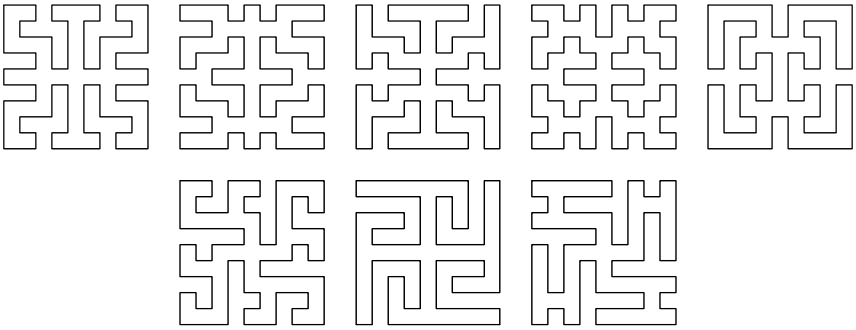
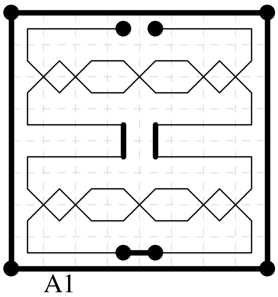
⌘ 2

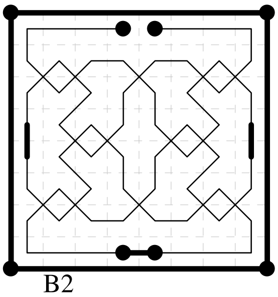
Walker Tours



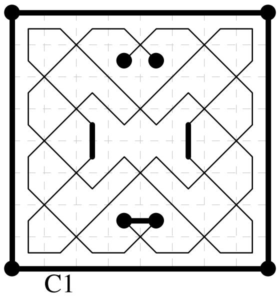
by G. P. Jelliss



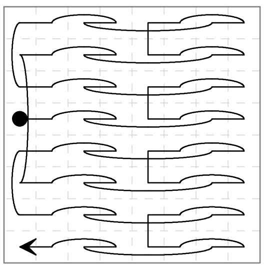
A1

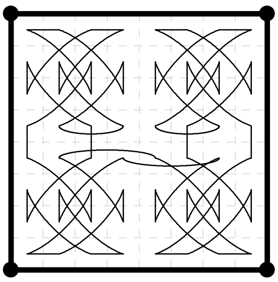


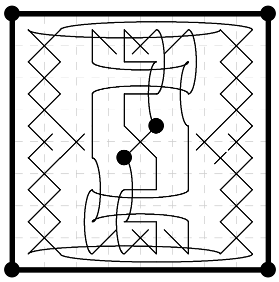
B2



C1







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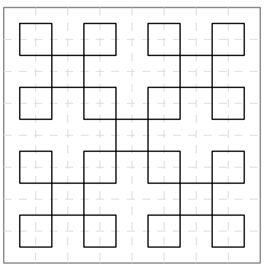

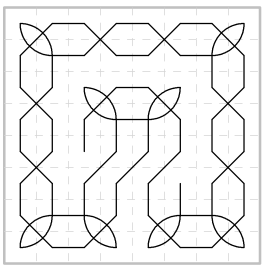

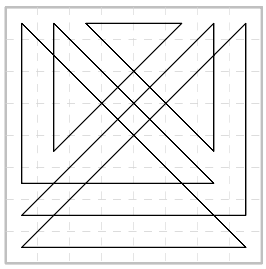

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Title Page Illustrations:
10×10 Wazir Tours.
8×8 Diagonally Magic King Tours
8×8 Hyperwazir and Diagonally Magic Queen Tours
8×8 Rook, Moose and Queen Crossover Tours

Contents

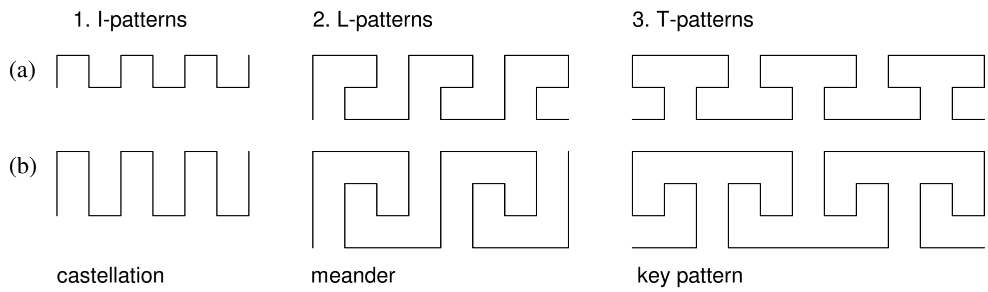
Knight's Tour Notes, Volume 2, Walker Tours.

Lateral Movers

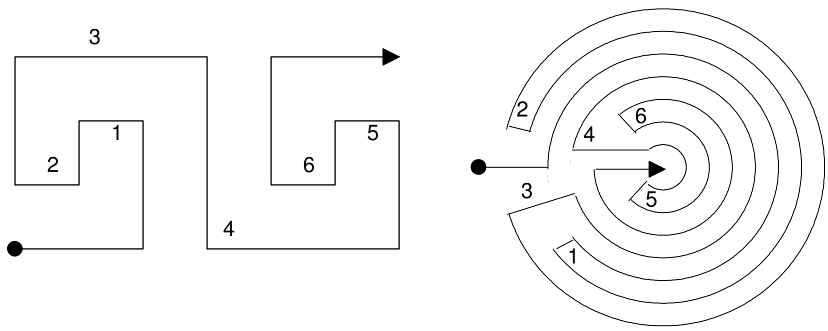
Labyrinths

Logically one would expect patterns using simple lateral and diagonal moves, to have been studied before those using skew moves such as that of the knight. This is in fact the case, though generally they have appeared not explicitly as tours but disguised as labyrinths or as elements in artistic design. Here we look briefly at the history of this topic which has ancient origins.

Greek Art. Non-intersecting rook tours are implicit in ‘boustrophedonal’ writing, found in early Greek inscriptions, running from right to left and left to right in alternate lines, in imitation of ploughing [*Brewer* 1974, p.142]. The ploughing of a field is in effect a rook tour. The plough was used in Mesopotamia around –3500 [*Times Atlas of World History* 1989]. Slightly more elaborate rook tours are seen in the wave-like patterns which are common as borders in Greek art. The ‘meander’ pattern, shown as a rook path in diagram 2(b) below is so named from the winding course of the river Maeander in Phrygia (part of modern Turkey). The meander design occurs prominently for instance on the shield of Philip II of Macedon (–382 to –336).



The Minoan Labyrinth. Spiral labyrinths and dances are also ancient. The term **labyrinth** is used for designs in which there is a single path to follow; as opposed to puzzle paths, called **mazes**, with branching paths and dead ends, which were a Renaissance innovation (c.1550). Rock-carved labyrinths are known from as far apart as Arizona, Sumatra and Scandinavia, but their dates are disputed [Nigel Pennick *Mazes & Labyrinths* 1990]. Most however are of similar design to the legendary labyrinth at Knossos, as represented on Minoan coins (c.–1600). The classical pattern of a labyrinth is formed by bending and stretching a double meander round in a circle, as illustrated here.

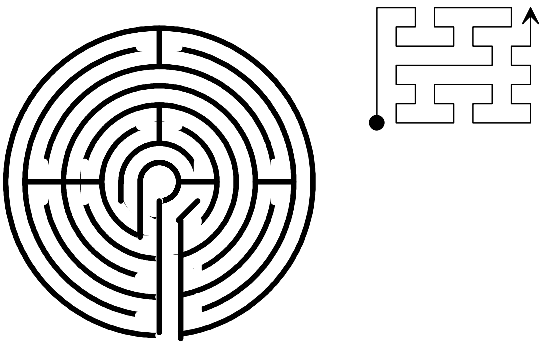


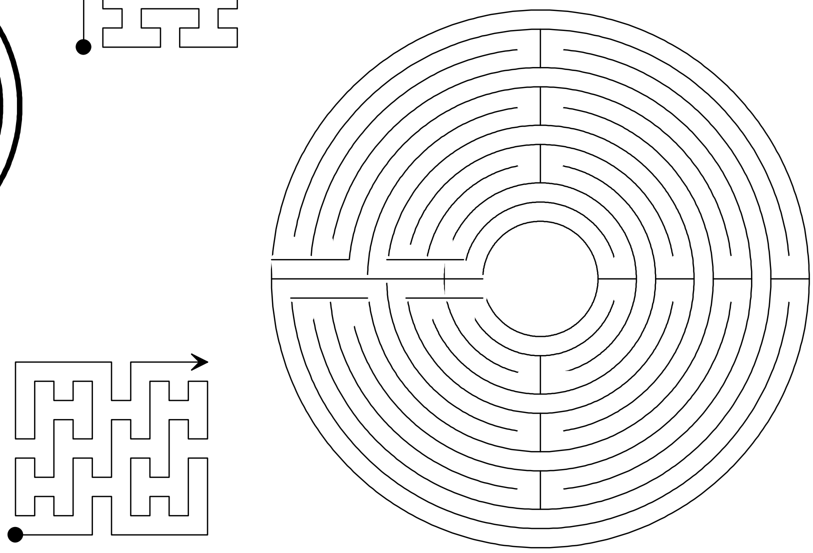
Greek myth attributes the design of the Minoan labyrinth to the proto-engineer Daidalos, who is also mentioned by Homer as maker of a ‘choros’ for Ariadne, which was a circling dance, or a place for dancing. The prefix ‘Dai-’ is said to mean cunning or curious and ‘Alos’ is a furrow, so perhaps his very name means ‘labyrinth’. [Liddell & Scott (1935)]

Pennick (1990) attributes the discovery of the topological equivalence illustrated above to Jeff Saward in lectures given in 1981-2, however *Brewer* (1974) states that the Maeander "is said to have given Daedalus his idea for a labyrinth" so knowledge of the property can probably be traced to earlier sources (if indeed it was ever lost).

A triple spiral design occurs inside the Newgrange passage tomb in Ireland, illumined by sunlight at the winter solstice, and also occurs on entrance stones. It is somewhat roughly carved, but is easily interpreted as an endless labyrinth and has been dated to c.–3200.

Pavement Labyrinths. During the middle ages the classical labyrinth design was elaborated. An early example is the pavement labyrinth in the Church of San Vitale, Ravenna (c.530). The underlying single-step rook (wazir) tour is shown here alongside the labyrinth diagram.



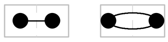


The famous pavement labyrinth at the Cathedral of Notre Dame, Chartres (c.1250) is even more elaborate. For more examples see H. E. Dudeney (1917).

Wazir Tours

The simplest lateral mover, and the simplest single-pattern leaper, is the {0,1}-mover known as a Wazir. Since the Wazir itself is not used in modern chess, except as part of rook, king, queen and pawn moves, its study has been neglected. Wazir tours, besides being of interest in themselves, are also useful in analysing the structure of knight's tours on boards whose sides are double those of the wazir tour. Since the king can make wazir moves, all wazir tours are also king tours.

If we regard the wazir move as a leap, representing it by a curved line, it can be argued that a closed tour is possible on the 1×2 board, but on the other hand if the wazir is considered to move along the straight line joining the two points this involves a **switchback**, going over a part of the path again, which is not usually allowed in counting a path as a tour. It is really just a matter of convention, or precisely which definition of the wazir move we adopt.



On a longer 1×n board the only tour by a wazir is the straight open path of n–1 steps from end to end. The general problem of enumerating wazir tours on rectangular boards, one would think should not be intractable to normal mathematical methods, such as the derivation of recurrence relations, but as far as I know it has only so far been tackled successfuly in a few special cases.

The shortest wazir paths equivalent to an (m,n) move obviously use m+n moves. The number of shortest wazir paths equivalent to an (m,n) move is (m+n)!/(m!·n!) These are special cases of results reported for more general journeys in the Theory section (p.11-14).

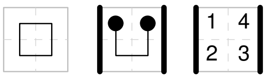
A closed wazir tour is possible on any rectangular board of an even number of cells and with sides greater than one. The shapes outlined by the tours are of course 'ominoes'. The area enclosed by any closed wazir tour on the m×n board is (m·n)/2 – 1, i.e. one less that half the area of the board. This is true for any non-crossing tour of the m×n board using any straight-line moves, and can be deduced from Pick's theorem which states that the area of a simple polygon joining points on a grid is i + b/2 − 1 where i is the number of internal nodes, b the number of nodes on the polygon. For any such tour i = 0 (since it is a tour) and b = m·n since it passes through every cell.

Wazir tours 2×n

On any board 2×n (n >1) the wazir has one closed tour, a walk around the edges.

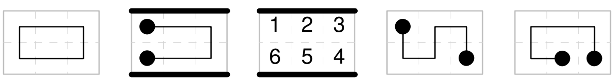
On the 2×n board we denote the number of geometrically distinct wazir open tours by G(n). The formula for G(n) given below has been found by drawing out the tours for the cases 2×2 to 2×6, as follows, and then deducing a general rule and using it to calculate the other cases.

2×2 Board: There is one wazir open tour and one closed tour.



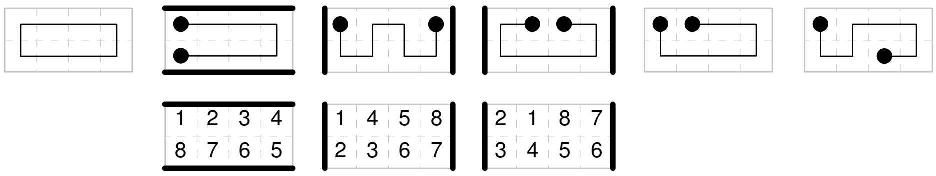
It can also be noted that when numbered 1 to 4 in the sequence visited by the piece the tour, having axial symmetry, adds to the same sum (1 + 4 = 2 + 3 = 5) in the lines perpendicular to the axis. This is a general rule for axially symmetric tours. We call such tours semi-magic, and this is indicated by the dark border lines on the sides parallel to the axis.

2×3 Board: There is one wazir closed tour and three open tours, one axially symmetric, one centrosymmetric and the other asymmetric.

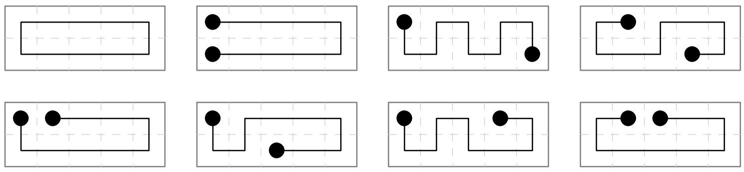


The axisymmetric tour, marked with dark border line, is semi-magic, adding to 7 in the files.

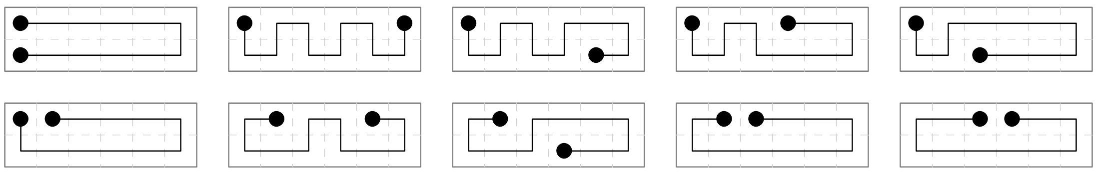
2×4 Board: There are five wazir open tours, three symmetric. Those with axial symmetry are semi-magic adding to 9 in the files or 18 in the ranks. There is of course also one closed tour.



2×5 Board: There is one closed wazir tour and seven open tours, three symmetric.



2×6 Board: There is one closed and 10 open wazir tours of which 4 are symmetric.



In drawing the tours we find there are (when n > 2) n tours with an end at a corner, then n–3 with an end one in from a corner, then n–5 with an end two in from a corner, and so on, until we reach 1 or 2 as last term. Thus G(n) = 1 + (1 + 3 + 5 + ... + (n–1)) when n is even, and G(n) = 1 + (2 + 4 + 6 + ... + (n–1)) when n is odd. Denoting by h and k the nearest whole numbers such that h ≤ (n/2) ≤ k, and using the properties that the sum of the first s successive odd numbers is s^2 and the sum of the first s even numbers is s·(s+1) we have G(n) = 1 + (n/2)^2 for n even and G(n) = 1 + [(n–1)/2]·[(n+1)/2] for n odd, which can be expressed concisely as **G(n) = 1 + h·k** in both cases. However for n = 2, when the board is square the 2 reduces to 1 due to rotation being possible.

The symmetric tours consist of one U-shaped tour with both ends in the first rank (as oriented in the diagrams above) and a lengthwise axis of symmetry, plus when n is even (and > 2) n/2 other tours that have a breadthwise axis, but when n is odd there are instead (n–1)/2 that have 180° rotational symmetry. Thus **S(n) = h + 1**, except for n = 2 when we must again reduce to 1.

A curiosity of the symmetric case is that if the two end-points of a wazir tour are symmetrically placed then the tour itself must be symmetric, and when it exists it is uniquely determined. It happens also that the number of reentrant tours equals S(n); this set includes the U-shaped tour and one other symmetric tour when n is even.

Table of numbers of wazir open tours on boards 2×n

n	3	4	5	6	7	8	9	10	11	12
open	3	5	7	10	13	17	21	26	31	37
symm	2	3	3	4	4	5	5	6	6	7
n	13	14	15	16	17	18	19	20	21	22
open	43	50	57	65	73	82	91	101	111	122
symm	7	8	8	9	9	10	10	11	11	12
n	23	24	25	26	27	28	29	30	31	32
open	133	145	157	170	183	197	211	226	241	257
symm	12	13	13	14	14	15	15	16	16	17

The total of tour diagrams now can be calculated, if desired, by the usual formula for rectangles: T(n) = 4·G(n) – 2·S(n) = 4·h·k – 2·h + 2. The successive values are: 1, 1, 8, 14, 22, 32, 44, 58, 74, 92, 112, 134, 158, 184, 212, 242, ...

Wazir tours 3×n

For a 3×n closed tour n must be even. Every 3×(2·m) closed wazir tour forms a circuit round the edge with m–1 ‘indentations’, each of which may be in either of the longer edges (2 choices). The number of tour diagrams is thus T(n) = 2^(m–1).

To find the number of symmetric tours we separate the cases of m even and odd. When m is even a tour can always be oriented with the middle indentation in a chosen edge. Then there are 2^(m/2 – 1) ways of indenting on each side of this, giving this number of tours with an axis of symmetry. When m is odd there is no central indentation, but we can always reflect if necessary so that the first indentation is in the chosen edge. The other indentations in that half can be made in 2^((m–1)/2 – 1) ways, giving this number of reflective and the same number of rotative tours. Thus S(n) = 2^(g–1) where g is the nearest whole number g ≥ n/4.

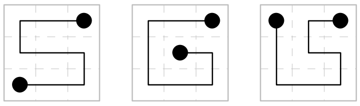
We can now calculate G(n) = T(n)/4 + S(n)/2 = 2^(m–3) + 2^(g–2).

The formulas for closed tours given above lead to the following values.

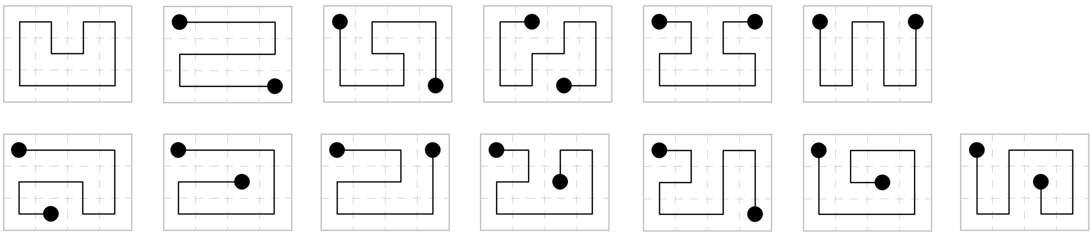
Table of 3×n wazir closed tours:

n	4	6	8	10	12	14	16	18	20	22	24
G(n)	1	2	3	6	10	20	36	72	136	272	528
S(n)	1	2	2	4	4	8	8	16	16	32	32
T(n)	2	4	8	16	32	64	128	256	512	1,024	2,048

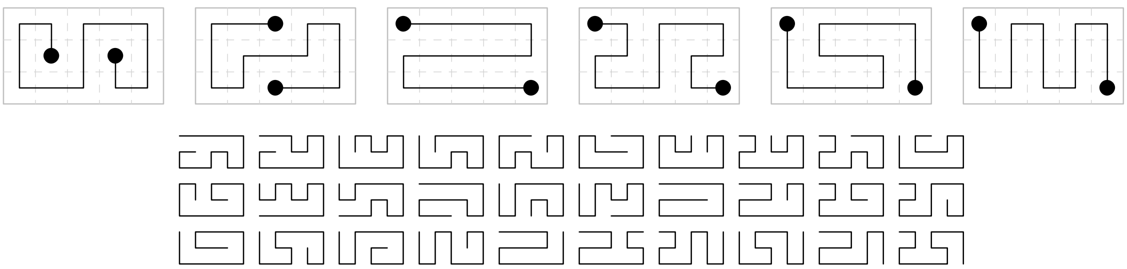
3×3 Board: There are three open wazir tours, one symmetric.



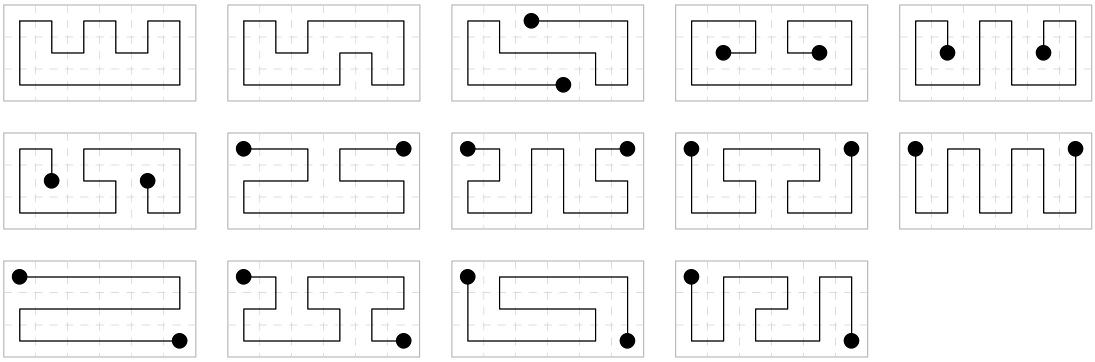
3×4 Board: There is one closed wazir tour, which gives 7 reentrant tours (two symmetric) by deleting one move, and there are 12 other nonreentrant tours (5 symmetric).



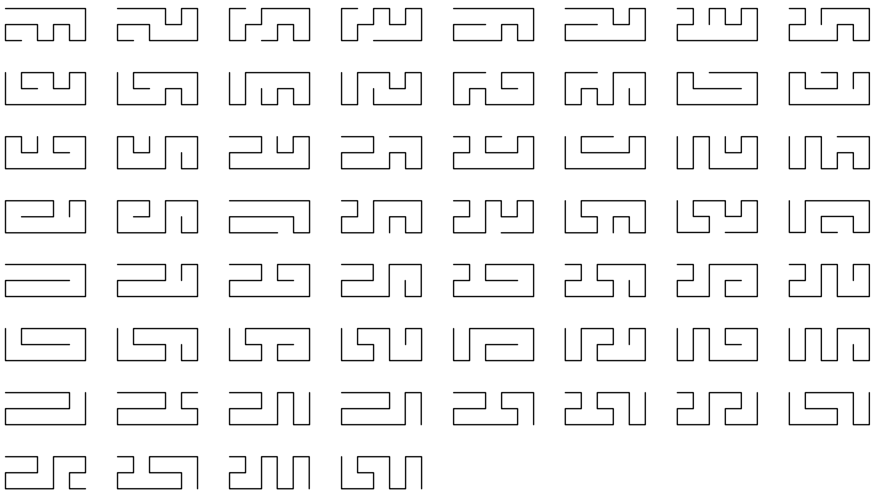
3×5 Board: There are 36 wazir tours, 6 symmetric and 30 asymmetric (grouped here according to the separation of their end-points)



3×6 Board: There are two closed wazir tours (shapes E and S). These give 19 reentrant tours (two symmetric). And there are 72 other open tours of which 12 are symmetric. Making 91 in all.



The 60 asymmetric and non-reentrant cases in smaller form:



I have no general formula for the number of open wazir tours 3×n. The individual cases 3×3 to 3×6 above show the following results. 3×3: G = 3, S = 1 rotative, T = 20. 3×4: G = 19 (7 reentrant), S = 7 (4 reflective, 3 rotative), T = 62. 3×5: G = 36, S = 6 rotative, T = 132. 3×6: G = 91 (19 reentrant), S = 14 (8 reflective, 6 rotative), T = 336.

Wazir tours 4×n

A recurrence relation for closed wazir tours on 4×n boards with $n > 4$ was given by C. Flye Sainte-Marie (in *L'Intermédiaire des Mathematiciens*, vol.11, 1904, p.86–88), namely:

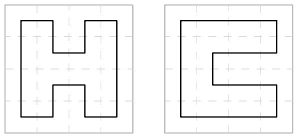
$T(n) = 2\cdot T(n-1) + 2\cdot T(n-2) - 2\cdot T(n-3) + T\cdot(n-4)$,

with initial values $T(1) = 0$, $T(2) = 1$, $T(3) = 2$, $T(4) = 6$.

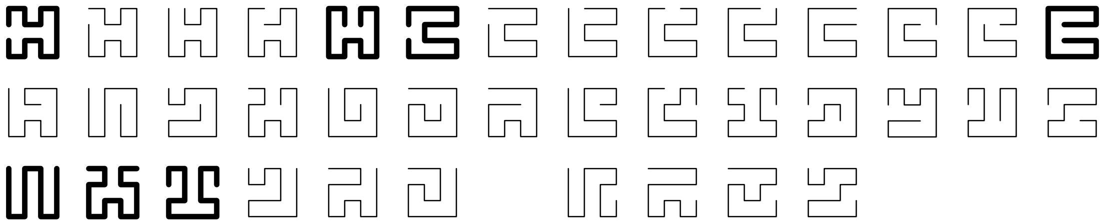
The succeeding values for n = 5, 6, 7, 8 are: T(n) = 14, 37, 92, 236.

. KNIGHT'S TOUR NOTES

4×4 Board: The two wazir closed tours in the shape of H and C (hot and cold) are well known. The total T(4) = 6 arises since the H pattern can occur in two orientations and the C pattern in four.

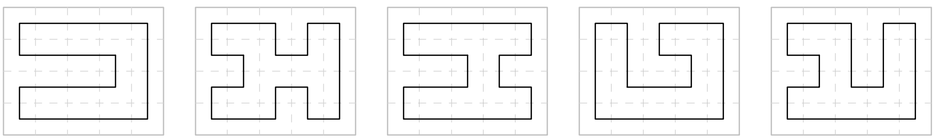


There are G = 38 geometrically distinct open tours, with S = 7 symmetric (shown bold below). Of these R = 14 are reentrant, 9 fron the C tour and 5 from the H, two in each case symmetric.

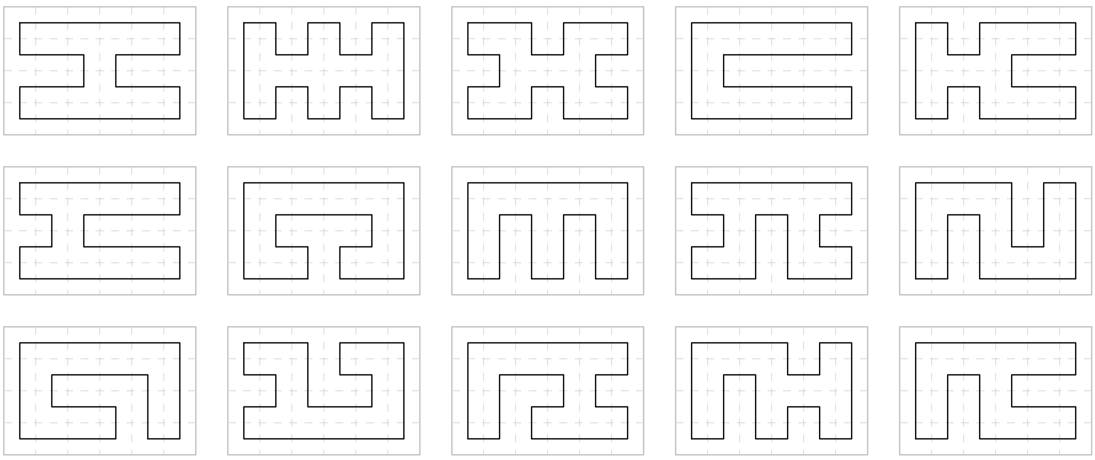


Of the 24 non-reentrant tours, 3 are symmetric. The 10 reentrant and 21 nonreentrant asymmetric tours are arranged by separation of end-points.

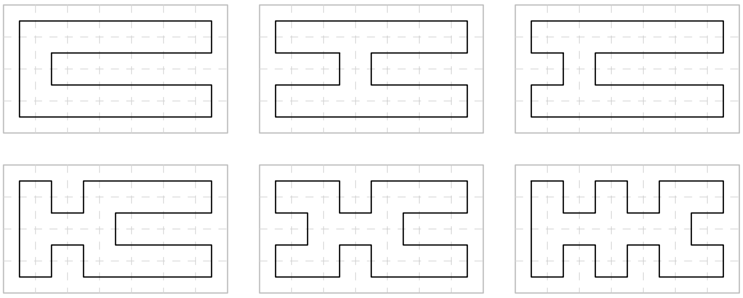
4×5 Board: The Sainte-Marie recursion gives closed tours T(5) = 14. The geometrically distinct are G(5) = 5, of which S(5) = 3 are symmetric and A(5) = 2 are asymmetric. T = 2S + 4A = 4G − 2S.



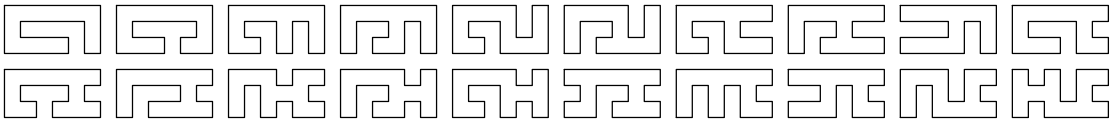
4×6 Board: The Sainte-Marie recursion gives closed tours T(6) = 37. The number geometrically distinct is G(6) = 15, of which Q(6) = 3 have quaternary symmetry, S(6) = 7 have binary symmetry and A(6) = 5 are asymmetric. Thus T = Q + 2S + 4A.



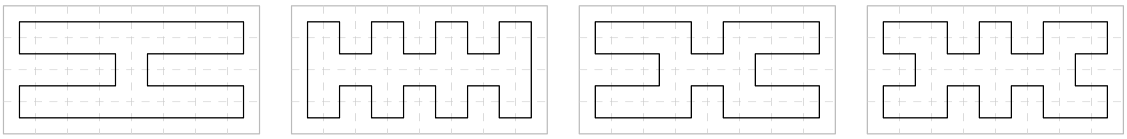
4×7 Board: The Sainte-Marie recursion gives closed tours T(7) = 92 and G(7) = 26, of which S(7) = 6 have binary symmetry A(7) = 20 are asymmetric. We show the 6 symmetric tours.



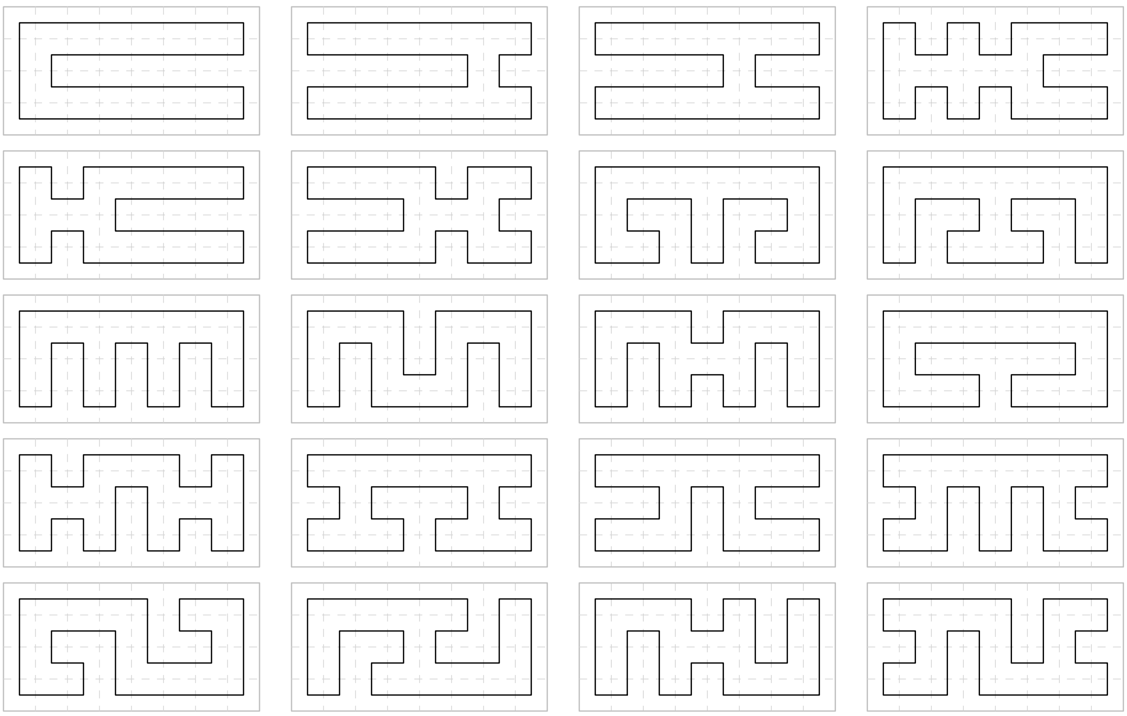
and 20 asymmetric in reduced size. Thus T = 2·S + 4·A.



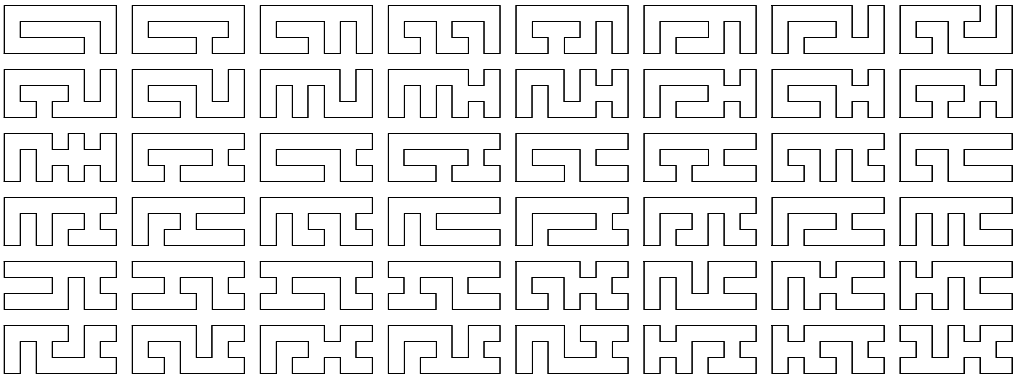
4×8 Board: The Sainte-Marie recursion gives the total for wazir closed tours as T(8) = 236. The number geometrically distinct is G(8) = 72, of which Q(8) = 4 have quaternary symmetry, S(8) = 20 have binary symmetry and A(8) = 48 are asymmetric. Thus T = Q + 2·S + 4·A. We show the 24 symmetric, beginning with the four quaternary.



Here are the 20 with binary symmetry, 6 with long axis, 10 with short axis, 4 with centre.

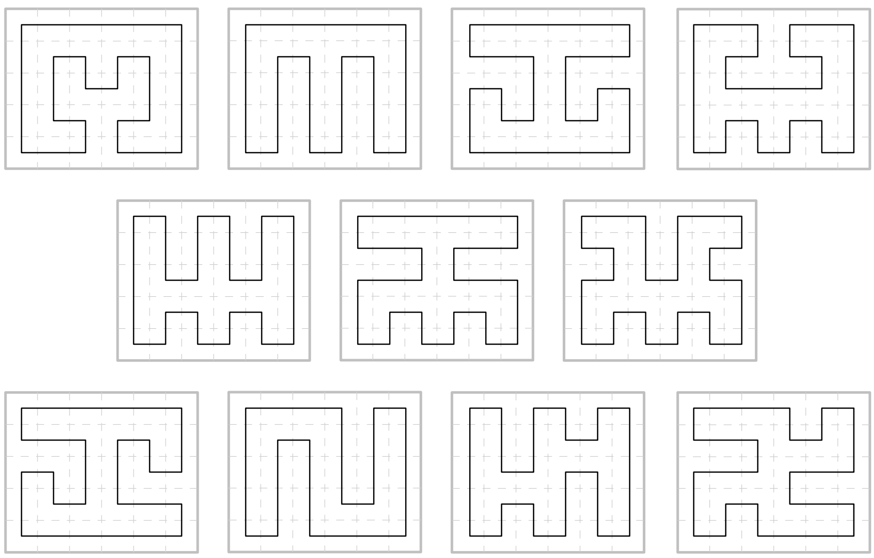


48 asymmetric:

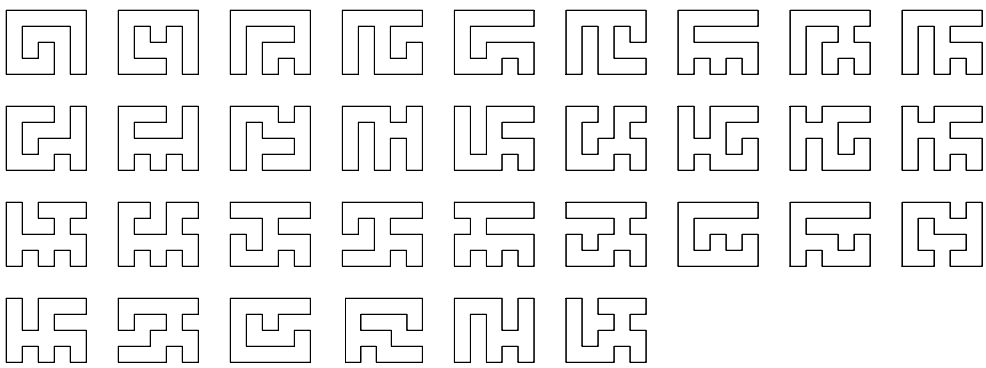


Wazir tours 5×n

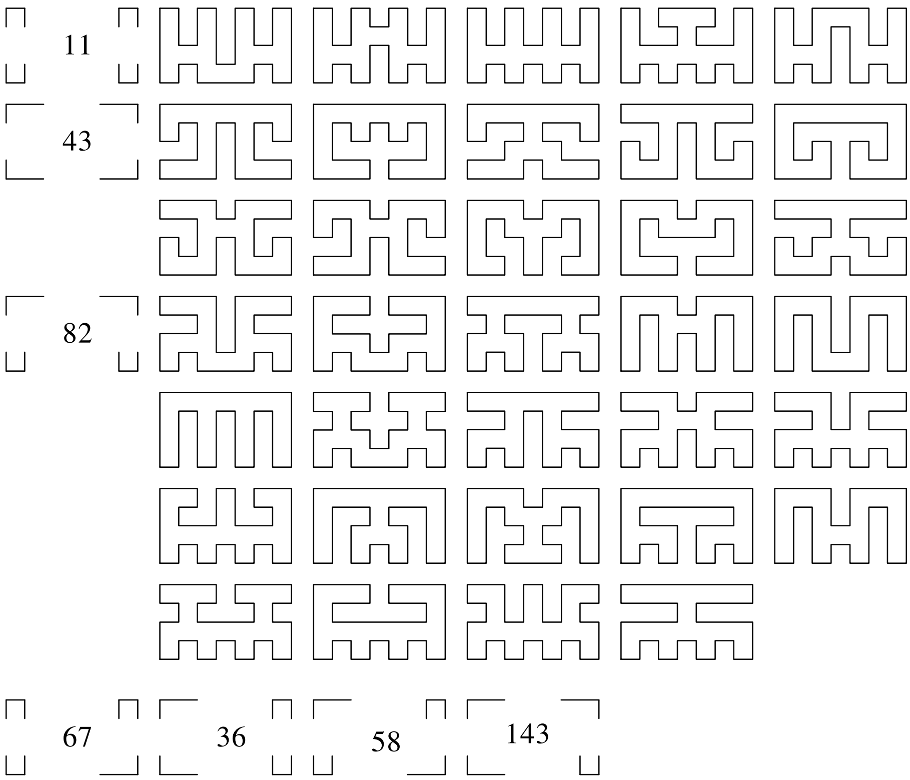
5×6 Board: I find 44 closed wazir tours, 11 symmetric, with 7 reflective and 4 rotative.



plus 33 asymmetric.



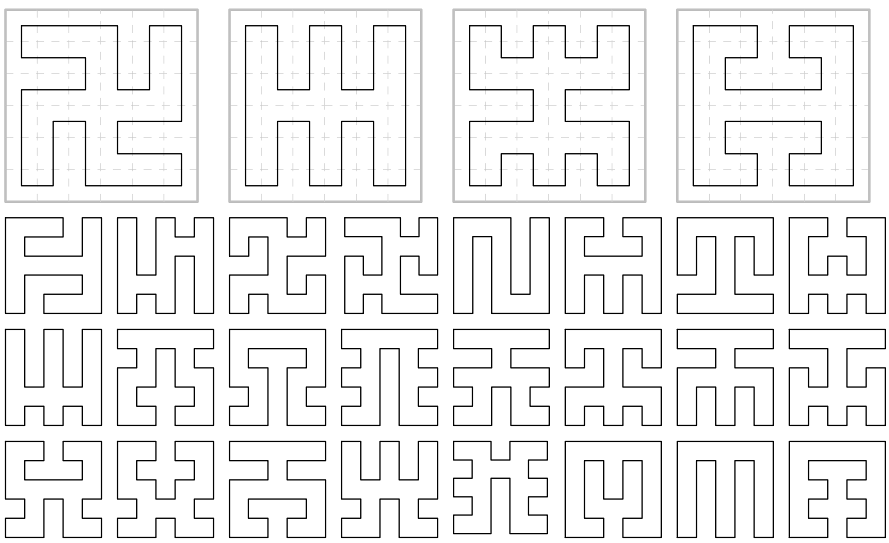
5×8 Board: On this I find 440 closed wazir tours (total not independently checked). These may be classified by corner patterns, and include 34 symmetric tours, as shown below, all having binary symmetry about the short axis.



Wazir tours 6×6

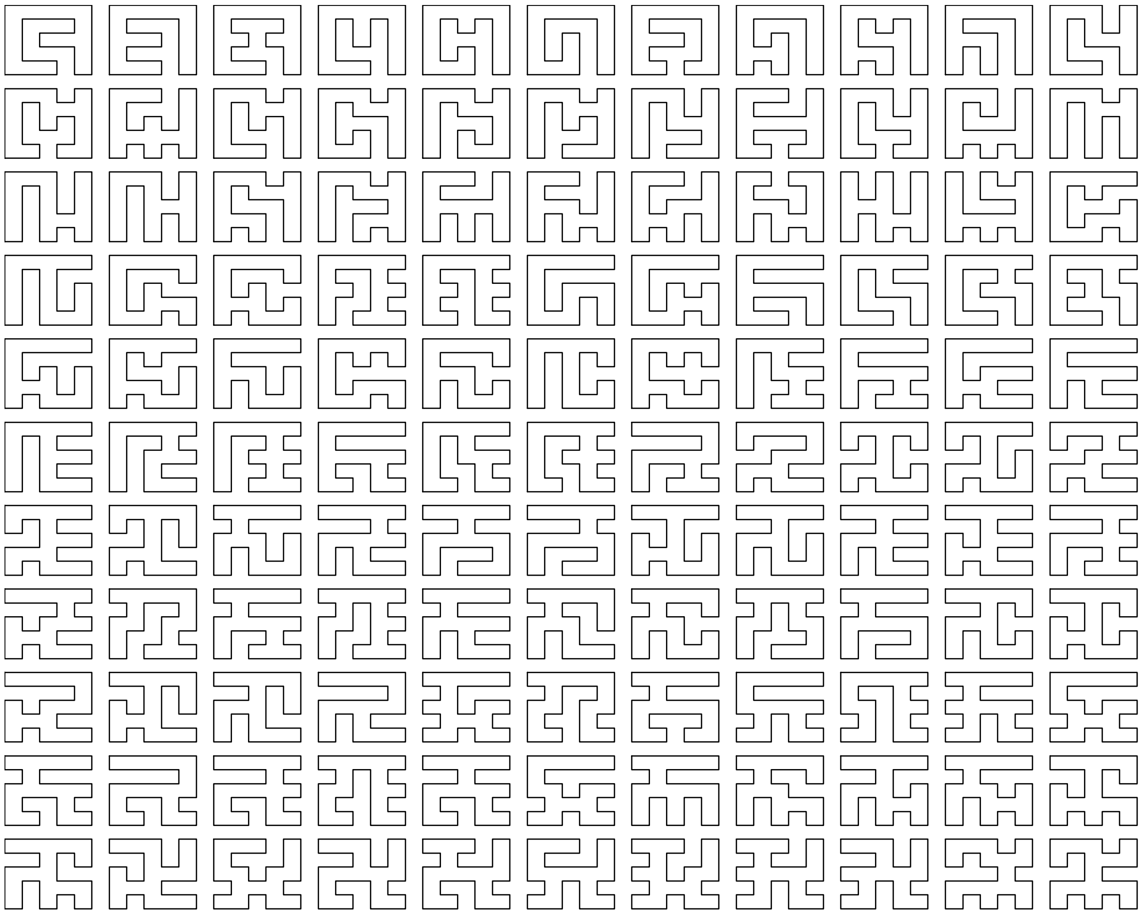
These can be classified according to number of indents, 0, 1, 2 in each side. The numbers found in each roup are: 0001 (13), 0011 (17), 0012 (12), 0101 (13), 0102 (13), 0202 (5), 0111 (27), 0112 (18), 0121 (9), 0212 (4), 1111 (10), 1112 (6), 1212 (2). Single indents can be central or offset.

Total 149 closed tours, 28 symmetric (1 quaternary rotative, 3 quaternary reflective, 5 binary rotative, 19 binary reflective). Symmetric cases shown.



(binary at reduced scale)

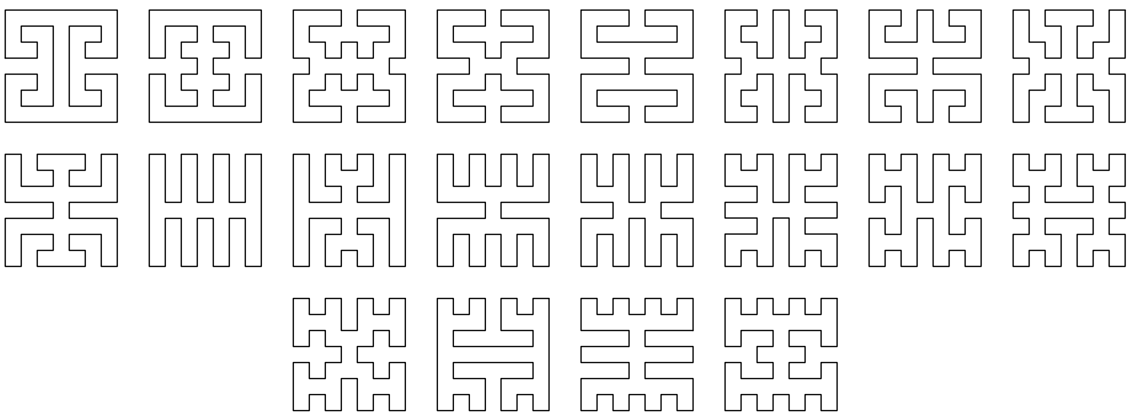
And there are 121 asymmetric.



Classification by various features:
Straight edges: 0 (18), 1 (58), 2 (60, formed of 29 adjacent, 31 opposite), 3 (13).
Indents: 1 (9), 2 (34), 3 (52), 4 (42), 5 (10), 6 (2). No centre (34).

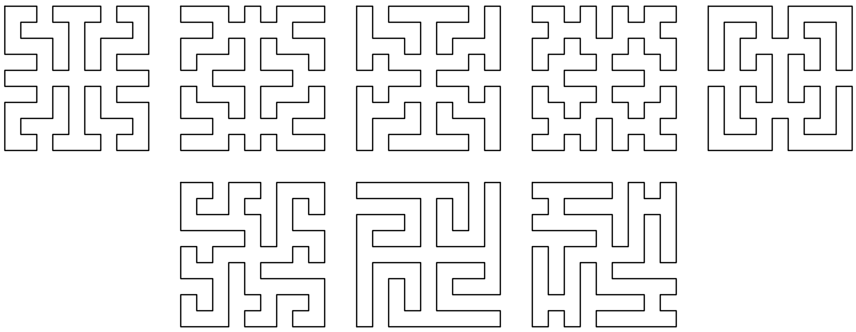
Wazir tours 8×8

On the standard 8×8 chessboard my most recent count of the wazir tours with 180° rotation (6 June 2004) found 374, of which none of course have 90° rotary symmetry but 20 (shown here at reduced scale) are biaxial. My previous count missed the sixth diagram shown here.



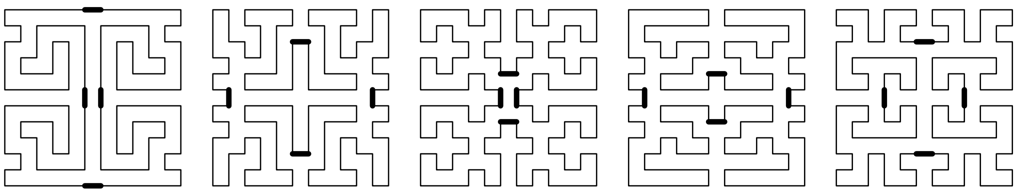
Wazir tours 10×10

Wazir tours in quaternary symmetry on the 10×10 total 224 reflective, 101 rotative (totals not independently checked). The reflective cases can be classified according to the distance in from the edges of the moves that cross the two axes, thus: (0,0) 52, (0,2) 38, (0,4) 86, (2,2) 14, (2,4) 34. One example of each class, at reduced scale, and some assorted rotative examples.



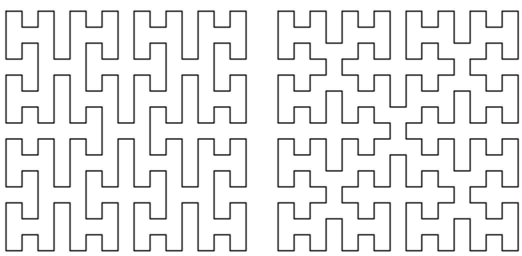
Wazir tours 12×12

Just a few examples in biaxial symmetry. They consist of four copies of an open tour 6×6, with ends in adjacent edges, joined by pairs of horizontal and vertical links across the medians.

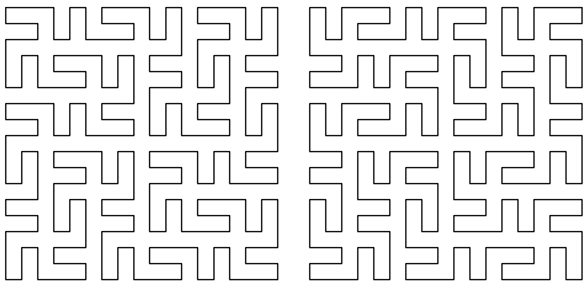


Wazir Tours on Larger Boards

The 4×4 H-pattern can be repeated in 2×2 array and the four components joined up by an H-pattern of linkages to generate a quaternary reflective tour 8×8. This in turn can be used to generate a tour a factor of 2 larger, that is 16×16 as shown on the left below, and the process can be continued to larger boards. More variety in the pattern can be obtained by making the joining H-connections horizontally instead of vertically as on the right.



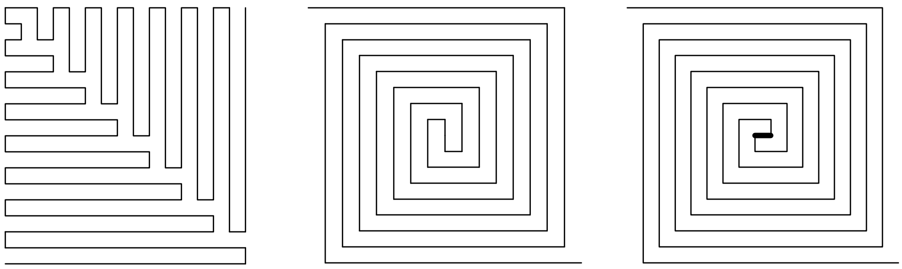
The method can be applied similarly to boards with side not a multiple of 4. The 6×6 swastika pattern can be repeated in 3×3 array and the nine components joined up by a swastika pattern of linkages to generate a quaternary rotative tour on the 18×18 board, as on the left below. Alternatively the anti-clockwise swastikas can be joined clockwise as on the right. These methods in turn can be used to generate a tour a factor of 3 larger, that is 54×54, and so on.



Many varied patterns of this type can be constructed by systematic linking of tours. Tours of this type are similar to patterns encountered in the study of ‘fractals’ or ‘space-filling curves’.

The following wazir tour designs can also be extended to fill boards of indefinite size. The first has indents of lengths 1, 3, 5, ... down the left and indents of lengths 2, 4, 6, ... in the top. The first double spiral is formed of two spirals consisting of sections of lengths 1, 1, 3, 3, 5, 5, ... while in the second double spiral the segments of the spirals are of lengths 1, 2, 3, ... with a join across the centre.

The first diagram provides a one-to-one correspondence between the doubly infinite succession of signed integers ..., -3, -2, -1, 0, +1, +2, +3, ... and the pairs of positive or non-negative integers (r,s), with (0,0) at top left, while the second diagram, taking (0,0) in the centre, does the same for the pairs of signed integers, showng that they are sets of the same infinite size.



Non-crossing Rook Tours

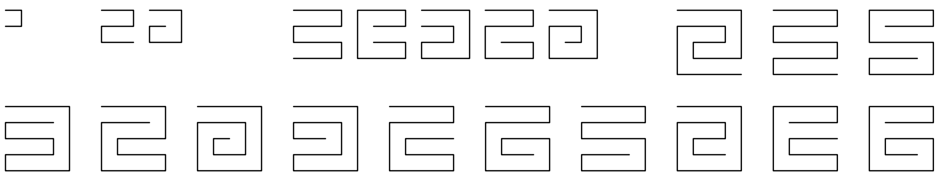
The problem of finding a rook tour of fewest moves, not crossing its own path, is the same as that of finding a wazir tour with fewest turns.

On a board n×n, if there are moves in every rank and file then we have at least 2·n moves. On the other hand, if there is a rank or file without a move along it, then it must be crossed or met by n mutually parallel moves, and these will require a further n–1 rook moves to join them into a single path, or n to make a closed circuit. The minimum is therefore 2·n rook moves for a closed tour and 2·n – 1 for an open tour. In an open tour the first and last moves must be parallel, since they are among the n parallel lines. We take these horizontal. There are always at least two full-rank moves, top and bottom. To solve the problem on a rectangle n×m (m > n) we just stretch the n parallel moves along the ranks, so we can confine our attention here to square boards.

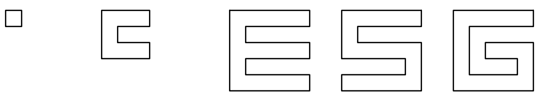
The problem of finding a closed tour with just two turning points in each rank and file is impossible; there must be more than two turns in at least one of the edges. There are however pseudotours with this property: the set of concentric rectangles on an even board.

We consider wazir tours with the minimum number of right-angled turns, which are equivalent to non-intersecting rook paths with the minimum number of rook moves.

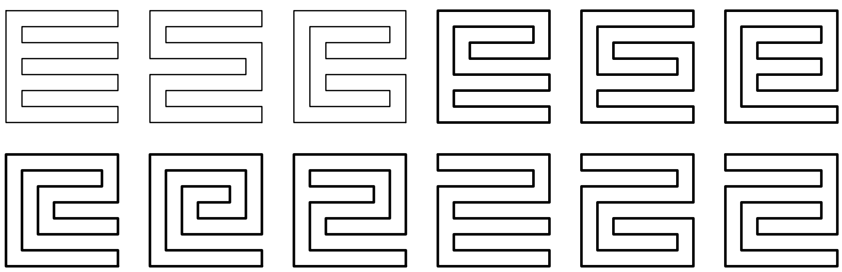
2×2 open: G = S = 1, T = 4.
3×3 open: G = 2, S = 1, T = 12.
4×4 open: G = 5, S = 2, T = 32;.
5×5 open: G = 13, S = 3, T = 92. .



2×2: closed: G = S = T = 1.
4×4 closed: (the C-shaped tour) G = S = 1, T = 4.
6×6 the total of open tours is not known, but there are 21 with end at corner (2 corner to corner, 1 symmetric) the closed tours (illustrated): G = 3, S = 2, T = 16.



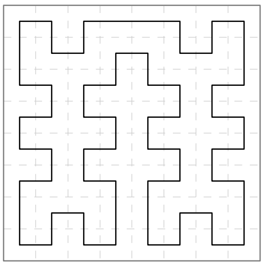
8×8: closed (illustrated): G = 12, S = 3, T = 84.



The popular problem of a 16-move non-intersecting rook closed tour of the standard chessboard was proposed and solved with one example, the ‘four-pronged’ pattern, by Sam Loyd (*Chess Strategy* 1881) [A. C.White 1913]. H. E. Dudeney (*Tit Bits* 1897) gave a problem solved by a 15-move open tour c2-b2 [*The Canterbury Puzzles* 1904] ensuring a unique solution by a barrier between d1 and e1. T. R. Dawson gave similar problems in (*Chess Amateur* 1909) and (*British Chess Magazine*, 1943).

. KNIGHT'S TOUR NOTES

W. E. Lester (*Fairy Chess Review* vol.3 #13 Aug 1938 p.140 ¶3250) gives this tour of binary symmetry with 56 turns (the links d8-e8, d6-d7-e7-e6 can be inverted d8-d7-e7-e8, d6-e6).



10×10 Board: G = 58? This is an isolated note.

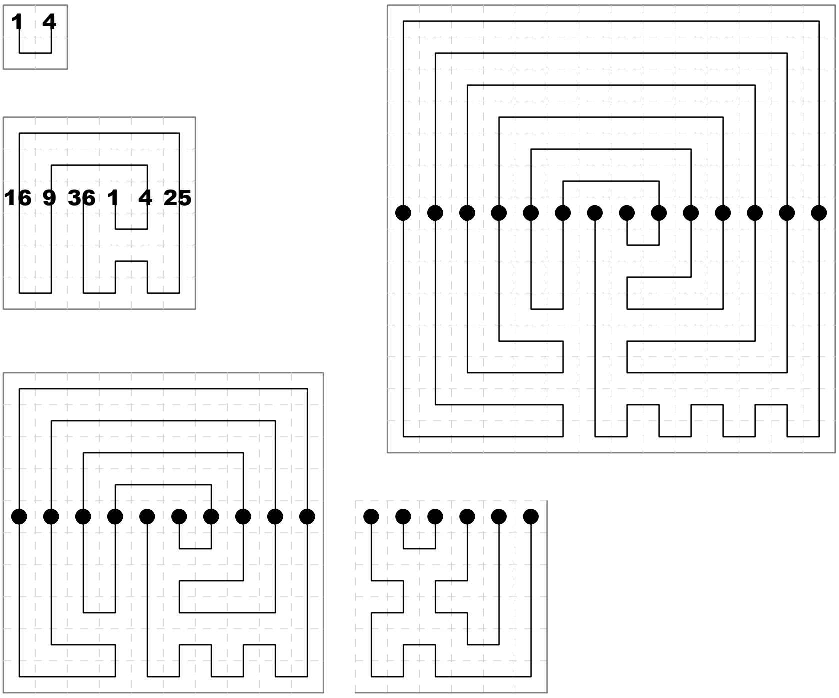
(For non-crossing knight tours see ⌘ 11.)

==

Figured Wazir Tours

In the 2×2 wazir tour, numbered 1 to 4, it so happens that the square numbers, 1 and 4, are in the same row. This may seem a trivial observation, but when we try to extend the feat to larger boards it begins to appear much more interesting. The feat of having all the square numbers in a row cannot be accomplished on boards of side 3, 4 or 5, but when we consider side 6 it turns out to be possible once more, and moreover the solution is unique!

The 2×2 case must form the basis for any larger tour with the squares in a row property. To get the next square, 9, in line with the 1 and 4 there are three ways but only the method for the 6×6 board extends to larger square boards of side 10, 14 and so on, but the routes on these larger boards are not completely determinate - an alternative corner route for the 10 case is shown.

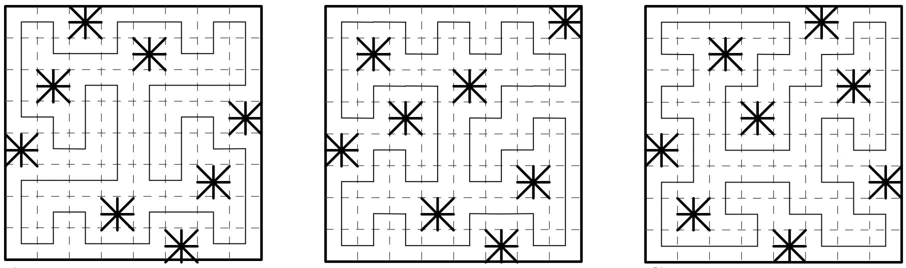


These results obviously have some connection with the fact that the differences of successive squares are successive odd numbers - an elementary result of number theory. The number of cells to be visited by the wazir between z^2 and $(z+1)^2$ is 2z and these must all lie on the same side of the row of square numbers. These tours are from my article in *Chessics* (#21 p.56 1985), with the alternative route noted in *G&PJournal* (#8+9 p.143 1988-9).

Puzzle: Construct 4×4 wazir tours with the square numbers (a) at corners of a square, (b) at corners of an oblong. *G&PJournal* #11 p.178 1989)

Rook around the Rocks

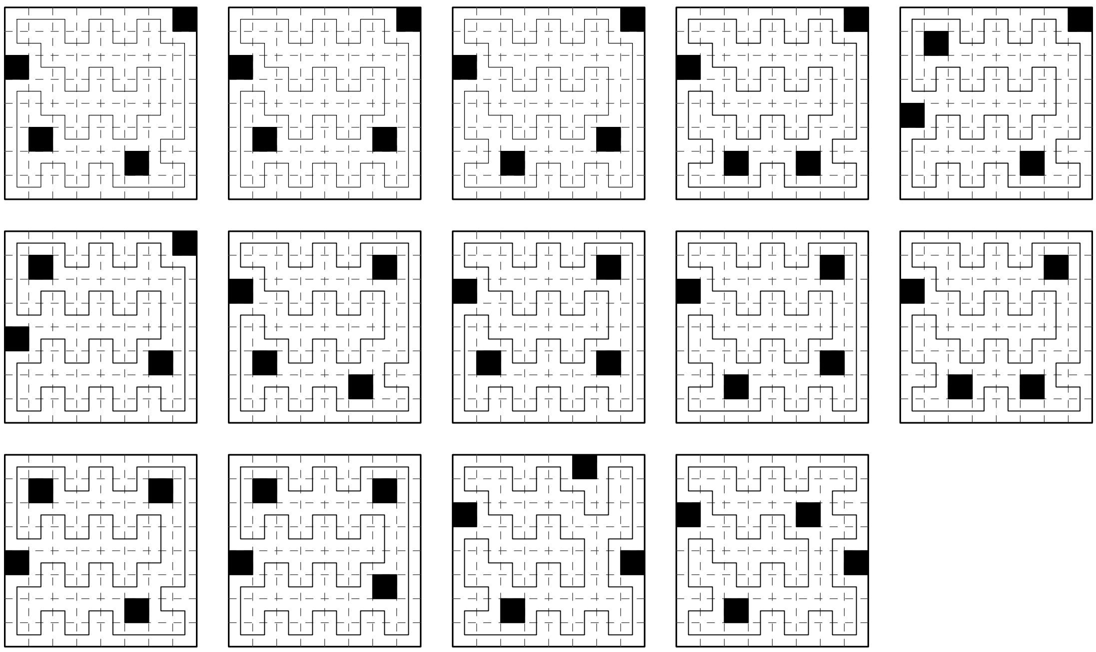
This is really another Wazir question. In 1978 I came across some amusing chess puzzles by T. R. Dawson involving rook journeys and knight tours on the vacant squares of a board on which eight white queens are standing in one of their famous twelve mutually non-guarding positions. These compositions set me wondering whether a closed non-intersecting rook tour (i.e. a wazir tour) was possible on the 56 unobstructed squares.



A B C

I quickly found that a symmetrical tour around the symmetrically arranged queens is possible, as shown in diagram A. In fact 14 such symmetrical tours (and about 150 asymmetrical ones) are possible round this particular setting of the queens. Of the other 11 solutions to the 8Qs problem nine do not admit of any wazir cruise around the queens. The arrangement shown in B allows just 20 different tours. The remaining case C however provided a surprise, sice the tour is in this case uniquely determined!

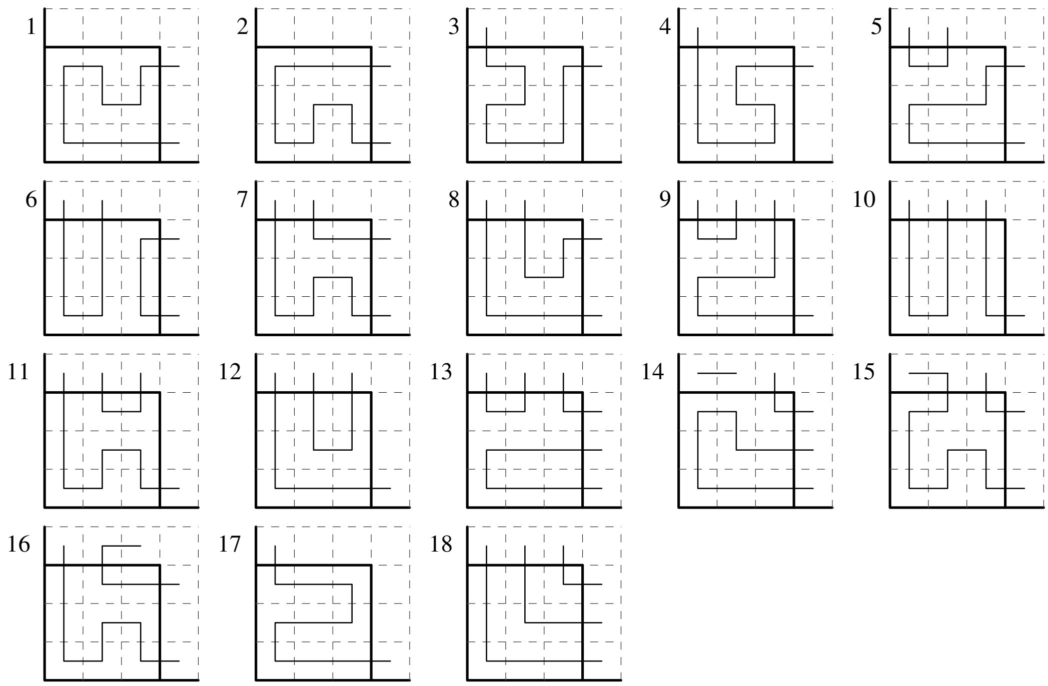
This discovery naturally led me to enquire if the number of blocks needed to fix the wazir tour could be reduced. The number of blocks must be even, since the wazir moves alternately to dark and light cells, so in a closed tour there must be the same number of each. The number was quickly reduced to six (a6, c2, d6, e6, f2, h6). Then further search led to the unexpected discovery (Feb 1978) that four blocks suitably placed on the chessboard are sufficient to determine a unique closed wazir tour of the remaining squares.



Three four-block arrangements were published in *The Problemist* [vol.10, #22 (Nov 1979) p.376]. In 1981 Tom Marlow sent me seven more solutions, which stimulated me to find four more, and these were published in *Chessics* [vol.1, #12 (1981) p.7-9, 12-13]. The 14 known solutions are as shown above. The last one shown is the only one with a block within the central 4×4.

THEOREM: On the 8×8 board the minimum number of blocks to force a unique closed wazir tour of the remaining cells is four, and one block must appear in each quarter of the board.

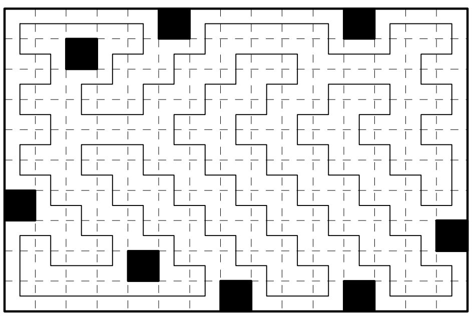
Proof: [*Chessics* 1981] If there are only two blocks then at least one 4×4 quarter of the board must be free of blocks. We can take it to be the a1 quarter. Now consider the 3×3 cells in the a1 corner, and all possible routes of the wazir through these nine cells.



The above diagrams show the possible paths. Diagrams 1-2, 3-4, 5-6, 7-8, 9-10, 11-12, 13-14, 15-16 are pairs in which the entrances and exits to the 3×3 are the same but the routes followed are different; in other words the tour is not uniquely determined in these cases. Diagram 17 similarly pairs with its own reflection in the a1-d4 diagonal. Diagram 18 can be settled by considering how the wazir tour outside the 3×3 must link up the entrances and exits to the 3×3 without crossing over or forming detached circuits. The links must be either a4-b4, c4-d1, d2-d3 or a4-d3, c4-d4, d1-d2. If we delete the interior of the 3×3 we can reconnect in accord with pattern 3, or its reflection. QED.

The general problem is: How few blocks are needed on a board m×n so that there is a uniquely determined closed wazir tour of the remaining cells? On the larger boards it is as if the blocks direct the wazir across the open board in lateral or diagonal zigzagging ‘wazir waves’. 2×n (0). 3×3 (1), 3×5 (1). 4×n (2), e.g. on any board 4×2m a solution is (a1, y2), where y represents the penultimate file. 3×4 (2), 3×6 (2), 3×8 (2)., 3×7 (3), 3×9 (3), 3×11 (3), 3×10 (4), 5×6 (4), 6×6 (4), 8×8 (4), and generally 8×2m, (4) e.g. a3, c6, y2, y6. 7×7 (5), 12×12 (6), 9×9 (7), 10×10 (8), 14×14 (8), 16×16 (8), 11×11 (9). T. H. Willcocks (*G&P Journal* 1988) gave larger solutions derived from smaller ones.

The 10×15 case appeared in the *Journal of Recreational Mathematics* 1999 and I was able to solve it in 8 blocks as shown below.



Rook Tours

In chess the Rook is able to move any distance along the ranks and files of the chessboard. It is thus a composite piece able to make moves of type {0,n} that is {0,1}{0,2}{0,3}{0,4} and so on, though in chess it is not allowed to pass over other pieces in its way, which means it is a 'rider' rather than a 'leaper'. However this distinction is irrelevant here since we are only concerned with the paths followed by a single rook and not with its interactions with other pieces.

One-Rank Tours

At first sight the idea of a tour of a one-rank board would not seem likely to lead to anything very interesting. Apart from the 1×2 board where a wazir can arguably execute a closed tour its only tour on a 1×n board with n > 2 is an open tour consisting of a straight walk from one end to the other. A rook on the other hand makes a series of wazir moves in one ‘go’. It can be regarded as ‘visiting’ all the cells it passes through, or only those it lands on at the end of the go. The latter interpretation allows us to make open and closed tours of a one-rank board of any length.

Such sequences of moves, or their diagonal equivalent, can form components in more elaborate tours. Rook tours along a rank can be transformed into bishop moves along a diagonal and nightrider tours along a knight-line.

A rook can start on any cell and can enter the cells in any order. There are thus n-factorial ways of numbering the cells of a 1×n board in sequence 1 to n, since there are n choices for the first cell, n–1 for the second, n–2 for the third and so on, making n.(n–1).(n–2) ... 3.2.1 ways (denoted n!). Thus the number of open rook tour diagrams, each of which can be numbered from either end, is T(n) = n!/2 for n > 1. All of these open tours are reentrant, since the two end cells are always a rook move apart. We get T(2) = 1, T(3) = 3, T(4) = 12, T(5) = 60, T(6) = 360, T(7) = 2520, and so on. However these rook moves will be of two or more different lengths.

Every one-rank tour has the rank, trivially, as an axis of symmetry, so strictly speaking all the tours are symmetric. However, we count as symmetric only those tours that have a vertical axis of symmetry when we draw the moves as curves above the horizontal line. When n is even the number of symmetric open tours is S(n) = n.(n–2).(n–4)...6.4.2/2 since there are n choices for first cell, but this also fixes the last cell, then there are n–2 choices for second cell, but this also fixes the next-to-last cell, and so on, until we reach 2 choices for the cell numbered n/2. When n is odd the centre cell of the board must be the centre cell of the symmetric open tour. Taking this out we get the same formula for S(n) but with n–1 in place of n. If h is the integral part of (n/2) we have when n > 1 S(n) = [2^(h–1)].h! in each case. We get S(2) = S(3) = 1, S(4) = S(5) = 4, S(6) = S(7) = 24, and so on.

The number of geometrically distinct open tours is G(n) = [T(n) – S(n)]/2 + S(n) since each asymmetric tour can be reversed. That is G(n) = [T(n) + S(n)]/2. We thus get G(2) = 1, G(3) = 2, G(4) = 8, G(5) = 32, G(6) = 192, G(7) = 1272, and so on.

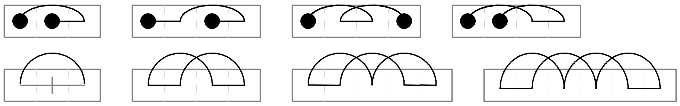
From one closed tour we can derive n open tours by deleting one move. So the closed tours total T(n) = (n–1)!/2 for n > 2. The number of symmetric closed tours on an odd-length board is the same as the number of symmetric open tours on the same board, since they are formed by joining the ends of the open tours, thus S(2h+1) = [2^(h–1)].h! The number on an even-length board is n/2 + 1 times the total for the previous odd board: S(2h) = 2^(h–1).(h+1)! The added ‘1’ in the factor counts the tours with no symmetric move, while the h part counts those with two symmetric. The geometrically distinct closed tours are calculated as before: G(n) = [T(n) + S(n)]/2.

Some diagrams of these one-rank tours are included in the following sections.

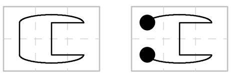
Two-Move Rooks

{0,1}{0,2}

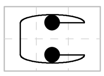
Wazaba or 1-2 Rook: One-rank open and closed non-wazir tours by this piece.



On the 2×3 this piece has a closed axial tour that can be semi-magic in two ways.

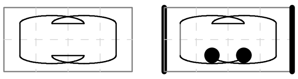


1	3	2
6	4	5

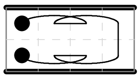


3	1	2
4	6	5

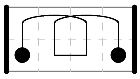
On the 2×4 it has a biaxial tour semi-magic in two ways, and a semi-magic open tour.



6	4	5	3
7	1	8	2

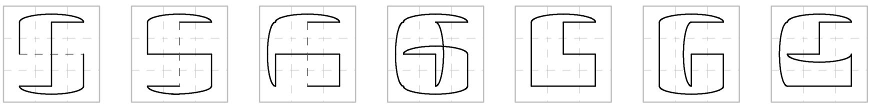


1	3	2	4
8	6	7	5



2	6	3	7
1	5	4	8

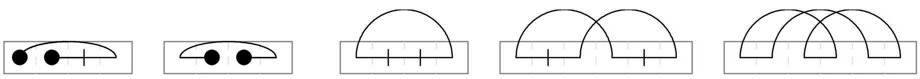
On the 3×3 board the rook can make 7 geometrically distinct closed tours (the board edges restricting it to 1- or 2-step moves of course). The first two illustrated below differ only in the stop at the centre being made on the vertical move or the horizontal move through the centre.



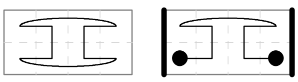
The dabbaba moves are shown by broken or curved lines. The first two have centro-symmetry. The third gives an appearance of diagonal axial symmetry, but a dabbaba move is not quite the same as two wazir moves.

{0,1}{0,3}

Skip-Rook. One-rank open and closed tours, excluding simple wazir paths:

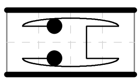


One biaxial 2×4 closed tour semi-magic in two ways.



4	3	6	5
1	2	7	8

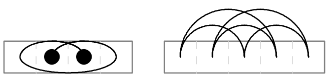
7	8	5	6
2	1	4	3



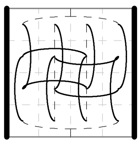
{0,2}{0,3}

Ski-Rook. The 2-3 mover is an amphibian: it can reach any cell on the board, though its components cannot. One-rank open and closed tours:

And a 4×4 semi-magic tour (Jelliss 1985 *Chessics* #24 p.95):



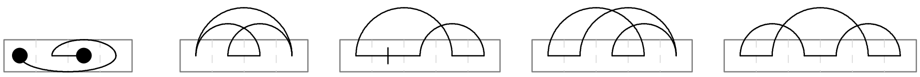
8	12	5	9
2	14	3	15
7	11	6	10
1	13	4	16



Three-Move Rooks

{0,1}{0,2}{0,3}

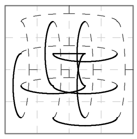
Here is a one-rank open tour and closed tours using all three moves.



A puzzle consisting of 16 square arrowed tiles, 1 with 1 dot, 8 with 2 dots, 7 with 3 dots was mentioned by MathsJam on Twitter (May 2017). The aim is to arrange the tiles in a 4×4 so that following the indicated moves a closed tour results. A solution is:

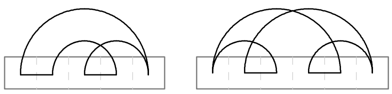
3>	2v	3v	3v
2v	2>	1<	3<
2>	2>	2^	3<
3^	3^	2^	2<

5	9	14	6
3	1	16	2
12	10	13	11
4	8	15	7



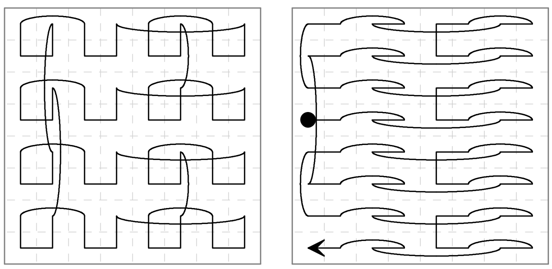
{0,1}{0,2}{0,4}

Closed one-rank tours using all three moves:



The **hyperwazir** is a restricted {0,1}{0,2}{0,4} leaper on the 8×8 board. The rule for movement of a strict hyperwazir is that it can make {0,4} moves freely, {0,2} moves within any quarter of the board, and {0,1} moves within any sixteenth of the board (i.e. any quarter of a quarter). The moves of the hyperwazir on the chessboard constitute a representation of the 6D hypercube. See the Space Chess section for more on this (p.731).

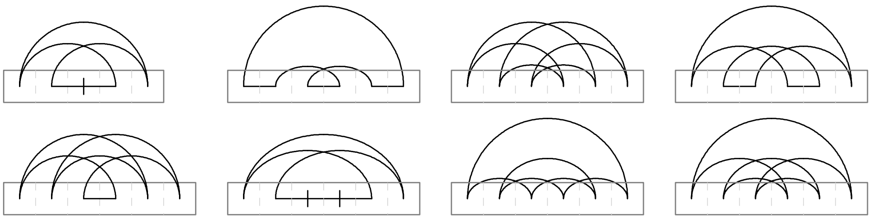
A hyperwazir has 6 moves available at any position on the board, and in fact is capable of making tours of the board. A tour is shown (*Chessics* #13, p.13, 1982) that uses the maximum number of 48 {0,1} moves. Similar tours can be constructed showing 48 {0,2} moves or 48 {0,4} moves, though these are difficult to show clearly in diagrams because of the need to curve the move lines.



The other tour shown relates to the puzzle toy known as the Chinese Rings or Tiring Irons, in the form of six rings attached to a bar in such a way that a ring can be removed from or put on the bar only when the next lower ring is on and all other lower rings are off. Any configuration of the rings can be represented by a sequence of six binary digits (0 for off, 1 for on), and the most difficult task is to get from 100000 = 32 to 000000 = 0. Surprisingly, to get the ring nearest the handle off the bar we have to put all the others back on! Numbering the squares of the board in natural order (0 on a1, 1 on b1, to 63 on h8) and converting the numbers to binary notation, the sequence of positions of the rings can be represented visually by the hyperwazir tour shown (*Chessics* #19, 1984, p.30-31). Remarkably, all 64 possible arrangements of the six rings, on or off the bar, are used in the solution.

Other Three-Move Rooks

At the left are closed tours on the 1×5 and 1×6 using 1-3-4 moves. On the 1×6 board there are also the following rook closed tours using three lengths of move: One 1-2-5 tour. One 2-3-4 tour. One 1-3-5 tour. One 1-4-5 tour. Two tours 2-3-5.

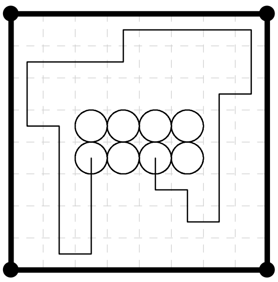


Four-Move Rooks

{0,1}{0,2}{0,3}{0,4}

Magic Rook Tours. The following work on 8×8 board magic rook tours restricted to 4 move lengths (1,2,3,4) was inspired by a magic rook tour by Joachim Brügge (*Die Schwalbe* August 1985) that used six lengths of move (see the next page).

24	62	63	8	25	34	35	9
6	44	45	7	26	52	53	27
23	22	46	47	50	51	11	10
5	4	64	48	49	33	29	28
60	61	1	17	16	32	36	37
42	43	19	18	15	14	54	55
59	21	20	58	39	13	12	38
41	3	2	57	40	31	30	56

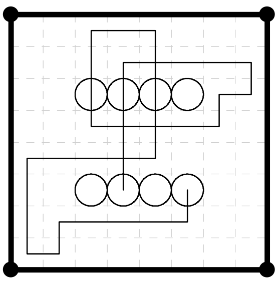


56	30	31	8	57	34	35	9
6	44	45	7	58	20	21	59
55	54	46	47	18	19	11	10
5	4	32	48	17	33	61	60
28	29	1	49	16	64	36	37
42	43	51	50	15	14	22	23
27	53	52	26	39	13	12	38
41	3	2	25	40	63	62	24

I have a note that I found the first tour here on 20 May 1986. Each quarter forms a 'bisatin' of one circuit, using two cells in each rank and file. The circles mark the ends of the quarters where the numbers 1, 16, 17, 32, 33, 48, 49, 64 occur. It was only some time later that I realised that the same system could be numbered in a second way, as shown on the right, where the link 16-17 is taken along the file instead of the rank. Both numberings are diagonally magic. I call each numbering a 'semi-rotation' of the other. Half the numbers in each rank and file are kept and the others changed. In the first the files are made up of complements adding to 65 while in the second it is the ranks.

The next two tours were found the day after the above example. These biaxial tours remain magic in ranks and files when the origin of numbering is shifted to the next quarter, i.e. 17 becomes 1, 18 becomes 2 and so on. The two tours shown below also remain diagonally magic when renumbered in this way, but the tour above loses the diagonal property; this is because it has more than two numbers ≤ 16 in a diagonal. The semi-rotated versions of these two numberings are not shown.

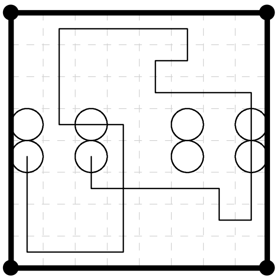
53	52	8	24	9	25	45	44
30	51	47	2	31	50	46	3
29	28	48	64	33	49	5	4
54	27	7	42	55	26	6	43
11	38	58	23	10	39	59	22
36	37	17	1	32	16	60	61
35	14	18	63	34	15	19	62
12	13	57	41	56	40	20	21



37	36	56	8	57	9	29	28
14	35	31	50	15	34	30	51
13	12	32	48	17	33	53	52
38	11	55	26	39	10	54	27
59	22	42	7	58	23	43	6
20	21	1	49	16	64	44	45
19	62	2	47	18	63	3	46
60	61	41	25	40	24	4	5

In the next tour the rotated version does not merely remain magic in the diagonals but retains the same numbers in each diagonal, this happens when the two numbers ≤ 16 add to 17. Here they are 2+15 and 4+13. The semirotations are not shown.

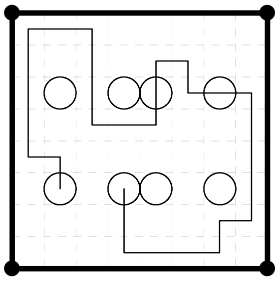
63	6	26	62	35	7	27	34
44	45	25	24	9	8	52	53
22	46	50	23	10	47	51	11
64	5	49	4	29	48	28	33
1	60	16	61	36	17	37	32
43	19	15	42	55	18	14	54
21	20	40	41	56	57	13	12
2	59	39	3	30	58	38	31

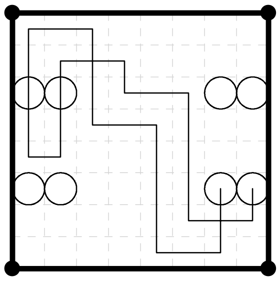


47	54	10	46	19	55	11	18
28	29	9	8	57	56	36	37
6	30	34	7	58	31	35	59
48	53	33	52	13	32	12	17
49	44	64	45	20	1	21	16
27	3	63	26	39	2	62	38
5	4	24	25	40	41	61	60
50	43	23	51	14	42	22	15

Here are some more of these magic rook tours found later. For reasons of space only one numbering in each case is shown. In these the small numbers in the diagonals are 4+13 and 7+10

4	46	5	50	47	28	51	29
44	45	24	25	8	9	52	53
22	64	23	49	48	10	33	11
62	63	6	26	7	27	34	35
3	2	59	39	58	38	31	30
43	1	42	16	17	55	32	54
21	20	41	40	57	56	13	12
61	19	60	15	18	37	14	36





10	47	11	46	51	22	50	23
34	7	35	6	27	62	26	63
33	48	29	5	28	4	49	64
56	57	12	20	13	21	40	41
9	8	53	45	52	44	25	24
32	17	36	60	37	61	16	1
31	58	30	59	38	3	39	2
55	18	54	19	14	43	15	42

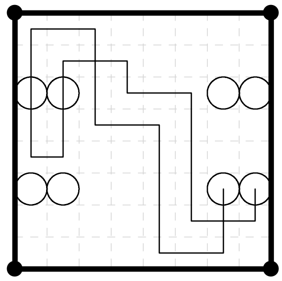
An enumeration of the possible bisatins that could be used in this type of tour would be the next step, if anyone with computing ability would like to take it on.

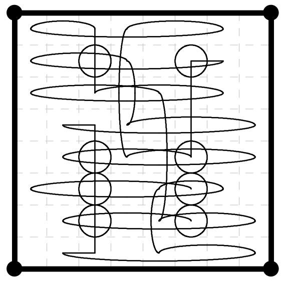
More-Move Rooks

{0,1}{0,2}{0,3}{0,4}{0,5}

Five-Pattern Magic Rook Tour. Here is another of my diamagic 8×8 rook tours (R12345 type) using a 5-leap (16-17 and 48-49) in addition to the four shorter moves. As before only the first quarter path is diagrammed. The other quarters repeat the route. Only the one numbering is shown, but it remains diagonally magic when renumbered cyclically or by semirotation. The small numbers in the diagonals are 4+13 and 7+10.

10	47	11	46	51	22	50	23
34	7	35	6	27	62	26	63
33	48	29	5	28	4	49	64
56	57	12	20	13	21	40	41
9	8	53	45	52	44	25	24
32	17	36	60	37	61	16	1
31	58	30	59	38	3	39	2
55	18	54	19	14	43	15	42





10	54	9	12	53	56	11	55
19	47	48	20	45	17	18	46
6	58	8	60	5	57	7	59
43	23	24	21	44	41	42	22
51	15	49	13	52	16	50	14
31	35	33	36	29	32	30	34
62	2	64	61	4	1	63	3
38	26	25	37	28	40	39	27

{0,1}{0,2}{0,3}{0,4}{0,5}{0,6}

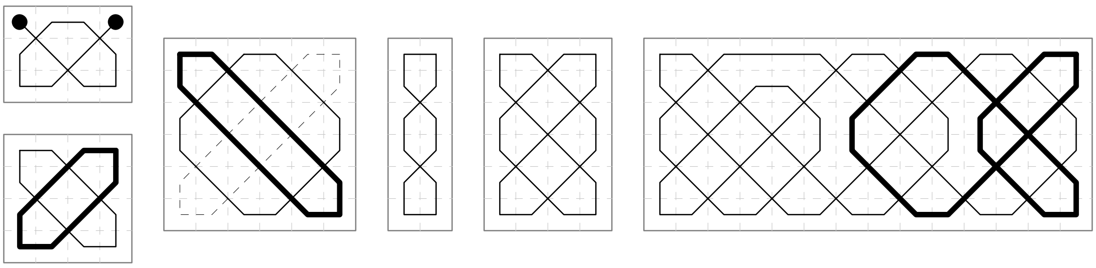
Six-Move Magic Rook Tour. This (above right) is the earliest 8×8 diamagic rook tour I know of, by Joachim Brügge *Die Schwalbe* Aug 1985. It employs 6 move types. It is not of bisatin structure but the numbers 1-16 and 17-32 have four in each rank, and combine to provide four in each file. Half the tour is diagrammed.

Diagonal Movers

Knots

Celtic Art. One of the defining characteristics of ‘Celtic Art’ is the use of interlace patterns depicting a ribbon passing alternately over and under itself. Many interlacings can be represented graphically by a path of king moves on a lattice of squares, the cross-overs being made by diagonal moves of the king, though this does not necessarily imply that they were designed this way.

These underlying king paths can be tours or pseudotours (formed of two or more superimposed circuits). It may be noted that these king tours avoid sharp turns (of 45°) since this would result in small interstitial areas, not allowing room for the definite width of the ribbon. My simplified drawings leave out the over-and-under interlacing.



One of the earliest and simplest examples is the Stafford Knot used as an heraldic symbol on the arms of Stafford and Staffordshire and dating back to Anglo-Saxon times, 7th century or earlier. It is shown here as a 3×4 open king tour. Pozzi (1998) notes that the 4×4 ‘Solomon's Knot’ pattern of two interlinking circuits occurs widely. He also notes the 6×6 king pseudotour shown here.

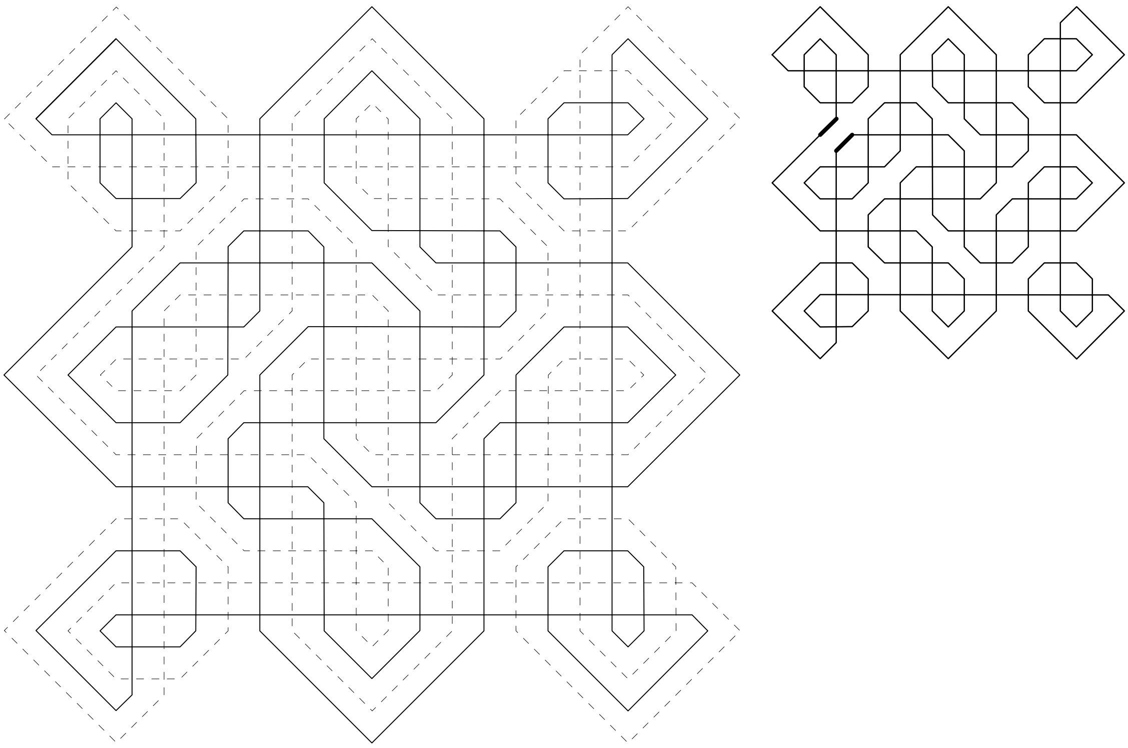
The cover of “A strange object known as the *soiscél Molaise* ... made around 1000 AD as a *cumdach* or book-shrine for the Gospel of St Molaise, using pieces of a house-shaped shrine of the early eighth century” as described by Lloyd and Jennifer Laing (*Art of the Celts*, 1992, p.146) uses various interlacing patterns to fill gaps in the design. Among these are patterns that may be represented by king paths on 2×6, 4×6 and 6×14 boards, as shown above. The second of these is a ‘prime knot’ in the modern mathematical theory of knots (Adams *The Knot Book* 1994, p.24, 281.)

A ‘knot’ in mathematical terms is a closed curve in three-dimensional space. If one knot can be deformed so that it is congruent to another without passing through itself anywhere, the two knots are counted of the same type. In particular a knot that can be simplified into a circle is a ‘trivial knot’ (or an ‘unknot’). It is by no means always obvious that a given knot is an unknot. A ‘compound’ knot is one formed from two knots by deleting a small part of each knot and joining the ends in such a way that the joining lines do not cross each other or any parts of the existing knots. A ‘prime’ knot is one that cannot be formed by joining two knots in this way.

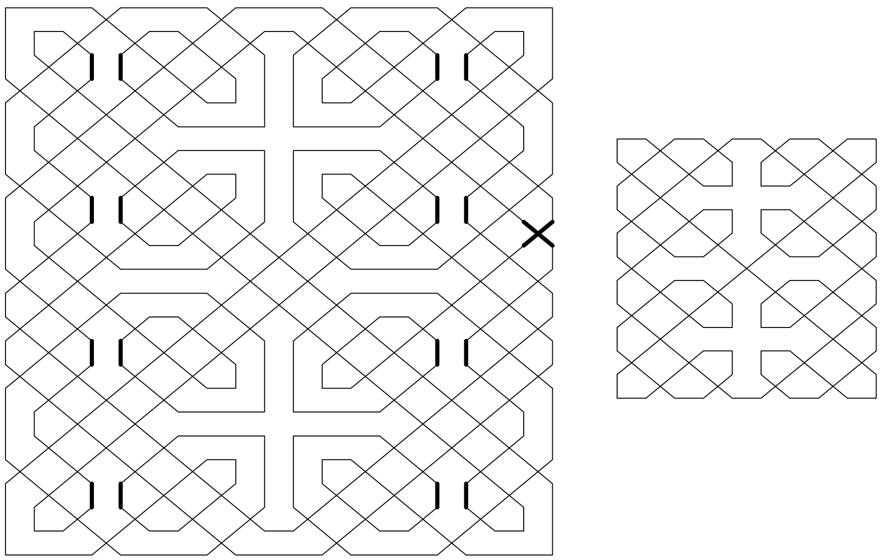
Similar but more elaborate designs are used in the upper and lower borders on a page illustrating an Eagle symbol in the *Book of Dimma*, late 8th century, and in the left and right borders on the Lion symbol page of the *Book of Durrow* of the mid 7th century, also described by the Laings (1992, p.142, 174). These simplify to pseudotours 4×28 and 4×34 as the basis for the designs.

The ‘binding knot’ from the *Book of Kells*, late 8th or early 9th century is possibly the most elaborate of these designs. For the full pattern see Pennick (1990, p.45) and Bain (1951, p.46). By redrawing the knot design on squared paper, I found that this knot becomes two circuits of king moves, on a board in the shape of a quadrate cross (formed of thirteen 8×8 squares) as shown below.

The following quotation from Alcuin (735–804), cited by the Laings (1992, p.181), is perhaps what the design illustrates: “the four rivers of the virtues flowing out of one bright and health-giving paradise, irrigating the whole breadth of the christian church”. The river enters from the left, or the West side.



Examination of the knot shows that the two threads of which it is constructed follow each other in parallel throughout, thus the pair of threads define a single path. Replacing the two king paths by one line midway between them results in a single king tour of a quadrate cross formed of thirteen 4×4 squares, as shown in the smaller diagram. This cross tour explains why the threads come to a point at the four cardinal points instead of just curving round: They correspond to the corners of the square part of the quadrate cross. It seems evident to me that in this case at least the king tour was consciously the basis of the design of the Kells knot.



A rectangular king tour underlies the design of a panel from the Saint Madoes stone, Perthshire, described by George Bain [1951, p.46]. This is a 20×24 king tour, which can be formed by ‘doubling’ a simpler king tour 10×12 and altering some links to turn the pseudotour into a true tour.

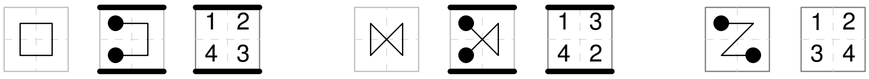
King Tours

The king combines wazir {0,1} and fers {1,1} moves, one step laterally or diagonally. The king is the only double-pattern mover with all coordinates less than 2. King tours, as noted in the preceding section, can be seen to underlie some of the interlacing designs in Celtic Art, but a conscious study of king tours does not seem to have begun until the late nineteenth century.

It was only recently (2018) that I came across magic king tours in the French newspaper columns of the 1880s where many of the magic knight tours were first published. Previously I had thought their study began in the first decade of the twentieth century. A thorough study of 8×8 magic king tours with biaxial symmetry was made by myself and Tom Marlow in 1986 and 1997.

King Tours 2×n

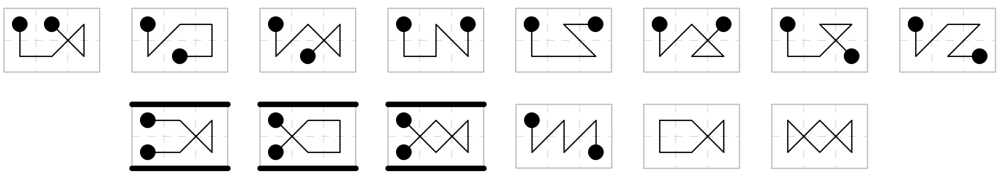
2×2 Board To form a magic array we need a number M that can be expressed as the sum of smaller numbers in several ways. The first case of this is 5 = 1 + 4 = 2 + 3. Any arrangement of 1, 2, 3, 4 in a square has two of its lines additively magic, and the other lines subtractively magic. The rows, columns, or diagonals either add to 5 or have constant difference 1 or 2.



On the 2×2 board the king has two closed tours (one of wazir type) and three open tours (again one of wazir type). We will normally leave out the wazir tours in this section. The centrosymmetric open tour adds to 5 in the diagonals but the rank and file sums are all different.

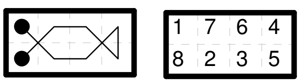
2×3 Board The next cases of magic pairs are 6 = 1+5 = 2+4 (omitting 3, so that no tour is possible in this case) and 7 = 1+6 = 2+5 = 3+4 (familar on the opposite faces of standard dice).

On the 2×3 board the king has three closed tours, and 15 open tours, four with axial and two with central symmetry (of these one closed and three open are wazir tours, not shown here).



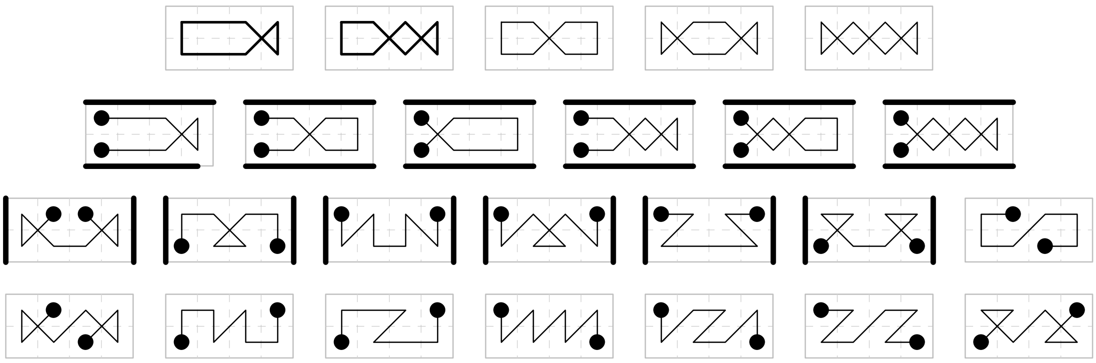
The axisymmetric tours, marked with dark border lines, are semi-magic, adding to 7 in the files, since the mth cell from the start is in line with the mth cell from the end (1 + 6 = 2 + 5 = 3 + 4 = 7). The closed tours are also semi-magic if numbered from appropriate points. The ‘bottle’ shaped king tour is the first to have two different semi-magic numberings.

2×4 Board The next two-term magic constants are 8 = 1+7, 2+6, 3+5 (not 4 so no tours), 9 = 1+8, 2+7, 3+6, 4+5. Odd cases use every number less than M, while in even cases M/2 is omitted. The 2×4 board is the first case where we encounter a fully **magic** tour with both ranks and files equal summed. It is also a tour of alternating fers and wazir moves.

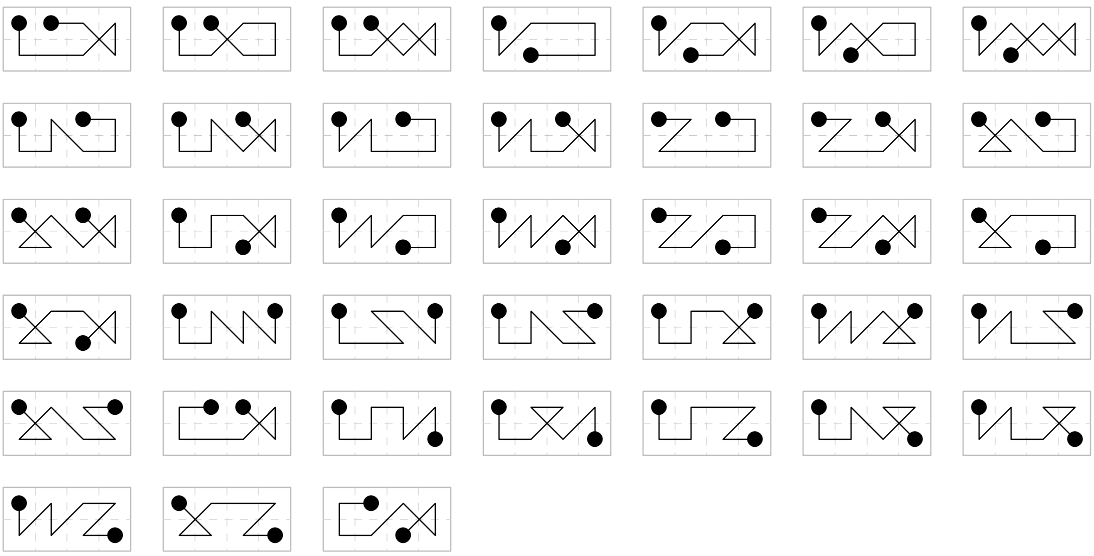


This is the start of a series of magic king tours on rectangular boards 2×2n. The files consist of pairs of complements, adding to 4·n + 1, the first and last entries, 1 and 4·n, being in the first file, and the middle entries, 2·n and 2·n + 1, in the last file.

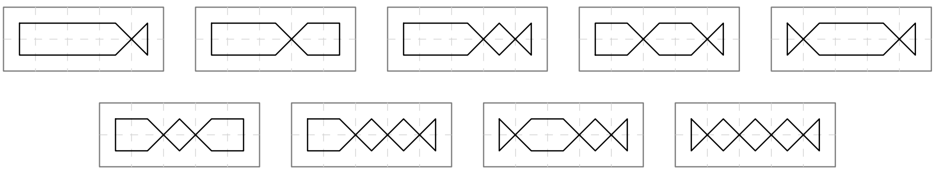
On the 2×4 board the king has six closed tours (one of wazir type) and 64 open tours (including 5 wazir tours), of which 24 are symmetric shown below (3 of wazir type). Those with axial symmetry are semi-magic in the ranks or in the files.



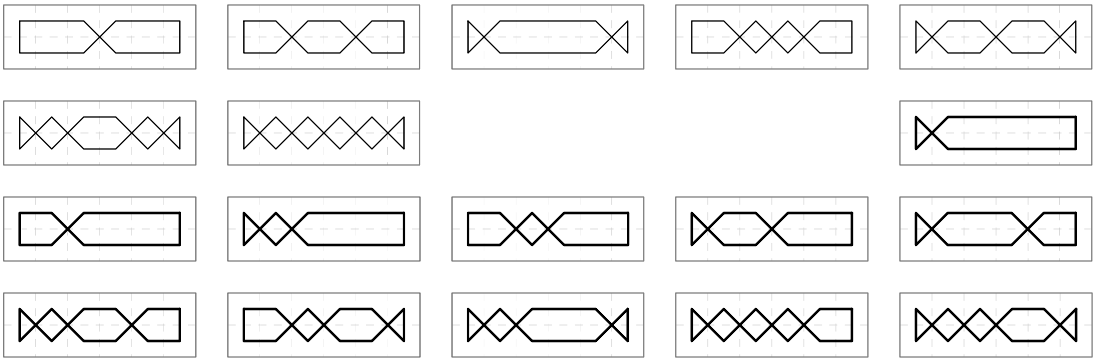
There are 40 asymmetric (38 excluding wazir).



2×5 Board. Additionally to the wazir tour there are 9 other geometrically distinct closed king tours, three of which are symmetric about both axes.

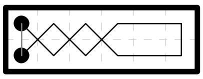


2×6 Board. There are 20 closed king tours, 8 of which are symmetric about both axes (including the wazir tour). There is one with alternating wazir and fers moves (top right).



There is one alternating king tour on any board 2×2h and it has biaxial symmetry.

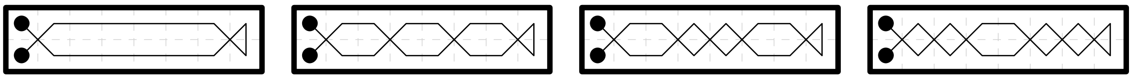
It is perhaps surprising that there is only one magic king tour on this board.



1	11	3	9	8	7
12	2	10	4	5	6

2×7 Board. There are 36 closed king tours, 8 biaxial (including the wazir case).

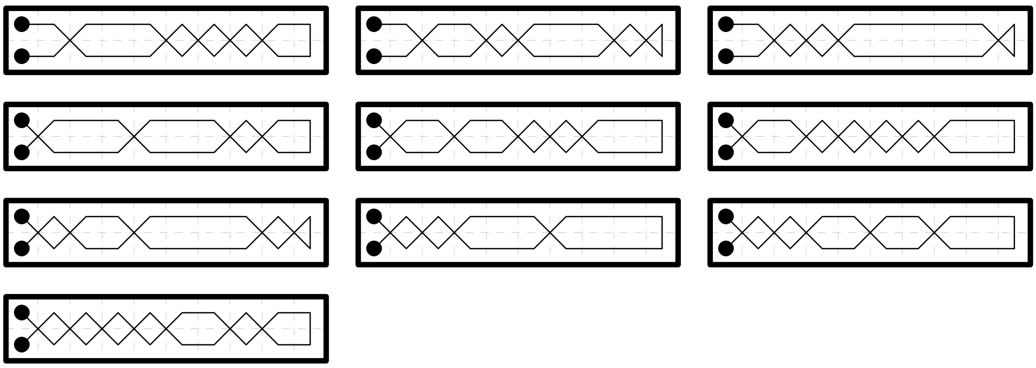
2×8 Board. There are 72 closed king tours of which 16 are biaxial. There are four magic king tours, magic constants 17 and 68. Including the alternating tour.



The Kashmir poet Rudrata (c.900) gave an open ‘elephant’ tour of a two-rank board in the form of a saw-tooth pattern of alternating wazir and fers moves.

2×9 Board. There are 136 closed tours with 16 bisymmetric (including the wazir tour).

2×10 Board. There are 272 closed king tours of which 32 are symmetric about both axes (count includes the wazir tour, and one with alternating wazir and fers moves). There are 10 magic king tours. These have magic constants 21 and 105 (*Chessics* #26 1986 p.117 and #30 p.163).



Enumeration of 2-Rank Closed King Tours

We now look at the general problem of counting king tours on 2-rank boards of any length. On all 2×n boards (n > 2) the link between two adjacent files in a diagram of a king's closed tour is either a pair of lengthwise moves or a pair of diagonal moves; two choices in each of n–1 cases, hence the number of closed tour diagrams (including the wazir tour) is T(n) = 2^(n–1). However this formula breaks down for n = 2 since the board is then square and there is an extra diagram because the bow-tie shaped tour diagram can be rotated 90 degrees to figure-of-eight shape. All these tours are symmetrical about the horizontal axis and therefore semi-magic in the files.

Let H(n) be the number also symmetric about the vertical axis. We have two choices at each pair of files in one half, and in the centre pair when n is even, so the number of these biaxial tours is H(n) = 2^h, where h is the integral part of n/2, that is n/2 when n is even, (n−1)/2 when n is odd. The total for 2·h + 1 is the same as for 2·h. The number of axial tours is A(n) = T(n) – H(n) = 2^(n–1) – 2^h.

Table of results for closed tours

n =	1	2	3	4	5	6	7	8	9	10	11	12
T =	1	3	4	8	16	32	64	128	256	512	1024	2048
h =	0	1	1	2	2	3	3	4	4	5	5	6
H =	1	2	2	4	4	8	8	16	16	32	32	64
A =	0	0	1	2	6	12	28	56	120	240	496	992
G =	1	2	3	6	10	20	36	72	136	272	528	1056

The total T(n) is made up of 1 from each biaxial tour, 2 from each axial tour, so the number of geometrically distinct tours G(n) for n > 2, is H(n) + A(n)/2 = T(n)/2 + H(n))/2. Thus: G(n) = 2^(n−2) + 2^(h−1). The total for an even number is twice that for the preceding odd number.

Enumeration of 2-Rank Open King Tours

This problem, more difficult than the closed tour case, was first considered by T. H. Neal in *Chess Amateur* in 1908, showing that the numbers were related to the Fibonacci series. The number of geometrically distinct reentrant tours (that is open tours where the ends are a king move apart) can be calculated from the table for closed tours. Since each 2×n closed king tour is symmetric about the horizontal axis we can derive only n+1 geometrically distinct reentrant tours from an axial tour by deleting a move: n−1 from the = or × moves and 2 from the end moves, while from a biaxial tour we can derive h+1. So the number of geometrically distinct reentrant tours is R = (n+1)·A + (h+1)·H, though n=2 is again an exceptional case. This gives:

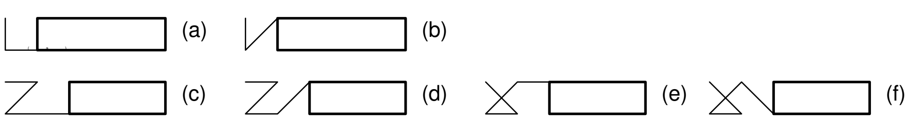
n =	1	2	3	4	5	6	7	8	9	10	11	12
R =	1	3	8	22	48	116	256	584	1280	2832	6144	13344

We now look at recursion relations and formulae to calculate the numbers of open tours generally (reentrant tours being a particular case). An open king tour with both ends in the same file is only possible if the file is an end-file of the board and, since joining the ends produces a closed tour, the number of such tours on a board 2×n is the same as the number of closed tours, 2^(n−1).

Any 2×n open king tour with ends on files p and q (p < q) consists (except when p = 1 or q = n) of reentrant tours of the end sections (with both their terminals on the p and q files) joined by an end-to-end tour of the middle section (with terminals on the p+1 and q–1 files). The reentrant sections total 2^(p–1) and 2^(n–q), from the formula for closed tours.

To complete the enumeration we need to know the number L(n) of tour-diagrams on a 2×n board, with one end at the top of the first file and the other end in the last file.

The sequence L(n) has the initial values L(1) = 1 and L(2) = 4 and obeys the recurrence relation L(n) = 2·L(n–1) + 4·L(n–2), since the initial section can be either of the two forms (a) or (b), derived from a tour of length n–1, or any of the four forms (c), (d), (e) , (f), derived from a tour of length n–2.



From this recursion we can calculate that $L(3) = 12$ (2 symmetric + 2×5 asymmetric), $L(4) = 40$ (12 symmetric + 2×14 asymmetric), $L(5) = 128$ (8 symmetric + 2×60 asymmetric), which agree with the totals found by construction. Thus the total of tour diagrams with ends in the p and q files is $2^{(p-1)} + 2^{(n-q)} + 8\cdot L(q-p-1)$ (Where $1 < p < q < n$.).

It is possible to express $L(n)$ in terms of the Fibonacci numbers as $L(n) = [2^{(n-1)}]\cdot F(n+1)$, where $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$, giving the series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... though this doesn't seem to make the calculation any easier.

Half of these L-type diagrams end at the top right and half at the bottom right, but when counted geometrically it appears that the numbers are equal in the even cases but differ in the odd cases since the number of symmetric tours are different, e.g. in case 3 there are 6 diagrams in each class, but the first consists of 3 asymmetric tours, the second of 2 symmetric and 2 asymmetric (2×3 = 6 and 2 + 2×2 = 6); and in case 5 there are 64 diagrams in each class, 32 asymmetric in the first class, 8 symmetric plus 28 asymmetric in the second class (2×32 = 64 and 8 + 2×28 = 64).

The number of king open tours from a given corner of a 2×n board, $K(n)$, follows the similar recurrence: $K(1) = 1$, $K(2) = 6$ and $K(n) = 2^{(n-1)} + 2\cdot K(n-1) + 4\cdot K(n-2)$, since diagrams (a) to (f) still apply, the additional term being the number of reentrant tours with both ends in the initial file. From this recurrence we calculate $K(3) = 20$, $K(4) = 72$, $K(5) = 240$, agreeing with the totals found by construction. This is also solved in Fibonacci numbers as: $K(n) = (2^n)\cdot F(n+1) - 2^{(n-1)}$, a formula given by T. H. Neal (1908).

We can now determine the number of geometrically distinct symmetric tours. For axial tours: the number with horizontal axis for $n>2$ is $2^{(n-1)}$ to which must be added $K(n/2) + 2\cdot K((n-2)/2)$ with vertical axis when n is even. For rotary tours: $2\cdot K((n-1)/2)$ when n is odd, $K(n/2) + 2\cdot K((n-2)/2)$ when n is even. So the total of symmetric tours is $S = A + Z$.

The open tours with one end at the top of the pth file (p not 1 or n) consist of either a reentrant tour of the files p to n, followed by a tour from a corner of the files p−1 to 1, or a reentrant tour of the files p to 1 and a tour from a corner of the files p+1 to n. Denoting this total by $K(p, n)$ we have, from the above results: $K(p,n) = [2^{(n-p)}]\cdot 2\cdot K(p-1) + [2^{(p-1)}]\cdot 2\cdot K(n-p)$, where the K-value is doubled since there are two possible links from the reentrant section to the other part. This simplifies, in another formula due to T. H. Neal (1908), to: $K(p, n) = (2^n)\cdot[F(p) + F(n+1-p) - 1]$. Of course $K(1,n) = K(n,n) = K(n)$ and $K(p,n) = K(n+1-p,n)$.

We can now calculate the total number of 2×n king open tours, $T(n)$, by summing the $K(1,n)$ from 1 to n, with the help of the Fibonacci property that the sum of $F(m)$ over m from 1 to n is $F(n+2) - 1$. This gives $T(n) = (2^n)\cdot[4\cdot F(n+1) - (n+3)]$. The number of geometrically distinct tours can then be calculated for each n (>2) by $G = (T + 2\cdot S)/4$.

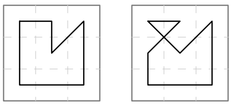
Table of results for open tours:

n =	1	2	3	4	5	6	7	8	9
F =	1	1	2	3	5	8	13	21	34
L =	1	4	12	40	128	416	1344	4352	14080
K =	1	6	20	72	240	800	2624	8576	27904
A =	0	2	4	16	16	64	64	240	256
Z =	0	1	2	8	12	32	40	112	144
S =	1	3	6	24	28	96	104	352	400
T =	1	12	48	208	768	2752	9472	32000	106496
G =	1	3	15	64	206	736	2420	8176	26824

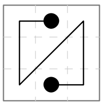
Here F = Fibonacci numbers, L = Tour diagrams from given corner to board end, K = Tour diagrams from given corner, A = axial, Z = rotary, S = symmetric, T = Total of tour diagrams and G = geometrically distinct. The figures in the first five columns of the table have been enumerated by construction as above, the other figures have been calculated from the recursions or formulae, but have not been independently checked.

King Tours 3×n

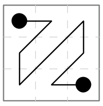
3×3 Board. The 3×3 king tours, open and closed, were enumerated by T. R. Dawson (1949). There are 2 closed tours 3×3; one polygonal, the other with one self-intersection. Dawson reported 51 geometrically distinct open king tours (3 of wazir type), of which 4 are symmetric (1 of wazir type) and 18 are reentrant (1 of wazir type: the reentering move being a non-wazir move).



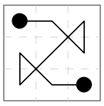
There are four centro-symmetric king open tours of the 3×3 board. They all have the property that the sums of their ranks and files form arithmetic progressions, i.e. sequences with a fixed common difference. I call tours with this numerical property **arithmic** (from *arithmos* meaning number).



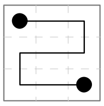
2	1	6
3	5	7
4	9	8



1	2	6
3	5	7
4	8	9



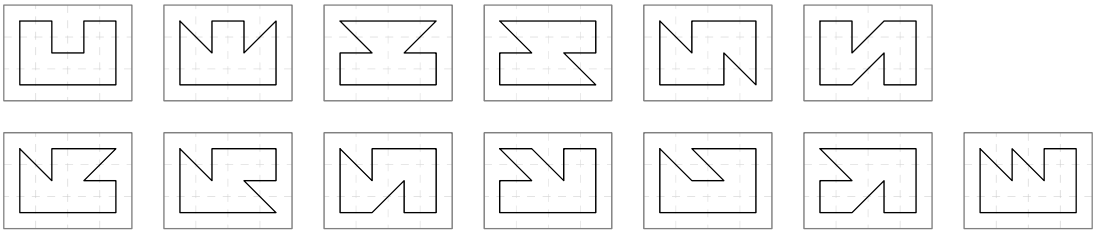
1	2	4
7	5	3
6	8	9



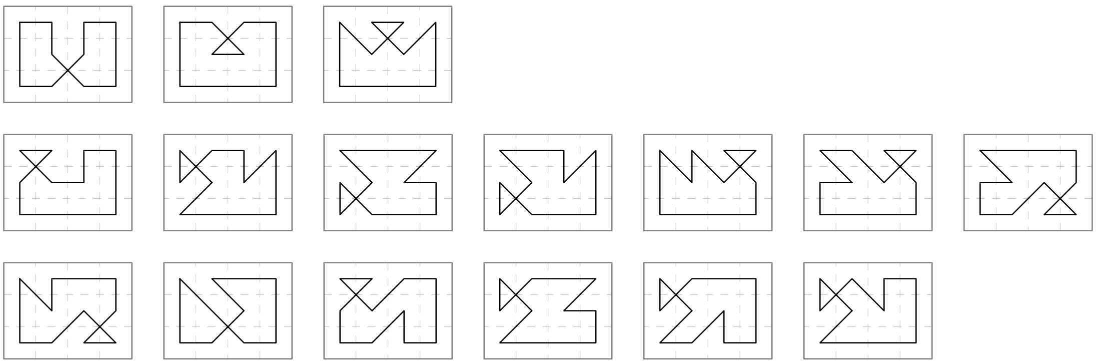
1	2	3
6	5	4
7	8	9

The first has ranks and files adding to the same progression 9, 15, 21 (cd 6). The others have one progression in the rank sums and another in the file sums. The second has ranks 9, 15, 21 (cd 6) files 8, 15, 22 (cd 7). The third has ranks 7, 15, 23 (cd 8) files 14, 15, 16 (cd 1). The fourth has ranks 6, 15, 24 (cd 9) files 14, 15, 16 (cd 1). In all of them, because the symmetry places the middle number 5 in the middle cell and places pairs of complements adding to 10 in the diametrally opposite cells the lines through the central cell all add to 15. Transposing the figures 2 and 8 in the first tour gives us the familiar 3×3 magic square. To make the second and third king tours magic we cycle 1-2-9-8-1. To make the wazir tour magic requires more manipulation. I have found this type of process of transforming arithmic king tours effective in generating magic tours on various boards.

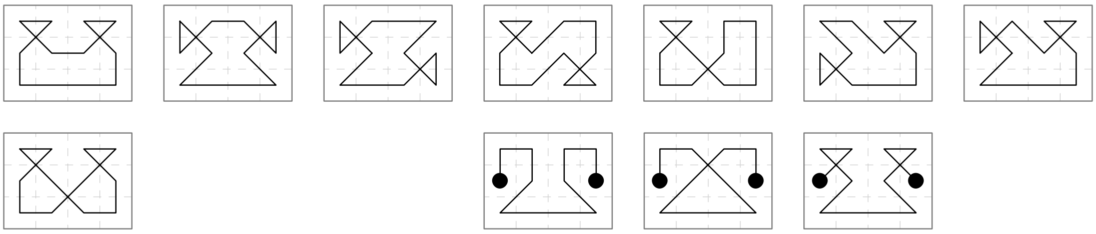
3×4 Board. I find 37 closed king tours including the wazir tour; 13 polygonal,



16 with one intersection,



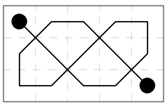
7 with two intersections, and 1 with three intersections.



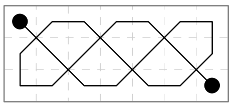
Of these 14 are symmetric; 5 centrally and 9 (including the wazir tour) axially.

The closed 3×4 tours give rise to 366 reentrant open tours, so counting the full number of open tours is likely to be very large, but among these are 40 with axial symmetry: 14 reentrant, derived from the seven axially symmetric closed tours, including the wazir case, that cross the axis twice at right angles. Of the 26 others, 23 differ from 4×4 closed tours only by omission of a straight base line, the 3 other cases are shown above.

3×5 and 3×7 Board. In the 3×5 and 3×7 arithmic king tours below the moves are completely regular, being diagonal except where they meet a board edge when the king takes a lateral step along the edge and then resumes its diagonal moves as if reflected from the edge. The ranks of the 3×5 add to the progression 39 40 41 and the files to 22 23 24 25 26 (both cd = 1), the middle numbers being the magic constants for rank and file. The ranks of the 3×7 add to 76 77 78 and the files to 30 31 32 33 34 35 36 successively. Arithmic king tours of this type seem to be possible on any odd-sided oblong where the sides have no common factor.

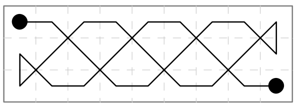


1	12	13	7	6
11	2	8	14	5
10	9	3	4	15



1	13	12	18	19	6	7
14	2	17	11	5	20	8
15	16	3	4	10	9	21

3×9 Board. An arithmic symmetric king tour, files 38 39 40 41 42 43 44 45 46 (cd 1), ranks 106 126 146 (cd 20). Not of the regular pattern.

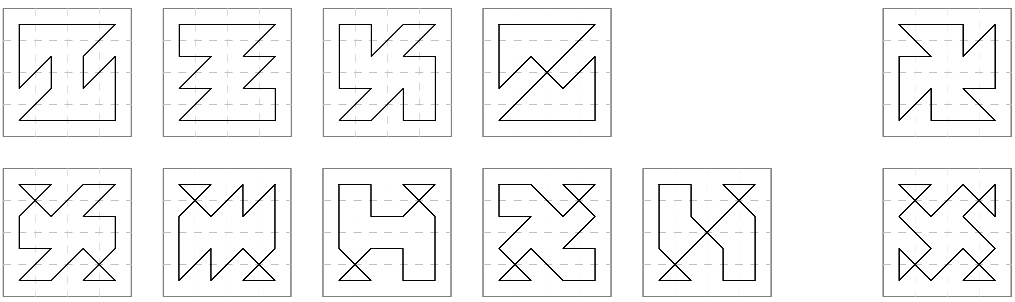


1	2	16	15	23	24	7	8	10
19	17	3	22	14	6	25	11	9
18	20	21	4	5	13	12	26	27

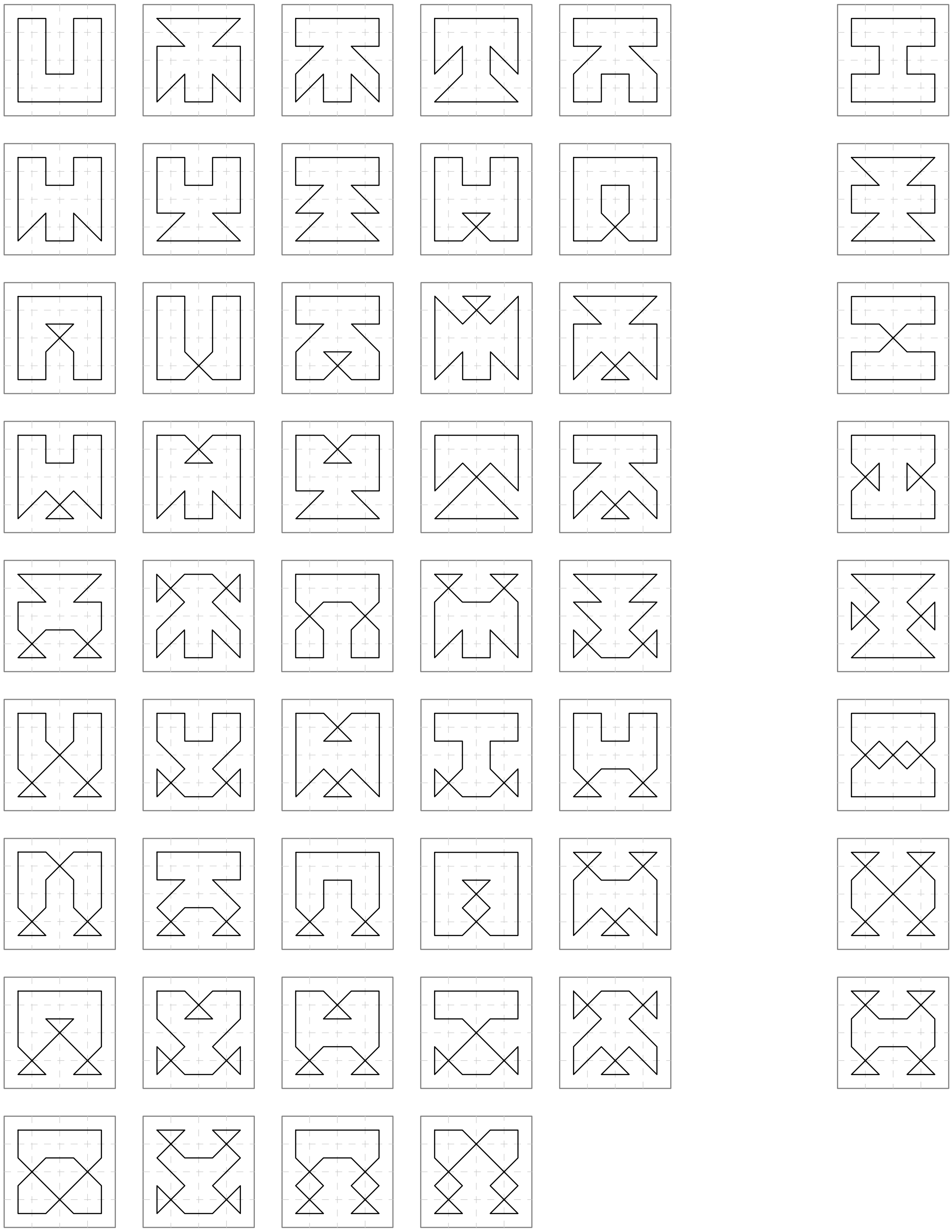
King Tours 4×n

4×4 Board. I find 368 closed king tours, of which 63, shown here, are symmetric.

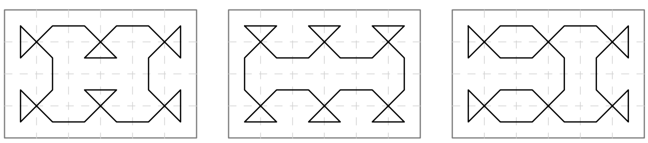
There are 9 rotary and 2 birotary



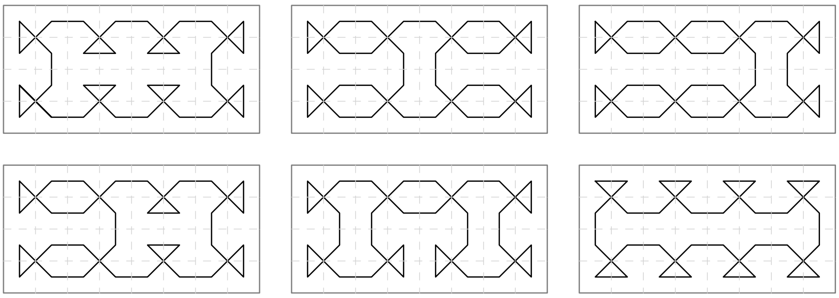
There are 44 axial and 8 biaxial (one has alternating wazir and fers moves),



4×6 Board. There are three alternating tours.



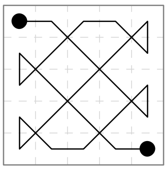
4×8 Board. There are six alternating tours.

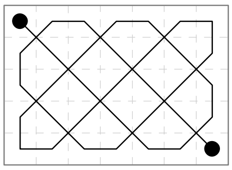


King Tours 5×n

5×5 Board. Symmetric open semi-arithmic king tour. The files add to 63 64 65 66 67 (cd 1), but the ranks are irregular, adding to 53 73 65 57 77.

1	2	18	17	15
21	19	3	14	16
20	22	13	4	6
10	12	23	7	5
11	9	8	24	25

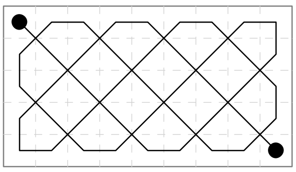




1	30	31	11	10	20	21
29	2	12	32	19	9	22
28	13	3	18	33	23	8
14	27	17	4	24	34	7
15	16	26	25	5	6	35

5×7 Board. Arithmic king tour is possible as for the 3×5 and 3×7 cases. The ranks add to the successive numbers 124 125 126 127 128 and files to 87 88 89 90 91 92 93.

5×9 Board. An arithmic king tour is again possible. The ranks add to 205 206 207 208 209 and the files to 111 112 113 114 115 116 117 118 119 (207 and 115 being the magic constants).

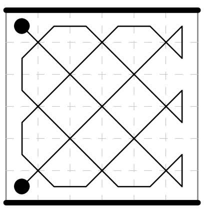


1	20	21	40	41	31	30	11	10
19	2	39	22	32	42	12	29	9
18	38	3	33	23	13	43	8	28
37	17	34	4	14	24	7	44	27
36	35	16	15	5	6	25	26	45

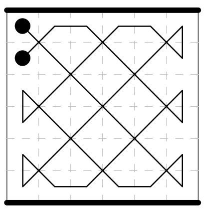
King Tours 6×n

6×6 Board. On this board I have noted some semi-magic king tours of similar structure to the above arithmic tours. The files add to 111 (three pairs of complements adding to 37). The ranks of the first add to 118, 120, 110, 112, 102, 104, and of the second to 122, 124, 112, 110, 98, 100.

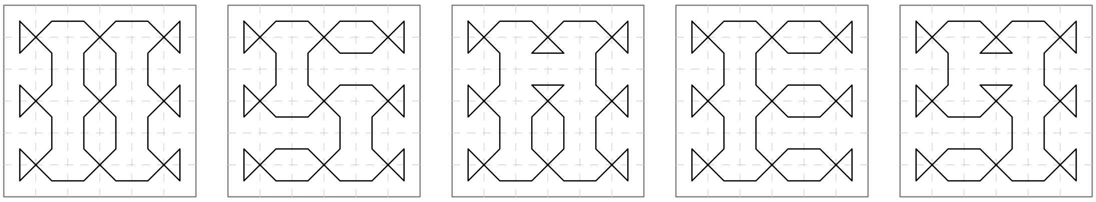
1	14	15	28	29	31
13	2	27	16	32	30
12	26	3	33	17	19
25	11	34	4	20	18
24	35	10	21	5	7
36	23	22	9	8	6



1	35	34	16	17	19
36	2	15	33	20	18
12	14	3	21	32	30
13	11	22	4	29	31
25	23	10	28	5	7
24	26	27	9	8	6

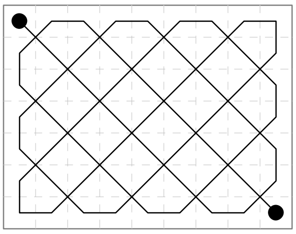


I find 5 alternating solutions, all with different symmetry, including a first asymmetric example.



King Tours 7×n

7×9 Board. This diagram shows an arithmic King Tour in which the ranks sum to all the successive values from 285 to 291 and the files to the successive values from 220 to 228.

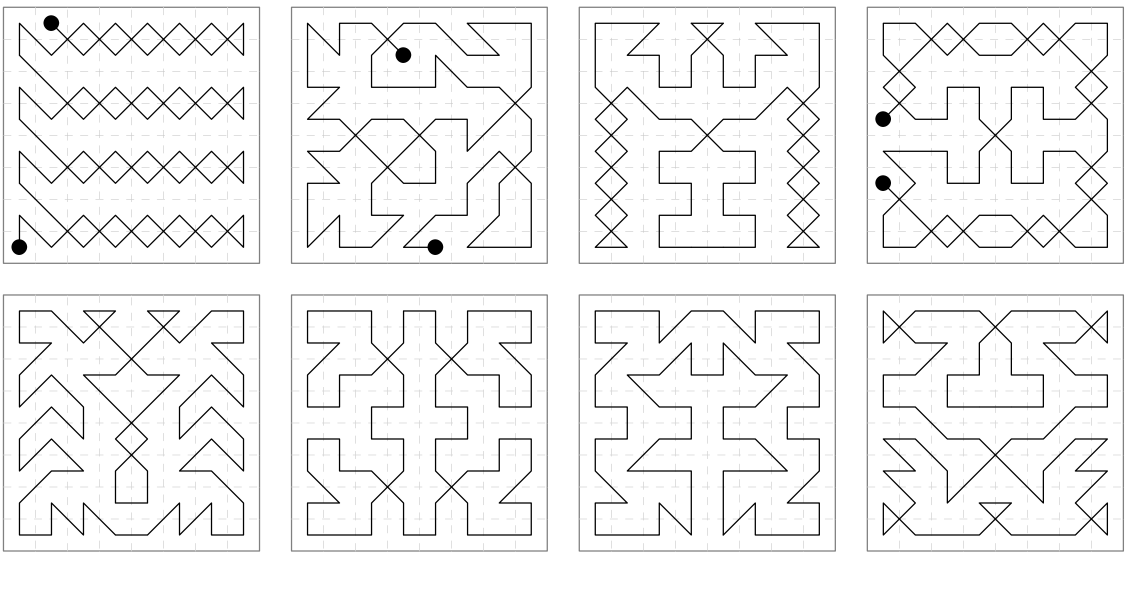


1	56	57	15	14	42	43	29	28
55	2	16	58	41	13	30	44	27
54	17	3	40	59	31	12	26	45
18	53	39	4	32	60	25	11	46
19	38	52	33	5	24	61	47	10
37	20	34	51	23	6	48	62	9
36	35	21	22	50	49	7	8	63

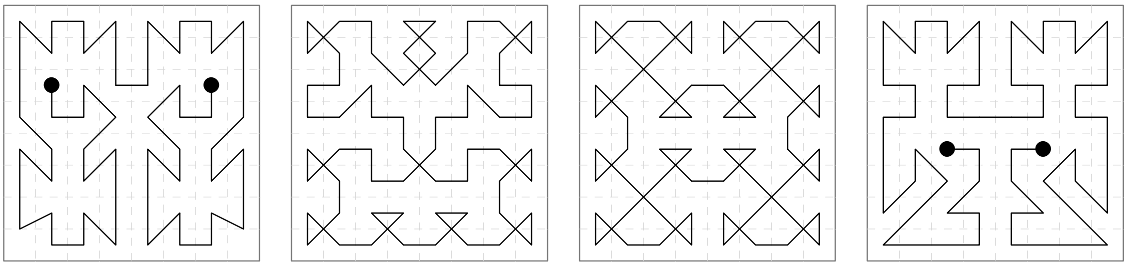
The middle rank and file are naturally magic, consisting of pairs of complements (adding to 64) plus the middle average number 32, giving the required totals of 288 and 224.

King Tours 8×8

The earliest examples of 8×8 king tours I have found appear as solutions to cryptotours in French newspapers of 1876-7. (1) *Le Siecle* ¶34 (15 Dec 1876) open. (2) *Le Gaulois* ¶47 (29 May 1877) an irregular open tour by 'J. Faer'. (3) *Le Siecle* ¶190 (8 Jun 1877) axial tour by M[onsieur] Clerville. (4) *Le Galois* ¶68 (19 Jun 1877) an approximate biaxial tour by 'Mme Celina Fr' (Francony). and (5) ¶82 (4 Jul 1877) axial by 'Aristide Lejuste'. (6) *Le Siecle* ¶232 (27 Jul 1877) biaxial by Clerville. (7) ¶250 (17 Aug 1877) axial by A. Béligne. (8) *Le Gaulois* ¶124 (17 Aug 1877) axial.

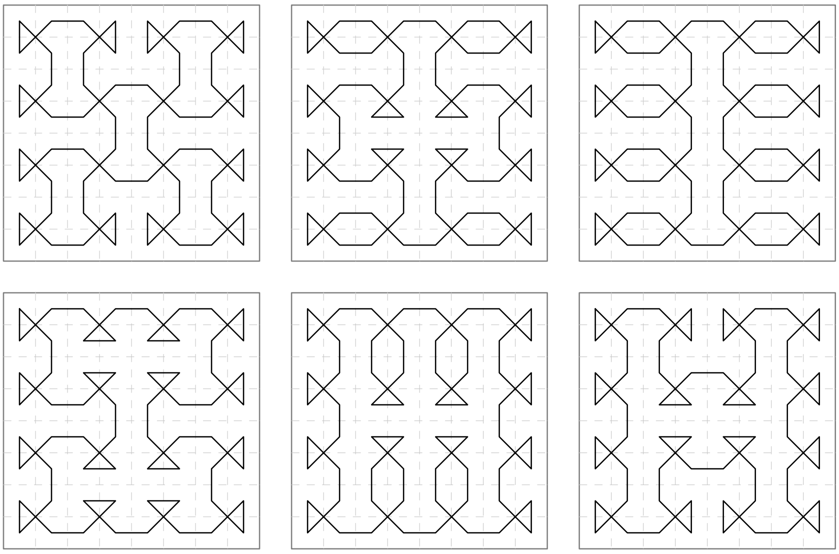


(9) *Le Gaulois* ¶145 (9 Sep 1877) axial open and (10) ¶187 (24 Oct 1877) axial closed. (11) *Le Siecle* ¶310 (26 Oct 1877) biaxial. (12)*Le Gaulois* ¶208 (15 Nov 1877) axial open. In the closed tours I have joined the end-points since their positions seem irrelevant.

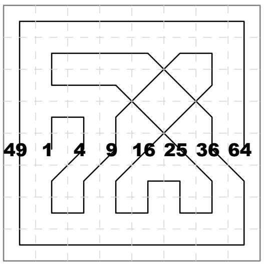


Axial tours are semi-magic in the ranks if the end-points are symmetrically placed, but this was not pointed out in the above problems. Such tours are easy to construct so not detailed further here.

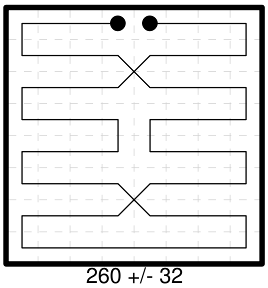
Alternating 8×8 King Tours. Rudrata (c.900) gave an alternating wazir-fers open tour on two ranks, but I have found no others before my own work. On the 8×8 there are over a hundred of these closed tours, but there are only 6 with biaxial symmetry. The first is the first one I found in the 1970s and published in *Chessics* #18 (1984). The last three shown here were among 20 sent to me by Clive Grimstone (30 Apr 1983) calling them ‘ant’ tours. Observation: In closed rectangular tours, the wazir moves at a corner must be followed by a series of wazir moves along the edge, and the same on the opposite edge. This property is not true in open tours.



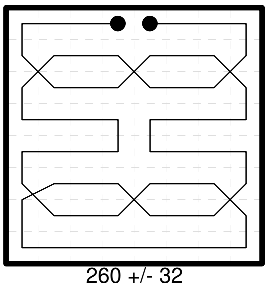
Figured king tour Jeepyjay Diary 12 October 2014. All numbers in sequence is apparently not possible. With the condition 'minimum crossovers' it is probably unique. It includes a 6x6 solution.



Magic 8×8 King Tours. We begin with non-diagonal examples.

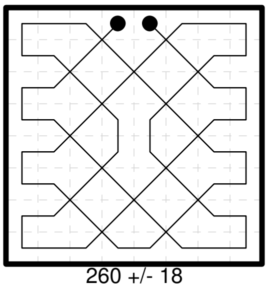


260 +/- 32

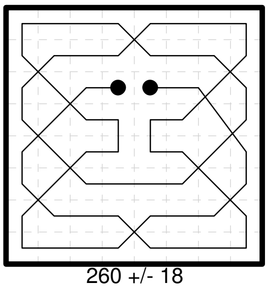


260 +/- 32

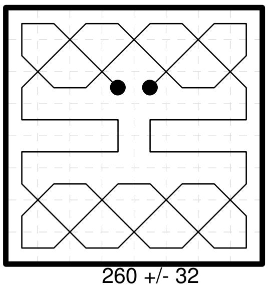
The two Magic king tours above are by M. B. Lehmann (1932). These and the following are king tours of biaxial type that are magic in ranks and files. The deviations from magic in the diagonals are noted. The following are magic king tours of my own construction (Feb 1986).



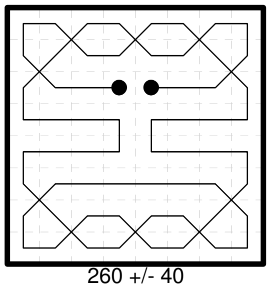
260 +/- 18



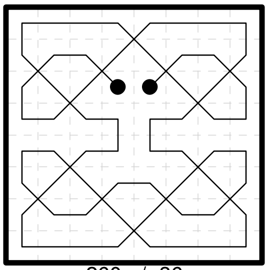
260 +/- 18



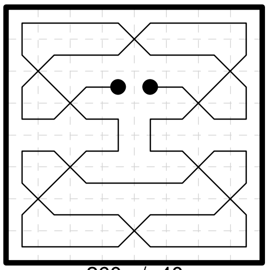
260 +/- 32



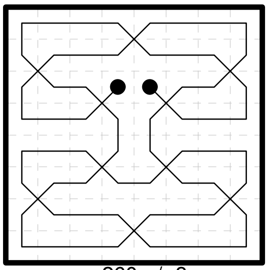
260 +/- 40



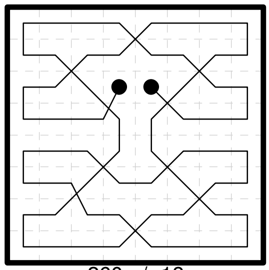
260 +/- 36



260 +/- 40

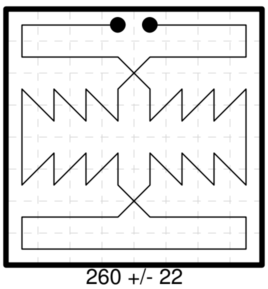


260 +/- 8

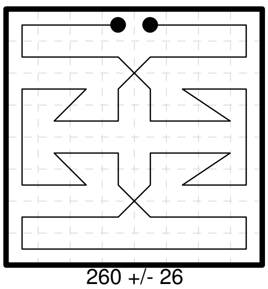


260 +/- 18

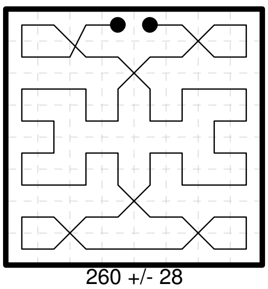
Requiring the diagonals to also add to the magic sum allows less variety in the designs.



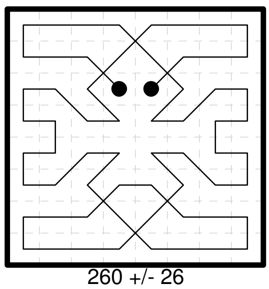
260 +/- 22



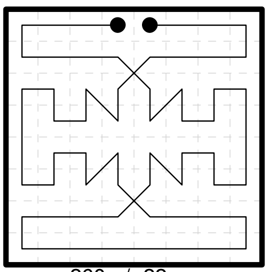
260 +/- 26



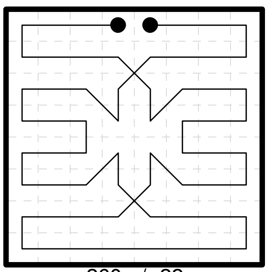
260 +/- 28



260 +/- 26



260 +/- 22



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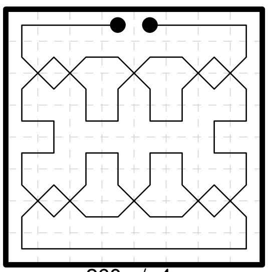
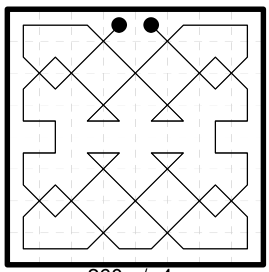
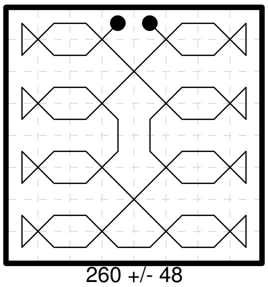


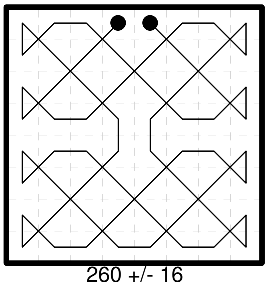
260 +/- 4

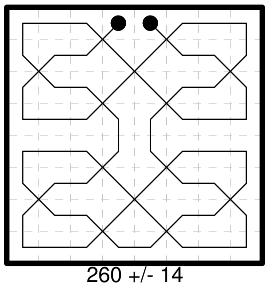


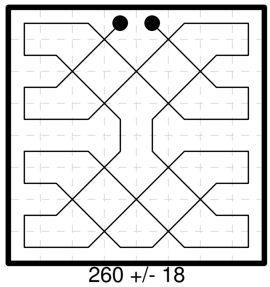
260 +/- 4

More magic king tours of my own construction (March 1986)


260 +/- 48


260 +/- 16


260 +/- 14

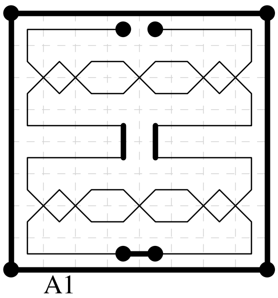

260 +/- 18

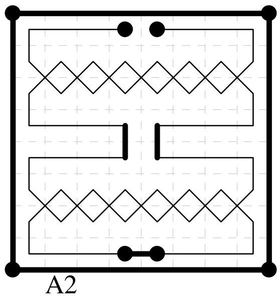
Diagonally Magic 8×8 King Tours: Biaxial Type.

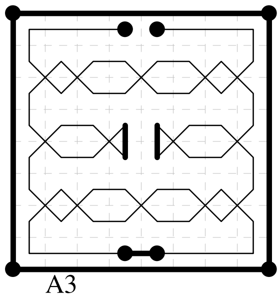
Based on information in Rouse Ball (1917) and Lehmann (1932) I published 16 diagonally magic king tours with biaxial symmetry in *Chessics* in 1986, which Tom Marlow extended to 47 in *The Games and Puzzle Journal* in 1997. However I have now found that this work is anticipated at least in part by tours published in French newspaper puzzle columns from 1882 onwards.

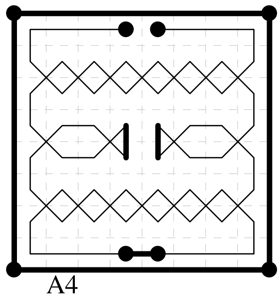
A catalogue of all 47 recorded examples follows, classified according to the numbers in the diagonals and the positioning of the end-cells of the quarters. The dark links are where the quarters join (16-17, 32-33, 48-49).

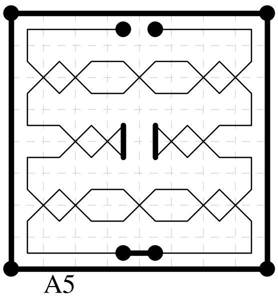
A, B and C Groups. A1 and A2 are two of the four solutions to ¶1021 by 'X à Belfort' (Reuss) in *Gil Blas* 9 Oct 1882 and A7 is one of the four solutions to his ¶1042 (30 Oct 1882). A1 was in Rouse Ball (1917) and the other A patterns can be formed from A1 by interchanges of non-diagonal elements, as indicated by M. B. Lehmann (1932). The diagonal with the smallest number in it is: 4, 54, 55, 49, 17, 23, 22, 36. To find the entries in the other diagonal subtract from 65.

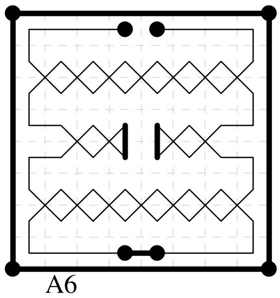

A1

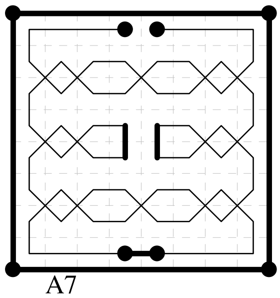

A2

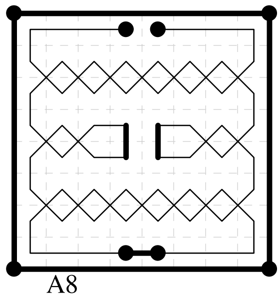

A3

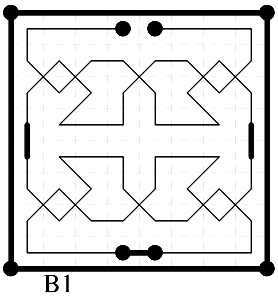

A4

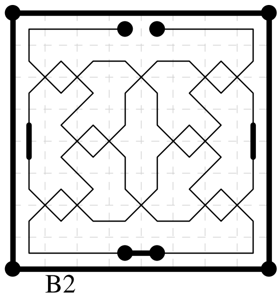

A5

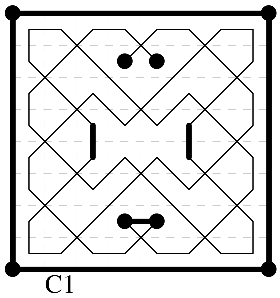

A6

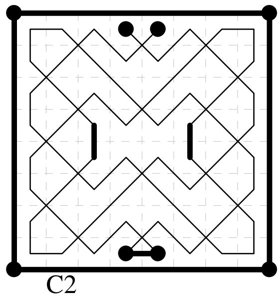

A7


A8


B1


B2

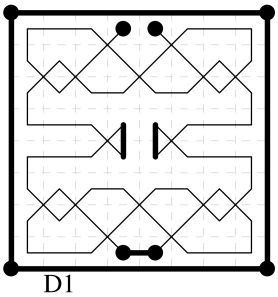

C1

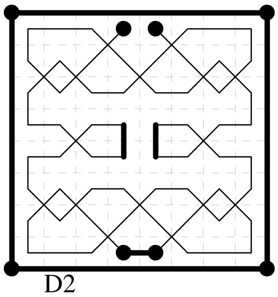

C2

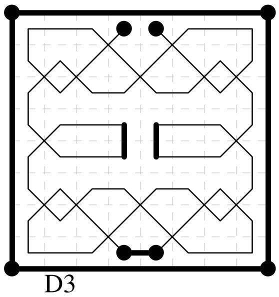
Other biaxial magic tours can be formed by slight variation, for instance by transposing 16-17 and 48-49 in A1, but in these the diagonals are no longer magic.

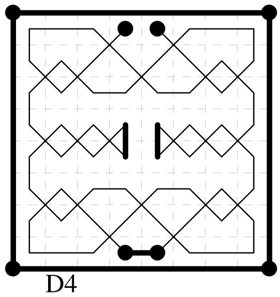
The first B case is due to J. Brügge (1985), and from this I derived the other, the diagonal is: 4, 51, 52, 55, 23, 20, 19, 36. C1 is the second solution in ¶1013 by 'Mme Celina Fr' (Francony) in *Gil Blas* 1/8 Oct 1882, the diagonal in C is: 45, 4, 25, 24, 56, 57, 36, 13.

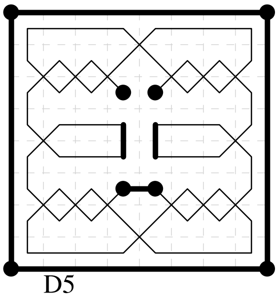
D Group. D2 is one of the four solutions to ¶1042 by 'X à Belfort' (Reuss) in *Gil Blas* (30 Oct/6 Nov 1882). The diagonal is D: 5, 54, 55, 48, 16, 23, 22, 37.

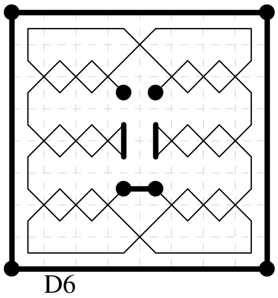

D1

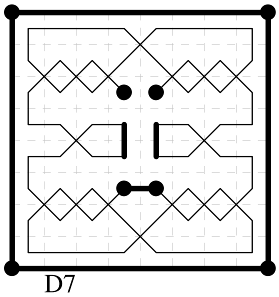

D2

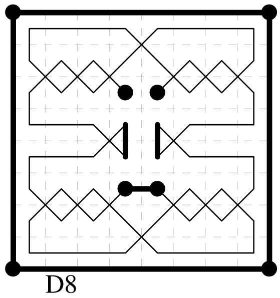

D3


D4

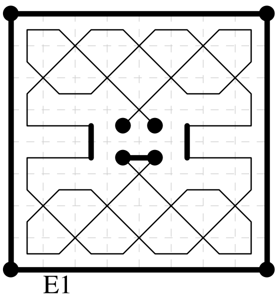

D5

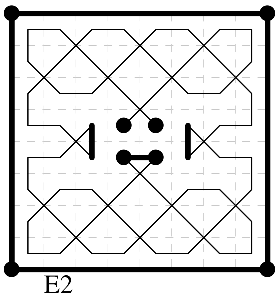

D6

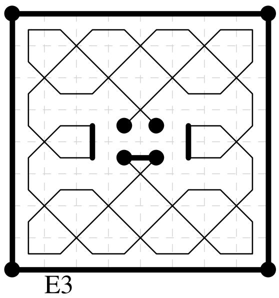

D7

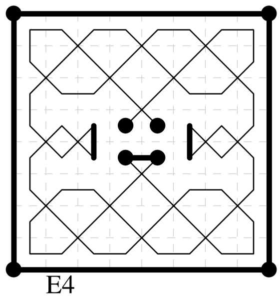

D8

E Group. E1 and E3 are the two other solutions to ¶1021 by Reuss in *Gil Blas* 9/16 Oct 1882. Also E1 and E4 are the solutions to his ¶1064 in *Gil Blas* 21/28 Nov 1882. The complete set is due to Marlow (1997). The E diagonal is: 60, 12, 57, 1, 33, 25, 44, 28.

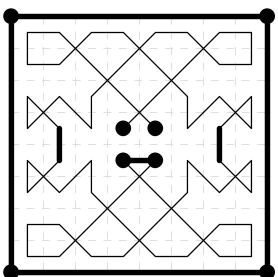

E1

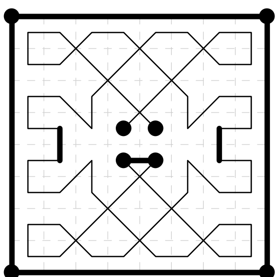

E2

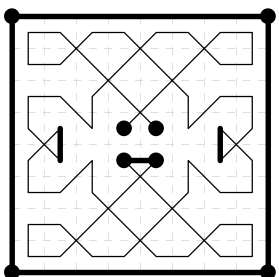

E3

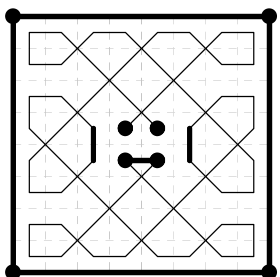

E4

F Group. The diagonal in all is: 60, 58, 11, 1, 33, 43, 26, 28. There are twelve cases with vertical joins inset from the edge.


F1

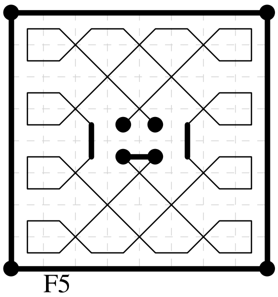

F2

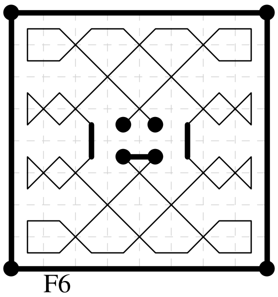

F3

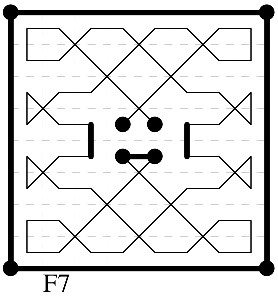

F4

continued

F5 is the solution to ¶999 in *Gil Blas* 17/24 Sep 1882 by ‘X a Belfort’ (Reuss). F12 is the solution to ¶441 in *Le Gaulois* 17/22 Oct 1882 by ‘Adsum’ (Bouvier).


F5


F6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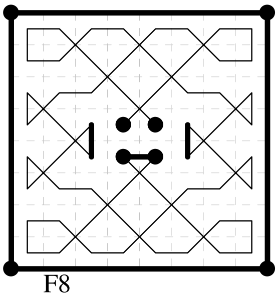
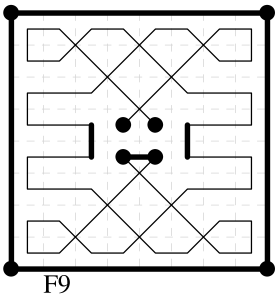
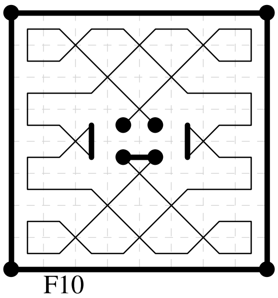
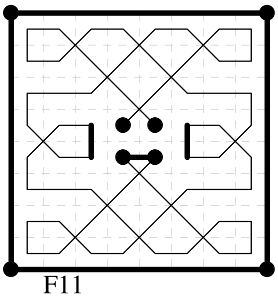
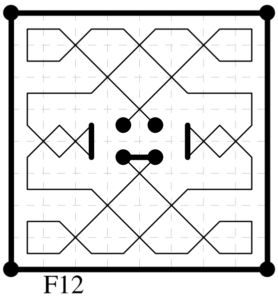

F8

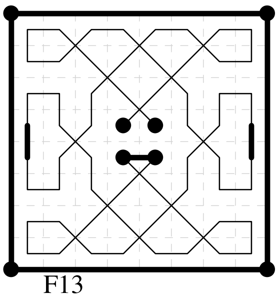

F9

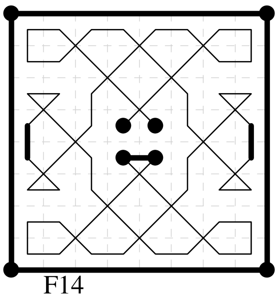

F10

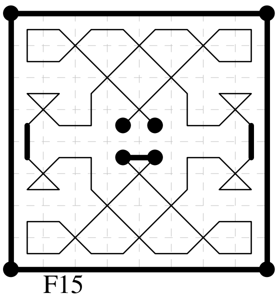

F11

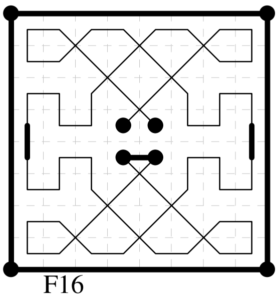

F12

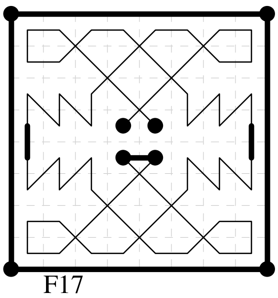
There are eleven cases with the join in the edge. F20 and F21 are two of the four solutions to ¶1042 by ‘X à Belfort’ (Reuss) in *Gil Blas* (30 Oct/6 Nov 1882).

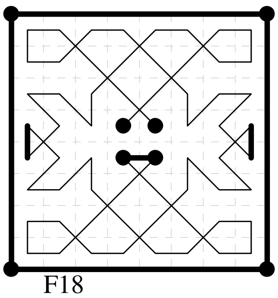

F13

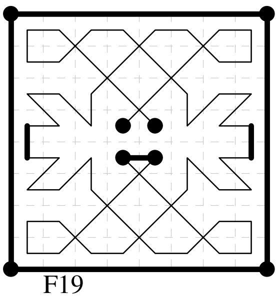

F14


F15


F16


F17


F18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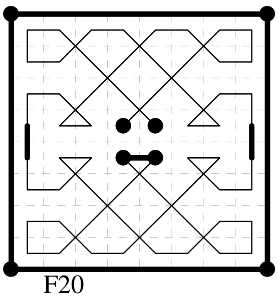
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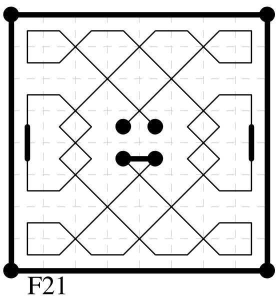
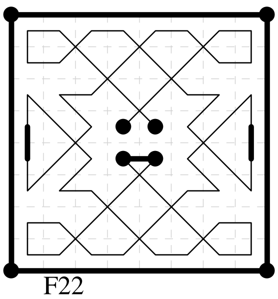
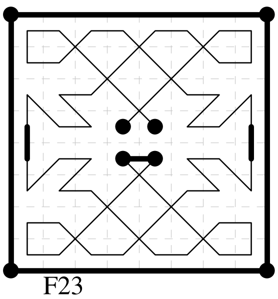

F21

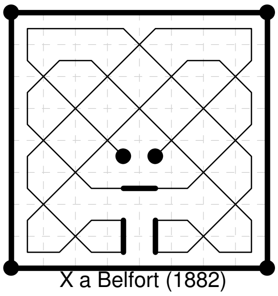

F22

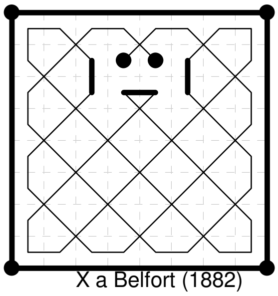

F23

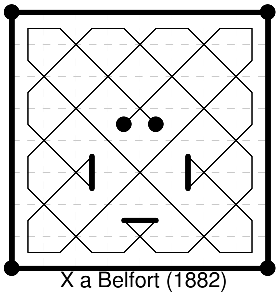
Note: The lettering and numbering of the tours is different from the original 1997 article.

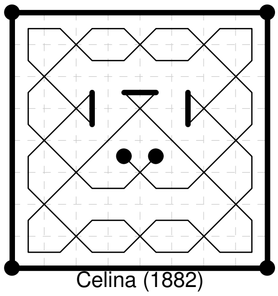
Diagonally Magic 8×8 King Tours: Axial Type.

It is possible to construct diagonally magic king tours that are not biaxial, as in these axial examples from French newspapers of the 1880s. Four from *Gil Blas*, three by 'X a Belfort' (Reuss) ¶992 (10 Sep 1882) and ¶1006 (24 Sep 1882) with two solutions, and one by 'Mme Celina Fr' (Francony) ¶1013 (1 Oct 1882) the second solution to this problem is the biaxial case C1.

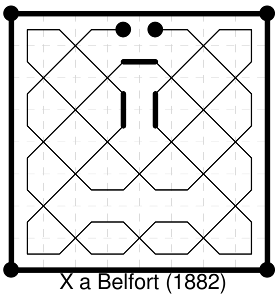

X a Belfort (1882)

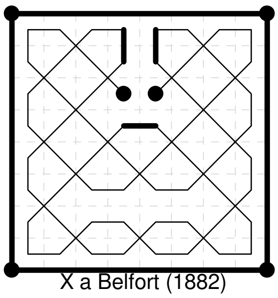

X a Belfort (1882)

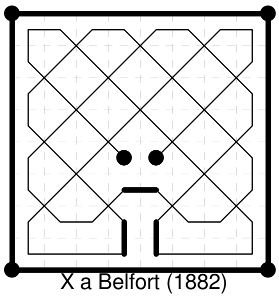

X a Belfort (1882)

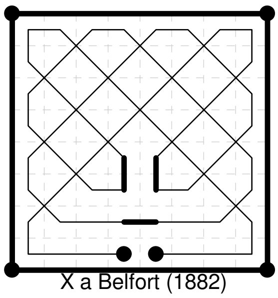

Celina (1882)

From *Gil Blas* four tours by 'X a Belfort' (Reuss) ¶1028 (16 Oct 1882) with four solutions.

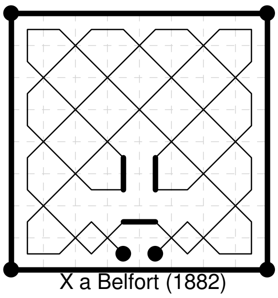

X a Belfort (1882)

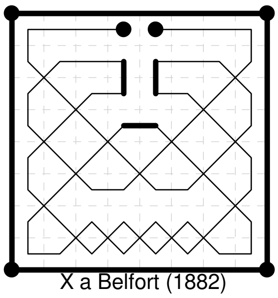

X a Belfort (1882)

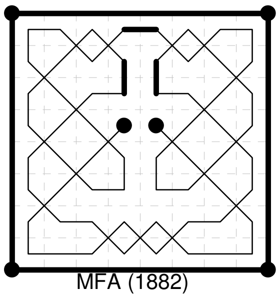

X a Belfort (1882)

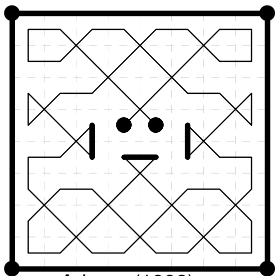

X a Belfort (1882)

Four from *Le Galois*. Two by 'X a Belfort' (Reuss) ¶34 (9/23 Sep 1882) and ¶41 (23 Sep/12 Oct 1882) in the Jeux D'Esprit section. One by by 'MFA' ¶411 (18 Sep 1882) and another by 'Adsum' (Bouvier) ¶466 (13/29 Nov 1882).

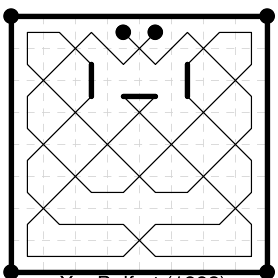

X a Belfort (1882)

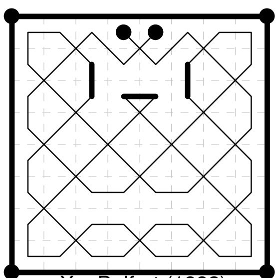

X a Belfort (1882)

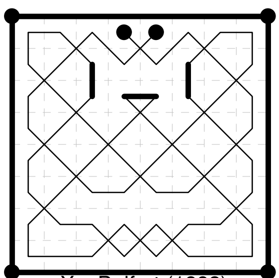

MFA (1882)

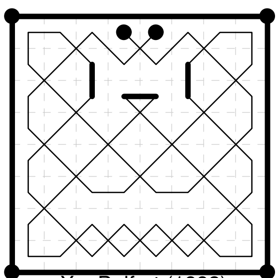

Adsum (1882)

Four by 'X à Belfort' (Reuss) in *Gil Blas* ¶1035 (23 Oct 1882) with four solutions.

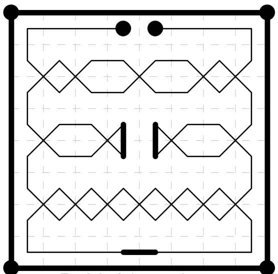

X a Belfort (1882)

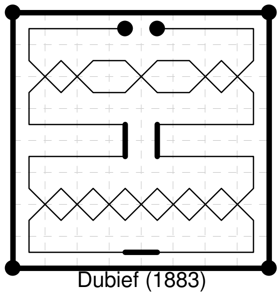

X a Belfort (1882)

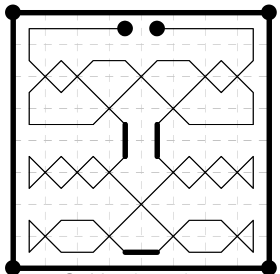

X a Belfort (1882)

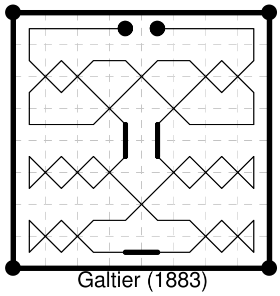

X a Belfort (1882)

The patterns in many of these examples evidently follow to a great extent the same scheme as the arithmic king tours shown earlier. Two magic king tours by R. Dubief ¶85 (15/29 Jan 1883) and two by Galtier ¶106 (26 Feb/12 Mar 1883) in *Le Gaulois* follow.

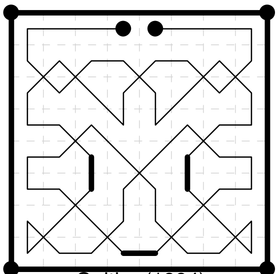

Dubief (1883)

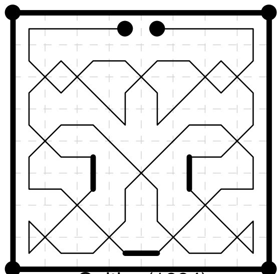

Dubief (1883)

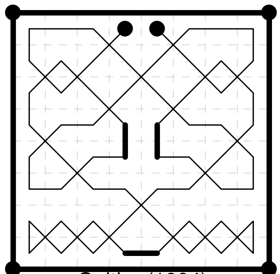

Galtier (1883)

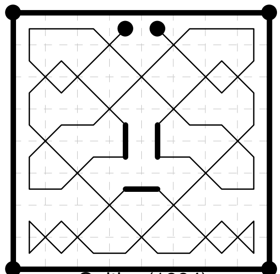

Galtier (1883)

Four more by Galtier from *Le Gaulois* ¶267 (14/28 Jan 1884) and ¶288 (25 Feb/10 Mar 1884) each problem having two solutions.

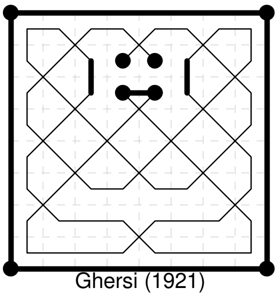

Galtier (1884)


Galtier (1884)


Galtier (1884)

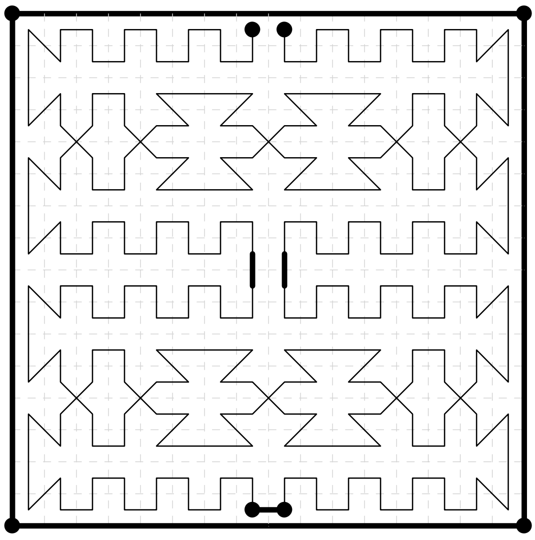

Galtier (1884)

A later example with single axis of symmetry is given by Italo Ghersi (1921).


Ghersi (1921)

Magic King Tours on Larger Boards

Our final example (Jelliss 1986) is a diagonally magic king tour on a 16×16 board, derived from the A1 tour. Many more are of course possible. The magic constant for the 16×16 board is 2056.



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Queen Tours

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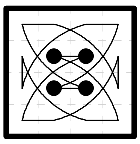
Any piece with lateral and diagonal moves at least one being longer than a king move can be regarded as a restricted queen. We arrange the tours here according to the number of different moves and secondarily by the size of the board used.

Two-Move Queens

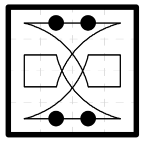
{0,1}{2,2}

Wazir+Alfil. In *Chessics* #26 (p.119, 1986) I noted the following two magic tours (necessarily non-diagonal) by this double-pattern mover. They show biaxial symmetry when the end-points of the path are joined up to give a closed path. The black dots mark 1, 8, 9, 16.

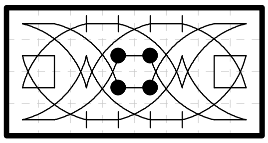
2	3	14	15
12	9	8	5
13	16	1	4
7	6	11	10



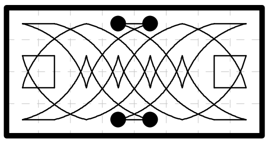
2	1	16	15
12	11	6	5
13	14	3	4
7	8	9	10



Biaxially symmetric 4×8 magic tours. Results from a partial enumeration (18 Oct 2014). The alfil moves are curved to avoid overlapping.



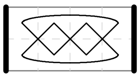
3	2	27	28	5	6	31	30
23	24	13	16	17	20	9	10
26	25	4	1	32	29	8	7
14	15	22	21	12	11	18	19



3	4	27	32	1	6	29	30
23	24	15	12	21	18	9	10
26	25	2	5	28	31	8	7
14	13	22	17	16	11	20	19

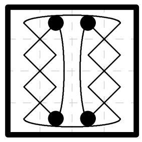
{0,3}{1,1}

Frog. The {1,1} mover is restricted to cells of one colour, and the {0,3} mover to one in nine of the cells on a large board. But the {1,1}+{0,3} mover can reach every cell. The name comes from a leapfrog puzzle by Pearson (1907 part 3 p.62). It is the simplest possible 'amphibian', so the name Frog is appropriate. On the 2×4 board there is one biaxial tour. It is semi-magic. In *Chessics* #26 I noted the following 4×4 magic frog tour (necessarily non-diagonal) showing biaxial symmetry.

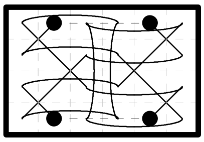


1	6	3	8
5	2	7	4

4	16	9	5
15	3	6	10
2	14	11	7
13	1	8	12

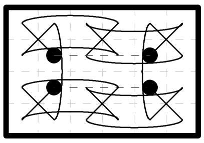


5	13	19	6	12	20
14	4	10	15	21	11
23	9	3	22	16	2
8	24	18	7	1	17



2	18	12	1	19	11
17	3	21	16	10	20
8	22	4	9	15	5
23	7	13	24	6	14

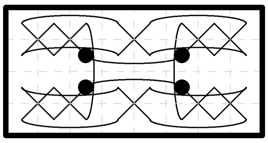
2	19	22	3	6	23
20	1	4	21	24	5
17	12	9	16	13	8
11	18	15	10	7	14



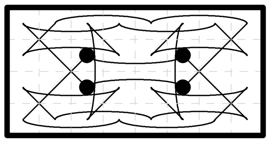
5	12	9	4	1	8
11	6	3	10	7	2
14	19	22	15	18	23
20	13	16	21	24	17

Two 4×6 biaxial magic Frog tours that I found 10 Aug 1991.

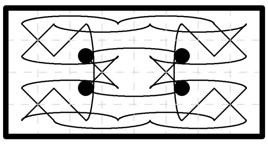
File sum 50. Rank sum 75. The first is also given by Trenkler (1999). Besides the magic numberings they each have a quasi-magic numbering, the rank sums of the first being 63 and 87, and the second 39 and 111. The {0,3} moves are curved or dashed to avoid overlap. These were among a series of 29 by two-pattern movers that I found. The same is possible on the 4×8 board.



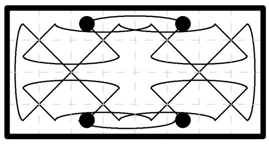
6	31	8	5	28	25	2	27
30	7	32	29	4	1	26	3
19	10	17	20	13	16	23	14
11	18	9	12	21	24	15	22



3	28	25	4	29	8	5	30
27	2	16	26	7	17	31	6
22	15	1	23	10	32	18	11
14	21	24	13	20	9	12	19



6	29	8	5	28	25	4	27
30	7	17	31	2	16	26	3
19	10	32	18	15	1	23	14
11	20	9	12	21	24	13	22

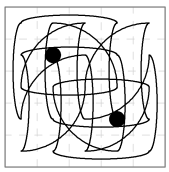


8	20	17	12	21	16	13	25
19	7	11	18	15	22	26	14
30	10	6	31	2	27	23	3
9	29	32	5	28	1	4	24

{0,3}{2,2}

Threeleaper + Alfil. A 5×5 (non-magic) symmetric open tour by another amphibian.

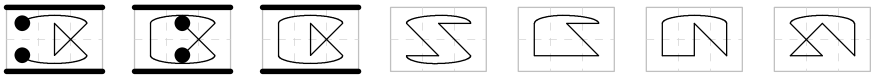
3	17	20	2	14
11	25	5	10	22
19	8	13	18	7
4	16	21	1	15
12	24	6	9	23

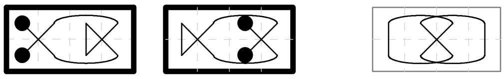


Three-Move Queens

{0,1}{0,2}{1,1}

King + Dabbaba. On the 2×3 board besides the two {0,1}{0,2} wazaba tours there are two other semi-magic queen tours. When closed these open tours give only one geometrically distinct axially symmetric closed tour. There are also four other closed tours. One rotary and three asymmetric.



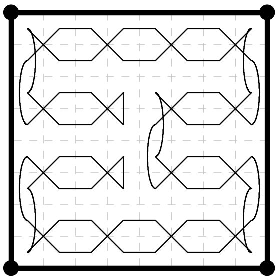


1	6	2	5
8	3	7	4

3	6	2	7
4	1	5	8

On the 2×4 board by permuting the files of the magic king tour we can form two magic queen tours. The numbers in the ranks are 1,4,6,7 and 8,5,3,2 which are the only sets of four adding to 18 that do not consist of two complementary pairs adding to 9, which are needed for the files. There is also a biaxial tour 2×4 tour, which can be numbered to be semi-magic in either ranks or files.

Magic 8×8 tour Jelliss (2001) derived from the {0,1}{0,2}{1,2} tour (see ⌘ 10) by Wallis (1908) by interchanging 2-3 and 6-7 files which transforms knight {1,2} moves to fers {1,1} moves.

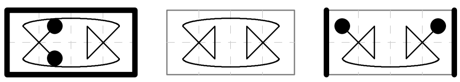


5	59	58	8	9	55	54	12
60	6	7	57	56	10	11	53
4	62	63	1	16	50	51	13
61	3	2	64	49	15	14	52
29	35	34	32	17	47	46	20
36	30	31	33	48	18	19	45
28	38	39	25	24	42	43	21
37	27	26	40	41	23	22	44

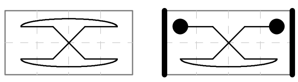
{0,1)(0,3}{1,1}

King + Threeleaper. By permuting the files of the 2×4 magic king tour we can form a magic biaxial tour by this piece when numbered about the short axis. About the long axis it is only magic in the ranks. There is also another biaxial tour which is semi-magic.

2	8	5	3
7	1	4	6



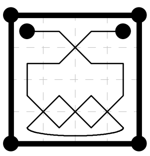
1	3	6	8
4	2	7	5



1	2	7	8
5	6	3	4

One of the 4×4 magic squares (#2 in the Frénicle list) that is a diamagic queen tour.

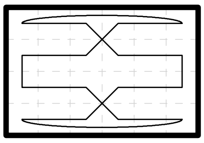
1	2	15	16
13	14	3	4
12	7	10	5
8	11	6	9



{0,1}{0,5}{1,1}

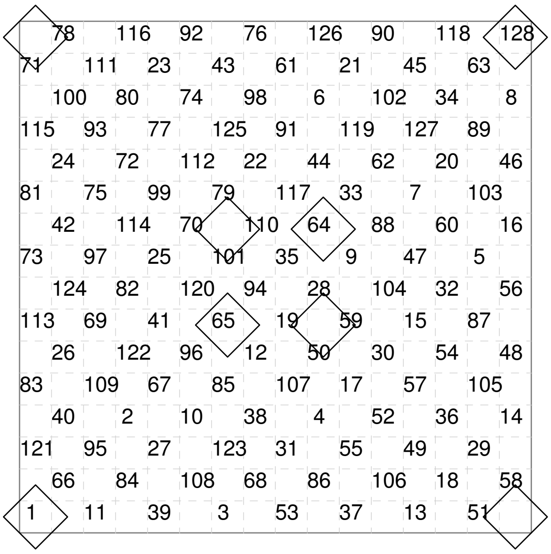
King + Fivemover. Magic biaxial rectangle tour 4×6 by L. S. Frierson in Andrews (1917, Fig.272 reflected). Sums 50 and 75.

1	2	3	22	23	24
19	20	21	4	5	6
18	17	16	9	8	7
12	11	10	15	14	13



{0,15}{3,3}{5,5}

Pterodactyl. This is the smallest three-pattern amphibian. Half of a closed 16×16 tour (Jelliss *Chessics* #24 1985 p.98) is shown. This half-tour has 180 degree rotary symmetry, the middle move 64-65 passing through the centre. The second half begins with 129 on the top left or bottom right corner, and reflects the first half, so the tour overall has axial symmetry.



The name pterodactyl was proposed because of the spiky nature of the tour.

The middle move of the second half is 192-193. Diametrally opposite cells add to 129 or to 385, while axially opposite cells add to 257. The 128-129 move and the closure move 256-1 are the only {0,15} moves used.

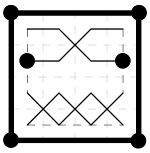
Four-Move Queens

{0,1}{0,2}{0,3}{1,1}

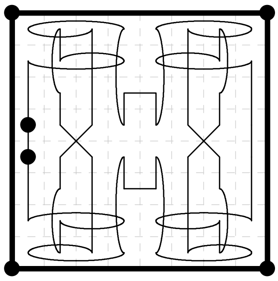
One of the 4×4 diagonally magic squares (#619 in the Frénicle list) is a magic queen tour. It has axial symmetry. The longer moves of two or three steps are shown here by broken lines.

Diagonally magic 8×8 tour by four-move queen (Jelliss 26 Feb 1986). The files consist of pairs of complements adding to 65. The ranks are formed of pairs adding to 97 and 33. The {0,2} and {0,3} moves are shown curved, to avoid overlapping.

4	3	14	13
16	15	2	1
5	10	7	12
9	6	11	8



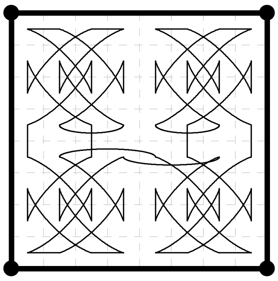
13	59	12	14	19	21	38	20
62	60	11	61	36	22	37	35
63	58	10	16	17	23	39	34
64	57	9	15	18	24	40	33
1	8	56	50	47	41	25	32
2	7	55	49	48	42	26	31
3	5	54	4	29	43	28	30
52	6	53	51	46	44	27	45



{0,1}{0,2}{0,3}{2,2}

Jelliss (2001) diamagic queen alternating rook-bishop tour derived from Wallis (1908) alternating knight-rook tour (see ⌘ 10) by interchange of the 2-3 and 6-7 ranks, and rotation 90° to conform to the Frénicle convention.

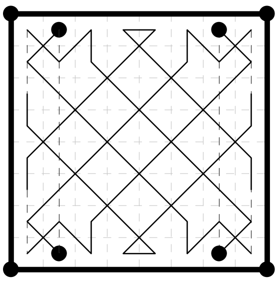
5	4	60	61	29	28	36	37
58	63	7	2	34	39	31	26
9	62	6	3	35	38	30	27
8	1	57	64	32	25	33	40
9	16	56	49	17	24	48	41
54	51	11	14	46	43	19	22
55	50	10	15	47	42	18	23
12	13	53	52	20	21	45	44

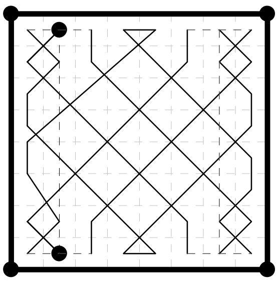


{0,1}{0,2}{0,7}{1,1}

Two diamagic four-move queen tours (Jelliss *Chessics* #26 p.120 1986) showing mainly diagonal moves. The files consist of pairs of complements adding to 65. The ranks are formed of pairs adding to 33 and 97. The lateral moves {0,2} and {0,7} are shown by dashed lines. The {0,7} moves form the middle link 32-33 and the closure link 64-1.

23	64	25	49	48	8	33	10
63	24	26	47	50	7	9	34
22	62	46	27	6	51	35	11
21	45	61	5	28	36	52	12
44	20	4	60	37	29	13	53
43	3	19	38	59	14	30	54
2	41	39	18	15	58	56	31
42	1	40	16	17	57	32	55



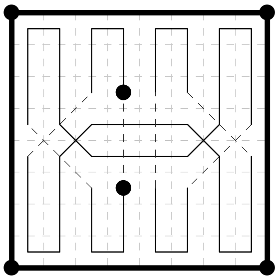


The second version transposes the two numbers at the end of each diagonal, e.g. 9-10 to 10-9.

{0,1}{0,3}{1,1}{2,2}

Another diagonally magic four-move queen tour (Jelliss, *Chessics* #26, p.120, 1986) showing mainly lateral moves (1-3-rook, 1-2-bishop). The {2,2} and {0,3}moves are shown by dashed lines. The files consist of pairs of complements adding to 65. The ranks are formed of pairs adding to 97 and 33. The {0,3} moves are the middle 32-33 and closure 64-1.

10	11	61	62	35	36	22	23
9	12	60	63	34	37	21	24
8	13	59	64	33	38	20	25
7	14	50	49	48	47	19	26
58	51	15	16	17	18	46	39
57	52	6	1	32	27	45	40
56	53	5	2	31	28	44	41
55	54	4	3	30	29	43	42

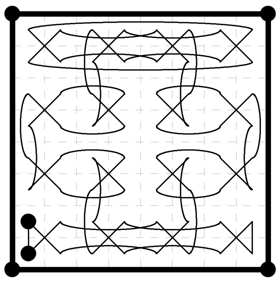


Five-Move Queens

{0,1}{0,2}{0,3}{0,7}{1,1}

Diamagic queen tour using 5 move types (R1237 + B1) by J. Brügge *Die Schwalbe* Aug 1985. In the diagram the three types of longer rook moves are curved.

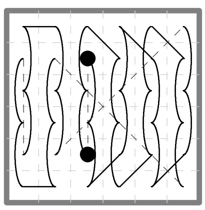
16	50	20	46	51	13	47	17
49	15	45	19	14	52	18	48
24	42	21	43	54	12	55	9
41	23	44	22	11	53	10	56
25	39	28	38	59	5	58	8
40	26	37	27	6	60	7	57
1	63	29	35	62	4	34	32
64	2	36	30	3	61	31	33



{0,1}{0,2}{0,3}{1,1}{4,4}

Magic 6×6 queen tour (Jelliss 16 Mar 1986) with diagonals 60 and 162 (that is 111±51).

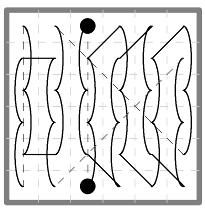
21	22	3	31	9	25
18	13	36	4	30	10
20	23	2	32	8	26
17	14	35	5	29	11
19	24	1	33	7	27
16	15	34	6	28	12



{0,1}{0,2}{0,5}{1,1}{4,4}

Magic 6×6 queen tour (Jelliss 16 Mar 1986) with diagonals 60 and 162 (that is 111±51).

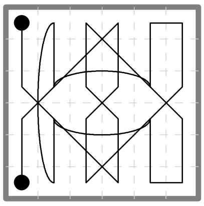
19	24	1	33	7	27
16	15	34	6	28	12
20	23	2	32	8	26
17	14	35	5	29	11
21	22	3	31	9	25
18	13	36	4	30	10



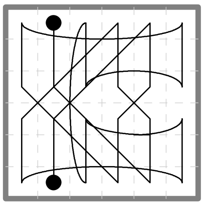
{0,1}{0,3}{0,5}{1,1}{3,3}

Magic 6×6 queen tour (Jelliss 16 Mar 1986) with diagonals 60 and 162 (that is 111±51) and with diagonals 96 and 126 (that is 111±15).

1	19	9	33	24	25
2	20	8	32	23	26
3	21	7	31	22	27
34	16	30	6	15	10
35	17	29	5	14	11
36	18	28	4	13	12



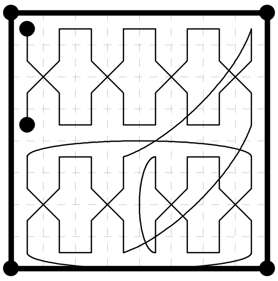
25	1	19	9	33	24
26	2	20	8	32	23
27	3	21	7	31	22
10	34	16	30	6	15
11	35	17	29	5	14
12	36	18	28	4	13



{0,1}{0,3}{0,7}{1,1}{4,4}

Diamagic queen tour with 5 move types by L. S. Frierson in Andrews *Magic Squares and Cubes* (1917) Fig.268, consists mainly of 8 repetitions of an 8-cell king-path.

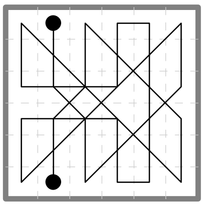
1	61	60	8	9	53	52	16
2	62	59	7	10	54	51	15
63	3	6	58	55	11	14	50
64	4	5	57	56	12	13	49
24	44	45	17	32	36	37	25
23	43	46	18	31	35	38	26
42	22	19	47	34	30	27	39
41	21	20	48	33	29	28	40



{0,1}{0,3}{1,1}{3,3}{5,5}

Magic 6×6 queen tour (Jelliss 16 Mar 1986) with diagonals 102 and 120 (that is 111±9).

27	1	31	22	21	9
26	2	32	23	20	8
25	3	33	24	19	7
12	34	4	13	18	30
11	35	5	14	17	29
10	36	6	15	16	28

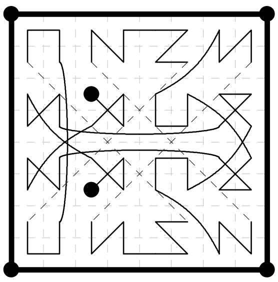


This is one of a series of 10 magic queen tours I constructed on the above date using five or six move types. This has {5,5} moves along the diagonals, crossing at the centre.

{0,1}{0,5}{1,1}{2,2}{5,5}

Diamagic queen tour using 5 move types (6 including the closure move) by J. Brügge.

35	34	22	24	25	26	48	46
36	33	21	23	27	28	47	45
5	7	64	62	49	52	11	10
6	8	61	63	50	51	9	12
59	57	4	2	15	14	56	53
60	58	1	3	16	13	54	55
29	32	44	42	38	37	18	20
30	31	43	41	40	39	17	19



Die Schwalbe Aug 1985.The {5,5}moves are dashed. The {2,2} and {0,5} moves are curved.

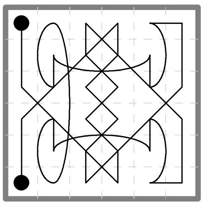
More-Move Queens

Six Moves

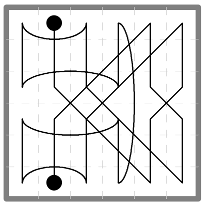
{0,1}{0,2}{0,3}{0,5}{1,1}{3,3}

Magic 6×6 queen tours (Jelliss 16 Mar 1986) with diagonals 110 and 112 (that is 111±1) and diagonals 102 and 120 (that is 111±9) .

1	19	9	33	24	25
2	21	8	32	22	26
3	20	31	7	23	27
34	17	6	30	14	10
35	16	29	5	15	11
36	18	28	4	13	12



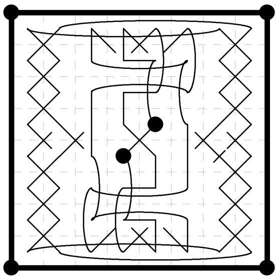
24	1	25	19	33	9
23	2	26	20	32	8
22	3	27	21	31	7
15	34	10	16	6	30
14	35	11	17	5	29
13	36	12	18	4	28



{0,1}{0,2}{0,3}{0,6}{1,1}{4,4}

An 8×8 diamagic queen tour with 6 move types by Joachim Brugge *Die Schwalbe* Aug 1985. Diametrally opposite cells add to 65 (i.e. Bergholtian symmetry). Four pairs of numbers in the files add to 33 and 97, and in the ranks to 49 and 81.

41	8	20	61	29	52	40	9
7	42	19	62	63	18	10	39
43	6	50	31	30	51	38	11
5	44	49	32	64	17	12	37
28	53	48	1	33	16	21	60
54	27	14	35	34	15	59	22
26	55	47	2	3	46	23	58
56	25	13	36	4	45	57	24

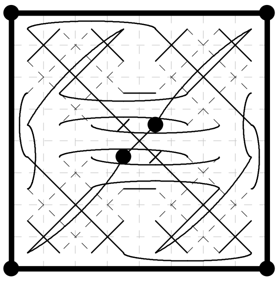


In the diagram only the end sections of the long diagonal moves are shown.

{0,1}{0,2}{0,4}{1,1}{2,2}{3,3}

8×8 diamagic queen tour by Joachim Brügge *Die Schwalbe* 1985, based on Durer's 4×4 square.

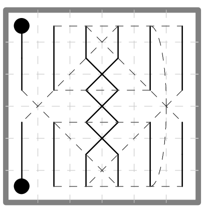
29	50	42	7	28	55	47	2
51	30	6	43	54	27	3	46
41	5	31	49	48	4	26	56
8	44	52	32	1	45	53	25
40	12	20	64	33	13	21	57
9	39	61	17	16	34	60	24
19	62	38	11	22	59	35	14
62	18	10	37	58	23	15	36



{0,1}{0,3}{0,4}{0,5}{1,1}{3,3}

Magic 6×6 queen tour (Jelliss 16 Mar 1986) with diagonals 114 and 108 (that is 111±3).

1	24	9	33	19	25
2	23	8	32	20	26
3	22	31	7	21	27
34	15	6	30	16	10
35	14	29	5	17	11
36	13	28	4	18	12

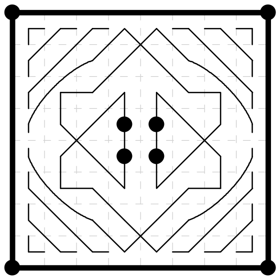


Seven Moves

{0,1}{0,2}{0,4}{0,6}{0,7}{1,1}{2,2}

This 8×8 diamagic queen tour with 7 move types is a slight modification of a non-diamagic tour given by J. Brugge (*Die Schwalbe* Aug 1985), it is based on spirals (Jelliss *Chessics* #30 p.163 1987).

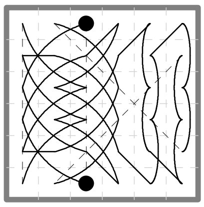
17	46	52	8	25	45	51	16
47	53	9	26	7	24	44	50
54	28	27	34	63	6	5	43
10	29	35	33	64	62	4	23
55	36	30	32	1	3	61	42
11	37	38	31	2	59	60	22
18	12	56	39	58	41	21	15
48	19	13	57	40	20	14	49



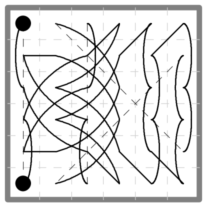
{0,1}{0,2}{0,5}{1,1}{2,2}{3,3}{4,4}

Magic 6×6 queen tours (Jelliss 16 Mar 1986) with diagonals 87 and 135 (that is 111±24) and diagonals 69 and 153 (that is 111±42).

19	24	1	33	7	27
16	15	34	6	28	12
2	32	20	23	8	26
35	5	17	14	29	11
21	22	3	31	9	25
18	13	36	4	30	10



1	24	19	33	7	27
16	15	34	6	28	12
2	32	20	23	8	26
35	5	17	14	29	11
21	22	3	31	9	25
36	13	18	4	30	10

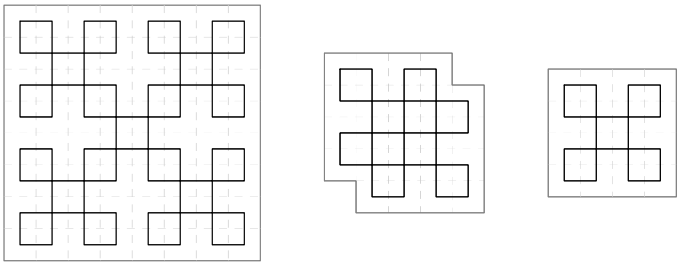


Rider and Hopper Tours

If we regard a cell passed through in two different directions as ‘visited’ then we find some distinctive tours by a ‘rider’, such as rook or queen, become possible. These are not strictly speaking ‘tours’ in the same sense as knight or king tours.

Rook Crossover Tours

A group of assorted games pieces of distinctive design were discovered in the bay of Uig in the Isle of Lewis in Scotland in 1831 and dated to the 12th century.

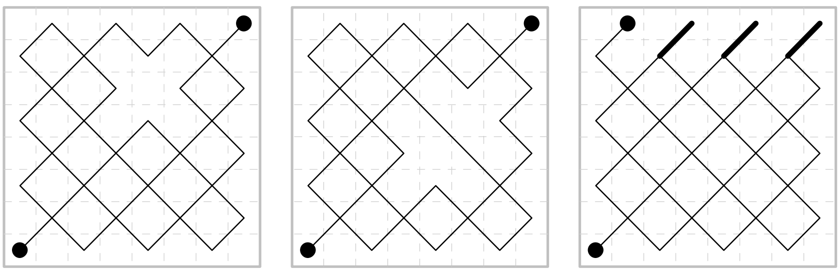


In an illustration to an article by Rodolfo Pozzi (1998, Fig 11), copied to me by Mike Pennell, the back of one of the Lewis chess pieces, a queen, is decorated with the pattern of a rook 8×8 crossover tour. This pattern, rotated 45°, is shown in Celtic fashion as a ribbon going alternately over and under itself, the bends being rounded rather than angular. A 4×4 version also occurs on the chair of a king. A similar design based on a 5×5 board with two corner cells omitted is found, rotated 45°, on Tuvinian rook and king pieces described by Pozzi.

To see this 8×8 pattern was a surprise to me since it was one of my earliest tour discoveries (noted as made on 23 Jul 1973) and published in *Chessics* (#10 p.7 1980 and #11 p.11 1981) as a ‘clockwork mouse tour’. The mouse (otherwise known as a rotating directed wazir) faces in a definite direction (North, South, East or West) and moves one step at a time directly forward, or turns where it stands to face 90° to the right or left. This rotation counts as a move. These moves can be shown simply by stating the new direction that the piece faces after the turn. The puzzle was to place a clockwork mouse anywhere on the chessboard, wind it up and allow it to wander round the board in such a way that it spends the same amount of time (i.e. number of moves) in every cell, and ends up where it started (facing in the same direction). Surprisingly, the path is uniquely determinate!

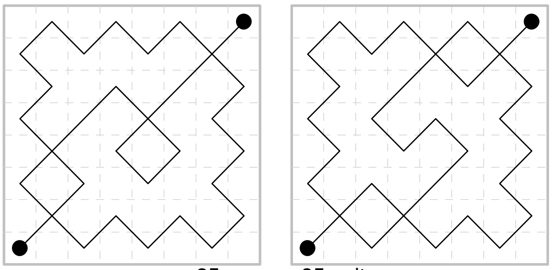
Bishop Crossover Tours

The Bishop requires 17 moves to tour the board without switchbacks. The first solution is by H. E. Dudeney (*Tribune* Dec 1906) and uses 45 bishop steps. The second is by Siep Korteling (*Games and Puzzles Journal* online #31) and uses 43 bishop steps, so is shorter than the Dudeney by geometrical length. Siep also found a solution in 16 moves if switchbacks are allowed, but the geometric length is 49 steps.



17 moves, 45 units | 17moves, 43 units | 16 moves, 49 units

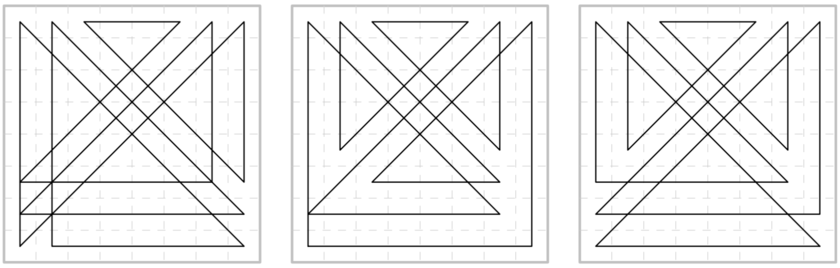
The solutions below use 25 moves but reduce the length to 35 units. The first is by G. Jelliss (*Games and Puzzles Journal* online #31) but I'm not sure of the source of the symmetric solution.



25 moves, 35 units

Queen Crossover Tours

The 14-move ‘Queen Tour’ problem was presented by Sam Loyd in *Le Sphinx* March 1867 and has appeared in numerous places since: I'm not sure if Loyd gave all three solutions as were shown by T. R. Dawson (Problem 194 in his Echecs Feeriques column in *L'Echiquier* Dec 1930 p1149 and 1249). The third solution is the best in my view since it has no K-type junctions which are both visited and passed through.



A version in *Sam Loyd's Puzzle Magazine*, April 1908 (#46 in *Mathematical Puzzles of Sam Loyd* by Martin Gardner, Dover 1959) starts the piece in the middle of a side (d1 or a5) in which case to pass through all cells in 14 straight moves and return to the start is impossible in queen moves. The solution in 14 moves given uses two nightrider lines, and one cell is visited twice.

The simple 15-move non-intersecting open queen tour from c6 to f3 was used in a puzzle by H. E. Dudeney (in *The Tribune* 3 Oct 1906, and later as problem 70 in *The Canterbury Puzzles*).

Bouncer

The Bouncer, invented by Peter Wong, moves along Queen lines until it reaches an edge of the board it then bounces straight back along the same line to twice the distance, measured from the centre of an imagined cell beyond the board edge. (It can also bounce off pieces, which makes it a sort of hopper, but that property is not used here.) The cells cf36 cannot be reached.

The 60-cell tour problem was proposed by Peter Wong (*Variant Chess* #3 Jul-Sep 1990 p.31. Solutions in #4 Oct-Dec 1990 p.47). The Wong tour is asymmetric and uses only three diagonal moves (12-13, 48-49, 60-1). The Jelliss tour is quatersymmetric and uses 16 diagonal moves.

41	48	54	13	34	55	27	20
42	43	49	14	35	56	28	21
9	12		1	8		5	4
37	44	50	15	36	57	29	22
40	47	53	18	33	60	26	19
10	11		2	7		6	3
39	46	52	17	32	59	25	24
38	45	51	16	31	58	30	23

Wong

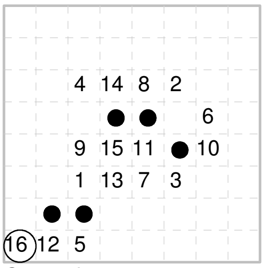
59	12	48	13	39	21	15	14
60	7	49	26	40	16	22	27
6	1		2	5		4	3
24	25	50	8	23	17	41	28
58	11	47	53	38	20	55	54
33	34		35	32		31	36
57	52	46	10	56	19	37	30
44	45	51	9	43	18	42	29

Jelliss

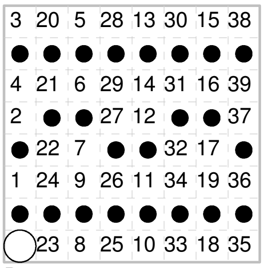
Grasshopper

The Grasshopper moves along queen lines to the cell beyond a hurdle. F. Cassani (*Die Schwalbe* Aug 1931, cited in *Chessics* #15 p.7 1983) shows 16 maximumming moves over five fixed hurdles. The G starts and ends at a1 T. R. Dawson (*FCR* Aug 1950 prob 8755) gave a grasshopper 39-move open tour over 24 fixed hurdles. The G starts at a1 and ends at h6. Each move, including the long move 20, being the mimimum possible to an unused vacant cell. This problem was set by Dawson 23 years previously in *Chess Amateur* (Jan 1927) but he wrote in 1950: "The reason I did not publish the solution in *CA* is because I solved it only a month ago". The stipulation of the problem asked for the "fewest stops ... no stop being used solely to fill a square".

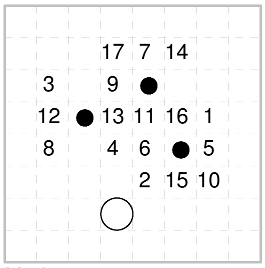
T. W. Marlow ('Grasshopper Gallery' *Chessics* 1983 #15 p.6-7 prob 567, solutions in #16 p.5) proposed the task of placing 2 to 5 hurdles so the G can visit as many cells as possible in as few moves as possible. With 2 blocks he finds 6 moves. However his other solutions allow cells to be visited twice. Without this 3 blocks allow 17 moves, 4 blocks allow 23 moves.



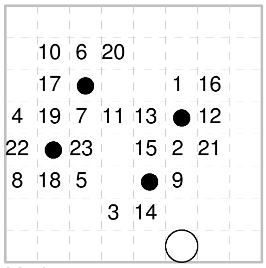
Cassani 1831



Dawson 1950



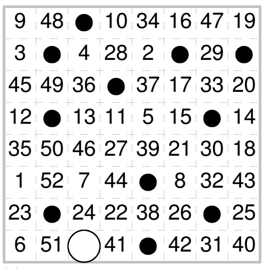
Marlow 1983



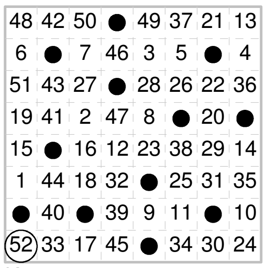
Marlow 1983

A similar problem but with the condition that each hurdle be used once only was proposed by G. Leathem (*FCR* Dec 1939 prob 3976). A complete tour of the 8×8 board under these conditions is impossible, since the number of cells used must be odd (initial cell of grasshopper and two for each hop). Dawson found 31-move solutions omitting one cell (*FCR* Feb 1940 and 1945), and also two closed solutions (which omit two cells). These are not shown here.

Vaclav Kotesovec *Dual-free Leaper and Hopper Tours*, Prague 2009 gives new results (reviewed by John Beasley in 'Grasshopper Tours' *Variant Chess*: #61 Jul 2009 p.115). For the Dawson task, without the minimummer condition, he finds a 52-move open tour over 11 hurdles and a 52-move closed tour over 12 hurdles. These are claimed to be computer proven maxima.



Kotesovec 2009



Kotesovec 2009

Grasshopper over Knight. The problem of a grasshopper tour over a knight as moving hurdle was raised by S. H. Hall (*FCR* 1938 prob 3107). He achieved a 61-cell open path (shown below). A 64-cell tour was considered impossible, but the problem was finally solved 50 years later by T. H. Willcocks (*Chessics* #23 1985 p.84) with two open tours. Closed tours formed in this way are termed by THW as **cyclic** if the hurdle's journey is also reentrant, since then the hopper can get back to where it started and the whole tour can be repeated. The second diagram below shows a cyclic 32-move tour with rotary symmetry. T. H. Willcocks explained in *Chessics* that his method of finding solutions was to construct tours of 16 cells, cyclic if possible, and then combine them into 32 and then 64. The third diagram shows his 64-cell open tour that includes a 62-cell cyclic tour (cells 2 to 63). Recently Kotesovec has found a 64-cell closed tour. The challenge of a full cyclic tour remains.

56		50	17	35	20	49	16
2	26	45	60	25	37	44	59
51	14	54	31	48	15	30	23
9	42	19	57	43	58	18	36
52	27	46	13	28	21	**61**	12
3	41	7	32	24	38	6	35
		53	10	47	11	29	22
8	33	4	40	5	34	**1**	39

Hall 1938 G / S

26		22		23		14	
		18	28			**17**	4
21		25		13		24	
	31			**32**	3		11
27		19	**16**			15	
	8		29		9		5
20	**1**			12	2		
	30		7		6		10

Willcocks 1985 G / S

14	45	10	54	11	53	**2**	57
35	48	6	16	59	49	5	24
9	55	13	44	**1**	56	12	52
42	19	34	**63**	20	23	62	31
15	46	7	4	47	50	3	58
36	28	40	17	60	29	39	25
8	21	33	43	32	22	**64**	51
41	18	37	27	38	26	61	30

Willcocks 1985 G / S

31	34	26	41	32	35	25	40
44	55	62	20	45	9	2	19
27	23	30	49	24	39	48	36
61	17	43	54	33	18	42	8
28	50	14	21	46	51	13	37
60	56	63	4	57	10	3	7
15	22	29	53	16	38	47	52
1	5	59	12	64	6	58	11

Kotesovec 2009

Grasshopper over Rook. The problem of grasshopper over rook tour was solved by T. R. Dawson (*FCR* Apr 1938 prob 3179). Dawson started Ga1 Rb2. Beasley (*VC* 2009) renumbers to start Ga7 Rb2 to make clear the repeated 16-cell pattern.

50	48	53	47	55	36	49	34
1	26	**8**	28	**11**	30	**10**	32
52	46	51	45	54	35	56	37
7	27	**6**	25	**9**	29	**12**	31
63	44	61	41	57	38	59	39
5	24	**3**	22	**13**	19	**14**	20
64	42	62	43	60	40	58	33
2	17	**4**	23	**15**	21	**16**	18

Dawson 1938 G / R

62	52	61	50	42	48	41	46
57	55	59	56	37	35	39	36
64	51	63	49	44	47	43	45
58	53	60	54	38	33	40	34
2	**8**	**1**	**6**	22	28	21	26
13	**11**	**15**	**12**	17	31	19	32
4	**7**	**3**	**5**	24	27	23	25
14	**9**	**16**	**10**	18	29	20	30

Tylor 1983 G / K

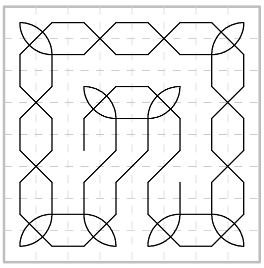
Grasshopper over King. The problem of a grasshopper tour over a king as moving hurdle on the 4×4 board was proposed by P. C. Taylor (*PFCS* Oct 1930 p.8 prob.45). Ten cyclic solutions were given, all starting and ending with Ga1, Kb1. A cyclic symmetric 8×8 solution, combining four 4×4 solutions, was given years later by C. M. B. Tylor (*Chessics* #5 1978 p.8). Diagram above. Ga6 Kb5.

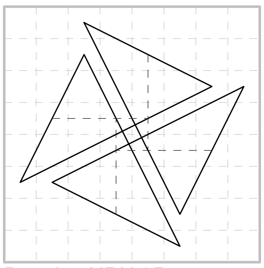
Moose and Nightriderhopper

Moose over Moose. C. M. B. Tylor *Chessics* #5 p.7 Jul 1978 found an amusing moose over moose tour. In this the pieces do not move alternately. Sometimes, as in the corners, one moose makes two successive hops. Each moose covers 32 cells. In effect each hop is a knight move made up of a fers followed by a wazir move, or vice versa, the other moose occupying the cell passed over. No long-distance moose moves are used. The moves of the first moose are shown. The other starts at c5. Both follow the path illustrated The c4 moose uses the cells aceg3478 and bdfh1256. The c5 moose uses the other 32 cells.

25		11		27		9	
12		26		10		28	
	24		2		18		8
	13	○	19		3		28
23		(1)		17		7	
14		20		4	○	30	
	22		16		(32)		6
	15		21		5		31

Tylor 1983 M / M





Douglas NRH / R

Nightriderhopper over Rook. This piece was investigated by T. R. Dawson and F. Douglas in *Chess Amateur* Aug 1928, looking at paths of NRH over R. They gave a shortest closed path of 8 moves, and a longest closed path of 56 moves (some cells are entered twice).

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Puzzle Solution

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PUZZLE: **Solutions to Wazir Figured Tours**: (a) squares at corners of a square, (b) at corners of an oblong (Jelliss *Games & Puzzles Journal* #11 p.178 1989); (c) in a knight path 4×4 (d) 6×6, (e) forming an AP with c.d 7. (Jelliss, *Mathematical Spectrum* v.25 1992/3 #1 p.18).