetry in a knight's tour is rotary symmetry, that is invariance to 180° rotation.

**Piecewise Symmetry.** Symmetric knight's tours of even square boards of side $n = 2 \cdot h$ have the property that, when the cells are numbered in the sequence in which they are visited by the knight, from any initial cell, then numbers in diametrally opposite cells differ by $n^2/2 = 2 \cdot h^2$ (i.e. 18 on the 6×6 and 32 on the 8×8). A related problem is to construct knight tours which can be numbered so that the numbers in diametrally opposite cells have a constant difference other than $2 \cdot h^2$. The constant must be even, since diametrally opposite cells are of the same colour and a knight path joining them must be of an even number of moves, since each move is to a different colour. If the constant is $2 \cdot k$ then the tour must consist of $h^2/k$ symmetric circuits, each of $4 \cdot k$ cells, one move being deleted from each circuit and the loose ends joined up to give an asymmetric tour. I call such tours **piece-wise symmetric**. To find these consider first the possible patterns of circuits, forming pseudotours. On the $2 \cdot h \times 2 \cdot h$ board with $h > 2$ a piecewise symmetric tour with constant difference $2 \cdot h$ is always feasible, since h divides $h^2$. Other constants are only possible when h is a composite number.

# Magic

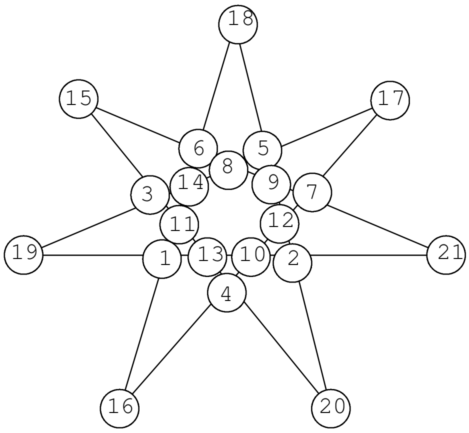
## Magic Arrays

The term **magic** is used in the recreational mathematics literature to refer to a spatial arrangement of numbers (not necessarily positive whole numbers) in which there is a correspondence between arithmetical and geometrical features. The arithmetical property calculated is usually the sum of the numbers in particular subarrays. Magic arrays involving multiplication and other operations can be found in books on mathematical recreations, but we stick to summation magic here.

The array is formed of **cells** where individual numbers are placed. If C is the number of cells and T is the total of the numbers used and A is the average of the numbers then $T = C \cdot A$ and $A = T/C$.

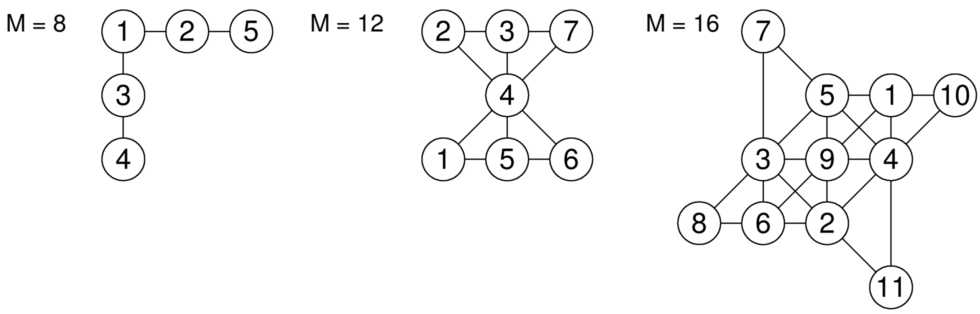
The number to which a sub-array adds is termed a **magic constant**. An array can have several different magic constants. Usually all subarrays of the same geometrical type have the same constant. The attractiveness of such patterns and the improbability of their arithmetical properties led them to be regarded as 'magical' and used as designs on amulets supposed to ward off ill luck.

Here is a magic 7-point star with the numbers 1 to 21 arranged with 6 in each line adding to M = 66. I have a note that I constructed this as long ago as 7 September 1975.




THEOREM: If a magic array can be partitioned into s magic subarrays each with r elements adding to a magic constant M then $M = r \cdot A$. *Proof*: If s magic subarrays, each adding to M, form a partition of the array (i.e. they are non-overlapping subsets, and include every cell) then $T = s \cdot M$, so $M = T/s = C \cdot A/s$. If the subarrays are also all of the same size r then $C = r \cdot s$ and so $M = r \cdot A$. QED.

We call a magic array **normal** if any magic subarray of r entries adds to $r \cdot A$, where A is the average. Thus any equipartitioned magic array is normal. The converse however is not true. A normal magic array may be unpartitioned, as in the M=66 example above or the M=12 example below.




A simple example of an **abnormal** magic array is the gnomon or cross formed of 1, 2, 5 across and 1, 3, 4 down, adding to 8. The average is 3, so the normal magic constant would be 9. The M=16 example is also of this type. This is numbered 1 to 11 with average 6, but 3 times this is 18.

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# Magic Rectangles

A **magic rectangle** is a rectangular array of numbers (of any type) in r ranks and s files (denoted r×s) in which the ranks and files are magic subarrays. In this case we have $C = r \cdot s$ and $A = T/(r \cdot s)$.

THEOREM: <u>All magic rectangles are normal magic arrays.</u> *Proof*: If the r ranks each of s elements add to a constant R then we must have $R = T/r = s \cdot A$. Likewise for the file sum $S = T/s = r \cdot A$. The magic constants are multiples of A. QED.

This also follows from the theorem about partitioned magic arrays being normal, since magic rectangles are partitioned into ranks and into files. The magic constants R and S are not independent, $R = s \cdot A$ and $S = r \cdot A$, so we always have $R/s = S/r = A$, and $s \cdot S = r \cdot R = T$, and $S/R = r/s$.

A **semi-magic** rectangle is one in which the lines in one direction, either the ranks or the files, are magic, but those in the other direction add to two or more values. In diagrams of semi-magic arrays in these Knight's Tour Notes we usually follow the convention of showing the magic lines vertically (i.e. as files) and drawing the top and bottom borders with bold lines (suggesting an addition sum). On the same principle, magic rectangles are drawn with all four sides bold.

It is natural to construct semi-magic arrays on those boards on which magic arrays with desired properties are impossible or have not been found. Special types of semi-magic array include **quasi-magic** in which the ranks add to two different values and **near-magic** in which they add to the magic constant and two other values. Also of interest are **demi-magic** squares where the files add to two totals and the ranks the same two.

It follows from our definition of a magic rectangle that a **magic square** is a square array in which the ranks and files are magic subarrays. When the rectangle is square we have r = s and R = S. A magic square in which the two main diagonals also add to the magic constant we call a **diagonally magic** square (or **diamagic** for short).

[Caution: In some literature on magic squares the diagonal condition is taken for granted. A non-diagonal magic square is then sometimes misleadingly termed 'semi-magic'.]

It is possible in some cases for the **broken diagonals**, which can be converted into long diagonals by a cyclic permutation of the ranks or files, to also add to the constant, we then have a **pandiagonal magic square** (or **panmagic** for short). Terms like 'diabolic' or 'nasik' are also used.

Another interesting extra condition on a magic square is that it have a form of supersymmetry. For example a 7×7 square can have **H-supersymmetry**, where certain areas of 7 cells in the shape of an H (upright or rotated) add to the magic constant.

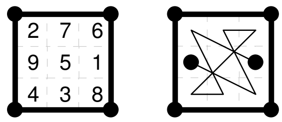
THEOREM: <u>From any magic rectangle r×s we can derive r!s!/4 magic rectangles by permutation and transposition of its ranks and files</u>. *Proof*: Permuting the ranks (which can be done in r! ways) or files (s! ways), or transposing them (to form an s×r rectangle from the r×s one, by reflection in a diagonal) or leaving the same, does not affect the uniform-sums property, so it is possible to derive from any magic rectangle of size r×s a total of 2.r!.s! magic rectangle diagrams. Some of these will be the given array and its rotations and reflections which form a set of 8. So the total geometrically distinct is 2.r!.s!/8 = r!.s!/4. QED.

These factorial products rapidly increase. For example from one 6×6 array we can derive 2.6!.6! = 1,036,800 diagrams, and we are already over a million! We must therefore be more selective in the patterns we study. The restriction to diagonal magic squares is a popular example.

From the geometrical point of view we are particularly interested in tours that use only a limited number of geometrically different types of moves. A tour that uses only knight moves for instance is a **knight tour**. The piece that makes the tour is defined by the various different moves used. When more than four different moves are used the subject moves into a more general realm where the tour path is of less interest. An open tour may be termed **reentrant** if the move connecting its end point to its start point is of the same geometrical type as at least one of the moves used in its path. Thus for example a knight tour is reentrant if its ends are a knight move apart.

# Some History of Magic Squares

**The 3×3 Magic Square.** The arrangement of the first nine numbers in a square so that all eight lines of three add to the total 15 is the oldest known magic square, and is said by many sources to have been known in China in –2200, or even earlier [e.g. Müller *The Yi-King* 1882] According to legend the design was found in a pattern of spots on the back of a river turtle. However, the earliest literature of definite date in which it is cited is by Shu Ching (–650) who refers to the *Lo Shu* {River Plan} which was the name given to the pattern [Singmaster 1996]. The square can be presented in eight different orientations by rotation and reflection (but keeping the number symbols the right way up!). There are eight sets of three of the numbers 1 to 9 that add to 15 and they are all used in the lines of the 3×3 square. This is a feature of completeness that is not possessed by magic squares of larger sizes. The sets are of course: (1,5,9) (1,6,8) (2,4,9) (2,5,8) (2,6,7) (3,4,8) (3,5,7) (4,5,6).



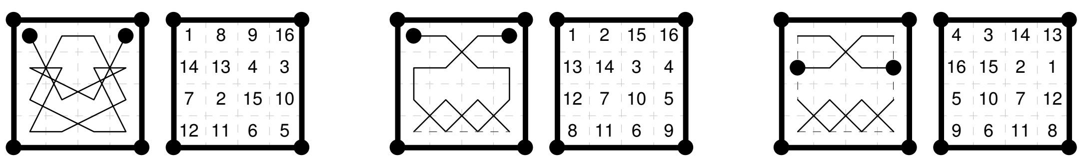

If the successive numbers in the 3×3 magic square are joined by straight lines the result is a tour of the 3×3 board by a piece making moves with coordinates {0,1}, {1,1} and {1,2}. Among players of chess and its variants pieces with these moves are known as ‘wazir’, ‘fers’ and ‘knight’. The closure move uses a fourth type the ‘dabbaba’ {0,2}. According to Pavle Bidev [1986, part 2, p.34] an Arabic encyclopaedia edited at Basra in 898 by members of the philosophical association ‘Brothers of Sincerity’ (*Ikhwan al-safa*) describes the method of constructing the 3×3 magic square by moves of knight, fers and pawn.

The knight's move, which some take to be the defining element of chess, may perhaps have first been seen in the path followed by the successive numbers in the 3×3 magic square. The tour is a geometrical pattern which is ‘centrosymmetric’, unaltered by 180° rotation. The vertical black lines indicate that the ranks add to the magic constant, and the horizontal black lines that the files add to the constant. The dots at the corners indicate that the diagonals also add to the constant in this case.

**The 4×4 Magic Squares:** There is a well-known 10th-century 4×4 magic square on display in the Parshvanath Jain temple in Khajuraho, India [Bidev, 1986, part 2, p. 34] and the subject is said to have been known in India since the first century. Another famous example is shown by Albrecht Durer in his *Melancholia* engraving of 1514, showing the date in the bottom row.

Around 1660 **Bernard Frénicle de Bessy** (c.1605-1675) compiled a catalogue of all possible diagonally magic squares of order 4. This was eventually published posthumously in 1693. [*Des Carres Magiques, Divers Ouvrages de Mathem. et de Physique* (Par Messieurs de l'Academie Royale des Sciences de Paris 1693) p.423-507, and reprinted 1729] A computer print-out of the list is given in *New Recreations with Magic Squares* by William H. Benson and Oswald Jacoby (1976).

Very surprisingly all but one of the diagonal magic squares require three or more types of move. The single exception is number 100 in the list. This uses only wazir and knight moves, and in variant chess terms is thus an ‘emperor’ tour. This special case does not seem to have been noticed until I reported it in *Chessics* (#26, 1986, p.19). As I also noted there, two of the magic squares (2 and 619 in the list) can be interpreted as magic queen tours, but these use three or four different moves. For clarity in these diagrams the longer moves of two or three steps are shown dashed to avoid confusing them with the shorter moves. These three tours all have axial symmetry.



**The Frénicle Convention**: For the purposes of comparision and cataloguing it is useful to have rules for displaying a tour or numbered array in a standard orientation. A scheme often adopted is that used by Frénicle. This is to reflect or rotate the array so that the smallest corner entry is at the top left. In the case of a square then the rule also applies that the number to right of the corner should be less than the number below the corner. To save space it is usually most convenient to place an oblong array with its longer dimension horizontal. However no scheme is entirely satisfactory for all purposes. The main disadvantage of this rule, from the point of view of tours, is that the first cell, if not in the top left corner, can appear in any quadrant, and the first move can be in any direction.

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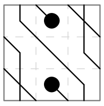
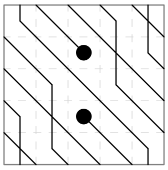
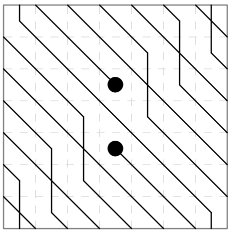
## The Step-Sidestep Method

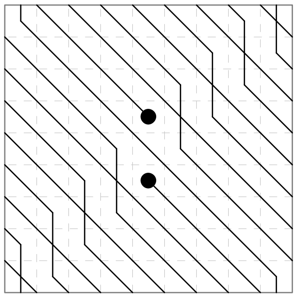
A systematic way of constructing magic squares on odd-sided boards is to begin from any cell and make a seres of moves of one type, the step, in a straight line, continuing across the board edges and reentering on the opposite edge as if the board is bent round to form a cylinder or torus. When no further step of this type is possible a sidestep move is interpolated.

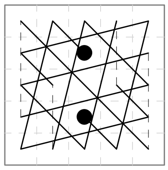
The earliest account of methods of this type seems to have been published by **Manuel Moschopoulos** (born circa 1265 at Constantinople) who wrote a work on magic squares around 1315. [See J. C. McCoy (1941) and P. G. Brown 'The Magic Squares of Manuel Moschopoulos' *Loci* (Jul 2012) and on the (MAA) website.]. Beginning with 1 at the cell below the centre, move diagonally down to the right. When you meet the edge of the board you reenter at the opposite edge. When you meet an already used cell you move two cells down and then resume the diagonal moves. This results in all the ranks and files and the two diagonals adding to the same sum.











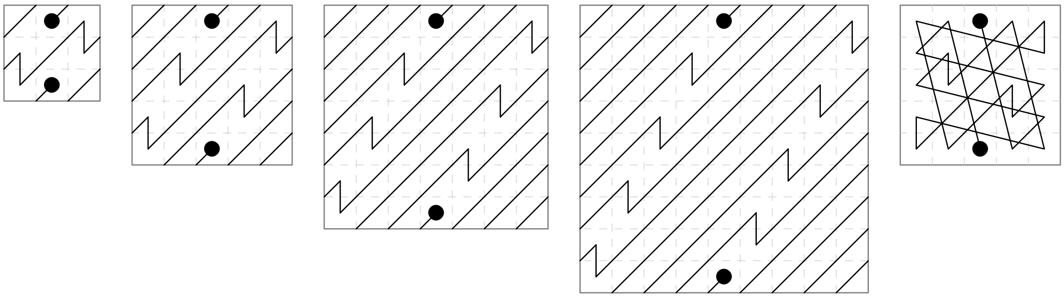
Starting with the 1 in a different cell still gives a square that is magic in the ranks and files, but usually without the diagonals being magic. The 3×3 magic square is the simplest case.

When considered as moves on an ordinary board of side n, instead of a torus, the number of types of moves used increases from two to four, the extra two being moves of types {1,n–1} and {0,n–2}, as shown for the 5×5 case on the right using moves {1,4} and {0,3}.

Moschopoulos also describes a second method for odd sided squares in which the start cell is the one in the middle of the top rank, the step is a knight move down to the right, and the sidestep is a move of 4 units down (which on the 3-board is equivalent to a move of 1 unit). He gives examples of size 3, 5 and 7. The 3 case being of course the usual 3×3 magic square. However the results on larger squares by this method are not symmetric. His methods for even-sided squares cannot be described in terms of simple tours.

A later account of a similar method was published by **Simon de la Loubère** (1642-1729) who describes a version of the step-sidestep method which he learnt while French envoy to Thailand in 1687-88. [*Du Royaume de Siam* English Translation, London 1693 vol.2 p.227-247, cited in Rouse Ball (1939)] The rule is to place 1 in the middle of the top rank, to step diagonally upwards to the right one cell (regarding the board as a torus) and to sidestep one cell down when blocked. Again the 3×3 magic square is a particular case.

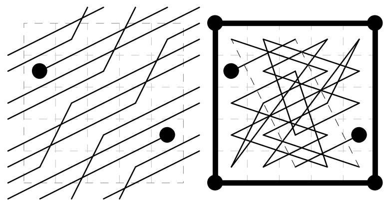
Considered as a tour on the normal board of course three types of move are used. The third being a {1,n–1} leap across the n×n board, as shown for the 5×5 case on the right. Also the closure move is of a fourth type {0,n}



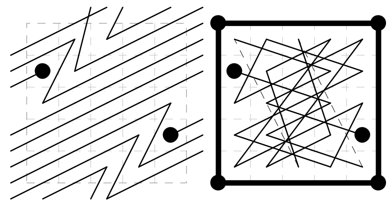
The book by William Symes Andrews [*Magic Squares and Cubes* 2nd edition 1917, reprint 1960] includes, as Figures 20 and 19 on page 11, the two examples below (though differently oriented) in which both the step and the sidestep are knight moves. Some sources attribute these tours to Moschopoulos but this does not seem to be correct. They are a much later development.

On the normal board the {1,2} moves become in addition {1,3}, {2,4} and {3,4} moves where they cross the edges of the torus. The diagrams are shown here with the middle number, 13, in the centre cell, and oriented according to the Frénicle rule. The magic sum is 5×13 = 65.

| | | | | |
|---|---|---|---|---|
| 3 | 11 | 24 | 7 | 20 |
| 25 | 8 | 16 | 4 | 12 |
| 17 | 5 | 13 | 21 | 9 |
| 14 | 22 | 10 | 18 | 1 |
| 6 | 19 | 2 | 15 | 23 |



| | | | | |
|---|---|---|---|---|
| 4 | 15 | 21 | 7 | 18 |
| 25 | 6 | 17 | 3 | 14 |
| 16 | 2 | 13 | 24 | 10 |
| 12 | 23 | 9 | 20 | 1 |
| 8 | 19 | 5 | 11 | 22 |



Magic squares of all sizes were constructed during the middle ages, but mostly the same ones were repeated. We show many other examples in ⌘ 10 classified by move types.

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## Natural Magic

The ‘natural numbering’ of a rectangular board r×s is to start with 1 at the top left corner, and following the same convention as for writing, proceed along the row, then jump back to the first cell in the next row, and so on. This numbering can be seen as a case of the step-sidestep construction. In terms of moves it proceeds in {0,1} wazir moves along the rank followed by a leap of type {1,s−1} from the end of the rank to the start of the next, making it a tour by a two-pattern leaper. Alternatively, if we regard the board as a cylinder, in which the right-hand side of the sheet is curved round to join the left-hand side then the {1,s−1} step becomes a {1,1} fers move on the cylinder, and the natural numbering is then a King tour. If we number the files, x, from 1 to s and the ranks, y, from 1 to r, so that each cell is specified by the ordered pair of coordinates (x,y) then the number assigned to cell (x,y) is $x + s\cdot(y - 1)$.

Because it is so familiar, the magic properties of the natural numbering are often overlooked. The ranks and files do not add to a magic constant but other sets do. The main magic property is that in a square the numbers in any **satin** (a set containing one cell from each rank and file) add to the same total (the magic constant). Thus in the case of the chessboard the magic constant is 260 and any arrangement of eight rooks or queens so as to guard all the cells of the board except those on which they stand has the property that the numbers of the cells on which the pieces stand add to 260.

THEOREM: <u>In a natural-numbered square every satin adds to the same total</u>. *Proof*: A satin in a square m×m is a set of m cells with one in each rank and one in each file. Thus in a satin in a naturally numbered square we have m numbers of the form $x_i + m\cdot(y_i - 1)$ where i takes the values 1, 2, ..., m. Since there is one entry in each file this ensures that each value of x occurs exactly once. Similarly each value of y occurs exactly once. The sum of the numbers in the satin is

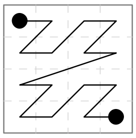
$$\Sigma(i)[x_i + m\cdot(y_i - 1)] = \Sigma(i)[x_i] + m\cdot\Sigma(i)[y_i - 1] = [1 + 2 + ... + m] + m\cdot[0 + 1 + ... + (m - 1)]$$
$$= m\cdot(m + 1)/2 + m\cdot[m\cdot(m - 1)/2] = m\cdot(m^2 + 1)/2,$$ which is the magic constant.

This property of the satins all adding to the same total is unaltered if the ranks or files of the square are permuted, or of the square is reflected in a diagonal (that is, transposed).

Any array with the sums of satins property I call a '**satinic**' square. Natural numberings and their permutes are thus particular cases of satinic squares. In any satinic square the pairs of numbers at opposite corners of any rectangle add to the same total, since in a satin any pair of entries can be replaced by the entries at the other corners of the same rectangle and still leave a satin, and the new satin still adds to the same total.

Also in any satinic square two parallel lines (ranks or files) differ by the same number in every pair of cells. Here is a satinic square that is not a permute of a natural numbering:

| | | | |
|---|---|---|---|
| 1 | 2 | 5 | 6 |
| 3 | 4 | 7 | 8 |
| 9 | 10 | 13 | 14 |
| 11 | 12 | 15 | 16 |



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## Semi-Natural Magic

Some magic squares can be constructed by a simple transformation of the natural numbering. The simplest examples are 'semi-natural' magic squares which have half their numbers in natural order and half in the reverse of natural order. In 1997 I was sent copies of a magazine *Irregular News* by Ricardo Calvo, published in Madrid. Issue 6 contained diagrams of two 8×8 diagonal magic squares named the 'Safadi' and 'Mercury' squares, shown below. The Mercury magic square occurs as Figs.53 and 91 and the Safadi square as Fig.94 in Andrews (1917) though not under those names. By studying these I found there is a third diamagic square of this type, which I call the 'Intermediate'. (Magic squares having been worked on for centuries I don't expect this is an original result.)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 62 | 61 | 60 | 59 | 7 | 8 |
| 9 | 10 | 54 | 53 | 52 | 51 | 15 | 16 |
| 48 | 49 | 19 | 20 | 21 | 22 | 42 | 41 |
| 40 | 39 | 27 | 28 | 29 | 30 | 34 | 33 |
| 32 | 31 | 35 | 36 | 37 | 38 | 26 | 25 |
| 24 | 23 | 43 | 44 | 45 | 46 | 18 | 17 |
| 49 | 50 | 14 | 13 | 12 | 11 | 55 | 56 |
| 57 | 58 | 6 | 5 | 4 | 3 | 63 | 64 |

Safadi

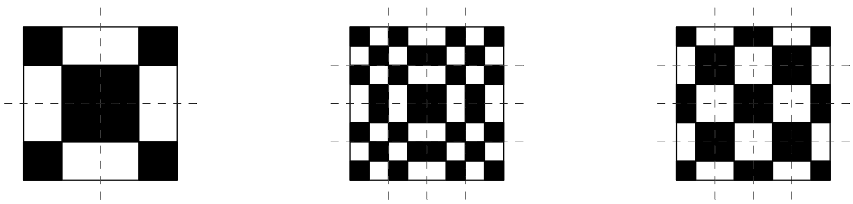
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 63 | 3 | 61 | 60 | 6 | 58 | 8 |
| 56 | 10 | 54 | 12 | 13 | 51 | 15 | 49 |
| 17 | 47 | 19 | 45 | 44 | 22 | 42 | 24 |
| 40 | 26 | 38 | 28 | 29 | 35 | 31 | 33 |
| 32 | 34 | 30 | 36 | 37 | 27 | 39 | 25 |
| 41 | 23 | 43 | 21 | 20 | 46 | 18 | 48 |
| 16 | 50 | 14 | 52 | 53 | 11 | 55 | 9 |
| 57 | 7 | 59 | 5 | 4 | 62 | 2 | 64 |

Intermediate

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 63 | 64 | 4 | 5 | 59 | 58 | 8 |
| 56 | 10 | 11 | 53 | 52 | 14 | 15 | 49 |
| 48 | 18 | 19 | 45 | 44 | 22 | 23 | 41 |
| 25 | 39 | 38 | 28 | 29 | 35 | 34 | 32 |
| 33 | 31 | 30 | 36 | 37 | 27 | 26 | 40 |
| 24 | 42 | 43 | 21 | 20 | 46 | 47 | 17 |
| 16 | 50 | 51 | 13 | 12 | 54 | 55 | 9 |
| 57 | 7 | 6 | 60 | 61 | 3 | 2 | 64 |

Mercury

The following are the patterns of the squares occupied by the moved and unmoved numbers:



In the Safadi the quarters are chequered boards, in the Mercury the sixteenths (i.e. the 2×2 blocks) are chequered; the Intermediate has both these properties. [I presented this item in *The Games and Puzzles Journal* (#15, p.252, 1997).] These magic squares use seven or more different move types which is the limit of our study. A magic square by H. J. Kesson (1881) similar to these but using only four move types appears in the section on Giraffe tours (see ⌘ 10).

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# Magic Tours

If we mark one of the ends of an open tour, or any cell of a closed tour, as its start or finish, and identify the direction of travel along the path, usually shown by an arrow, we have a **directed tour**. An open tour determines two distinct directed tours, and a closed tour of C cells 2·C directed tours.

By a **numbering** of an array of C cells we mean the assignment of the numbers 1, 2, ..., C to the cells. Any such numbering corresponds to a directed tour of the array. Conversely any directed tour defines a unique numbering. In other words there is a one-to-one correspondence between them. In most of the historical literature geometrical tours have been shown in this arithmetical form, since reproduction of geometrical diagrams was difficult and expensive until more recent developments.

The average value of the entries in a numbered array is A = (C+1)/2, the mean of the first and last entries. On any numbered array of C cells the first and last, 1 and C, the second and next to last, 2 and (C–1), the third and second from last, 3 and (C–2), and so on, always add to C+1 = 2·A.

Pairs of numbers adding to 2·A are called **complementary**. The **complement** u' of a number u is such that u + u' = 2·A. That is, u' = (2·A) – u. When C is odd A is the middle number in the sequence and is its own complement. When C is even A is fractional. The total of all the numbers 1, 2, ..., C is, by the well known rule for summing an arithmetic progression, the triangular number T = C·(C+1)/2 = C·A. The pairs of complements adding to 2·A are a trivial case of normal magic.

We can now combine the concepts of magic array and tour. A **magic tour** is of course a tour that in one or more of its arithmetical forms has non-trivial magic properties.

THEOREM: <u>The reverse numbering of a magic tour is also magic.</u> *Proof*: Reversing the numbering of a tour replaces each number u by its complement 2·A – u. Thus any set of entries $x_1$ to $x_k$ adding to M is replaced by 2·A – $x_1$ to 2·A – $x_k$ which add to k·2·A – M, which is still constant, but may be different from M. QED.

Reversibility also applies to semi-magic, quasimagic, nearmagic, and demimagic tours. It also follows that the reverse of a normal magic tour is normal. We have M = k·A so the constant in the reverse is 2·k·A – M = 2·k·A – k·A = k·A, the same.

THEOREM: <u>A magic tour renumbered in any other arithmetical progression is still magic</u>. *Proof*: Suppose 1, 2, ..., C replaced by c + d, c + 2·d, ..., c + x·d, ..., c + C·d. Then a set of entries $x_1$ ... $x_k$ adding to M in the original numbering is replaced by the entries c + $x_1$·d ... c + $x_k$·d whose sum is k·c + ($x_1$ + ... + $x_k$)·d = k·c + M·d which is the same for any set of k adding to M. QED.

This means there is no loss of generality in confining our study to magic tours with the usual ordinal numbering. It is however sometimes helpful to replace the ordinal numbering 1 ... C by the **cardinal** numbering 0 ... C–1 which may reveal patterns in the numbers more clearly.

THEOREM: <u>Any k numbers, not A, can occur in a normal magic set of 2·k or 2·k + 1 numbers</u>. *Proof*: Sets consisting of k pairs of complements (when n = 2·k) together with the central number A (when n = 2·k + 1) add to n·A, i.e. are normal magic sets. If n is even than any n/2 numbers can occur together in a normal magic set since any can be paired with its complement, or if its complement is already in the set then two of the empty spaces can be filled by any other pair of complements not so far used. If n is odd then we fill the extra space with the centre number. QED

We can extend this result. If n is even and n/2 < k < n then k numbers can occur together in a normal magic set provided they satisfy the inequalities: (n·A − sum of largest n−k numbers) ≤ (sum of the k numbers) ≤ (n·A − sum of smallest n−k numbers). Denoting the sum of the k numbers by Σk we require: (C+1)·(2·k − n)/2 + (n−k)·(n−k+1)/2 ≤ Σk ≤ (C+1)·n/2 − (n−k)·(n−k+1)/2.

Thus for example, when C = 64 and n = 8 we have 196 ≤ Σ7 ≤ 259, 133 ≤ Σ6 ≤ 257, 71 ≤ Σ5 ≤ 254 (and 10 ≤ Σ4 ≤ 250, which is true for any four numbers). When C = 36 and n = 6 we have 75 ≤ Σ5 ≤ 110, 40 ≤ Σ4 ≤ 108. These conditions allow one to eliminate many random arrangements of numbers as being non-magic because they have too many large or small numbers together in the same set. [This note is based on a more extensive account in Murray (1942).]

A cyclic renumbering of a tour, replacing u+1 by 1, u+2 by 2 and so on may also be magic in special cases. Thus a **cyclic** magic tour is a closed geometrical path that can be numbered from more than one cell to give two or more different arithmetical magic tours. The arithmetical tours derived from the same geometrical form can have different numerical properties; for example in one numbering of a square tour the diagonal sum may be maximum, while in another it may be minimum.

THEOREM: If a magic tour remains magic when cyclically renumbered from u+1 as 1 then u must be a multiple of the number of magic subarrays. *Proof*: The renumbering will increase a certain number i of entries in a subarray of e entries by C–u and decrease the other e–i entries by u. Thus for the subarray to remain magic we must have (e–i)·u = i·(C–u) that is e·u = i·C so since u is a whole number and i is less than e, then e must divide C that is u = i·(C/e), and C/e is the number of magic subarrays. QED. Thus on the 8×8 board u must be a multiple of 8.

A symmetric closed tour numbered 1 ... C/2, C/2 + 1 ... C can be renumbered from midway, C/2 + 1 to C/2, and remain magic, but this does not make it a cyclic tour since the resulting tour is not different from the original, merely a rotation or reflection of it.

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# Rectangular Magic Tours

A **rectangular magic tour** is a magic rectangle r×s numbered 1, 2, ..., C = r·s. In this case we have A = (r·s + 1)/2 = T/(r·s) and so T = r·s·(r·s + 1)/2, and R = s·(r·s + 1)/2 and S = r·(r·s + 1)/2.

If the magic constant for a square m×m board is M then m·M = T. And we have A = $(m^2 + 1)/2$ and T = $(m^2)\cdot(m^2 + 1)/2$ and M = $m\cdot(m^2 + 1)/2$. This can also be expressed M = $(m^3 + m)/2$.

THEOREM: A rectangular magic tour is impossible on a board that is odd by even. *Proof*: If the number of cells C = r·s is even then r·s + 1 is odd and so for the rank sum R = s·(r·s + 1)/2 to be a whole number s must be even. Similarly for the file sum S to be a whole number r must be even. On the other hand if r·s is odd then both r and s must be odd, since any product of an even number is even. So the board must be odd×odd or even×even. QED. It is however possible to have a semi-magic odd×even rectangle, the lines of even length being the magic ones.

**Axial Symmetry**: On a board r×s with r = 2·h a semi-magic tour (i.e. with files adding to the magic constant) using a piece that can make a rook move of an odd length, can easily be constructed by axial symmetry, since this ensures that 1 is in the same rank as C, 2 as C–1, and so on, adding in pairs to C+1 = 2·A, h pairs giving 2·h·A. The middle move (h·s to h·s + 1), and the closure move (C to 1), have to be rook moves along a file, but the other moves are less restricted.

It also follows that renumbering the tour cyclically from the halfway point (that is numbering from h·s + 1 as 1 to h·s as C) will not affect the semi-magic property. When the non-linking moves are all knight moves this type of tour is often called a '*two-knight tour*' though I prefer to classify it as emperor or empress tour since the piece has to be capable of both knight and rook moves. [See the Augmented Knights section p.642 for many examples]

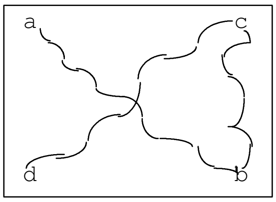
**Biaxial Symmetry**: A magic rectangle tour by a piece with rook-move component can be constructed when r = 2·h and s = 4·k using biaxial symmetry by (a) combining the above axial symmetry with symmetry about the vertical axis which ensures that 1 and 6·h·k, 2 and 6·h·k + 1, ..., 2·h·k and 4·h·k + 1, all adding to 6·h·k + 1, are in the same rank and 2·h·k + 1 and 8·h·k, 2·h·k + 2 and 8·h·k – 1, ..., 4·h·k and 6·h·k + 1, all adding to 10·h·k + 1, are also in the same rank, and (b) ensuring that k pairs of the first and second types occur in each rank, since two such pairs add to 16·h·k + 2 and the magic constant is k times this. This can be done without using any further rook moves. All that is necessary is that the route 1...2·h·k uses exactly k cells in each rank. A pattern of this type is known as a k-**satin**.

If all the moves are knight moves except the four linking rook moves the resulting empress tour is also called a '*four-knight tour*'. It follows from the biaxial construction method that in the case of a square the sum of the two diagonals will be twice the magic constant, because if x (in the range 1...2·h·k) is on a diagonal then so are 8·h·k + 1 – x and 6·h·k + 1 – x and x + 2·h·k which add to 16·h·k + 2, occurring k times. Many examples are shown in later sections.

In special cases a pair of rook move links may cross-connect by knight moves. In this case the four-knight tour converts into a two-knight tour magic in ranks and files. If the other pair of rook links can also be cross-connected however this results in a knight pseudotour of two circuits.

A diagonal magic tour on the above scheme is however not possible. This can be proved as follows [for the 12×12 case]. Six of the numbers 1-36 must occur on the diagonals, let them be a, b, c, d, e, f. Then if there are 3 (say a, b, c) on one diagonal and 3 (d, e, f) on the other (i.e. 3 odd and 3 even) the sum of the diagonal will be $a + (36+a) + b + (36+b) + c + (36+b) + (145–d) + (109–d) + (145–e) + (109–e) + (145–f) + (109–f) = 870 + 2\cdot[(a+b+c) – (d+e+f)]$. This can never be 870 since $(a+b+c) – (d+e+f)$ can never be zero, being a difference of odd and even numbers. Similarly if the numbers are distributed 4:2 the formula becomes $652 + 2\cdot[(a+b+c+d) – (e+f)]$. To give the total 870 the required difference is now 109, which cannot be the difference of two even numbers. The distribution 5:1 gives the formula $434 + 2\cdot[(a+b+c+d+e) – f]$ requiring an even difference of 218 but the difference can only be odd. The distribution 6:0 gives the formula $216 + 2\cdot[a+b+c+d+e+f]$ requiring a ‘difference’ of 327, but the sum of six odd or six even numbers is necessarily even. Thus if a diagonally magic 12×12 tour is to be possible it must deviate from this type of symmetry. [This proof is based on an argument by H. J. R. Murray in his 1951 manuscript.]

**Crossovers in Rectangular Magic Tours**: THEOREM: Every rectangular magic tour crosses its own path at least once. *Proof*: Let the four numbers in the corner cells of the board be $a < b < c < d$. If a and b are in opposite corners then so are c and d, and therefore the section a—b crosses the section c—d (as in diagram A). If a and b are in adjacent corners than so are c and d (as in diagrams B or C). If there is no crossover then all numbers in the edge a—b will be less than or equal to b and all numbers in the edge c—d will be greater than or equal to c; therefore the sums of these two lines cannot be equal, and the tour cannot be magic, contrary to hypothesis. QED.

A
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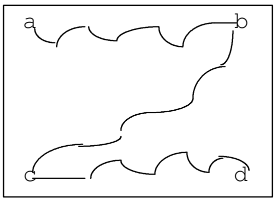


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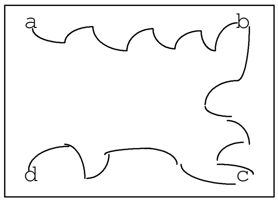


[I presented this as ‘Crossovers in Magic Rectangles’, Problems and Conjectures 2216, *Journal of Recreational Mathematics* vol 27, #1, p.62 (1995) solution vol 28, #1, p.77-78 (1996-97).]

CONJECTURE: Every rectangular magic tour crosses its own path at least twice. I think this is true but have not found a proof. The simple 2×4 magic king tour provides an example of a magic tour with exactly two crossovers (see ⌘ 2).

COROLLARY: A magic wazir tour is not possible on any rectangle, since it cannot cross its own path. Semi-magic tours without crossovers are however possible (e.g. axially symmetric rook tours).

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## Existence Theorems for Magic Leaper Tours

The following existence theorems were originally proved for the knight but remain true for any single-pattern free-leaper, that is with move {r,s} where r and s have no common factor and r+s is odd, which are the conditions for a leaper to be able to get from any cell to any other in a series of moves. We have shown above that a magic rectangle odd×even is not possible by any piece.

THEOREM (Kraitchik 1926): A magic leaper tour is impossible on a board odd×odd. *Proof*: Chequering the cells of the board white and black, a free leaper always moves to a cell of different colour to that on which it stands. All the cells of one colour will be odd numbered and all the other cells even numbered. In a board with an odd number of files, adjacent ranks contain an odd and an even number of odd-numbered cells. The rank sums would therefore be alternately odd and even. QED. Caution: More complex movers like king and queen, are not subject to this restriction.

THEOREM (Kraitchik 1926): A diagonally magic leaper tour is impossible on a square board whose side is not a multiple of 4. *Proof*: The previous theorem shows that the sides must be even. On a square board of singly-even side $4\cdot k + 2$, toured by a free leaper each rank or file contains an odd number, $2\cdot k + 1$, of odd numbers, and so the magic constant is odd. But any diagonal of $4\cdot k + 2$ cells contains numbers that are all even or all odd, and so must add to an even total. QED.

THEOREM (Jelliss 2003): In a magic leaper tour the number of entries of the forms $4\cdot x + 2$ and $4\cdot x + 3$ in a magic rank or file, counted together, must be even. *Proof*: We have shown that the board must have both sides even, so let them be $r = 2\cdot h$, $s = 2\cdot k$ then the rank sum $R = s\cdot(r\cdot s + 1)/2 = k\cdot(4\cdot h\cdot k + 1) = 4\cdot h\cdot(k^2) + k$. Thus if we write R and k in binary numeration the last two digits of R and k will be the same, since multiples of 4 affect only the higher digits. Each rank contains k odd numbers, so the sum of their 1-digits is k, which provides the required last two digits. Therefore the sum of the 2-digits of the $2\cdot k$ numbers must contribute 0 to the 2-digit position in R. The 2-digits of numbers of the form $4\cdot x$ or $4\cdot x + 1$ are 0 already. The 2-digits of numbers of the form $4\cdot x + 2$ and $4\cdot x + 3$ are 1, so to give 0 in this position when added there must be an even number of them.
The same argument applies with regard to the file sums. QED.

THEOREM (Jelliss 2003): A magic leaper tour is impossible on a board with singly-even sides. *Proof*: (a) The term singly-even refers to a number that is twice an odd number, that is of the form $2\cdot(2\cdot x + 1) = 4\cdot x + 2$, where x is any cardinal number (0, 1, 2, ...). Suppose the board sides are $r = 2\cdot h$ and $s = 2\cdot k$, where h and k are odd numbers. Now label the cells in the odd ranks A, B, A, B, … and the cells in the even ranks C, D, C, D, … (a super-chequering scheme):

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | A | B | A | B | | |
| C | D | C | D | C | D | | |
| A | B | A | B | A | B | | |
| C | D | C | D | C | D | | |
| | | | | | | | |
| | | | | | | | |

It will be seen that the As and Ds are of one colour in the usual chequering and the Bs and Cs are the other colour. Since the board is even by even we can rotate or reflect it if necessary to ensure that the odd numbers are on the A and D cells. There are $h\cdot k$ occurrences of each label, this number being odd, since it is the product of two odd numbers. This is also the number of entries of each of the four types $4\cdot x$, $4\cdot x + 1$, $4\cdot x + 2$ and $4\cdot x + 3$. Each rank and each file contains only two different letters.

(b) Now consider how the numbers of the forms $4\cdot x + 2$ and $4\cdot x + 3$ are distributed. The $(4\cdot x + 3)$s, being odd, are on the A and D cells, while the $(4\cdot x + 2)$s are on the B and C cells. Since the number of each is odd, when split into two groups one of the groups must be odd and the other even (possibly zero). Suppose there is an odd number of $(4\cdot x + 3)$s on the As and an odd number of $(4\cdot x + 2)$s on the Bs, then there will be an even number of $(4\cdot x + 2)$s on the Cs. This means that the total number of $(4\cdot x + 2)$s and $(4\cdot x + 3)$s on the A and C quarters is odd + even = odd. But for each AC file to be magic it must contain an even number of these (according to the previous theorem). All other possible distributions lead to the same contradiction, proving the theorem. QED.

**Background** The pattern formed by the $4\cdot x$ and $4\cdot x + 1$ cells contrasted with the $4\cdot x + 2$ and $4\cdot x + 3$ I call the ‘background’ of the tour. I have found this background pattern helpful in classifying the 8×8 magic knight tours. The above two theorems were published, for case of the knight, in the *Games and Puzzles Journal* #25 Jan-Feb 2003. The first is a lemma needed to prove the second.

**Quartes**: The above theorems show that, to adapt a terminology of Wenzelides (1850) any magic leaper tour $2\cdot h$ by $2\cdot k$ may be regarded as composed of a series of $(h\cdot k)$ four-cell chains (called quartes). In other words the segments of a magic tour numbered 1-4, 5-8, 9-16 and so on are quartes. When the move numbers are expressed in the form $4\cdot x + n$, where n is one or other of the four numbers 1, 2, 3, 4 then x ranges from 0 to $h\cdot k - 1$ and gives the number of chains which the leaper has already completed, and n gives the position of the leaper on the current chain.

**Contraparallel Chains**: This concept, as defined by H. J. R.Murray, is a generalisation of axial symmetry. He wrote: “Two chains a, b, ... and a', b', ... of equal length, forming parts of a single complete tour, are said to be contraparallel if a and a', b and b', ... each lie on the same column (or row) of the board and if the knight traverses the two chains in opposite directions. The importance of contraparallel chains in the construction of magic tours is that a pair of these chains contributes a constant sum to each column (or row) concerned. The contraparallel chains are said to be contiguous if a and a', b and b', ... are adjacent cells.” In axial symmetry the second half of the tour is contraparallel to the first half.

----------------------------------------------------------------------------------------------------------------------------

## Which Boards have Magic Knight Tours?

Some theorems that were originally proved for the knight, but remain true for any single-pattern free-leaper were presented above. Here we look at magic tour results specific to the knight.

We have shown that a magic knight tour requires a board even×even, and is impossible on a board with both sides oddly even (4·h +2)×(4·k + 2) , so at least one side must be a multiple of 4.

In the case of boards (4·h)×(4·k) it is now known that magic tours are possible in all cases except 4×4, 4×8, 4×12, 4×16.

On the 4×4 there are no tours at all. On the 4×8 it has been extablished that there are 8 semimagic tours with the lines of four all adding to 66, and 68 tours in which the lines of eight all give the total 132. But there is none magic in both ranks and files. These results were obtained by the present author in 2003 and I understand the same results were found by Gunther Stertenbrink around the same date. Work on these was also reported by Murray in 1917, though it is not clear if he found all the semimagic cases. The 4×12 and 4×16 cases were examined by Awani Kumar (2018) who has found magic tours 4×20, 4×24, 4×28 that can be extended to any length 4×(4·k).

The first magic tour 8×8 was found by Beverley in 1848, the first 12×12 by the Rajah of Mysore some time before 1868, and the first 16×16 by Wihnyk 1888. Tours of all boards larger than 8×8 and with sides a multiple of 4, that is on boards 8×12, 8×16, ...; 12×12, 12×16, ...; 16×16, ... can be constructed by simple braid extension of 8×8 tours (see ⌘ 4 and ⌘ 9 for examples and history).

This leaves boards (4·h)×(4·k + 2) to be considered.

It is now known that magic tours are impossible on boards 4×6 (where I examined all the 36 half-tours in 2003) and the cases 4×10 and 4×14 have been eliminated by Kumar (2018) who has constructed magic tours 4×18, 4×22, 4×26, which can be extended to any length 4×(4·k + 2).

Kumar has also shown the 8×6 case to be impossible, but the 8×10, 8×14, 8×18, ... cases are unresolved though the larger cases seem likely to have magic tours.

Kumar (2018) has constructed a magic tour 12×6 and this is easily extendable by braid to 12×10, and hence larger. This probably means all larger cases 16×6, 16×10, and so on can be solved.

Diagonal magic knight tours are possible on all square boards of side 4·k with k > 2. In 2003, thanks to the work of Stertenbrink, Meyrignac and Mackay in searching out the last missing magic tours on the 8×8 board it is now known that none is diagonally magic, as had long been suspected but not conclusively proven. The difficult 12×12 case was solved by Awani Kumar also in 2003. Examples on square boards of sides 16, 20, 24, 28, 32, 36 were constructed by T. H. Wilcocks and H. E. de Vasa in the 1950s, and it is clear that the same methods together with the braid extension method, can solve the problem on any of the larger square boards. (see ⌘ 9)

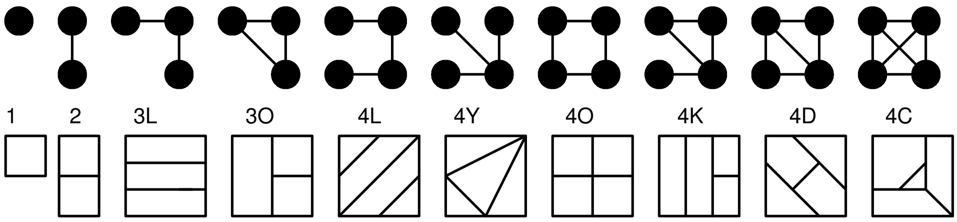
The larger the board of course the easier it is to construct magic tours and the more there are, so beyond the 8×8 we do not aim to make a complete catalogue or enumeration; instead we look for tours with extra structural properties, such as having symmetry, or containing smaller magic tours as components, or being formed by particular construction methods.

# Knight-Move Geometry

## Knight-Move Nets

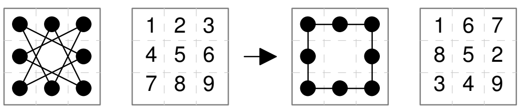
The complete network of knight moves on a rectangle m×n consists of 2·(m-1)·(n-2) + 2·(n-1)·(m-2) moves, found by counting those in the horizontal and vertical directions. This becomes 4·(n-1)·(n-2) on a square board, which works out at 168 on the standard chessboard.

It is natural to wonder whether such complex nets of moves can be simplified. It may be possible to represent the same net by a diagrams that looks rather different, but still connects the cell centres (or ‘nodes’ to use network terminology) in an equivalent way. This supposes that we are at liberty to arrange the nodes and links in our diagrams as we please, as if they were, to use the image of H. E. Dudeney (1917) “button and string configurations”. This includes untying the string and retying it (to the same button) after threading it differently through the other strings. Nets that can be represented by the same diagram have the same ‘topological’ structure. For example we do not distinguish topologically between three nodes connected in an L-shape and three nodes connected in a straight line. The following are diagrams of all the ‘topologically different’ nets of 1 to 4 cells.

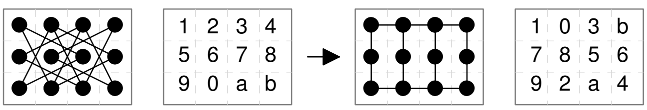



It may be that in a diagram of a net some of the links will cross one another at points that are not nodes. A simple example is the complete net of four nodes (4C), shown by the moves of a king on a 2×2 board, each node connected to the other four, the two diagonal moves crossing at the centre of the board. If we can eliminate such crossovers by suitably redrawing the diagram then the net is called **planar**. In fact 4C is planar, since one of the nodes can be moved to be within the triangle formed by the other three, eliminating the crossover, though the links are no longer king moves.

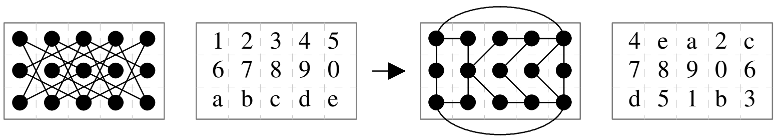
The equivalence of the knight's moves on the 3×3 board to a simple circuit of wazir moves was known to al Adli (c.840) since his work contained the puzzle (often ascribed to the later writer Guarini) of interchanging two white knights in one pair of adjacent corners with two black knights in the other corners. [H. J. R. Murray *A History of Chess* 1913 p.337.]




H. E. Dudeney gave a similar problem with three black and three white knights on a 3×4 board, and his solution shows that the net of knight moves on the 3×4 board is planar, by transforming it as shown in here. [*The Canterbury Puzzles* problem 94 p.140-141, 239-240.]



The planarity of the knight-move net on the 3×5 board is shown by this transformation:

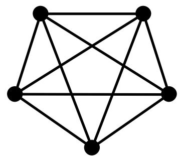



The remarkable 1930 Theorem of K. Kuratowski, which is proved in standard texts on graph theory, provides an exact criterion to determine whether a net is planar or not. It may be stated:

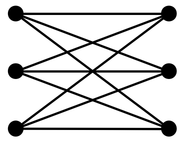
THEOREM (Kuratowski): <u>A net is planar provided it does not contain nodes connected (either directly or via distinct sequences of intermediate nodes) to form nets of type 5C or 6B</u>.

Net 5C is the complete net of five nodes, a **pentagram**. Net 6B is formed by connecting each of three nodes to each of three others, a **triglyph**.

5C
pentagram

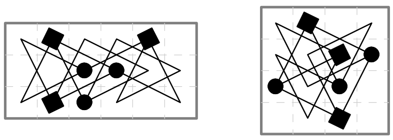


6B
triglyph

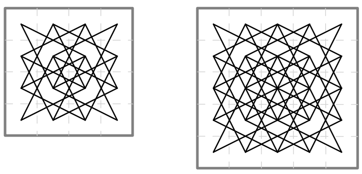


The non-planarity of the triglyph is the basis of the puzzle of the 'three utilities' which requires one to connect three houses to gas, electricity and water suppliers without the pipes or wires crossing. A solution is impossible without some 'cheat' such as passing lines for one house through another.

The non-planarity of the knight-move nets on the 3×6 and 4×4 boards (and thus on any larger boards) is shown from Kuratowski's theorem by the existence of triglyphs on these boards, as shown in these diagrams, where each of the three dots is connected to each of the three blocks by a separate path of knight moves.



**PUZZLE 13**: Redraw the knight-move nets on the 4×4 and 5×5 boards so that the number of crossovers of links is minimised in each case.

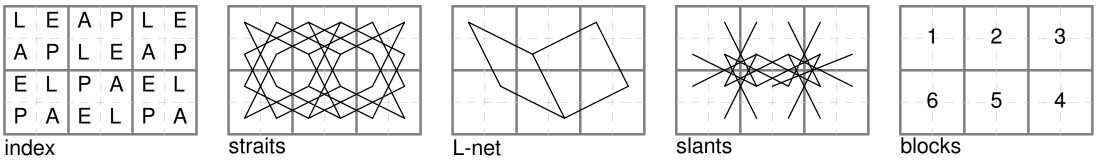


For solutions see end. Or can you do better?

## Straits and Slants

Any 2·h × 2·k board can be divided into h·k blocks, each 2×2, by grid lines drawn between every second rank and file. A knight placed anywhere on the board always moves from the block in which it stands to a neighbouring block, that is either to one with a side or one with a corner in common with the initial block. Knight's moves of these two types were usefully termed by **H. J. R. Murray** ‘straights’ and ‘slants’ respectively (though I am here proposing the spelling ‘strait’ to avoid a double meaning). An alternative way of defining them is that a strait knight move crosses one grid line, while a slant crosses two grid lines.

The strait knight moves form four separate **nets** of moves, each topologically equivalent to the moves of a wazir on a board h×k. Thus wazir paths on the h×k board (called the block diagram by Murray) correspond to knight paths of straits on the 2·h × 2·k board. It is because the straits form these tourable nets, and the wazir paths are easily counted, that this method is a useful one for classifying and enumerating tours on even × even boards.




If we letter the four nets L, E, A, P we arrive at the indexing system shown, which repeats itself in 4×4 blocks, and is applicable to any even×even board. Thus the indexing derives from the straits, rather than vice versa. This indexing method was first developed for the 8×8 board by P. M. Roget (1840) see the History section in ⌘ 6).

The straits are moves LL, EE, AA, PP indicated by two letters the same, and slants are moves LA, LE, PA, PE (or vice versa) indicated by two different letters, one a consonant and the other a vowel. The straits move the knight within a net, while the slants move it from one net to another. Moves of types LP and EA (or vice versa) connecting different consonants or vowels are impossible.

On the 4×4 board the nets consist of two ‘diamonds’ and two ‘squares’. On boards with both sides multiples of 4, the nets are always of two different shapes: the L and P nets each pass through two opposite corners while the E and A nets do not use corner cells. On the 6×6 board, and on oddly-even boards generally, the four nets are all of the same type. Nets are always made up of squares and diamonds linked together. The method is for this reason often confused with ‘squares and diamonds’ but is really distinct (see ⌘ 6).

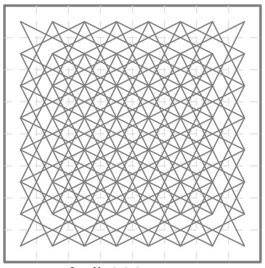
The number of slants in a tour provides a way of classifying tours according to their complexity of structure. The number of slants in a tour must be at least 3, since each change from one net to another (e.g. L to E) is made by a slant move, and there are four nets to be visited. In a closed tour the number of slants must be even, since entering and leaving a net always uses two slants, and the last slant must return to the initial net. Thus the least number of slants in a closed tour is 4.

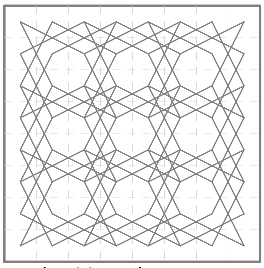
In any tour each net is divided into a number of **segments**, made up of one or more straits or consisting of a single cell. If l, e, a, p denote the numbers of segments into which the L, E, A, P nets are divided in a given closed tour then $l + e + a + p = s$, the number of slants, since the tour consists of a sequence of alternate segments and slants. The number of slants incident with a net, in a closed tour, is twice the number of segments into which it is divided (i.e. the number of segments is half the number of incident slants) since each segment has one slant at each end. Each slant has one end in a consonantal net and the other end in a vowel net, since there are no LP or EA connections. Thus the number of slants incident with the consonantal nets is s, and the number incident with the vowel nets is s, since each slant is counted twice, once from each end. It follows that $l + p = e + a = s/2$, and therefore s must be even. This implies the previous equation but is a stronger condition.

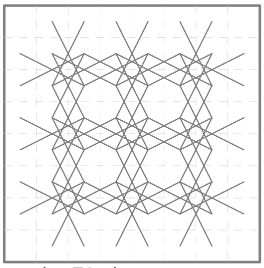
For further discussion of the method of straits and slants see the sections relating to the 6×8 board in ⌘ 4, the 6×6 board in ⌘ 5 and the 8×8 board in ⌘ 6.

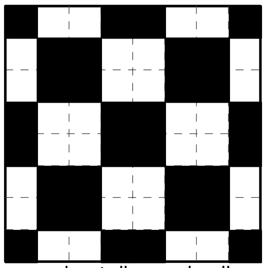
## Alternative Move Classifications

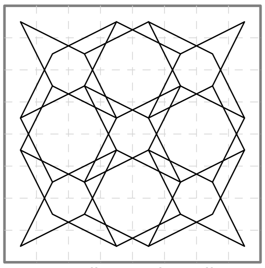
These diagrams show the straights and slants in relation to the pattern of diagonals of the quadrants. The slants form a pattern of nine octangles.

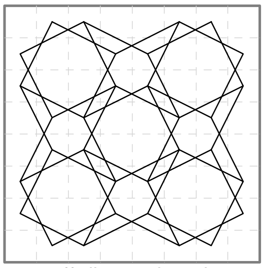

net of all 168 moves

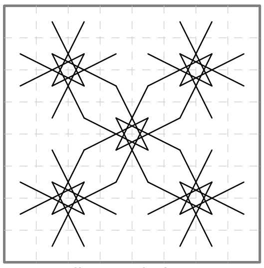

the 96 straits

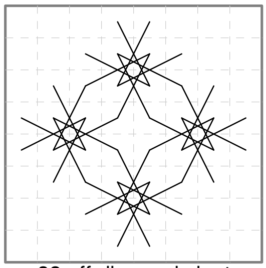

the 72 slants


quadrant diagonal cells

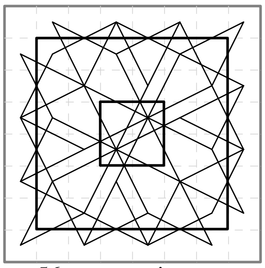

48 diagonal straiits

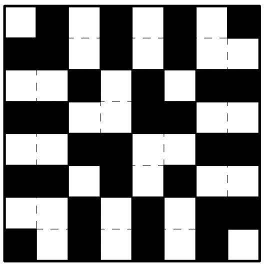

48 off-diagonal straits

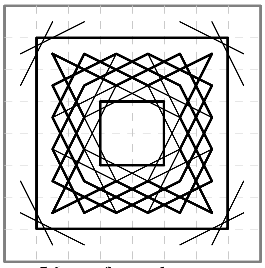

40 diagonal slants

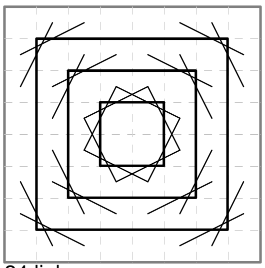

32 off-diagonal slants

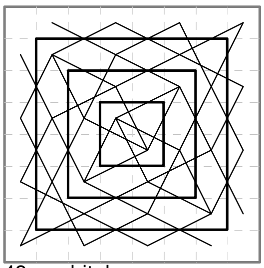
**Eccentrics and Conformals.** By working inwards from the ring of edge cells in steps of two, moves connecting cells within different regions we call **eccentric** (deriving from ‘edge-centric’, since in the 8×8 case they either have one end on an edge cell or one end in the centre) while other moves are **conformal** (connecting cells within the same region). On the 8×8 board the eccentric moves form two nets of 56 moves with no connection from one net to the other; they determine an alternative binary colouring of the board as illustrated. The remaining 56 moves are conformal, 48 within the middle ring and 8 edge-to edge links. For tours featuring these see the Graphic section in ⌘ 6.

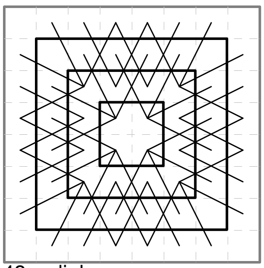

56 g-eccentrics

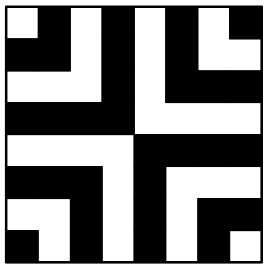



56 conformals


24 links


48 g-orbitals


48 radials



**Radials and Orbitals** relate to concentric rings of cells. The moves that link cells in non-adjacent rings, crossing two dividing lines we call **radials**. Moves that connect cells in adacent rings are **orbitals**. Other moves are **links**. On the 8×8 board there are 24 links, 48 radials and 96 orbitals. The orbitals form two separate nets. The radials and orbitals can be divided into horizontal and vertical, giving 7 classes each of 24. Little work has been done  with this idea so far.

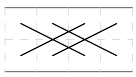
## Pseudotours

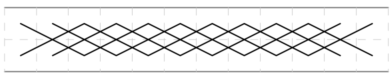
An arrangement of move-lines on a board with one or two moves incident at every cell is either a **tour**, i.e. a single path of moves that uses every cell of the board, or a **pseudotour**, made up of two or more superimposed paths. Pseudotour diagrams can often show a greater degree of symmetry than a tour on the same board. They can be studied for their own inherent interest or as the basis for the construction of tours, by deleting the minimal number of moves from them so that the resulting paths can be linked up to form a tour.

Just by looking at such a tour-pattern it is often not easy to decide whether it is a tour or a pseudotour, unless the paths cover separate areas or have multiple end-points. One way of making clear which diagrams are tours and which are pseudotours is to show one or more of the circuits in a distinctive line, heavier than the others or broken into dashes or dots, but this often loses a lot of the attractiveness of the pattern. An alternative method, adopted in these notes, is to use a distinctive border. We distinguish diagrams of pseudotours from tours by using a dashed line border.

## Braids and Borders

On one-rank boards 1×n no knight tours are possible, since the knight move is two dimensional. On two-rank boards 2×n with n > 3 the moves are in four separate **strands**, forming a **braid**. These appear as a common component in tours on larger boards.





The construction of tours by border methods, in which many of the moves occur in the two ranks or files adjacent to the board edge, forming braid patterns, was one of the earliest methods employed, and indeed such tours dominated the output until the middle of the 19th century. Haldeman (1864) dubbed them ‘Fillet-and-Field’ tours. Collini (1773) systematised the study of tours formed in this manner on the 8×8 board. Moon (1843) extended Collini's method (which he called the ‘Method of Annuli’) to square boards of any size, by placing successive borders of two-square width round a central square. He did not give any actual tours, but noted that in the case of odd-sided boards the moves within the annulus form two circuits instead of four, which makes the task of linking them up to form a tour easier. Borders are also important in extending magic knight tours to larger boards.

My notes here generalise these results to rectangular boards of any dimensions, not merely square boards. By a **border** added to a board r by s we mean an area two cells wide all round it. This border adds 4·(r + s + 4) extra cells and increases the board to (r + 4)×(s + 4). On the other hand a border subtracted from r by s contains 4·(r + s – 4) cells and surrounds an internal area (r – 4)×(s – 4). The internal area is zero if r or s equals 4. If the internal area is 4×4 or larger we can put in another border, and we can continue putting in concentric borders until the innermost area is zero or has one side of 1, 2 or 3 cells.

A border can always be completely filled with knight's moves in the form of a braid pattern, which will be either two or four circuits. When both dimensions are even the braid has four strands of equal length, each covering r + s + 4 cells. When one dimension is odd and the other is doubly even (that is, a multiple of 4) the braid is of two equal strands, each of 2·(r + s + 4) cells. When one dimension is odd and the other is singly even (that is, of the form 4·k + 2) the braid is of four strands, two being short and two long. If a board edge is increased by 4 the number of strands in the border is not altered.

Summary:

| | | |
|---|---|---|
| 4 strands, equal, | even×even | i.e. 4×4, 4×6, 6×6, 4×8, 6×8, 8×8, ...; |
| 2 strands, equal, | (4·k)×odd | i.e. 4×5, 4×7, 4×9, 8×5, 8×7, 8×9, ...; |
| 4 strands, unequal, | (4·k + 2)×odd | i.e. 6×5, 6×7, 6×9, 10×5, 10×7, 10×9, .... |
| 2 strands, unequal | odd×odd | i.e. 5×5, 5×7, etc. |

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# Simple Linking of Pseudotours

The process of covering a board with circuits, or partial tours, to form a pseudotour, and then deleting as few moves as possible from the circuits to allow the loose ends to be connected up into a tour, open or closed, provides a general way of looking at a number of methods of constructing tours that evolved separately. These include those described for the knight on the 8×8 board by Euler (1759), Vandermonde (1771), Collini (1773) and the Squares and Diamonds method (Warnsdorf 1823, F. P. H. 1825, von Schinnern 1826, and others).

If it is possible to link a pseudotour pattern to form a single tour by deleting just one move in each circuit, I call the process **simple linking**. The ideas expounded here on the use of pseudotours and linkage polygons to construct tours do not seem to have been explicitly developed before my own work, though various aspects of the ideas can be seen in the work of others from Euler onwards, particularly Laisement 1782, Laquière 1880 and Sharp 1925, but not as overtly expressed as here.

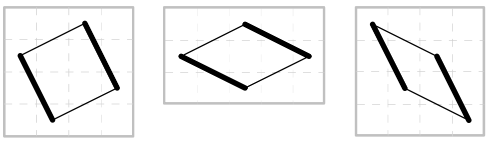
**Linkage Polygon**. In a tour formed by simple-linking of a pseudotour the deleted and inserted moves must form a knight's path, the ‘linkage polygon’. In a closed tour this is a closed path, and in an open tour it is an open path whose end cells are the same as the endpoints of the tour.

**Theorem**: <u>If we can link n circuits of knight moves to form a closed or open tour by deleting one move from each circuit and joining up the loose ends, then the deleted and the inserted moves must themselves form a single closed or open path of 2·n or 2·n – 1 moves respectively.</u>

*Proof*: Let a1, b1 be the two loose ends of the path formed from the first circuit in an open tour, or from any one of the circuits in the closed tour, after deletion of the move a1–b1. Call the end to which b1 is joined in the final tour a2, and the other end of the link deleted from this second circuit b2, then call the end to which b2 is joined a3, and the other end of the path formed from the third circuit b3, and so on. By this process we number the circuits 1, 2, 3, .... If at any stage we were to join bk to a1 we would form a short circuit. So, avoiding this, we eventualy reach bn which must in the closed case join to a1, the only remaining loose end. The moves a1–b1, b1–a2, a2–b2, b2–a3, ... , an–bn, which are alternately the deleted and inserted moves, thus form a path. QED.

The idea of a linkage polygon of alternately deleted and inserted moves also provides a method of possibly converting an open tour (on an even board) into a closed tour. If we can find such a path joining the two ends of the open tour, the deleted moves being moves of the existing open tour, then substituting the inserted moves for the deleted moves results in a pattern with two moves at each cell, i.e. either a closed tour or a pseudotour. The same procedure may also convert a pseudotour to a tour.

**Simple Linking of Two Circuits**. To link two circuits into one we need to find a pair of parallel moves, one in each circuit, whose end points are a knight's move apart. This was the method used by Vandermonde (1771, see ⌘ 6). In other words the deleted and inserted moves in this simplest case must form a linkage polygon that is in the shape of a rhomb, of one of the three possible shapes: square, lozenge and diamond (similar rhombs can be drawn for other leapers):



If we have a centrosymmetric pseudotour of two paths, then to connect them together to form a symmetric tour by a single deletion in each there must be a pair of parallel moves, one in each circuit, forming two sides of a rhomb whose centre coincides with the centre of the board.

If we start with four circuits we may be able to link them into one by first joining two pairs to give two circuits and then joining these two circuits. This is a two-stage linking process, which involves six deletions and six insertions, and so its overall result is not in general a simple linking, though it will be a simple linking (four deletions and four insertions) if we can arrange for two of the links inserted at the first stage to be those deleted at the second stage. In this case the three rhombs join up to form a polygon bounded by eight knight moves.

**Simple Linking of Three or More Circuits**. To find all the simple-linkages of three or more circuits requires a longer step-by-step approach. A method (indicated by Collini 1773, see ⌘ 6) is to give each circuit a label a, b, c, ..., giving the same letter to all cells visited by the same circuit. Now the problem is to find all the sets of n moves aa, bb, cc, ... one in each circuit, that can be joined to form a linkage polygon.

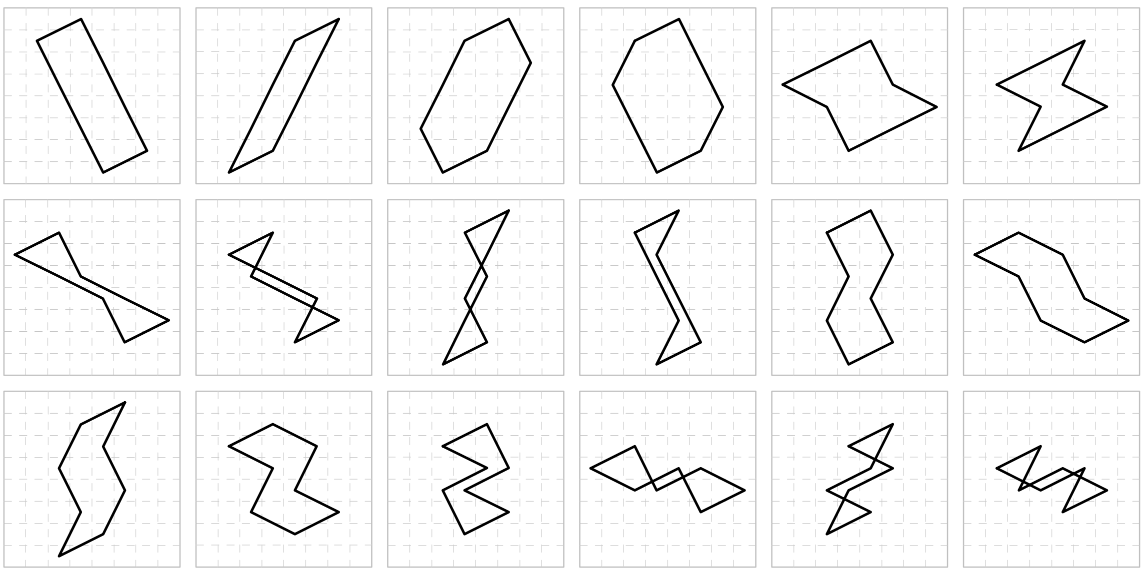
To find all possible linkages we start with one circuit, labelled ‘a’, and consider in turn all the moves of type aa. If the circuits are of different lengths it is best to take circuit ‘a’ to be one of the shortest, or else one in a corner where it has less linkages to other circuits. From each end of each chosen move aa we find out to which other circuits, if any, it will link. If we find a partial linkage polygon b–aa–c then we draw in all possible bb and cc links. Then we consider the ways in which these partial linkage polygons bb–aa–cc can be extended or completed.

In the case of three circuits we have only to look for a link b–c that will complete the linkage hexagon. The sequence of letters on the linkage polygon will be abbcca or its reversal accbba. In the case of four circuits we have to consider all the dd links, seeking for one that will join up with the two ends of one of our bb–aa–cc partial linkages. Six sequences of linkage are possible, occuring in pairs that are reversals of each other: abbccdda and addccbba, abbddcca and accddbba, accbbdda and addbbcca. In special cases there may perhaps be no linkages between two particular circuits, and this would reduce the number of possible sequences.

**Simple Linking of Symmetric Circuits**. In the case of a symmetric tour derived by simple linking from an even number of circuits forming a rotatively symmetric pseudotour (the circuits themselves being possibly asymmetric but symmetrically arranged) the linkage polygon is of course itself symmetrical, and centred on the midpoint of the board. The case where the linkage polygon is a rhomb is the simplest example of this.

In the case of four circuits, there are in fact 18 geometrically distinct centrosymmetric polygons that can occur, as shown above. All will fit on the 8×8 board. Note that though formed of eight knight moves two of the shapes are parallelograms, and eight are hexagons, rather than octagons.

Instead of the above process for finding a linkage polygon, an alternative procedure in the case of symmetric tours might be to try each of the 18 polygons in turn to see which will fit the given circuits. However, since each polygon can occur in four orientations there may be up to 4×18 = 72 cases to consider which is probably not practical.



**Double Linking.** If we begin with circuits that are already symmetric in themselves, as in Collini's method of borders, deleting one move from a circuit necessarily spoils its symmetry, so none of the tours formed by simple linking in this case can be symmetric. Symmetric tours can however be formed by ‘double linking’ that is deleting two moves, symmetrically placed, in each circuit and connecting up the loose ends symmetrically to form the tour.

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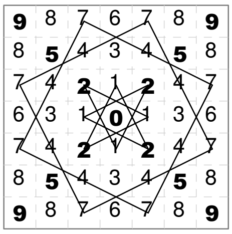
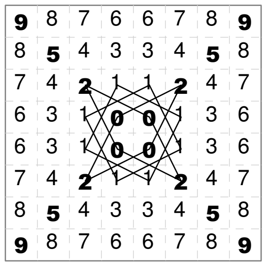
# Cell Coding

On square boards of side 2·k or 2·k – 1 there are k·(k+1)/2 differently placed cells. This is the kth triangular number 1 + 2 + ... + k and is the number of cells within or on the edges of the eight triangles formed by the diagonal and lateral axes of symmetry of the square. By rotation or reflection of the board any cell in one triangle can be brought to coincide with a similarly placed cell in any of the other triangles. It is useful for investigating symmetry to number the cells row by row from the centre outwards, beginning with 0. The numbers in the cells laterally outwards from the centre are 0, 1, 3, 6, 10 ... and those in the diagonal cells are 0, 2, 5, 9, 14 ... In the case of the 7×7 and 8×8 boards there are ten differently placed cells, and so this coding system conveniently labels them with the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

This scheme has the advantage that any sequence of numbers that represents a path of knight moves on a small square board also represents a path knight moves on any larger square board of the same parity. For example the formula 1-2 represents the 8-move star surrounding the centre on any odd board, and on any even square board the eight moves 1-1 and the eight moves 0-2 form the two squares and two diamonds in the central 4×4 area.




Using the cell coding the possible knight moves on the 7×7 and other odd-sided squares can be listed as here. A superscript indicates a choice of moves to cells with the same number.

$0 \rightarrow 4^8$

$1 \rightarrow 2^2, 3^2, 5^2, 7^2$ $\quad 2 \rightarrow 1^2, 4^2, 6^2, 8^2$

$3 \rightarrow 1^2, 4^2, 8^2$ $\quad 4 \rightarrow 0, 2, 3, 7^2, 9$ $\quad 5 \rightarrow 1^2, 6^2$

$6 \rightarrow 2^2, 5^2$ $\quad 7 \rightarrow 1, 4^2, 8$ $\quad 8 \rightarrow 2, 3, 7$ $\quad 9 \rightarrow 4^2$

On the 7×7 and larger odd boards there is an ambiguity in that the cells coded 4 and 7 define 16 knight's moves instead of 8. To distinguish the two cases we can write 4;7 for the move across a diagonal and 4:7 for the move across a median. Thus where 4-7 occurs in the formula for a tour there are two cases which are often strikingly different. This property is related to the fact that 4 is a knight move from the centre cell. (This effect can also be seen in the case of camel {1,3} and zebra {2,3} leapers on a board 11×11 or larger where the moves 7-12 and 8-16 are of this type. The cells 7 and 8 being a camel or zebra move from the centre cell.)

The list of moves on the 8×8 and larger even boards is somewhat different.

$0 \rightarrow 1^2, 2^2, 3^2, 4^2$

$1 \rightarrow 0, 1^2, 3, 4, 5, 6, 7$ $\quad 2 \rightarrow 0^2, 3^2, 6^2, 8^2$

$3 \rightarrow 0, 1, 2, 4, 7, 8$ $\quad 4 \rightarrow 0, 1, 3, 6, 7, 9$ $\quad 5 \rightarrow 1^2, 6^2$

$6 \rightarrow 1, 2, 4, 5$ $\quad 7 \rightarrow 1, 3, 4, 8$ $\quad 8 \rightarrow 2, 3, 7$ $\quad 9 \rightarrow 4^2$

There are 42 possible moves in any given direction, but there are two labelled 1-1, parallel to each other. It may be noted incidentally that the codes number the cells in the sequence of their geometrical distance from the centre point except that on the 8×8 board the 5 and 6 are at the same distance, $(\surd 50)/2$, measured from board centre to cell centre.

A pair of numbers u-v will indicate a single unambiguous move from any of the cells with the first number u except when u is on a diagonal or median, or in the case of the special moves 4-7 on odd boards and 1-1 on even boards, which form squares circlng the centre. In paths, sequences of the form u-v-u are possible only when v is a diagonal or medial cell (e.g. 4-9-4), otherwise u-v-u (e.g. 9-4-9) represents a ‘switchback’, using the same cell u twice, which is not allowed in a tour.

As a matter of historical interest, various ways of labelling the cells have been proposed for purposes of enumerating tours on particular boards, or for constructing tours with a particular symmetry. The works of Vandermonde (1771), Lavernède (1939), Roget (1840), de Hijo (1882), Bergholt (1918) and Murray (1942) all used different methods. For example:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 5 | 7 | 8 | 6 | 3 | 1 |
| 3 | 4 | 9 | 11 | 12 | 10 | 4 | 2 |
| 6 | 19 | 13 | 14 | 15 | 13 | 9 | 5 |
| 8 | 12 | 15 | 16 | 16 | 14 | 11 | 7 |
| 7 | 11 | 14 | 16 | 16 | 15 | 12 | 8 |
| 5 | 9 | 13 | 15 | 14 | 13 | 10 | 6 |
| 2 | 4 | 10 | 12 | 11 | 9 | 4 | 3 |
| 1 | 3 | 6 | 8 | 7 | 5 | 2 | 1 |

de Hijo

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | 15 | 5 | 23 | 27 | 13 | 3 | 9 |
| 3 | 21 | 11 | 17 | 1 | 7 | 21 | 15 |
| 13 | 7 | 25 | 29 | 18 | 25 | 11 | 5 |
| 27 | 1 | 19 | 31 | 31 | 29 | 17 | 23 |
| 23 | 17 | 29 | 31 | 31 | 19 | 1 | 27 |
| 5 | 11 | 25 | 19 | 29 | 25 | 7 | 13 |
| 15 | 21 | 7 | 1 | 17 | 11 | 21 | 3 |
| 9 | 3 | 13 | 27 | 23 | 5 | 15 | 9 |

Bergholt

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 4 | 5 | 6 | 6 | 5 | 4 | 0 |
| 4 | 1 | 7 | 8 | 8 | 7 | 1 | 0 |
| 5 | 7 | 2 | 9 | 9 | 2 | 7 | 5 |
| 6 | 8 | 9 | 3 | 3 | 9 | 8 | 6 |
| 6 | 8 | 9 | 3 | 3 | 9 | 8 | 6 |
| 5 | 7 | 2 | 9 | 9 | 2 | 7 | 5 |
| 4 | 1 | 7 | 8 | 8 | 7 | 1 | 4 |
| 0 | 4 | 5 | 6 | 6 | 5 | 4 | 0 |

Murray

A ten digit scheme for the 8×8 board was used by Murray (1942) numbering the diagonal cells inwards 0, 1, 2, 3 and the other cells 4, 5, 6, 7, 8, 9 row by row, but if the same scheme is used on larger boards a sequence representing a knight path on the smaller board will not also work on the larger board. To distinguish different symmetries Murray marked the numbers below the diagonal in the first quarter with primes, and also the numbers in the corresponding cells, related by rotation or reflection, as appropriate, but this now seems unnecessarily complicated.

In applying coding to enumerate 16-move paths in quaternary symmetry on the 8×8 we need to note that the diagonal cell numbers are used once and the others twice, and that the two occurrences of each off-diagonal number must have a diagonal number (0, 2, 5, 9) somewhere between them.

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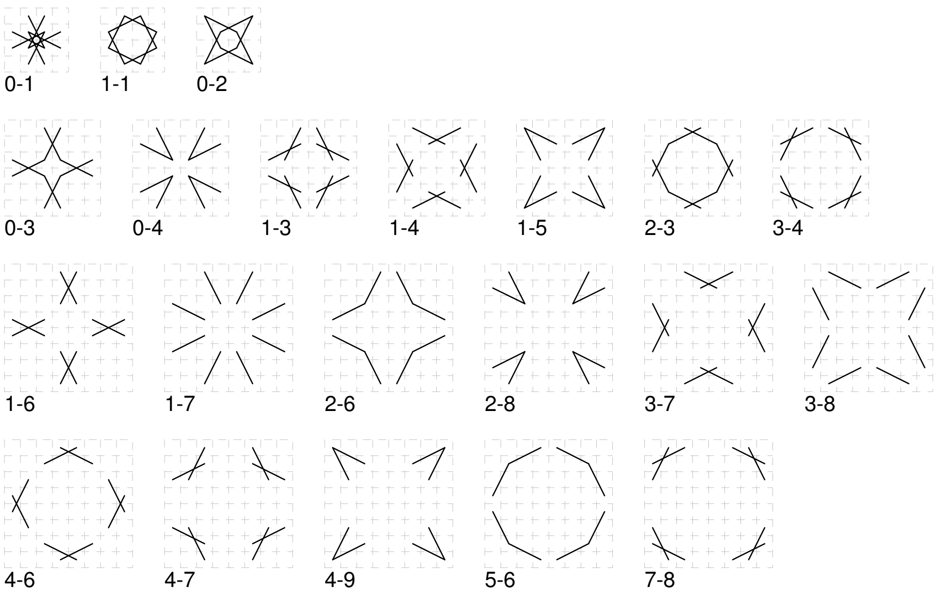
## Generic Moves

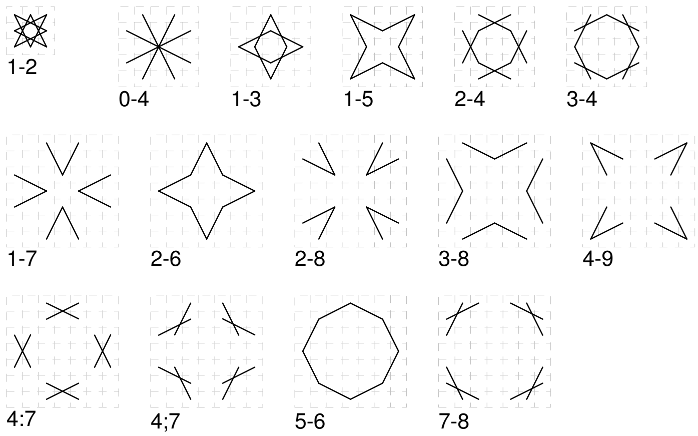
Using the cell coding the generic (geometrically distinct) knight moves can be listed, disregarding direction of move, as pairs of code numbers, as we do in the following diagrams.

Even Boards: 4×4 (3) and 6×6 (7) and 8×8 (11)




Thus on the 8×8 board there are 3+7+11 = 21 generic moves. Each move can occur in 8 different positions by rotation and reflection, giving these 21 octonary patterns. On the 10×10 board a further 15 must be added, on the 12×12 board a further 19, on the 14×14 board a further 23, on the 16×16 board a further 27, and so on, the numbers increasimg by 4 each time.

Odd Boards: 3×3 (1) and 5×5 (5) and 7×7 (9)

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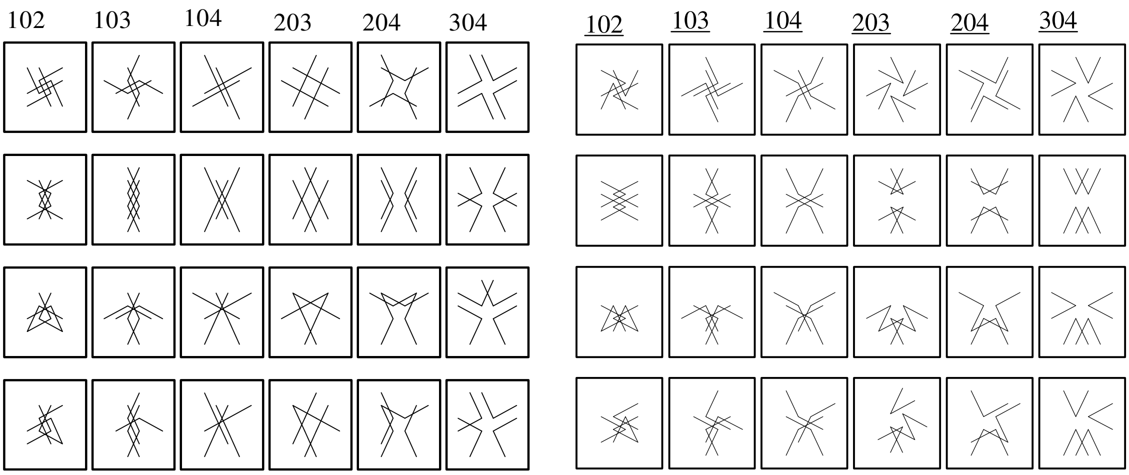
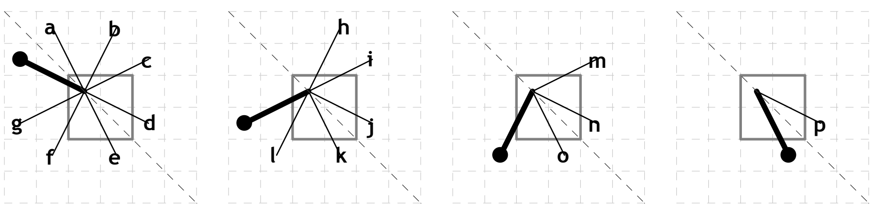
## Central Angles

The formation of a pattern in the central area of an even-sided board, to present the effect of a ‘picture’ within a frame is a common scheme for tours. The cell coding scheme can be employed to name the 16 different ways in which a pair of moves may pass through a central cell on an even-sided square board. Moves of type u0u are symmetric about a diagonal through the centre cell used. Moves of type u0v can be taken from a given ‘u’ to two different ‘v’s. We take <u>u0v</u> (underlined) and u0v (plain), to represent the moves which keep to one side of or cross the diagonal respectively. For example <u>102</u> is an acute angle with lateral bisector, while 102 is a right angle.

The following are charts of all the central patterns that can occur with four equal angles. In each case central quaternary, axial quaternary, axial binary and asymmetric formations are possible. Diagrams of the quaternary cases were given by Paul de Hijo (1882) giving each a descriptive name.




An alternative lettering method for the central angles which I developed originally in *Chessics* 1985 (#22 p.70) for classifying tours on the 6×6 board is explained in the diagrams below.



It works for any size boards: 404 = a, 403 = b, 402 = c, 401 = d, 401 = e, 402 = f, 403 = g, 303 = h, 302 = i, 301 = j, 301 = k, 302 = l, 202 = m, 201 = n, 201 = o, 101 = p.

In the case of a rotationally symmetric tour the angles in diametrally opposite cells will be the same. However the m case cannot occur since the two m angles combine to form a diamond circuit in the middle of the board. In the centre four cells there can be at most only two different angles. We can thus denote the central pattern by the names of these two angles, taken in alphabetical order. There are thus, excluding m, 15 cases with the four angles the same, ranging from (aa) to (pp), and (15×14)/2 = 105 with two different angles, ranging from (ab) to (op). Making 120 cases in all.

The angles a, h, m, p are symmetric by reflection in the diagonal that bisects them. In the case of the asymmetric angles u0v with u > v I call the first part u0 (i.e. the move nearest the corner, forming part of an a, h or m angle) the **leading arm** of the angle shown in bold in the above diagrams.

This enables us to distinguish two formations made with the same asymmetric angle as: direct (=) or oblique (~), according as the leading arms of the angles are similarly oriented (both vertical or both horizontal) or differently oriented (one vertical the other horizontal). Thus the angle pattern (bb) occurs in two forms (bb=) and (bb~). Roughly speaking the angles are related by reflection or rotation. It will be found that this distinction applies in all cases, even when the angles are different.

Thus we would now appear to have 240 cases in all. However tours involving the symmetric angles a, h, p occur in only one form, there is for example only one pattern (bh) with no distinction between (bh=) and (bh~), since a, h and p always have one arm vertical and the other horizontal. This effect reduces the total, since there are 6 cases using the angles a, h and p only: (aa), (ah) (ap), (hh), (hp), (pp), and there are 12 cases using each of a, h, or p with an asymmetric angle (42 cases in all). This makes the number of cases 240 – 42 = 198.

Angle 'a' cannot occur in a closed tour on the 8×8 board since it connects with the two moves at the corner forming a diamond, thus the number of central angle patterns in symmetric 8×8 tours reduces by 15 to 183 (see ⌘ 7 for example tours). On the 6×6 board angle 'p' cannot occur for the same reason, but further border effects mean that only 20 cases are in fact possible (see ⌘ 5).

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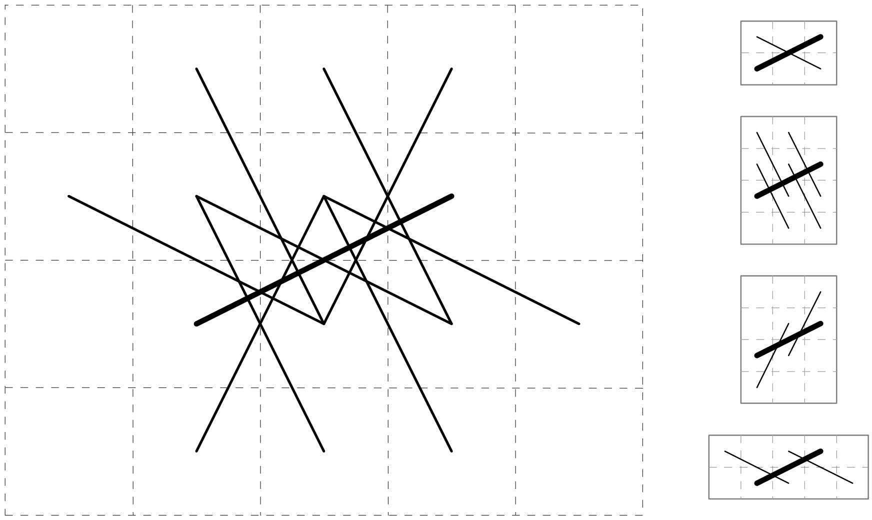
## Intersections

A single knight's move can be intersected by other knight moves in nine different places, as shown in the magnified diagram below. However, the maximum number of intersections that can occur on one move of a knight's path is seven. [*Chessics* #19 (1984)]



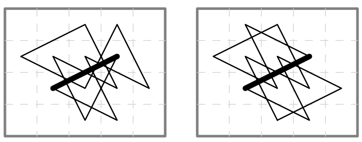
If we consider the types of crossovers that are possible between a pair of knight moves, there are four geometrically distinct cases, working from the centre of the reference move outwards:

(a) central cross, both moves divided 1/2 + 1/2;
(b) right cross, one move divided 2/5 + 3/5 and the other 1/5 + 4/5;
(c) diagonal cross, both moves divided 1/3 + 2/3;
(d) lateral cross, both moves divided 1/4 + 3/4.

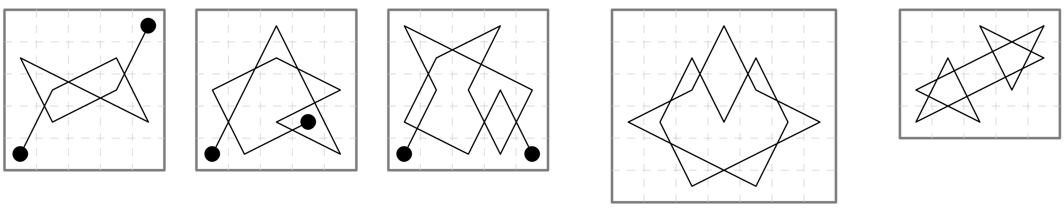
The fifth intersection repeats the right cross with the roles of the two moves transposed.

In considering the shapes that can be formed by knight's moves it is helpful to know that the nine points at which a move can be crossed by another move are at the fractional distances 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5 along the move.

T. R. Dawson constructed a number of short paths with intersection properties, as shown here. The first diagram is a shortest closed path, of 12 moves, with a maximum intersected move (*Sphinx at Play* Dec 1934). I found another, symmetric, solution.



Dawson also constructed four examples of knight-move chains with single intersection of every link, the ‘Lover's Knot’ problem The first three open paths, of 8, 10 and 12 moves, are from *British Chess Magazine* Dec 1930 and the closed path of 14 moves from *Evening Standard* 14 Aug 1933. His circuit of 10 moves with double intersection of every link (1937) is quoted in Murray's 1942 ms.

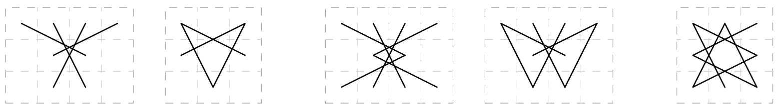


## Other Formations

We consider here other formations of a few intersecting knight moves within a restricted area that can often be found in tours. Some example tours are shown in the Graphic Tours section (p.446).

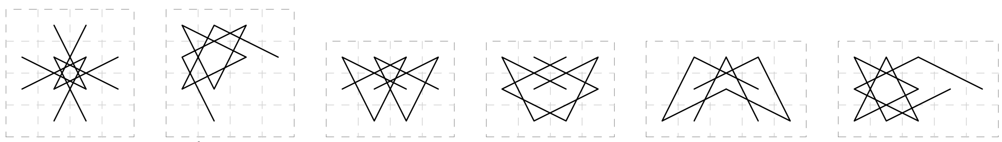
Four lines each intersecting the other three constitute a ‘complete quadrilateral’ in geometry, and four single moves in this formation I call a **quadrangle**. Another 4-move pattern is the **shield**.

Six lines can form overlapping quadrangles making a **spider** formation. When two of the loose ends of the quadrangle are joined we get a combination of two triangles of three successive moves making a formation I call the **moth**. The circuit of eight moves in a 3×3 area with one or two moves missing is another frequent feature of tours, particularly in the corners or at ends of narrow boards. We call the 7-move formation a **star** though Indian literature prefers to call them ‘ponds‘.



quadrangle shield spider moth star

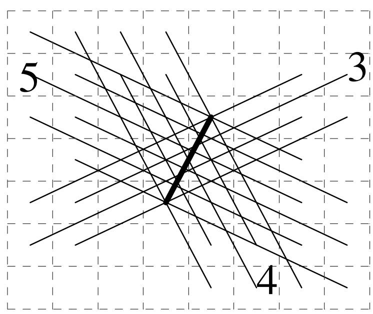
Eight moves formed of four overlapping acute angles form an **octangle** (often termed a ‘star’). This is a popular centre angle pattern (pp/pp) on even boards and can be seen as a combination of quadrangles. An 8-move star-like corner formation I call a **meteor**. Two shields overlapping form a **butterfly** (related to the moth above). Other formations of eight lines are the **cloud** and **dove**. A star extended to 9 moves is a **comet** forming a common end-pattern in three-rank tours (see ⌘ 4).



octangle meteor butterfly cloud dove comet

## Triangles

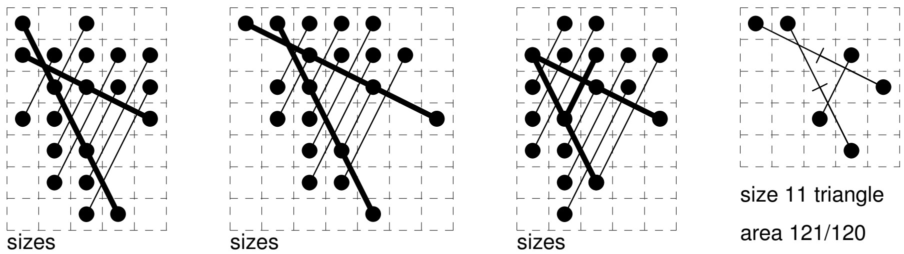
Despite what may appear on some hand-drawn knight-tour diagrams it is impossible to have three knight-lines in a tour concurrent, that is crossing at a point, and the same is true for any other {r,s} free leaper. They always form a triangle. (However, the same is not true for half-free leapers like camel or lancer.) Further, any two {r,s} leaper lines drawn on the board can only show the usual six angles between successive moves. Their intersections will not create any new angles. It follows that <u>all</u> triangles of Knight lines are 3:4:5 triangles, all triangles of Zebra lines are 5:12;13 type, and so on.




The following Puzzle Questions were posed by me in the *Games and Puzzles Journal* (issue 16, 1999, with answer in issue 17): (a) Taking a square of the board to be unit area, and numbering the triangles from the smallest area upwards, what is the area of a size k triangle? (b) Which triangle of knight-lines encloses an area as near as possible to a unit area, i.e. the area of a single cell of the board? (c) What is the area of a triangle of three successive knight moves?

THEOREM: If we number the triangles from the smallest upwards a size k triangle has area $k^2/120$. *Proof*: A set of close-spaced parallel knight-lines cuts a knight move crossing it into either 5, 4 or 3 equal parts, depending on the angle (see the magnified diagram above). Thus the distance between the two points where knight-lines at different angles cross another knight-line will be a multiple of sixtieths of a knight move, since the least common multiple of 3, 4 and 5 is 60. Denoting a 60th of a knight move, $\sqrt{5}/60$, by u, the sides of the size k triangle are $3 \cdot k \cdot u$, $4 \cdot k \cdot u$, $5 \cdot k \cdot u$ and the area ($\frac{1}{2} \cdot$base$\cdot$height) is thus $6 \cdot k^2 \cdot u^2$. Inserting the value for u and a little arithmetic gives $k^2/120$. QED

I communicated the above idea to Prof. Knuth in a letter of 4 Dec 1992, in response to his idea of a ‘Celtic tour’, which he defined as a tour having ‘no three lines nearly concurrent’; in other words no size-one triangles.



sizes 1, 4, 7, 10, ..., 22,... sizes 2, 5, 8, 11, ..., 23, ... sizes 3, 6, 9, 12, ..., 24, ... size 11 triangle area 121/120 = 1+1/120

The diagrams above show all the possible sizes of knight triangles from k = 1 to 24. The long heavy lines are those of the most acute angle. The triangle nearest to unit area is the k = 11 triangle, since $11^2/120 = 121/120 = 1 + 1/120$, while 10 gives $100/120 = 1 - 1/6$. The triangle of three successive knight moves, k = 12, has area $6/5 = 1 + 1/5$. (See the bold cross-line in the third diagram.). These answer questions (a) (b) (c) above.

The size number k for the big 3:4:5 triangle of knight moves (see p.10), calculated from $k^2/120 = 30$, is k = 60, as might be expected. The number 60 has other resonances with knight moves. T. R. Dawson's mss in the BCPS Archives include a chart of ‘The Unit of the Nightrider's Two-Move Domain 60×60’ which shows that any square can be reached in one, two or three nightrider moves.

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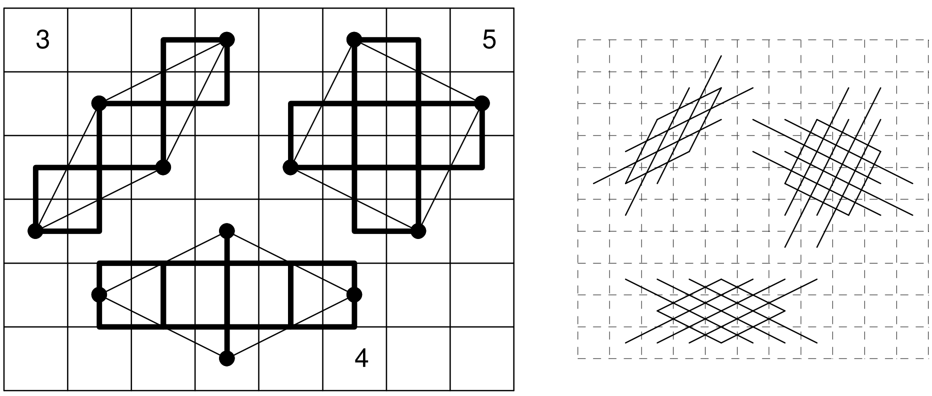
## Quadrilaterals

After triangles we naturally progress to quadrilaterals. Can we draw a quadrilateral of knight's moves which encloses a unit area (i.e. equal to the area of a square of the board)? [This was Puzzle Question 41 in the *Games and Puzzles Journal* (issue 17, 1999, with answer in issue 18).]




First note that the areas of the three types of rhomb formed by four knight moves, diamond, lozenge and square, are 3, 4 and 5 units respectively, as shown by the diagrammed dissections.

Thus the lozenge provides a way of making knight quadrilaterals with area 1 unit, solving the stated problem. The right-hand figure shows the various different shapes and sizes of rhombs that can be formed by knight lines. The areas of lozenges possible are 1/4, 4/4, 9/4, 16/4, and so on. The areas of squares are 1/5, 4/5, 9/5, 16/5, 25/5 and so on. The areas of diamonds are 1/3, 4/3, 9/3 and so on.

It is possible to form shapes of unit area made up of 3 smallest diamonds or 5 smallest squares or 4 smallest lozenges. The four lozenges can form a unit rhomb (4 being a square number).

If the squares formed by knight moves are numbered 1, 2, 3, ... according to size than the area of the kth square is $k^2/5$. The sides are multiples of 1/5 of a knight move (whose length is $\sqrt{5}$). The largest square possible in a knight tour of the chessboard is that outlined by four three-step lines and is a size 9 square with area 16 + 1/5 cells of the board.

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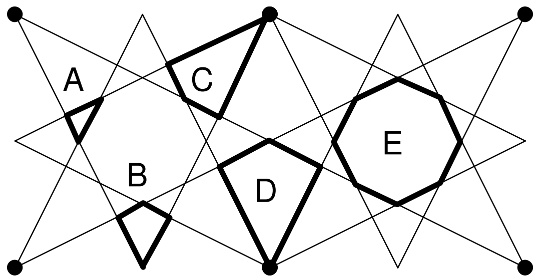
## Polygons

The complete network of knight moves divides up a board (except near an edge) into five basic 'irreducible' shapes (in the sense that no knight move can cut across them), namely the smallest triangle (A), three sizes of kite (B, C, D) and an octagon (E). The areas of these are 1/120, 1/40, 1/15, 1/10, 1/6 respectively, all 'aliquot parts' of the unit square.




In terms of triangles of size k whose area was proved above to be $k^2/120$, each kite can be seen to be the difference of two triangles: B = T2 - T1 area (4 - 1)/120 = 1/40. C = T3 - T1 area (9 - 1)/120 = 1/15. D = T4 - T2 area (16 - 4)/120 = 1/10. That the area of the octagon is 1/6 can be proved by noting that the square containing it, as proved earlier, is of area 1/5 = 24/120, from which four corners of size 1/120 are removed. In terms of 120ths the five areas are therefore 1, 3, 8, 12, 20.

For tour designs with shapes see the Graphic Tours in ⌘ 6 and Symmetric Tours in ⌘ 7.

# The Smallest Knight-Tourable Boards

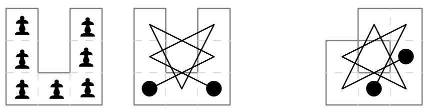
Results are gathered here for knight tours on the smallest boards of up to 12 cells as far as they are known. The tours are classified according to the types of symmetry they show, beginning with symmetries of higher degree and ending with asymmetric examples. Closed tours on these boards have been fully enumerated only on boards up to 12 cells, and even less work has been done on open tours, which allow more possibilities.

As noted in the introduction we require the cells of a board to be connected edge to edge and not attached only at a corner. Some examples were given there of knight circuits on loosely connected arrays of six or eight cells. We do not count these as boards. It must be possible for a wazir to get from any cell to any other in a series of steps making the board ‘wazir-connected’.

Knight tours on boards of more than 12 cells are catalogued in other volumes of the Knight's Tour Notes series. Shaped and Holey Boards in ⌘ 3, Oblong Boards in ⌘ 4, Odd and Oddly Even Square Boards in ⌘ 5, Evenly Even Square Boards in ⌘ 6, and others.

## 7 cells

The smallest knight-tourable boards are the two 7-cell boards (heptominoes) shown here, one a shaped board the other a holey board. Each is tourable by both wazir and knight uniquely. The tours have the same symmetry as the boards, namely binary symmetry of the direct type, unaltered by reflection. In the first the axis of symmetry is lateral and in the second is diagonal.

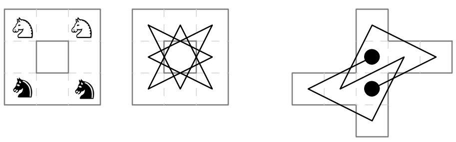


The lateral tour appeared in the earliest chess manuscripts, associated with the Shatranj player al-Adli ar Rumi (c.840). It was presented in the form of a puzzle where you are to place seven pawns round three edges of the 3×3 board, leaving only the top cell vacant, each new pawn being placed at a knight's move from the preceding one (Murray 1913 p.175-6,.336-7). The trick is to start at a knight's move from the vacant cell. A similar puzzle (not in the mss) can be set on the 7-cell board with hole.

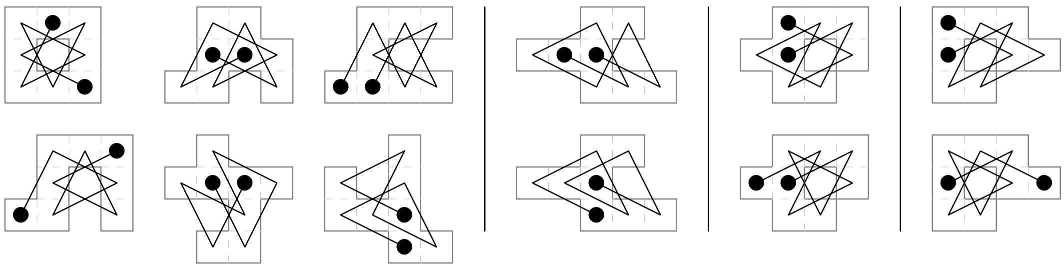
## 8 cells

The earliest manuscripts associated with al-Adli and as-Suli (c.900) include the puzzle, often attributed to the later writer Guarini (1512), of the Four Knights, White and Black, placed in the corners of the 3×3 board. These are to be interchanged by knight moves without having two on a cell at the same time. The knights go in procession round the star-shaped closed tour of the eight edge cells (taking 4, not 2, moves each).

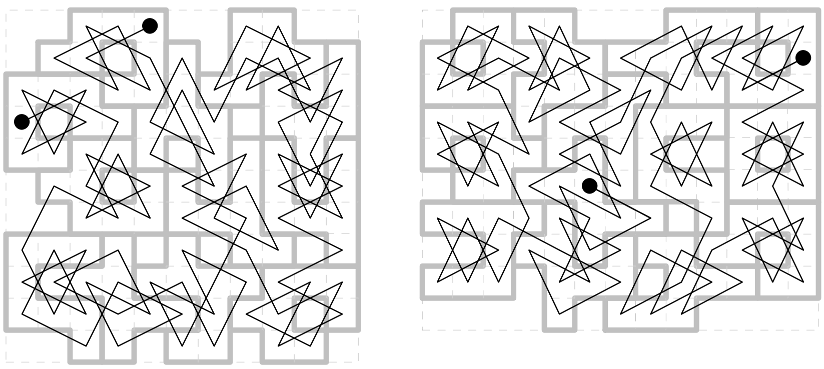
Thus the smallest board with a closed knight tour is the 8-cell centreless 3×3 board. In our coding system the centre cell is 0, the mid-edge cells are 1, and the corner cells are 2, so the moves are described by the formula: 1-2. Some examples of 8-cell tours on loosely connected arrays were shown in the introduction. Among the 8-cell open tours there is one with rotary symmetry.



We now encounter our first asymmetric tours. There are twelve distinct asymmetric open tours of 8 cells, forming nine board shapes, three of which can be toured in two ways. The only 8-cell board that admits a reentrant tour is thecentreless 3×3 board (delete one of the moves of the closed tour). This 7-move ‘star’ formation is a common component in larger tours.



These collages (Jelliss 1995) show the 8-cell open tours (including the symmetric case) joined to form a single open tour. The first, excluding the 3×3, fits them within an 11×11 area, the second, including the 3×3, requires a 10×13 area. Some voids, additional to the holes, seem unavoidable.



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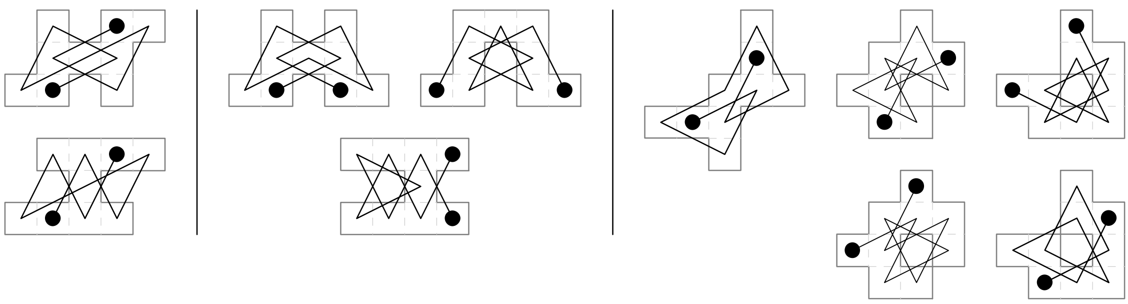
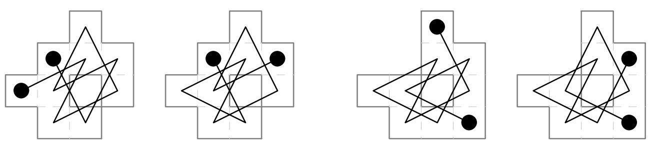
## 9 cells

There are 10 symmetric open knight tours on boards of 9 cells. Two have rotary symmetry, the others are axial, consisting of three with lateral axis and five diagonal. Of the diagonal tours one is on a shaped board, the others on two holey boards, each with two tours.

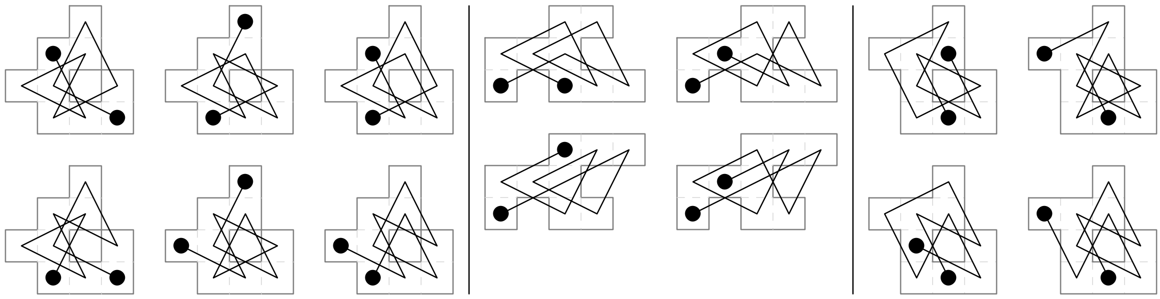


The two holey boards with axially symmetric tours also each have two asymmetric tours:

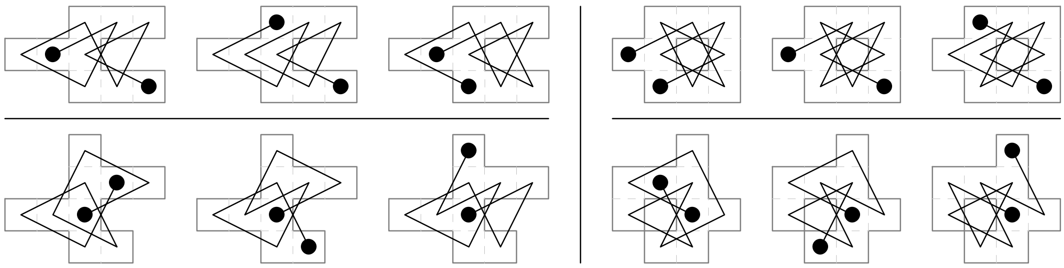


My last check on 9-cell knight tours (17 July 2013) found 93 tours (10 symmetric), using 55 boards (8 symmetric). There are 13 holey boards with 34 tours, and 42 shaped boards with 59 tours.

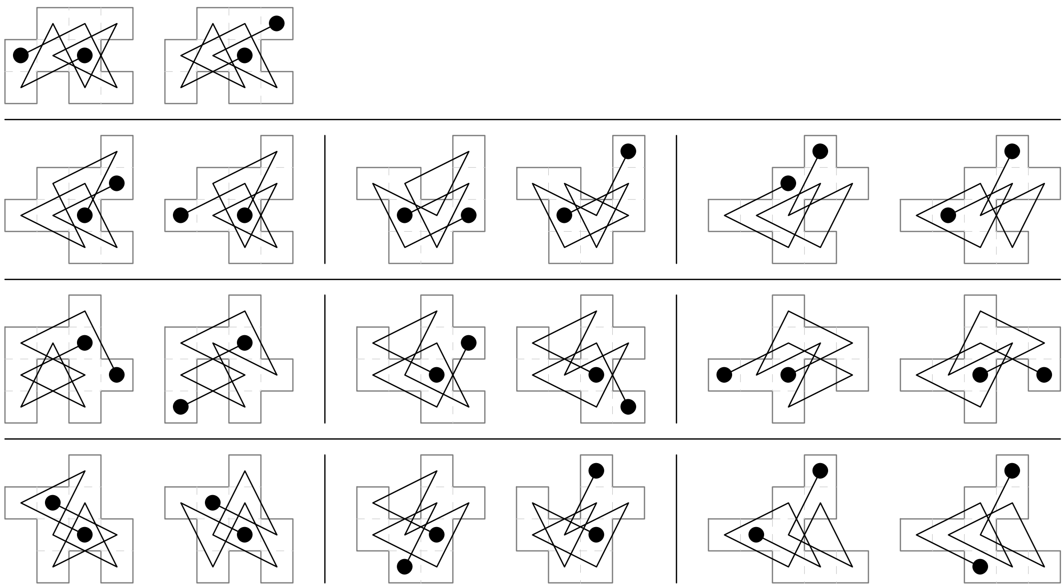
We next show other 9-cell boards with multiple tours. There is one that can be toured in 6 ways. Besides the two symmetric boards above, two other asymmetric boards can also be toured in 4 ways.



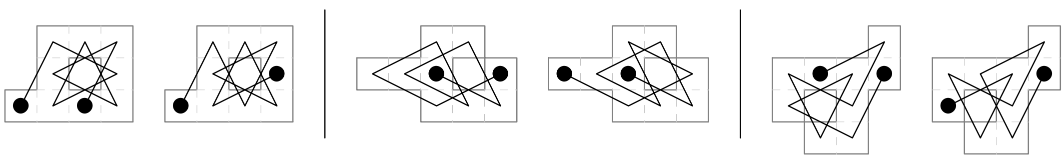
Then there are four boards that can be toured in three ways. Two of these are symmetric boards.



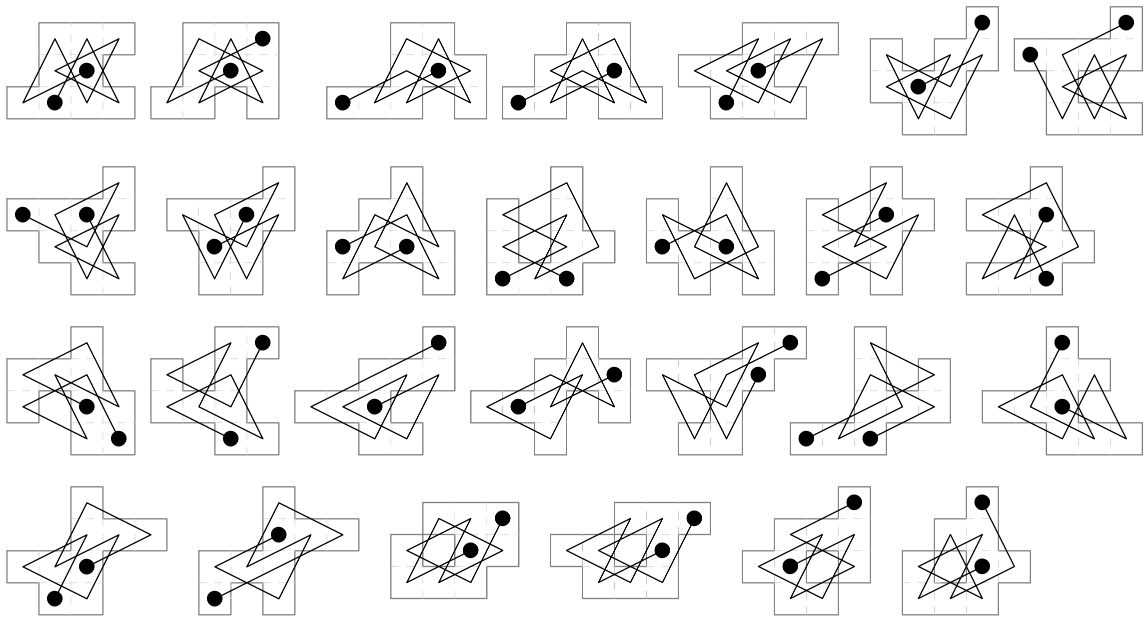
There are 13 board shapes that have two tours, 10 shaped (without holes):



and three holey:



Fnally we show the remaining 27 that tour uniquely.



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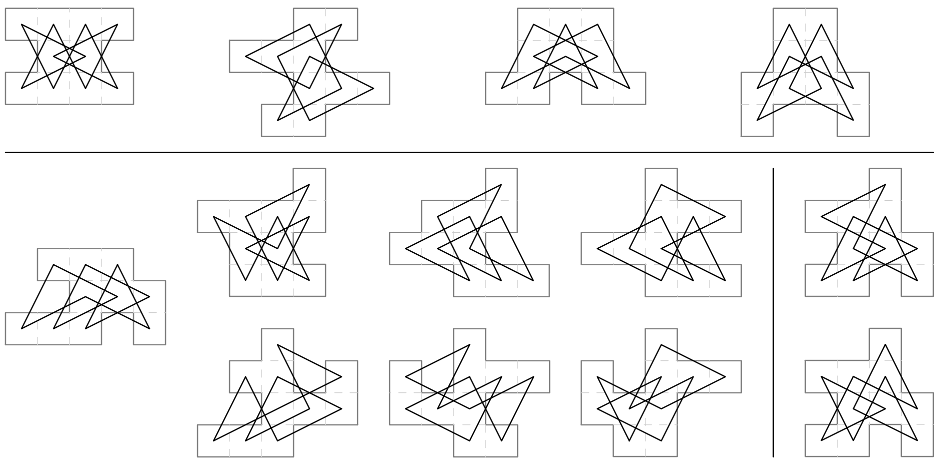
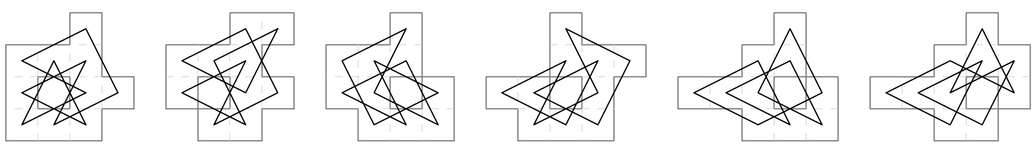
## 10 cells

This size of board is the first without holes on which a knight may play a closed tour. All tours of this type were enumerated by O. E. Vinje of Baltimore (*Fairy Chess Review*, ¶8149, Jul 1949 p.44 and Aug 1949 p.53). He found there are 13 tours on 12 board shapes. Four of the tours are symmetric. One is Biaxial, the others Eulerian, Murraian (axial with two cells on the axis) and Sulian (axial with no cells on the axis). Two of the nine asymmetric tours use the same board shape.

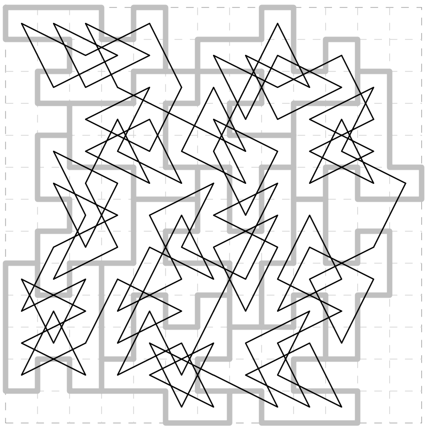


The biaxial tour was probably known to H. J. R. Murray earlier. It can be regarded as a combination of Bergholtian symmetry (where moves cross at the centre) with Sulian symmetry (where moves cross the axis of symmetry but do not enter any cell on the axis) and Murraian symmetry (where moves enter two cells on the axis).

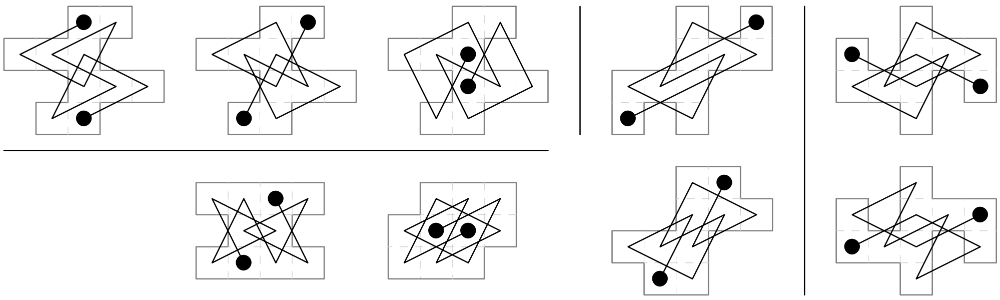
T. W. Marlow (1995) has noted six further 10-ominoes with holes that have closed knight tours.



This collage of the twelve 10-cell Vinje tours without holes, fitted within a 13×13 area to make a 120 move closed tour of a shaped board without holes, is by A. W. Baillie (*FCR* ¶8531 Dec 1949 p.84 and Jun 1950 p.110). As noted above a variant route is possible in one piece (replace d2-e4, f1-g3 by the other sides of the square).



The 10-cell board with a closed Eulerian tour also has three centro-symmetric open tours. There are a further 6 centro-symmetric open tours. Two use the same shape of board. One tour is reentrant (closure gives the 10-cell tour with biaxial symmetry).



The asymmetric open tours have not been enumerated.

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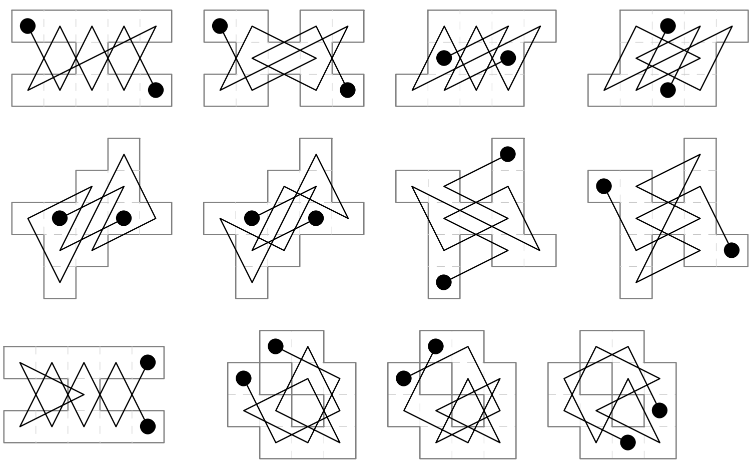
## 11 cells

On shaped boards of 11 cells there are 8 rotary tours on 5 different boards. One with lateral axis on a shaped board. Three with diagonal axis on a holey board.



Asymmetric tours on 11-cell boards have not been enumerated to my knowledge.

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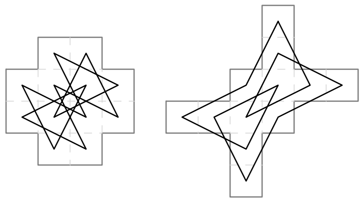
# 12 cells

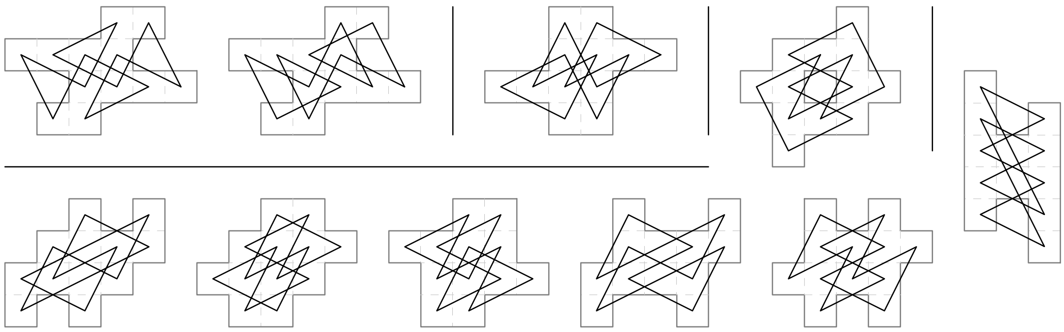
## The 12-cell symmetric closed tours

There are two 12-cell tours with biaxial symmetry of diagonal type. The one within the 4×4 area was found by Euler (1759), the one within the 6×6 was my own discovery which prompted an enumeration of 12-cell tours made by the author and T. W. Marlow in 1995. They can be described as doubly Murraian, entering two cells on each diagonal axis.

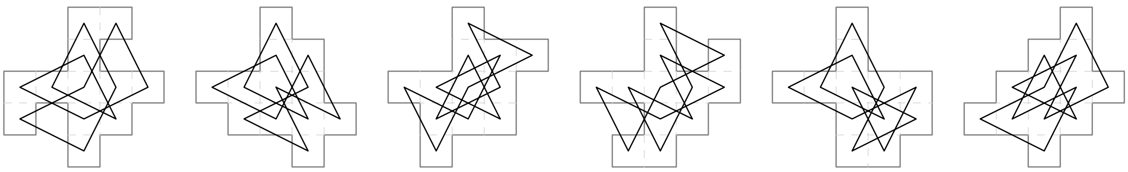


We found in all 27 symmetric closed tours of 12 cells on 24 board shapes, 8 with a hole.

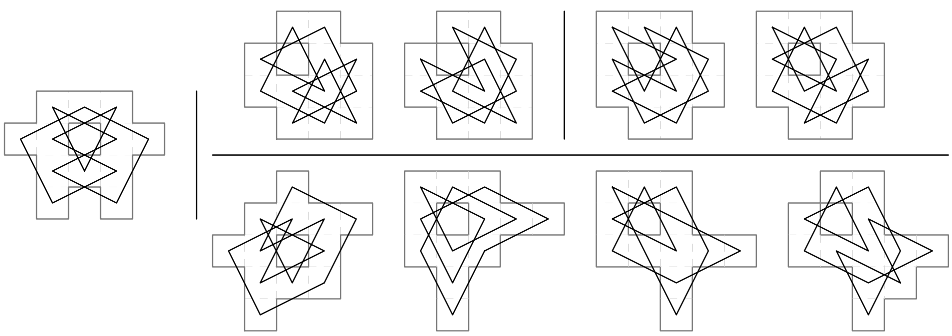
On 12-cell boards there are four closed tours with Eulerian rotary symmetry. Two on one board shape, and two on other boards, one having a hole. There are also six tours with Bergholtian rotary symmetry, on six differently shaped boards without holes.



There are also six tours with diagonal Murraian symmetry on shaped boards without holes.

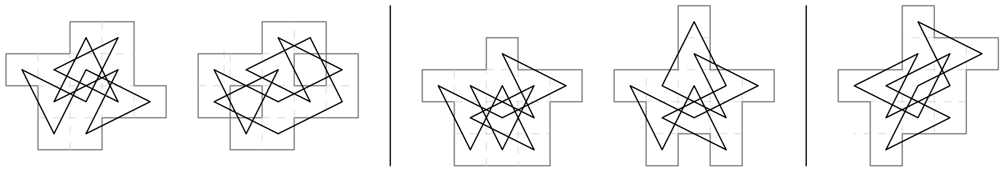


And nine more Murraian tours on seven holey boards (two with two tours). One tour has a lateral axis. The others have a diagonal axis. The first 4×4 area tour here was shown by Murray (1942).



## The 12-cell asymmetric closed tours

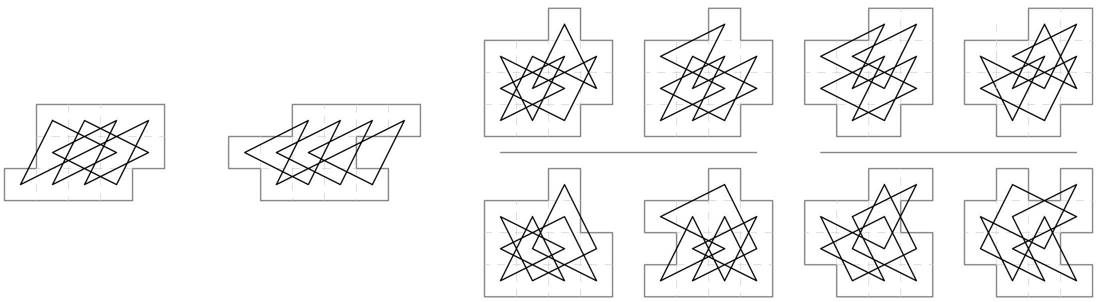
There are 5 asymmetric closed tours on symmetric boards, one with two holes. The first and last of these boards have a symmetric tour as well as the asymmetric one. The others have no symmetric tours, only the one asymmetric tour.



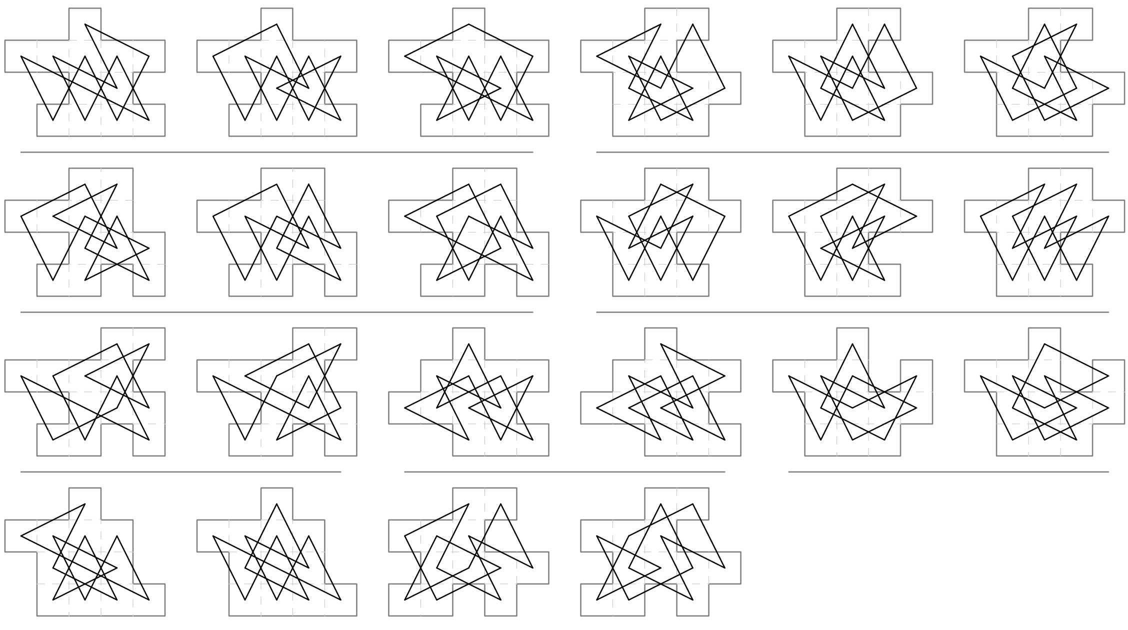
Each of the above tours can be seen in an alternative orientation (without the board being changed) by rotating the first two tours or reflecting the others. The two examples with lateral axis differ only in having a pair of moves, and the associated cell, folded up or down.

We now list the asymmetric closed tours on asymmetric boards according to the size of the containing rectangle.

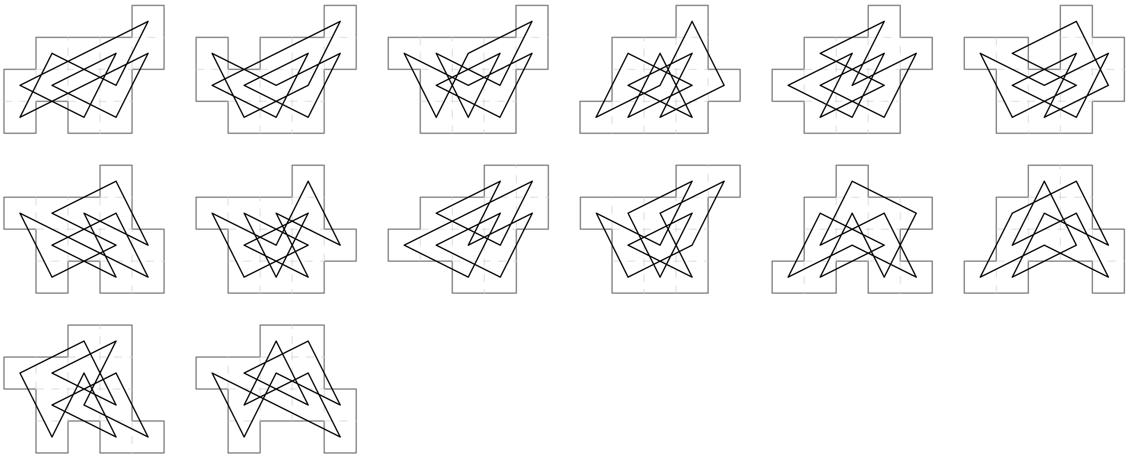
**3×5, 3×6 and 4×4 cases**



**4×5 case shaped**. (a) asymmetric boards with multiple tours: 4 with 3 tours, 5 with 2 tours.



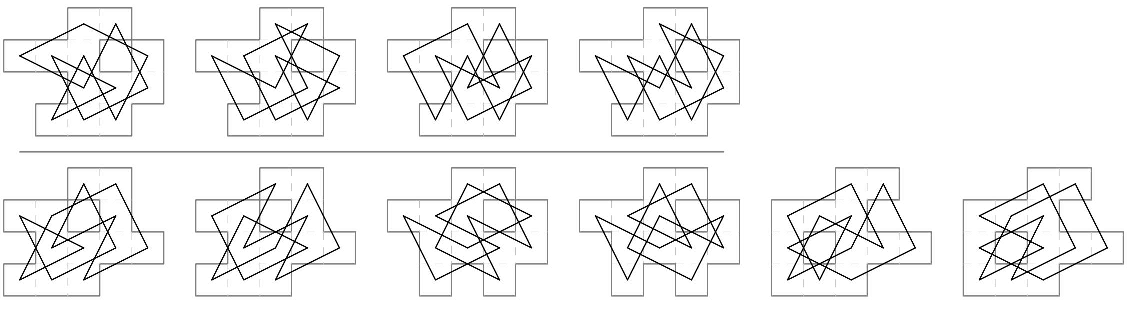
**4×5 case shaped**. (b) asymmetric boards with 1 tour and incomplete middle file.



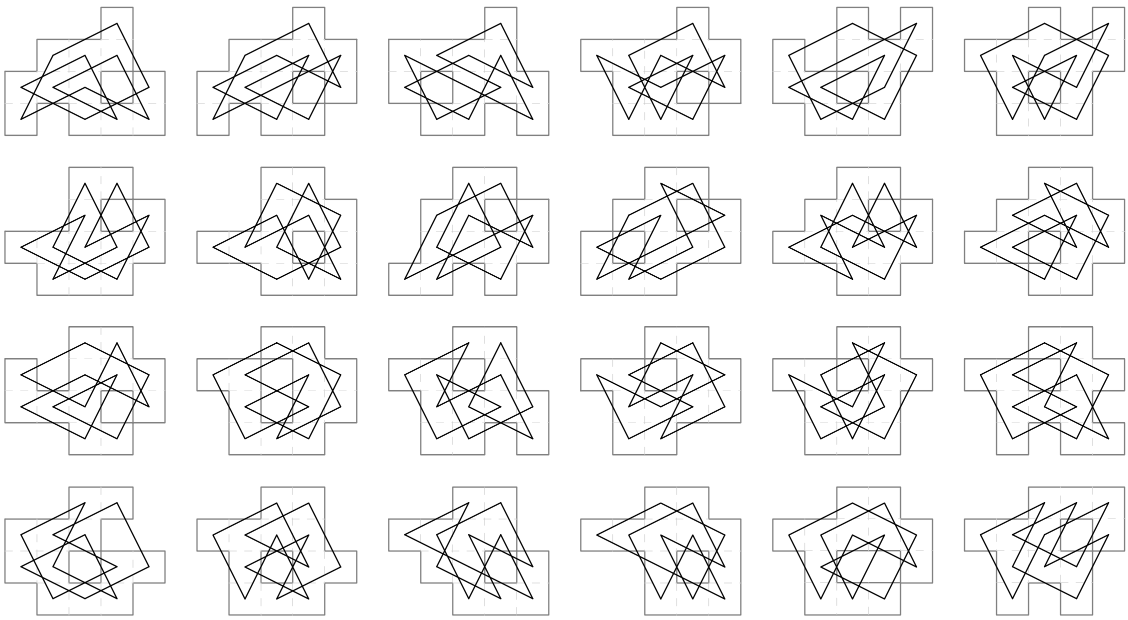
The sixth above is the only example where one of the centre two cells is cut away.

**4×5 case shaped**. (c) asymmetric boards with 1 tour and complete middle file.

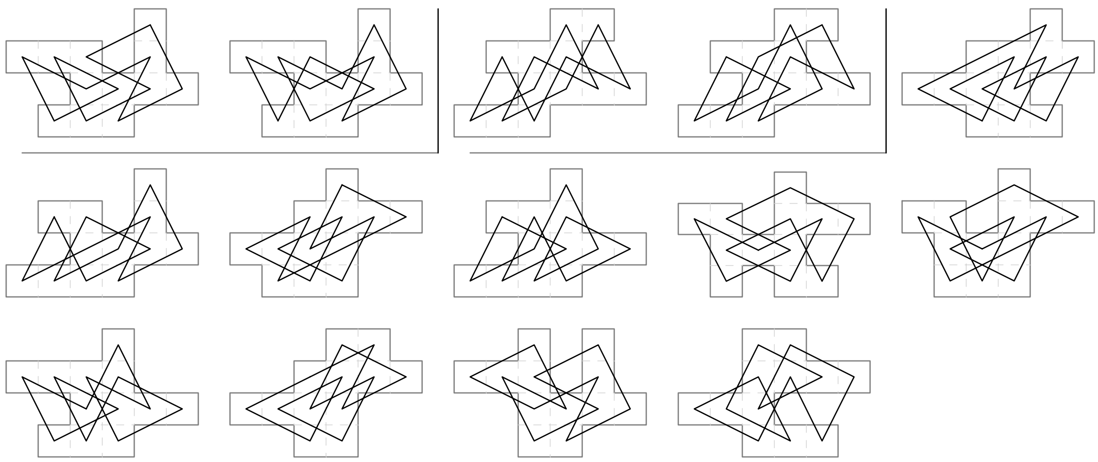
**4×5 case holey**. (a) asymmetric with multiple tours. One shape with four tours, and three shapes with two tours each.



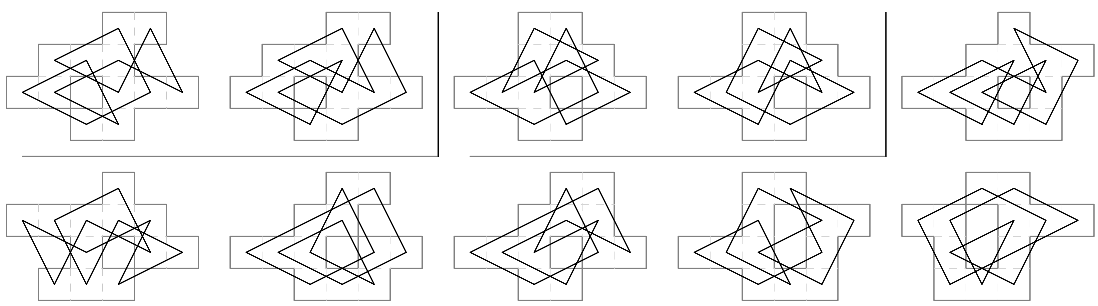
**4×5 case holey**. (b) Asymmetric boards with a single tour.



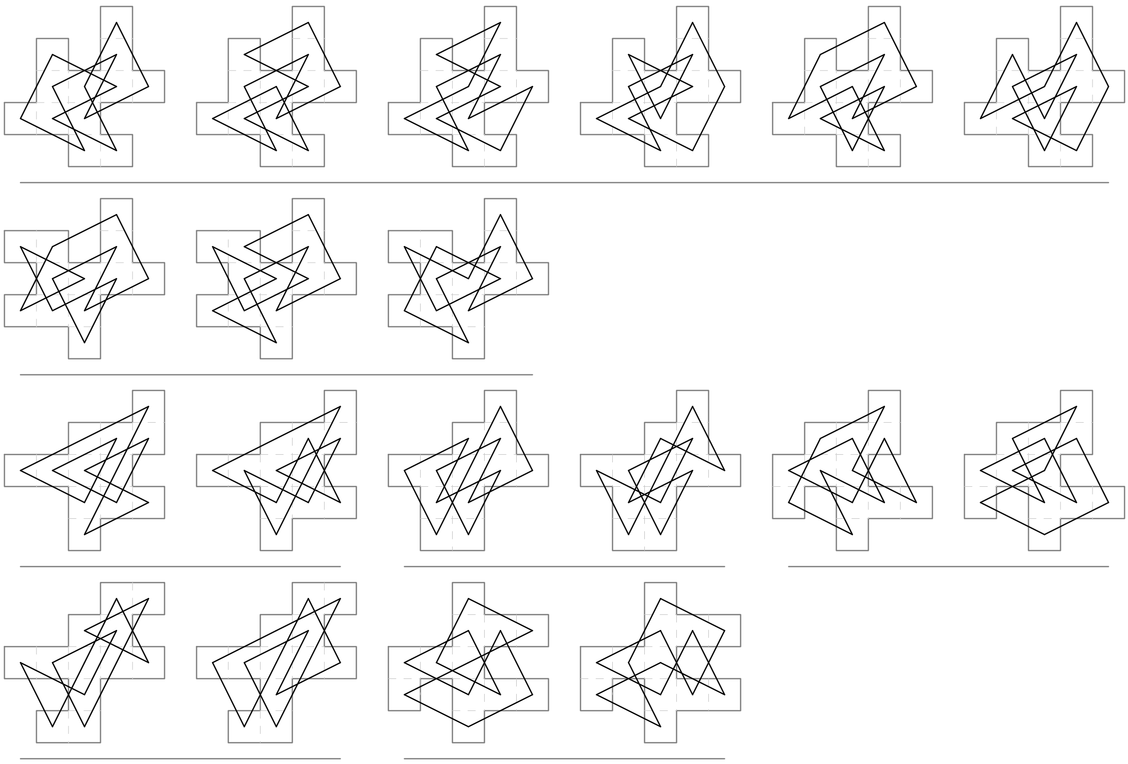
**4×6 case shaped**. Two shapes have two tours each. The other ten have a single tour.



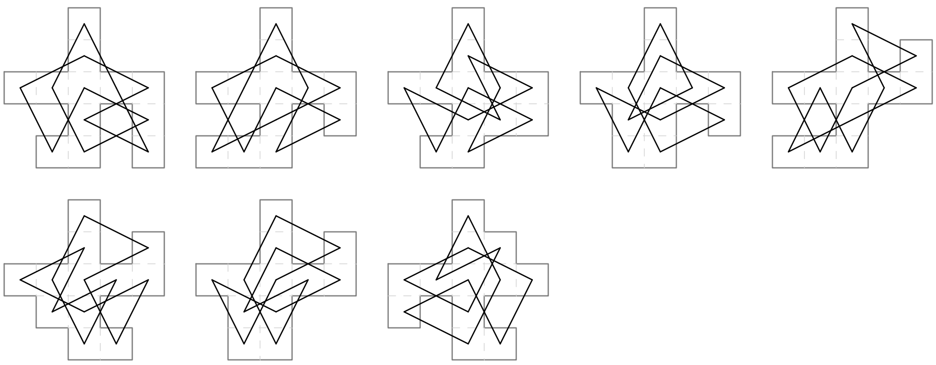
**4×6 case with holes**. Ten tours Two boards with two tours. Six boards with single tours.



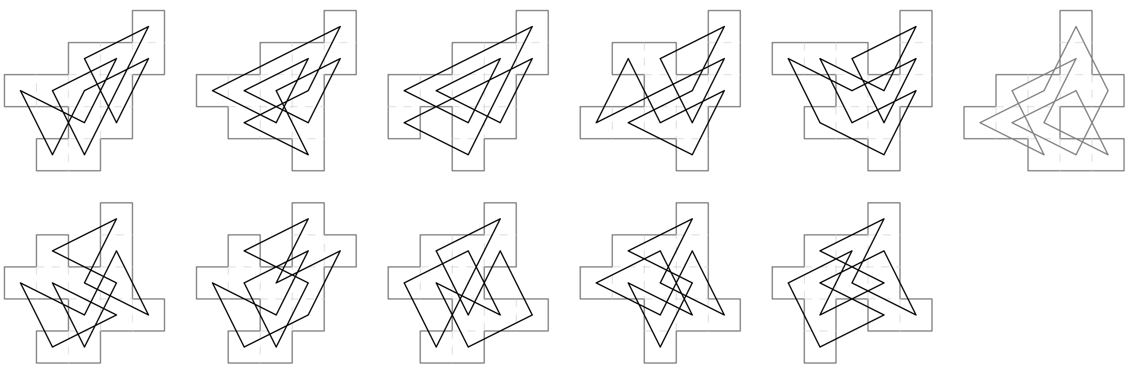
**5×5 case shaped**. (a) asymmetric boards with multiple tours:
There is one with six tours, one with three, and five with two.



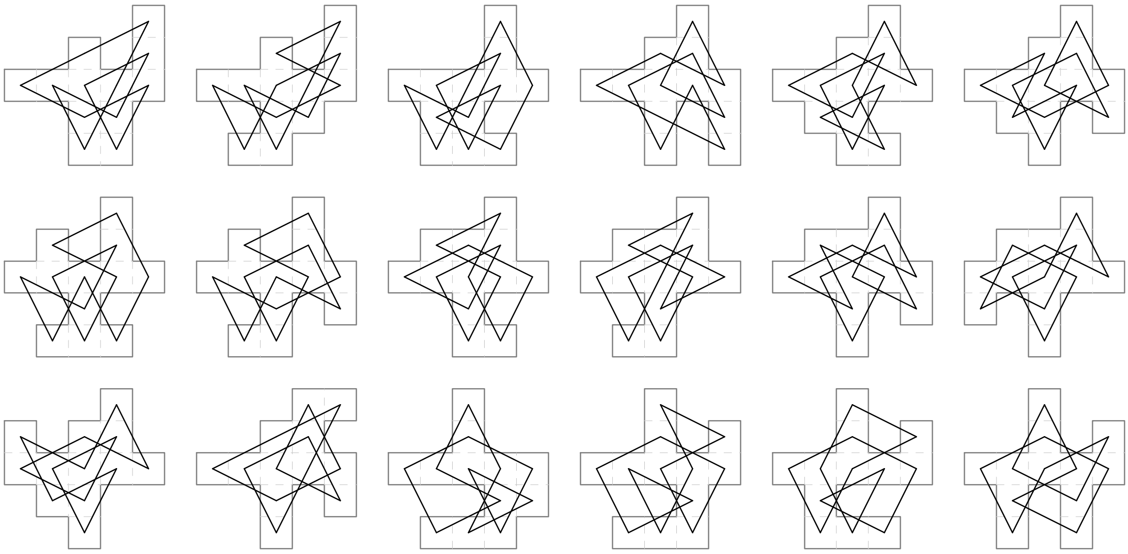
**5×5 case shaped**: (b) asymmetric with single tour: 8 with complete rank and file.



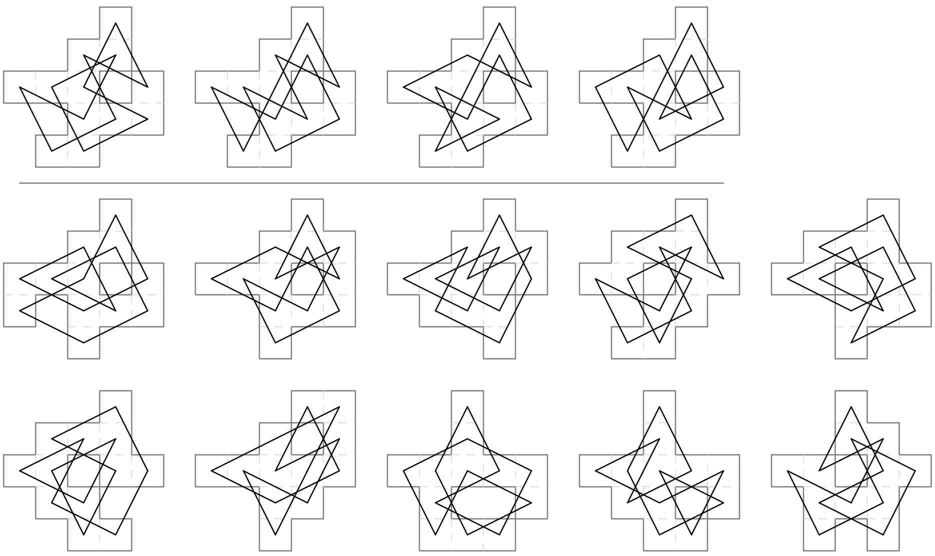
**5×5 case shaped**: (c) asymmetric boards with single tour: 11 with no complete rank or file. The third includes three two-move knight lines and also cycles round in one consistent direction.



**5×5 case shaped**: (d) asymmetric with single tour: 18 with complete rank but no complete file



**5×5 case holey**. One board has four tours. Ten with single tour.



**5×6 case** Six tours. Five boards, two with a hole, one having two tours.

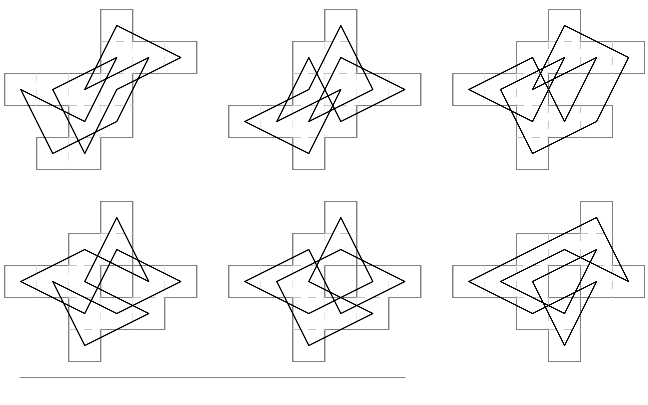


**Table of results for 12 cell closed tours**

In the preceding sections we have shown the 236 closed tours of 12 cell boards (27 symmetric and 209 asymmetric). This enumeration particularly of the asymmetric tours has proved to be a considerably stiffer problem than for 10 cells. An independent check may still be advisable.

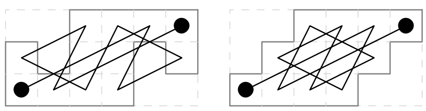
| Size | Boards | Tours | Symmetric | Holey |
|---|---|---|---|---|
| 3×5 | 1 | 1 | 0 | 0 |
| 3×6 | 2 | 2 | 1 | 0 |
| 4×4 | 9 | 11 | 3B/5T | 2B/4T |
| 4×5 | 83 | 103 | 8B/6T | 30B/36T |
| 4×6 | 22 | 27 | 2B/3T | 8B/10T |
| 5×5 | 67 | 83 | 12B/11T | 16B/19T |
| 5×6 | 5 | 6 | 0 | 2B/3T |
| 6×6 | 1 | 1 | 1 | 0 |
| Totals | 190 | 236 | 27B/27T | 58B/72T |

Thus for example within the 4×5 area there are 83 different 12-cell boards, of which 30 have a hole. Since some of these can be toured in two or more ways they give 103 tours, (36 on holey boards and thus 67 on shaped boards). There are 8 symmetric boards of this size but only 6 symmetric tours since two of the symmetric boards only have asymmetric tours (another has two tours, but only one of them is symmetric). In the 5×5 area there are 12 symmetric boards but one of these, the one with lateral axis of symmetry, has only an asymmetric tour, so there are only 11 symmetric tours. Again, one of the symmetric boards has two tours but only one of these is symmetric.

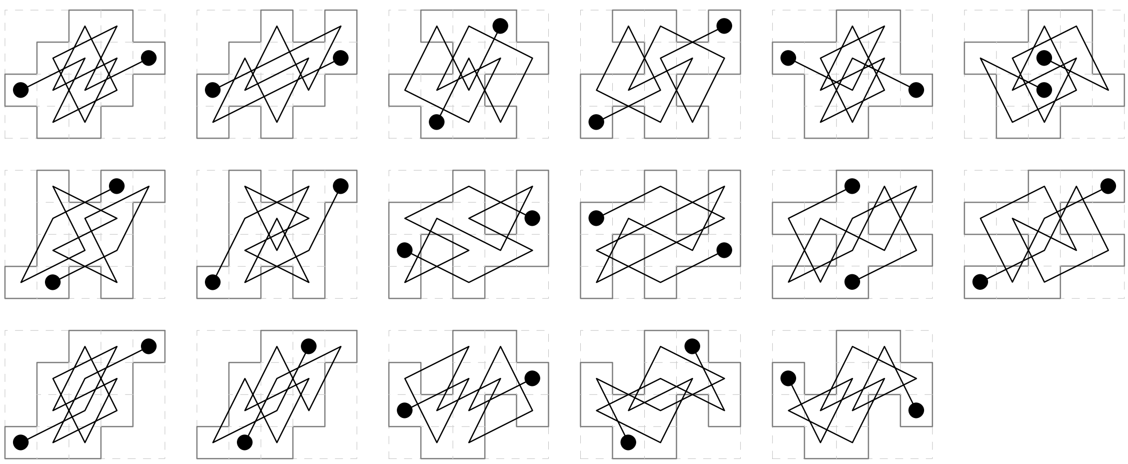
## The 12-cell symmetric open tours

There are 49 geometrically distinct symmetric 12-cell open tours. These consist of: two on the rectangular 3×4 board (found by Euler 1759) and 12 reentrant (derived from the six Bergholtian closed tours above by deleting one of the central moves) plus 35 other shaped-board tours, eight with holes (using 20 shapes of board, two with holes). Diagrams follow.

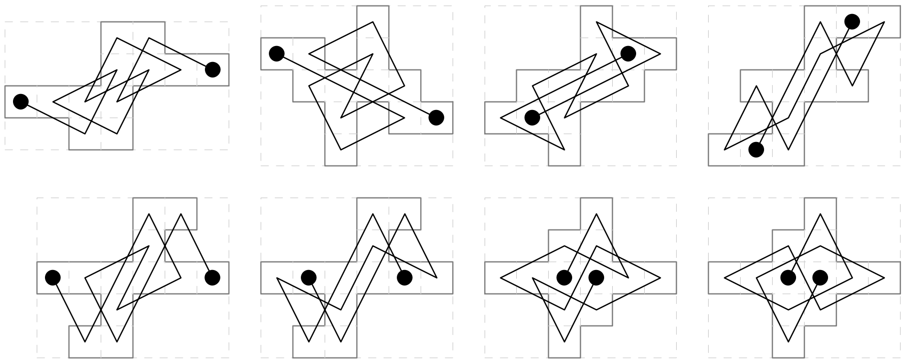
Besides the 3×4 tours there are two other three-rank symmetric open tours, within the 3×6 area. It may be noted that these two tours can be joined, jigsaw-style, to copies of themselves to form open tours within frames 3×10, 3×14, 3×18, and so on endlessly.



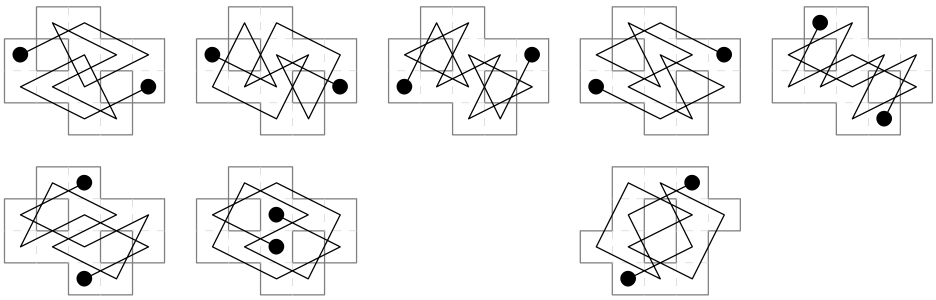
There are 10 shaped boards within the 4×5 area (4 with 1 tour, 5 with 2 tours, 1 with 3 tours).



The tours within the 4×7 area (1 tour), and 5×6 area (3 with 1 tour, 2 with 2 tours) are more spread out:



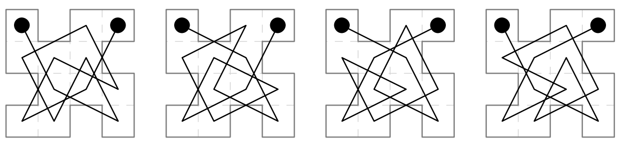
Here are the 8 symmetric open tours on holey boards (but using only 2 board shapes).



## The 12-cell asymmetric open tours

These have not been counted. There is the one on the 3×4 board. There is only one closed 12-cell tour within the 3×5 frame, and it is asymmetric, so produces 12 reentrant tours. In a partial enumeration not yet checked, I found 39 non-reentrant tours within this frame. These tours use 10 different board shapes. Removing the minority color square from a2 the other two cells removed can be a1, b2; a1, c3; a1, d2; a1, e1; a1, e3; b2, e1; d2, e1; e1, e3; and with b1 as the minority cell, the others are a3, d2 or a3, e1.

Within the 4×4 frame two diamonds and a square can be joned together to give four geometrically distinct three-quarter tours.



For more examples of tours on shaped and holey boards see ⌘ 3 of these Knight's Tour Notes.

# Glossary

Brief definitions are given here of terms used in special senses throughout the Knight's Tour Notes, and some comments on mathematical notations. Terms are also defined in more historical and analytical detail at appropriate points in the text. Author's names are included here only if used to describe some feature named after them. To locate biographical details see Bibliography in ⌘ 12.

**Adjacent**. Of cells with a side in common, i.e. connected by a wazir move (cf Neighbouring).
**Alfil**. {2,2}-mover.
**Antelope**. {3,4}-mover. See also Fiveleaper.
**AP** or **Arithmetical Progression**. A sequence of numbers of the form a, a + c, a + 2·c, ... with a 'common difference' (CD) equal to c.
**Arithmic**. Of a numbered rectangle tour in which the sums of the ranks and the sums of the files each consist of numbers in arithmetical progression, particularly with CD = 1.
**Array**. Any geometrical arrangement of cells.
**Asymmetry.** The property of being altered in appearance by any nonzero rotation or reflection. In the case of an **asymmetric** tour all orientations of its diagram will have different appearance.
**Axial**. Of a diagram having one axis of symmetry, but not two. (cf Biaxial)
**Axis** (plural **Axes**) of symmetry. A line in which a pattern may be reflected without alteration.

**Bergholtian**. Of a closed rotary tour which passes twice through the centre point of the board.
**Beverley**. Of a Regular Quarte not a Square or Diamond, or a Regulat Tour with such Quartes..
**Biaxial**. Unaltered by reflection in two axes, but not Octonary. Direct Quaternary symmetry.
**Binary.** Twofold. Binary symmetry can be reflective (Direct) or rotative (Oblique).
**Birotary.** Unaltered by 90° rotation, but not Octonary. Oblique Quaternary symmetry. (cf Rotary.)
**Bisatin**. An arrangement of cells having two in every rank and file of a square board. cf Satin.
**Bisatinic**. Describes a rook-move tour having one move in every rank and file.
**Bishop**. {1,1}-rider; Blockable {n, n}-mover for any value of n.
**Bishopwise**. Alternative term for Diagonal (adj).
**Block**. One of the 2×2 areas into which a board (2·h)×(2·k) can be dissected (cf Quad.)
**Block diagram**. Representation of knight path on (2·h)×(2·k) board as king-walk on h×k board.
**Board**. A connected area divided into cells, such as is often used in games. An array of small cells in which we can get from any one to any other in a series of moves between adjacent cells. This convention rules out arrays of squares where some are supposedly 'attached' only at a corner.
**Border**. Frame of two-cells width.
**Bracketing:** Curved brackets indicate ordered pairs, while curly brackets indicate unordered pairs. Thus (x,y) = (u,v) means x = u and y = v. In particular (x,y) = (y,x) only when x = y, whereas {a,b} = {b,a} always.

**Camel**. {1,3}-mover.
**CD**. Common Difference, in an arithmetical progression.
**Cell**. A point or area on which a piece may stand. Cells may be depicted as dots or as squares or hexagons, or other shapes on particular boards.
**Celtic Knot**. A ribbon design with alternate over and under crossings.
**Celtic Tour**. A knight tour with no minimal triangles (facilitating its depiction as a Celtic knot).
**Central symmetry.** Having a centre of symmetry. Centrosymmetry (cf Oblique, Rotary).
**Centre.** A point about which a pattern may be rotated without alteration. A centre of symmetry. If a centre exists it is unique. The centre of a rectangular board may be at the centre of a cell (odd by odd board), at the corner where four cells meet (even by even board), or at the middle of an edge where two cells meet (odd by even board).
**Chain**. Alternative term for open path.

**Chessboard**. Any board formed of square cells arranged edge-to-edge. Can be shaped and holey.
**Circuit**. Alternative term for closed path. See Short Circuit.
**Closed**. Describes a path or tour without end-points.
**Column**. Alternative term for File.
**Combinations.** The number of ways of selecting a set of r elements from a set of n (≥ r) is the combinatorial number nCr = C(n,r) = n!/[(n – r)!·r!].
**Common difference**. See Arithmetical Progression.
**Commuter**. {4,4}-mover.
**Connected**. Not divisible into two parts without a link between.
**Container**. The unique rectangle enclosing a chessboard and touching it on all four sides.
**Contiguous**. Of Chains a(1) ... a(k) and b(1) ... b(k) where a(r) and b(r) are in adjacent cells for all r.
**Contraparallel**. Of Chains a(1) ... a(k), and b(1) ... b(k) with a(r) and b(k+1−r) in the same file for each r, thus adding a constant amount to the file sums.

**Diagonal**. (adjective) In the direction of {n,n} moves. Bishopwise. (cf Lateral and Skew)
(noun) A diagonal line of cells. Also, line joining opposite corners of a four-sided figure.
**Diagonally magic**. Magic square in which the two long diagonals also add to the magic constant.
**Diagram.** A representation of a tour or other pattern presented on a page. Rotations and reflections of the diagram are regarded as representing the same tour or pattern.
**Diamagic**. Diagonally magic.
**Diamond**. Rhombus with diagonal angle bisectors. (cf Lozenge, Square)
**Direct**. Invariance to reflection. Having an axis of symmetry (cf Oblique).
**Direction.** Usually means parallel to a given line and in a given sense along the line, but sometimes not specifying the sense (better termed Line-mode in that case).
**Directed**. Of path or tour, having initial and final cell and direction of description specified.
**Dabbaba**. {2,2}-mover.
**Dummy**. Piece unable to move. A {0,0}-mover. Not the same as a Zero-mover.

**Emperor**. {0,1} + {1,2}-mover. Wazir + Knight.
**Empress**. Rook + Knight.
**End** or **Endpoint**. Cell where only one move (of given type) is incident.
**Eulerian symmetry**. Rotational symmetry not passing through the centre point.

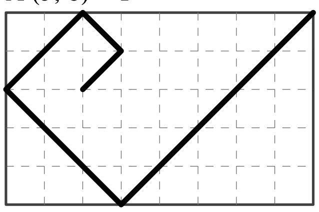
**Factorials.** The n-th factorial number is the product of the first n positive numbers n·(n – 1)·(n – 2)·...·3·2·1 and is denoted n!. It is usual to also define 0! = 1.
**Farzin**. Old name for Fers.
**Feasible**. Possible except in numerous special cases.
**Fers**. {1,1}-mover.
**Fibonacci Numbers.** The n-th Fibonacci number is defined by the recurrence F(n) = F(n–1) + F(n–2) with F(1) = F(2) = 1. The sequence runs: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
**Figured tour**. Tour in which, when numbered, selected numbers appear in geometrical formations.
**Fil**. Old name for Alfil.
**File**. Line of cells up or down a page, parallel to the margins (cf Rank).
**Five-leaper**. {0,5) + {3,4} leaper, making moves of length 5 units.
**Five-mover.** {0,5}-mover.
**Four-mover** or **Four-leaper**. {0,4}-mover.
**Frame**. Rectangular array r×s with rectangular hole (r–b)×(s–b) of the same breadth b throughout.
**Free leaper**. Leaper able to reach any cell from any other by a series of its moves on a sufficiently large board. On the 8×8 board the free leapers are wazir, knight, zebra, giraffe and antelope.

**Generic** = Geometrically distinct
**Geometrically distinct**. Counted without regard to orientation or direction of description.
**Giraffe**. {1,4}-mover.

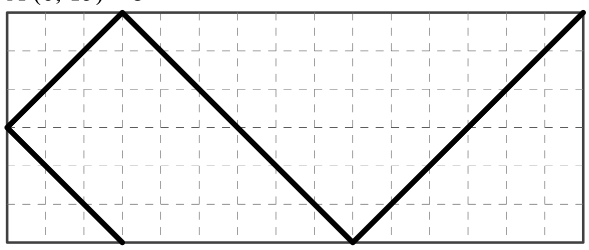
**Greatest Common Divisor**. See Highest Common Factor.
**Grid**. The lines dividing a board (2·h)×(2·k) into Blocks 2×2.

**Half turn**. A rotation through 180 degrees.
**Highest Common Factor (HCF):** The highest common factor (also known as 'greatest common divisor') of two cardinal numbers, is the largest number that exactly divides both of them, and we denote it hcf(r,s) or Λ(r,s). A method to determine the HCF, known as Euclid's algorithm, can be presented in the appropriately graphical form of a zizag path: Draw a rectangular board of r×s squares (r < s) and starting at a corner draw a diagonal. This diagonal is regarded as crossing out the square r×r of which it is the diameter, thus reducing the rectangle to a smaller rectangle r×(s–r). Now, from the point reached, draw a diagonal in the reduced rectangle. And so on. Each diagonal will be equal to or less than the preceding one in length, and the length of side of the square crossed by the final diagonal is the required HCF. Examples:

Λ (5, 8) = 1



Λ (6, 15) = 3



If the number of diagonal moves of the first size is m(1) then the r×s rectangle is reduced to a rectangle r(1)×r, where r(1) = s – m(1)r. Thus r/s = 1/[m(1) + r(1)/r] = 1/[m(1) + 1/(r/r(1))]. If the numbers of diagonal moves of successively decreasing lengths are m(1), m(2), m(3) ...
then the ratio is: r/s = 1/[m(1) + 1/(m(2) + 1/(m(3) + ... + 1/(m(n–1) + 1/m(n))...))].
This type of expression is known as a 'continued fraction'. If we calculate the continued fraction as far as 1/m(n–1) the resulting ratio p/q has the property that p·r – q·s = hcf(r,s). In particular, when hcf(r,s) = 1 this enables us to find a solution in whole numbers to the equation p·r– q·s = 1.
(Ref: G. Crystal, *Algebra*, 1900, vol.2, p.436, 474.)
**Hole.** A cell, or group of connected cells, omitted from a board internally, that is surrounded on all sides, but not necessarily at the corners, by cells of the board (cf Indentation).
**Holey.** Of a Board with one or more holes. It may be Rectangular or Shaped in outline.
**Horizontal**. Having the longest move component parallel to the ranks. (cf Vertical)

**Indentation**. A cell or group of connected cells omitted from a board externally, that is between the edge of the board and its Container (cf Hole, Shape).
**Indexing**. Lettering of the cells of a board to indicate its structure.
**Irregular**. Of a Quarte, not Square, Diamond or Beverley type.
Also: Unary without notable symmetry.

**King**. {0,1}+{1,1}-mover.
**Knight**. {1,2}-mover.

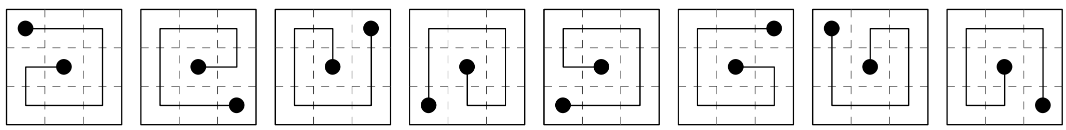
**Lancer**. {2,4}-mover; double-length knight.
**Lateral**. (adjective) Parallel to the edges of a rectangular board. In the direction of {0,n} moves. Rookwise. (cf Diagonal, Skew). Also (noun) A Rank or File.
**Leap**. Move {r,s} made in any orientation, and not blocked by intervening pieces.
**Leaper**. Piece having one or more leap patterns {r,s}.
**Line-mode.** Parallel to given straight line without sense of movement along the line. (cf. Direction)
**Link**. Line representing a move, without regard to direction.

**Lozenge**. Rhombus with Lateral angle Bisectors.

**Magic**. Of an array in which subarrays of a given type all add to the same constant value.
**Magic rectangle**. Arrangement of numbers in r ranks each adding to R and s files each adding to S. It follows that The total of all the values is T = r ·R = s ·S.
**Magic square**. Magic rectangle with equal sides, so that r = s and R = S. (Note: in some older literature ‘magic square’ is taken to imply ‘diagonal magic square’.)
**Magic tour**. Tour of a board which, when numbered from an appropriate cell, has magic properties.
**Median**. Middle line of a board, horizontal or vertical, may pass through or between cells.
**Move**. Transfer of a piece from one cell to another. On a rectangular board a move can be specified by the coordinates of the destination cell relative to the initial cell. A move from (x,y) to (x',y') is given in length and direction by the ordered pair of signed numbers (x'–x, y'–y).
**Move Type.** A move of type {r,s} takes a piece from cell (x,y) to any of the up to eight cells (x±r, y±s) or (x±s, y±r) within the confines of the board.
**Multiplication Signs**. Multiplication of numbers is shown by a raised dot (·). The cross sign (×) is short for ‘by’ as in dimensions ‘r by s’. Thus a rectangle of shape r×s has r·s cells.
**Murraian symmetry**. Axial symmetry with cells on the axis.

**Net**. A representation of a board, or moves of a piece on a board, as Nodes connected by Links.
**Neighbouring**. Of cells with side or corner in common, i.e. connected by king move. cf. Adjacent.
**Nightrider**. {1,2}-rider. That is blockable {n, 2·n}-mover for any n.
**Node**. Point representing a cell.
**Number notation.** Letters that take whole number values 0, 1, 2, 3, ... are written in normal upright type. Italic letters indicate that the numerical value may be negative or fractional.
**Numbered**. Of a series of things marked 1, 2, ... in the sequence they occur.

**Octonary.** Having eightfold symmetry. All eight orientations of the diagram look the same.
**Oblique**. Invariance to rotation but not to reflection. Having a centre of symmetry but no axis.
**Oblong.** Non-square rectangle.
**Open**. Describes a path or tour that has two ends.
**Oriented**. Arranged with a particular edge uppermost on the page.
**Orientation.** The way a tour is presented in a diagram. Assuming that rectangular diagrams are printed with their sides parallel to the sides of the page, one tour may appear in eight different orientations. From diagram (a), (b) is formed by a half-turn, (c) and (d) by a quarter turn, clockwise and anticlockwise, (e) and (f) by reflection in the horizontal and vertical medians, (g) and (h) by reflection in the principal and secondary diagonals.



(a) (b) (c) (d) (e) (f) (g) (h)

**Orthogonal**. At right angles (a term used by some writers to mean Lateral, since ranks and files are orthogonal to the edges of the board).

**Pandiagonal**. Describes a diagonal magic square in which the pairs of broken diagonals (which bcome diagonals when the files or ranks are cyclically permuted) also add to the magic constant.
**Panmagic** = Pandiagonally magic.
**Parent**. Describes a tour from which others are derived by addition of a frame.
**Path**. A connected set of moves in which no cell is incident with more than two moves.
**Piece.** A token with specified move powers, indicated by its shape, colour, or markings

**Piecewise symmetry**. Where sections of a tour show symmetry but the whole does not.

**Possible**. Can be done in all cases. (cf. Feasible)

**Powers.** To avoid small print, u to the power v is written u^v, though we also sometimes write the square u^2 and the square root u^(1/2) in the traditional forms $u^2$ and $\sqrt{u}$.

**Products.** We use the raised dot (·) to denote multiplication, reserving the cross symbol (×) as an abbreviation for 'by' in naming rectangular board sizes.

**Pseudotour**. A set of two or more paths that together use every cell of a board once. A pseudotour with no ends can be termed closed, one with two or more ends open.

**Quadrant**. A quarter of a board, bounded by medians.

**Quad**. One of the 4×4 areas into which a (4·h)×(4·k) board can be dissected.

**Quarte**. On a board even×even, a segment of four cells x+1, x+2, x+3, x+4 in a numbered tour where x is a multiple of 4 (or zero). Term introduced by Carl Wenzelides (1849) but applied by him only to quartes confined within a quarter of the 8×8 board.

**Quarter turn.** A rotation through 90 degrees.

**Quaternary.** Having fourfold symmetry. The eight orientations of the diagram occur in two sets of four alike. Quaternary symmetry may be Direct or Oblique. cf Birotary, Biaxial.

**Queen**. {0,1}+{1,1}-rider; Blockable {0,n}+{n,n}-mover for any n; i.e. rook + bishop.

**Rank.** Line of cells across the page (cf File).

**Rectangle.** Four-sided shape with right angled corners. Square or Oblong.

**Rectangular**. Of an Array of r·s cells in r ranks and s files with no cells missing.

**Reentrant**. Describes an open path by an X-piece whose ends are an X-move apart, i.e. a closed path can be formed from it. From a closed path of n moves we can form n reentrant paths. Care distinguishing between closed and reentrant is advisable when counting tours.

**Regular.** Of a Quarte with one cell in each rank and one in each file of a Quad. They are either of Square, Diamond or Beverley type.

**Rhombus.** Circuit formed of four equal moves. (cf Diamond, Lozenge, Square)

**Ride**. Move blockable if intermediate cells are occupied.

**Rider**. Piece able to make {k·r, k·s} rides for specified values of {r,s} and any value of k. (See Rook, Bishop, Nightrider, Queen.)

**Rook**. {0,1}-rider; Blockable {0,k}-mover for any value of k.

**Rotary**. Of symmetry invariant to 180° rotation, but altered by 90° rotation. (cf Birotary.)

**Row**. Alernative term for Rank.

**Satin**. An arrangement of cells having one in each rank and file of a square board.

**Semi-quarte**. Four-cell chain with one cell in each file and two in each rank, or vice versa.

**Semi-rotation**. A transformation of a bisatinic magic tour in which two quarters of the path are taken in the vertical instead of the horizontal direction, or vice versa.

**Shape**. Boundary of a connected area of cells.

**Shaped.** Of a Board that is non-rectangular and has no internal holes (cf Holey).

**Short Circuit**. A closed path covering part of a board, encountered in trying to construct a tour. If it cannot be avoided this shows that a tour of the whole board cannot be completed by that method.

**Skew.** Direction other than Lateral or Diagonal. Move {r,s} where $r \neq s$ and $r \neq 0$ and $s \neq 0$.

**Slant**. A knight move crossing two grid lines.

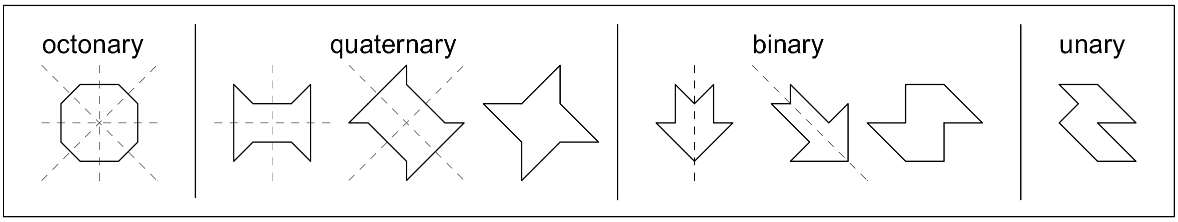
**Square**. Rhombus with right angled corners. (cf Diamond, Lozenge, Rectangle)

**Straight** or **Strait**. A knight move crossing one grid line.

**Sulian**. Of axial (binary direct) symmetry with no cells on the axis.

**Surround**. Frame of more than two cells width.

**Symmetry**. Invariance of a pattern to one or more transformations that preserve lengths. In the case of a bounded pattern this means invariance to certain rotations or reflections. In the case of a **symmetric** tour some of its eight orientations will look the same. See Octonary, Quaternary, Binary and Unary. Octagons showing these types of symmetry are illustrated.




**Threeleaper**. {0,3}-mover.
**Token**. A small shaped or marked object with unspecified powers (cf Piece).
**Tour**. A path that visits every cell of a specified board once. It can be open or closed.
**Triangular Number.** The n-th triangular number is the sum of the first n numbers.
$T(n) = 1+2+3+...+(n-2)+(n-1)+n = n\cdot(n+1)/2$.
**Tripper**. {3,3}-mover.
**Two-knight tour**. Two open knight paths connected by one (open) or two (closed) non-knight moves. If the connecting move is a rook move the whiole is an emperor or empress tour.

**Unary.** Not divisible into identical parts. An alternative term for asymmetric.
**Unit**. The width of a cell, is the unit in which lengths of moves are measured. Thus a wazir move is of length 1, a fers move of length $\sqrt{2}$, and a knight move is of $\sqrt{5}$ units.

**Vertical**. Having the longest move component parallel to the files. (cf Horizontal)

**Wazir**. {0,1}-mover. A single-step rook.
**Wild**. Applied to moves {r,s} in directions other than those of rook, bishop or nightrider.

**Zebra**. {2,3}-mover.
**Zero-mover**. Piece able to jump up and down on the spot in a turn of play. (cf Dummy)

# Puzzle Solutions

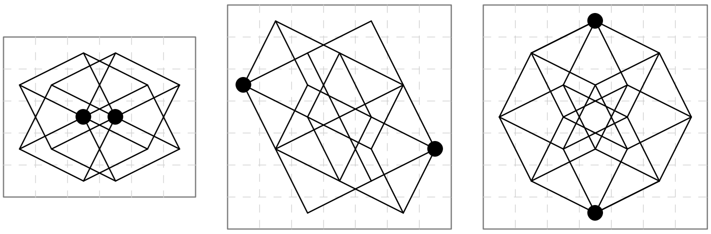
**PUZZLE 1** (p.6): What is the smallest triple-pattern amphibian? ANSWER {3,3}+{5,5}+{0,15} which I call a Pterodactyl because of its strangely spiky flight (see ⌘ 2 for example tour).

**PUZZLE 2** (p.7): Smallest board for {r,s} leaper to move from every cell? ANSWER 2r by 2s.

**PUZZLE 3** (p.7): Board for mobility of {0,s} leaper to equal 3? ANSWER Board of side 4s.

**PUZZLE 4** (p.7): Which is the smallest skew leaper that exhibits all eight mobility patterns? ANSWER The {4,6} mover, double zebra, on boards of sides 6 to 13.

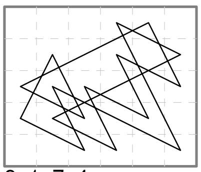
**PUZZLE 5** (p.10): How many 3-move knight routes from c3 to d3 on a 5×6 board? ANSWER 12. The paths delineate two interlocking cubes. This is from *L'Echiquier* Mar 1927 prob 183.



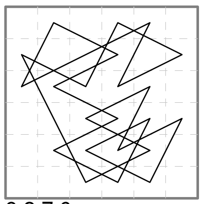
**PUZZLE 6** (p.10): How many 4-move routes a5 to g3 on a 7×7 board? ANSWER 18. The paths delineate a 2×2 square and a two-cube box-kite shape. This is from *L'Echiquier* Jun 1926 prob 115.

**PUZZLE 7** (p.10): How many 4-move paths to make a (0,6) journey d1 to d7 on 7×7 board? ANSWER 24. The moves form a ‘hypercube’ pattern (the 4-dimensional equivalent of the cube).

**PUZZLE 8** (p.10): Fit 8,1;7,4 and 9,2;7,6 journeys onto smallest possible boards. ANSWER as shown on 5×6 and 6×6 boards. Improvement may be Possible.

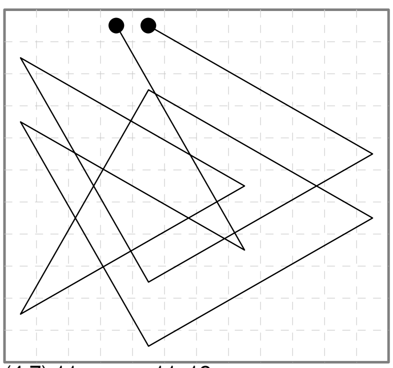


8, 1; 7, 4

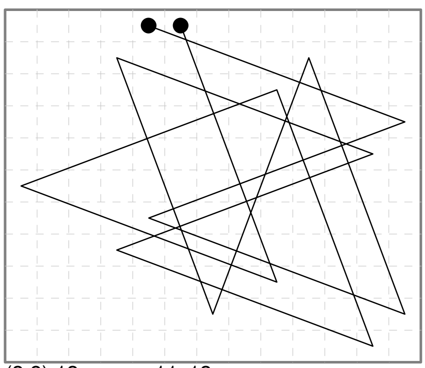


9,2;7,6

**PUZZLE 9** (p.13): To find the shortest paths by {3,8} and {4,7} movers to an adjacent cell and to fit the journeys onto the smallest possible boards. [*Chessics* #24 1985 p.98.]



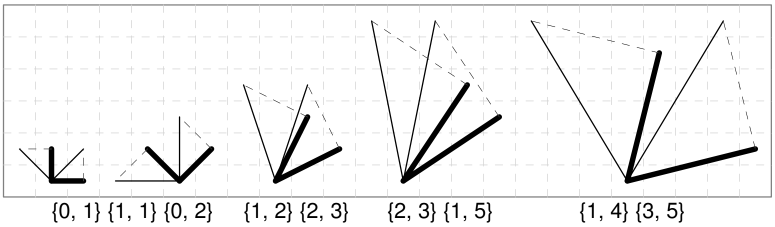
(4,7) 11 moves 11x12



(3,8) 13 moves 11x13

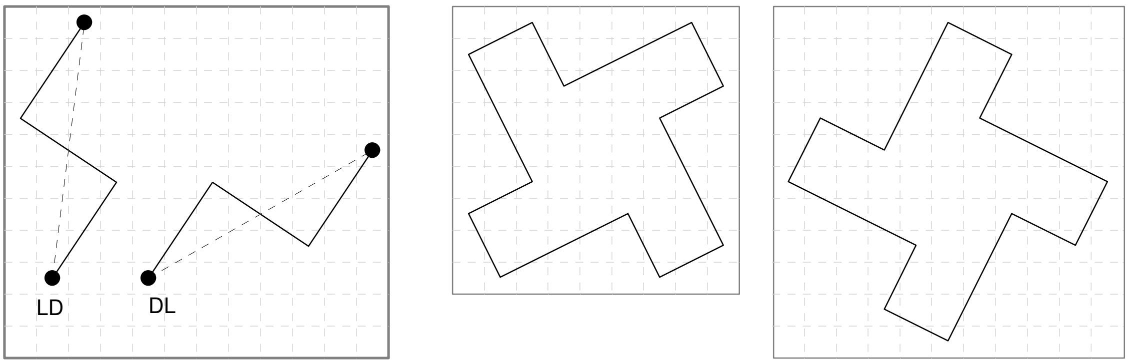
**PUZZLE 10** (p.14): For which leaper on the 8×8 board are α and β most nearly equal? ANSWER the {2,5} is slightly closer to equality than the {3,7} since 2/5 = 0.4 and 3/7 = 0.42857 while $\sqrt{2}-1 = 0.41421$, so the first is out by 0.01421 and the second by 0.01435.

**PUZZLE 11** (p.14): What leapers other than {k·r, k·s} can make the same angles as {r,s}? ANSWER The {s–r, s+r} leaper. This is by O. E. Vinje (*FCR* Dec 1940 prob 4656). It is obvious for{0,s} and {s,s} leapers, where the angles are multiples of 90°. Less obvious is the relation of {1,2} knight and {1,3} camel. Other pairs are {2,3}{1,5} and {1,4}{3,5}.

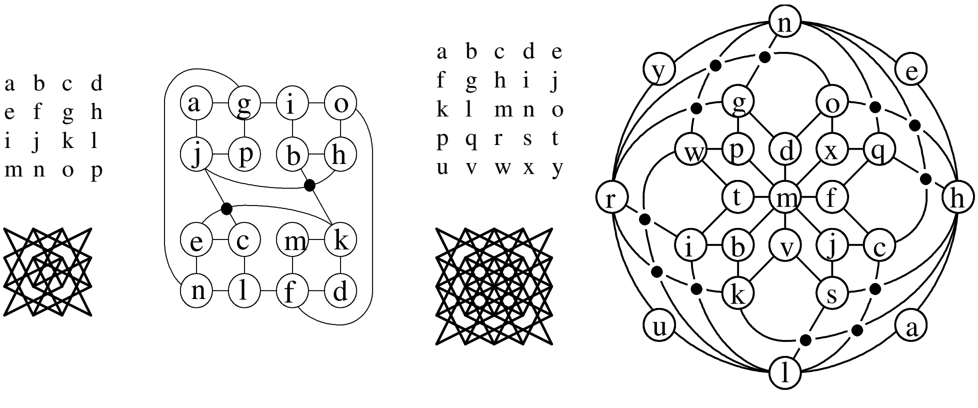



The sequence continues with {3,4} and {1,7}, which shows that the first two double-pattern leapers, the fiveleaper and root-fifty leaper, are also in this equal angle relationship. Two successive {r,s} moves at right angles are equivalent to an {s–r, s+r} move, so each {r,s} move corresponds to an {s–r, s+r} move at 45° to it. Rotating both arms of an angle equally does not alter the angle.

**PUZZLE 12** (p.15): Why are DL and LD of the same length but in different directions? If we begin with an (r,s) move, r<s, as shown for r=2 and s=3 below, then LD = (|2r-s|, 2s+r) and DL = (2r+s,2s-r) and these are always different because equality would entail either 2s+r = 2r+s or 2s+r = 2s-r, and these conditions imply r=s or r=0 respectively, contradicting that the leaper is skew. The equal length is obvious from geometrical congruence, but this can be confirmed by algebra as follows: $(2s-r)^2 + (2r+s)^2 = 5s^2 + 5r^2 = (2r-s)^2 + (2s+r)^2$. The length of the triple leap is $\sqrt{5}$ times the length of the (r,s) leap. The W. H. Cozens puzzle is from *Fairy Chess Review* Oct 1950, p.127 [solution Feb 1951 p.140]: Each path is a 45 degree rotation of the other.




**PUZZLE 13** (p.39): **Solution to the puzzle of the knight-move nets.** Redrawing the nets on the 4×4 and 5×5 boards with minimum crossovers, 2 and 12 respectively.




Eggleton & Eid (1984) give proofs that the ‘crossing number’ for the 4×4 and 3×6 case is 2.