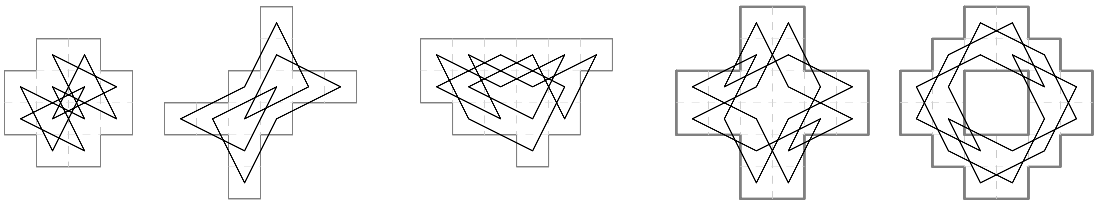
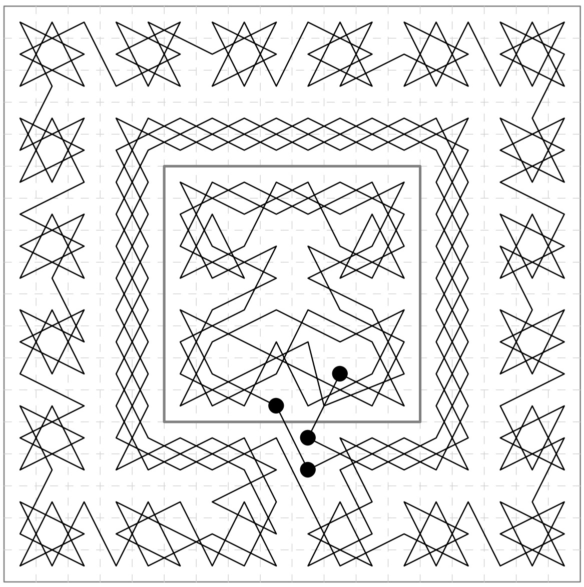
⌘ 3

Shaped & Holey Boards



by G. P. Jelliss



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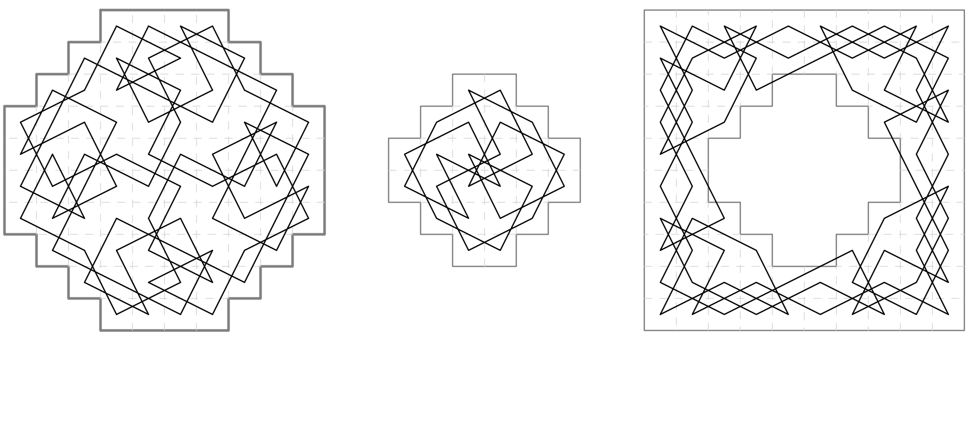


Title Page Illustrations:

Biaxial tours of 12 and 20 cell boards, the first of each being by Euler (1759). The asymmetric 16-cell tour covers a quarter of the standard 8×8 chessboard.

The large diagram is of a 320-cell tour by the Rajah of Mysore. The original, in a form of Indian Script, is shown on page 20 of *Das Schach und seine asiatischen Verwandten* (2008) by Maria Schetelich. The central 8×8 tour is the Rajah's magic tour, numbered with 1 at f2 and 64 at d1. At the centres of the 7-move stars in the outer border 20 cells of the 18×18 board are omitted. The tour as a whole is closed, cell 320 being a knight move from cell 1.

Rotary tours of a 24-cell board and of 76-cell boards formed by removing 24 cells from a 10×10 board, externally or internally.

Contents

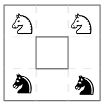
Octonary

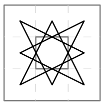
A pattern within a square area that appears unaltered by any of the transformations which leave the square unaltered has **octonary** symmetry. The diagonals and medians devide the pattern into eight triangles all of the same pattern. Such symmetry is also termed ‘square’ or ‘perfect square’ symmetry. For the cells to be considered a ‘board’ they must be properly connected edge to edge. Cells can be omitted in fours on the diagonals and medians, or in eights when off-axis. Tours with octonary symmetry thus all use a multiple of 8 cells (4 diagonal, 4 medial, each other position in sets of 8).

A knight tour with octonary symmetry on any even-sided board, such as the 8×8 chessboard, is impossible. In fact tours with octonary symmetry are not possible within any even-sided area, even with holes. Octonary tours all require boards within a containing square of odd side and omit the centre cell. Every octonary tour consists of eight equal paths, each path having one end on a median cell and the other end on a diagonal cell.

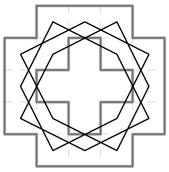
8 cells: The smallest board with a closed knight tour is of course the 8-cell centreless 3×3 board. In our coding system the centre cell is 0, the mid-edge cells are 1, and the corner cells are 2, so the moves are described by the formula: 1-2.

The earliest manuscripts associated with al-Adli and as-Suli (c.900) include the puzzle, often attributed to the later writer Guarini (1512), of the four knights. White and Black knights placed in the corners of the 3×3 board are to be interchanged by knight moves without having two on a cell at the same time. The knights go in procession round the star-shaped closed tour of the eight edge cells (taking 4 not 2 moves each).



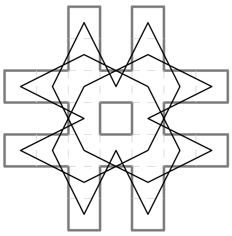


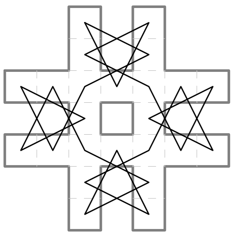
16 cells: The next case is a 16-cell tour within a 5×5 area, leaving a cross-shaped hole, tour formula 2-4-3. I call this ‘the bound cross’.



24 cells: The shortest octonary tours within the 7×7 area are two with 24 cells, both having the formula 1-7-4-2, where the 7-4 move can be taken in two different ways. We can write it 7:4 or 7;4 according as the move crosses a median or diagonal.

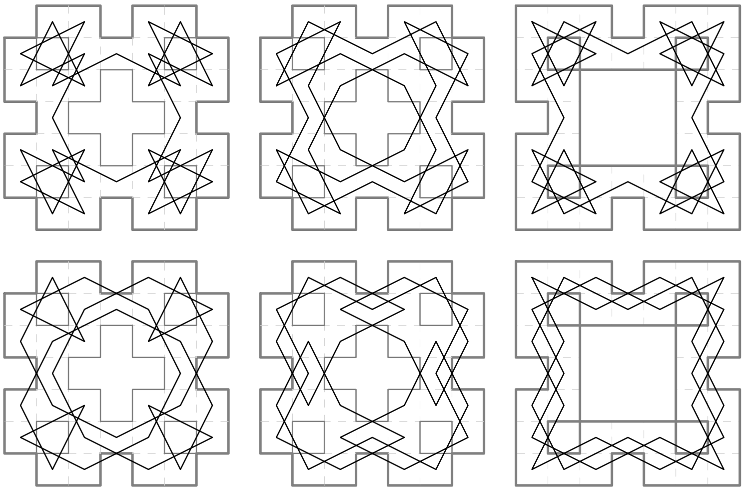
9	8	7	6	7	8	**9**
8	**5**	4	3	4	**5**	8
7	4	**2**	1	**2**	4	7
6	3	1	**0**	1	3	6
7	4	**2**	1	**2**	4	7
8	**5**	4	3	4	**5**	8
9	8	7	6	7	8	**9**



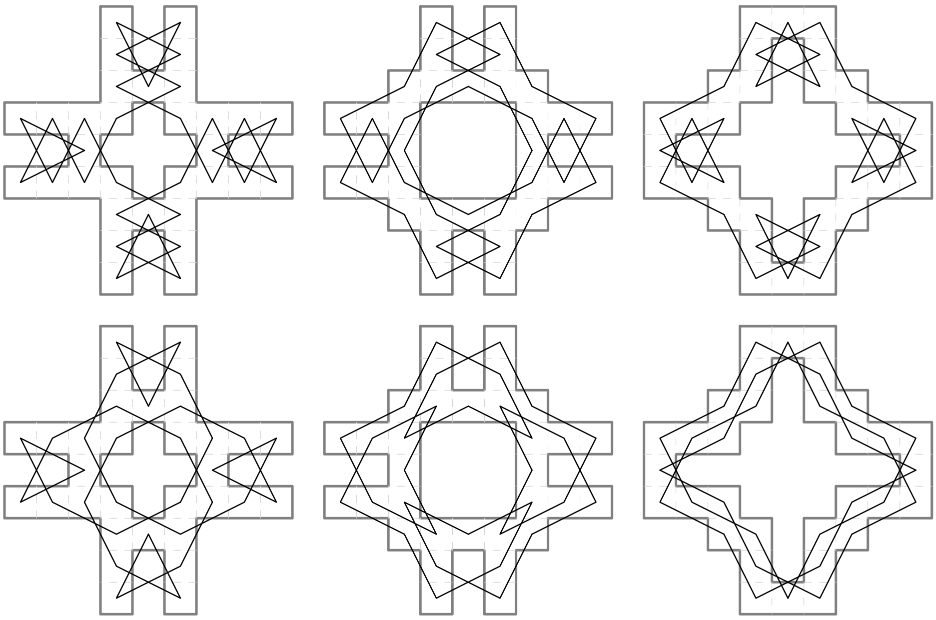


The inset diagram above shows how the cells are coded from the centre outwards. The 7-4 or 4-7 moves are the only cases where the coding of a move is ambiguous.

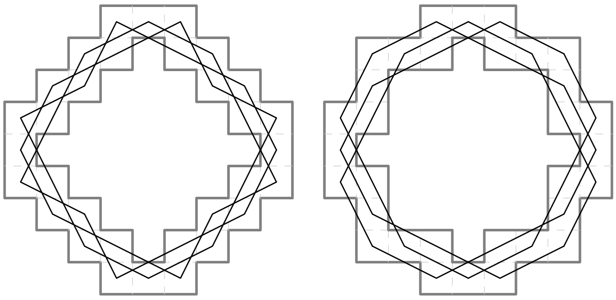
32 cells: There are 6 other octonary tours within the 7×7 area, on two board shapes, covering 32 cells. Their formulas are 3-4-7-8-2, 3-8-7-4-2, 3-8-7-4-9 with two of each according as the move 7-4 is across diagonal or median.



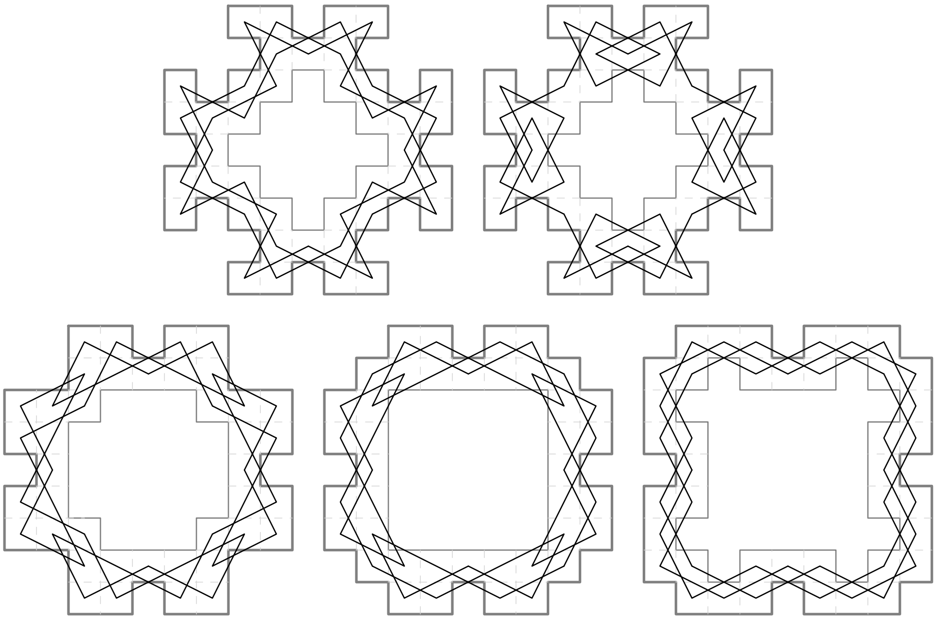
There are another 8 octonary tours of 32 cells within the 9×9 area using three board shapes. The formulas including the 7-4 link are 3-11-7-4-2, 10-4-7-11-5, 3-4-7-11-5 occurring in pairs.



The other two formulae are 10-8-7-11-5 and 10-8-7-11-9 on two board shapes, whose cells become ever more tenuously connected.

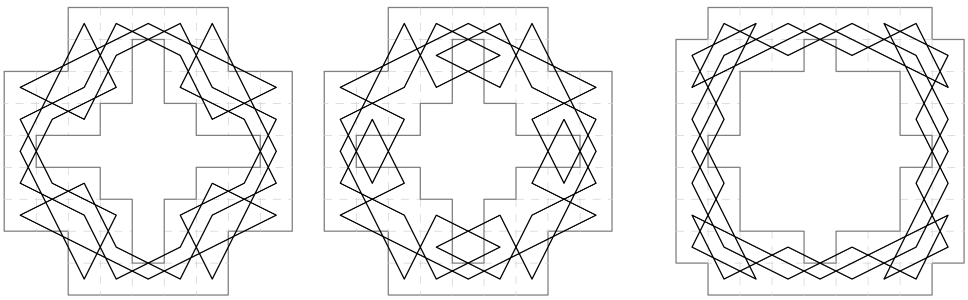


40 cells: These five examples using 40 cells are all within a 9×9 area.

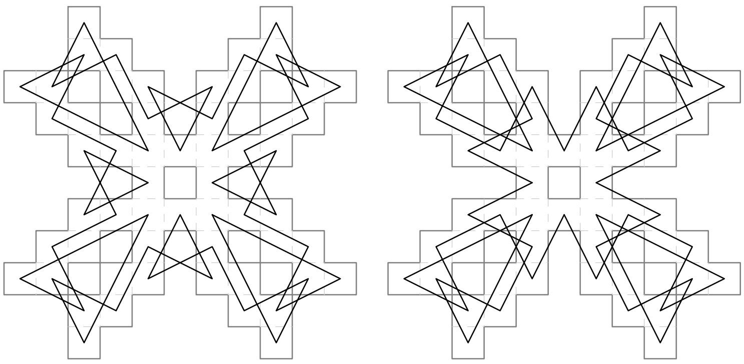


One pair has the formula 6-12-4-7-11-5 (with 4-7 taken two ways). The other three are 6-12-8-7-11-5, 6-12-8-7-11-9 (differing only at cells 5 and 9) and 6-12-13-7-11-9.

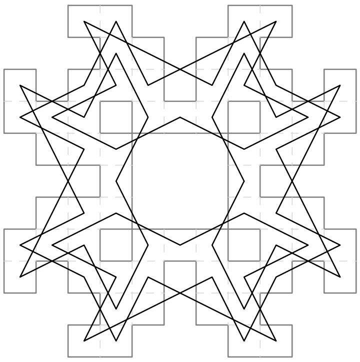
48 cells: Here are three octonary tours of 48 cells within a 9×9 area. A pair with formula 10-8-12-4-7-11-5, and another with formula 10-8-12-13-7-11-9.



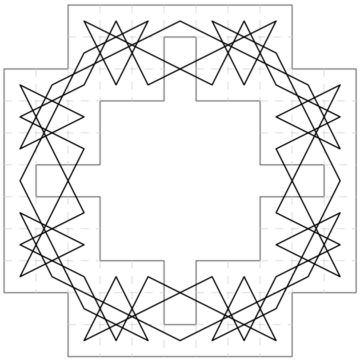
56 cells: A pair of octonary tours of 56 cells in an 11×11 area. Formula: 1-7-4-13-12-18-8-2.



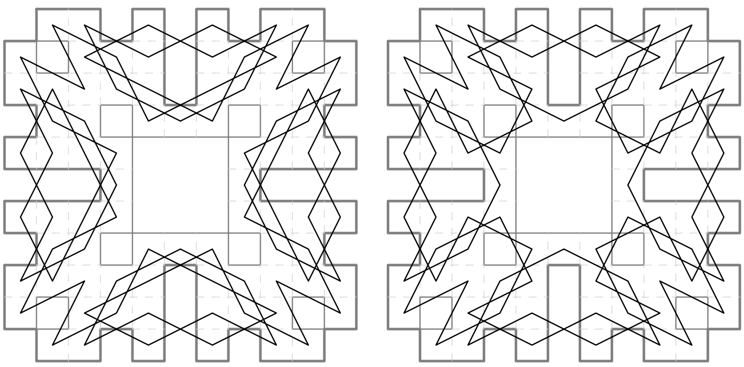
64 cells: Octonary tour of 64 cells within an 11×11 area. Formula: 3-4-12-8-18-11-7-17-9.



72 cells: Octonary tour of 72 cells within the 11×11 area. Formula 15-12-13-16-8-18-11-7-17-9.



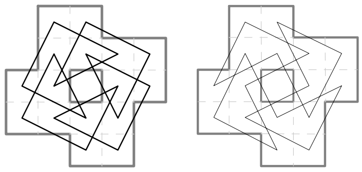
80 cells: A pair of octonary tours of 80 cells, the maximum possible within the 11×11 area. Formula: 3-8-18-11-16-13-7-4-12-19-9. These are the largest examples of this symmetry that I have constructed.



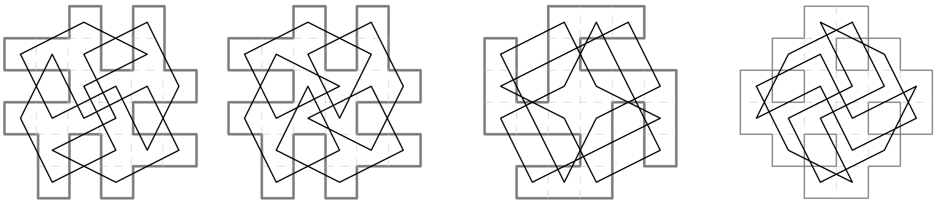
Birotary

Birotary symmetry, also known as ‘oblique quaternary symmetry’, is altered by reflection, but not by 90° rotation. This is also possible in tours on Square Boards of sides 6, 10, 14, 18, etc. (see ⌘ 5).

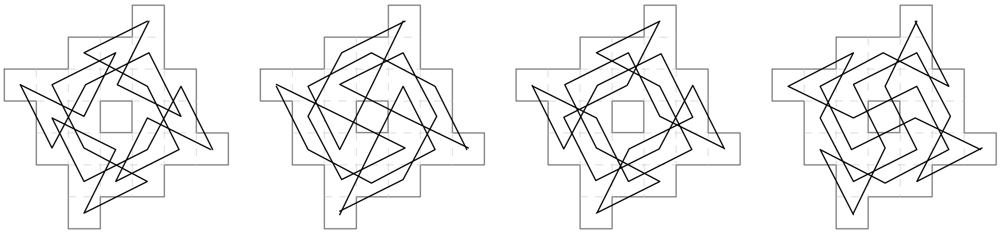
16 cells: Two birotary tours on a board formed of four 2×2 boards, surrounding a single-cell hole.



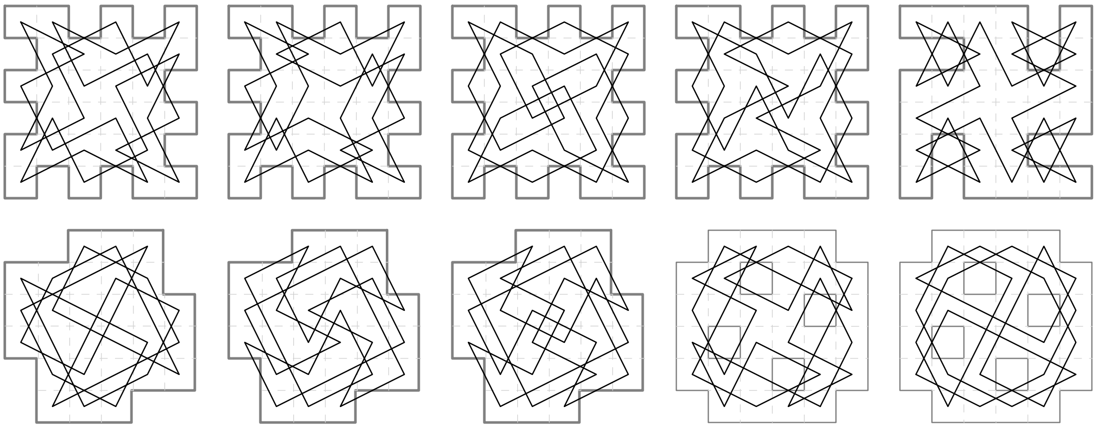
20 cells: Four birotary tours (Jelliss 2013). Two on one board. One with four holes.



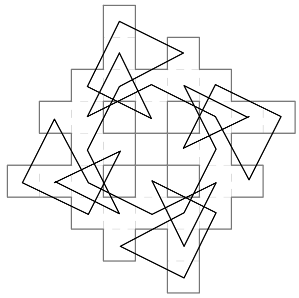
24 cells: Four birotary tours on a 5×5 with centre hole and with corner cells moved one step diagonally. The four corner cells cannot however be added at the middle of each side, to give a board with octonary symmetry, since this fails to maintain the correct balance of colours when chequered.



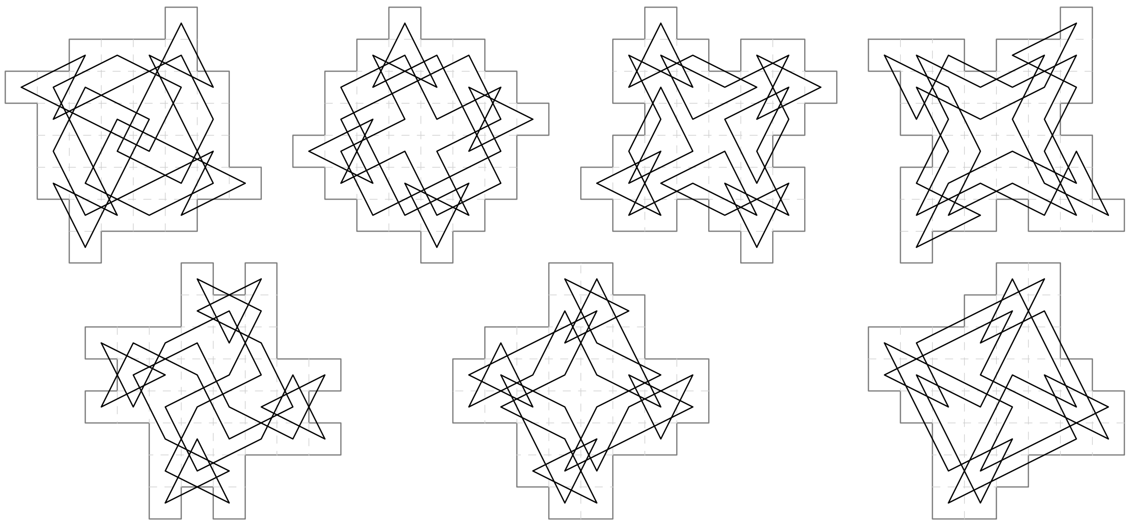
28 cells: Within a 6×6 area there are 10 birotary tours covering 28 cells, using four shapes of board, one with four single holes in a skew pattern.



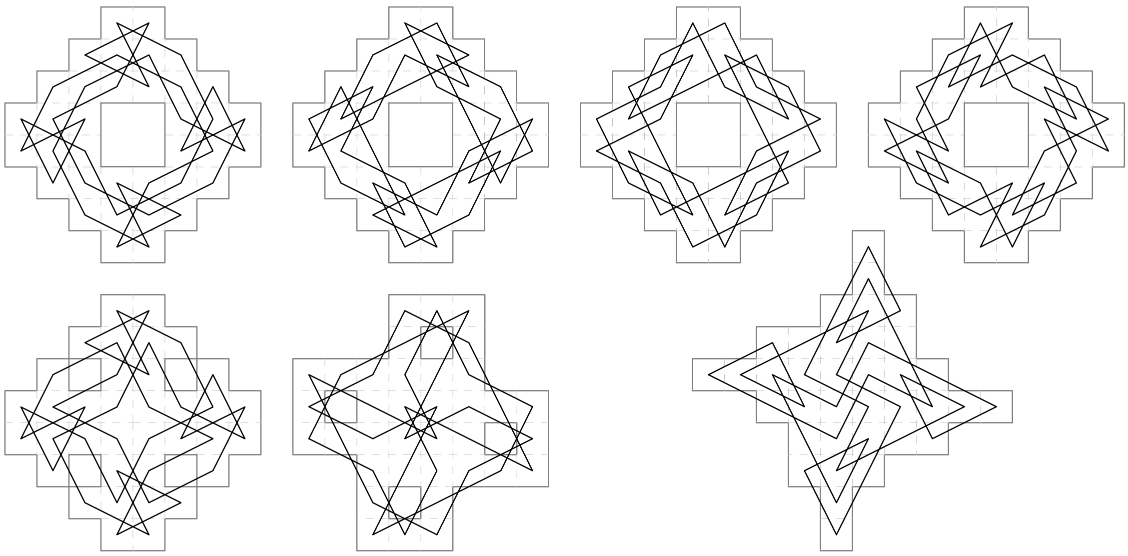
32 cells: Three boards of assorted shapes, with holes, showing 90 degree rotation.



36 cells: Birotary symmetry is possible on the 6×6 board. Here are some alternative 36-cell boards on which birotary tours are also possible. These boards can be regarded as 4×4 with a pentomino added on each edge or corner. There are other cases the reader might like to investigate.

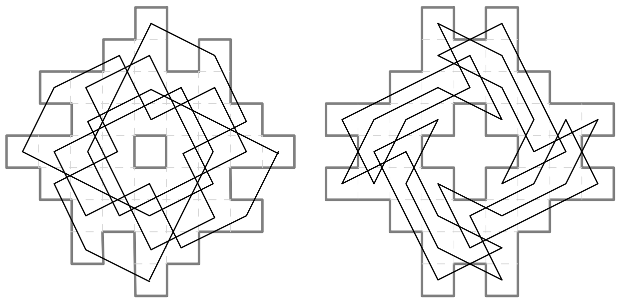


Quaternary symmetry is also possible on 36-cell holey boards. This board shape is used by the Rajah of Mysore (#45 in Harikrishna 1871) but his tour (not shown here) is asymmetric.

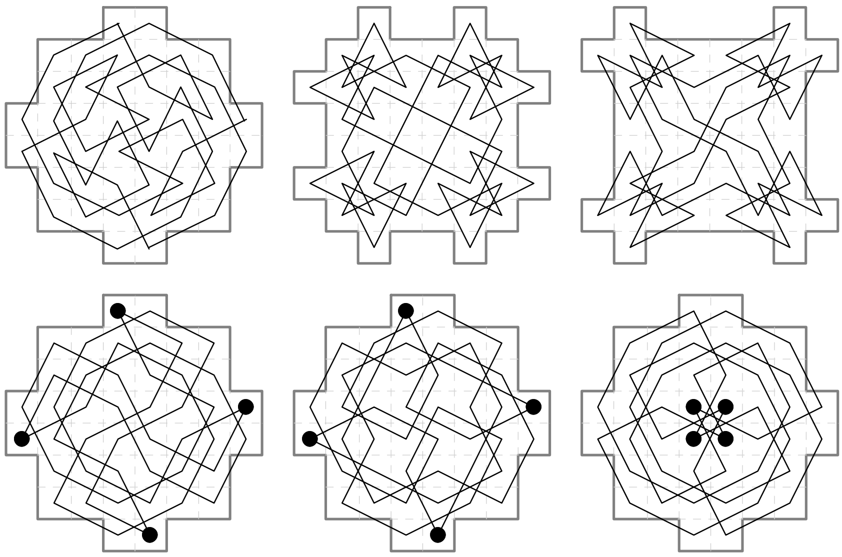


We add two with four single cell holes, and another pentomino example without holes.

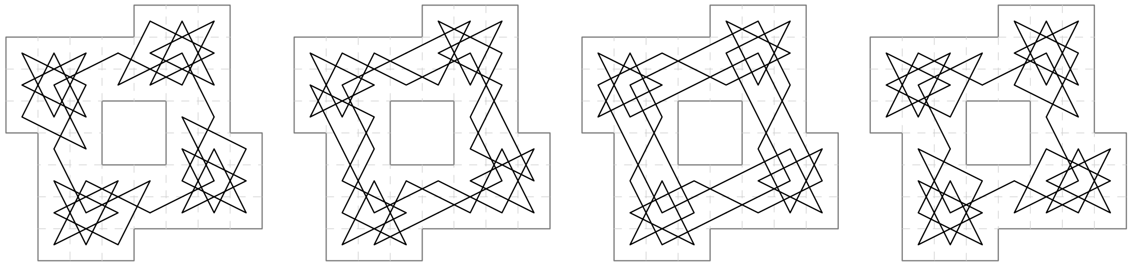
40 cells: Two assorted holey boards with birotary tours.



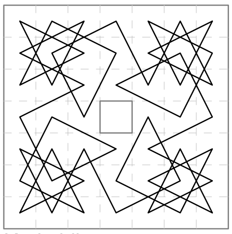
44 cells: Three octonary board-shapes can be formed by adding two cells to each side of the 6×6, making 44 cells, and each of these can be toured in birotary symmetry. In the three further tours on the first board (Jelliss 24 May 2013) there are only four acute angles (as shown by the dots).

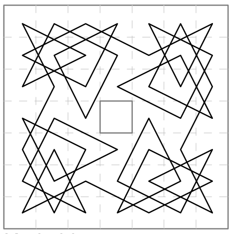


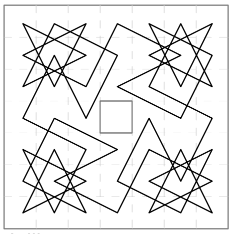
Here is an oblique quaternary board with four oblique quaternary tours (Jelliss 2013).

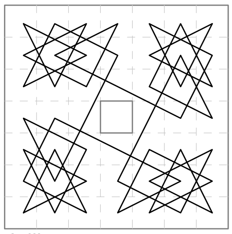


48 cells: According to W. H. Cozens (1940) there are 64 tours with 90° rotary symmetry on the 7×7 centreless board. The first five quaternary tours here are formed from a set of closed quarter-tours of the board by ‘simple-linking’ (a further 15 asymmetric tours can be formed from the same set by this process). The first two are by Kraitchik 1927 and the other three are my own work. These are followed by two other examples from Kraitchik 1927. The last is from Bergholt memorandum 24 Feb 1916.


Kraitchik


Kraitchin


Jelliss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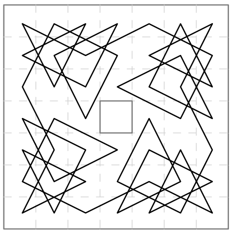
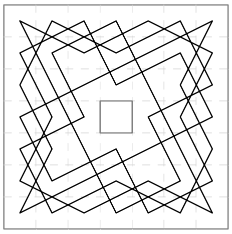
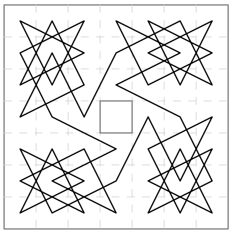
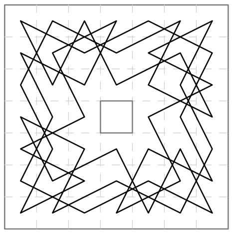
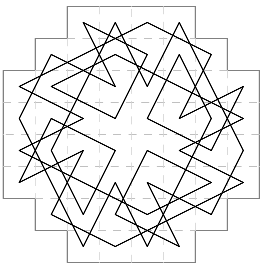

Jelliss

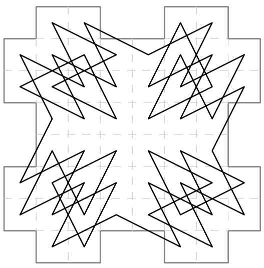

Kraitchik

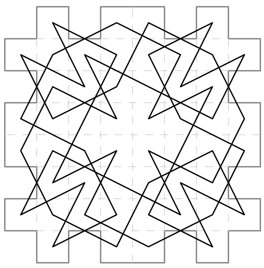

Kraitchk


Bergholt

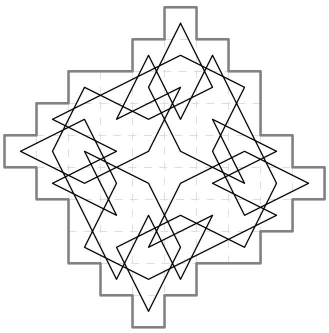
52 cells: Here are three 90° rotatory tours on boards formed from the 6×6 with four cells added to each edge in various formations (Jelliss 2013).

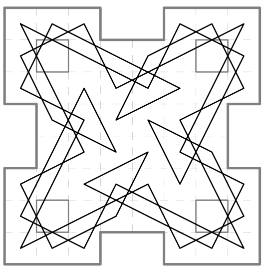


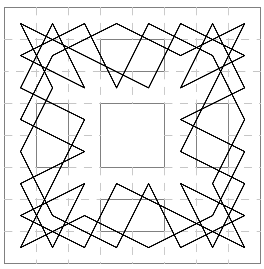




The fourth example, below, with a T-tetromino added to each edge, is by S.Vatriquant (*L'Echiquier* 1928).

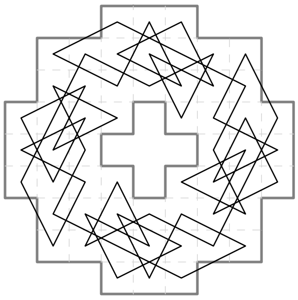


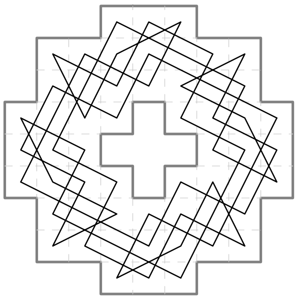




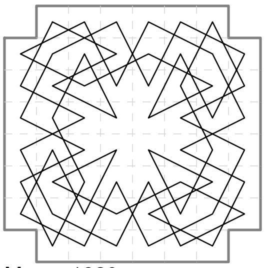
The example with four holes is from *Zürcher Illustrierte* (15 Jul 1932). This was collected by Murray (1942) who also gives one of his own with five holes.

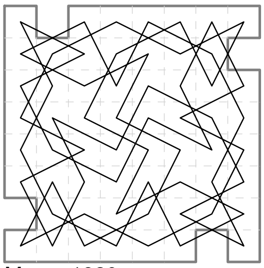
56 cells: Here are two very different birotary tours of an approximately circular board with cross-shaped hole (Jelliss 1998).

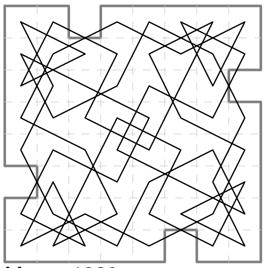


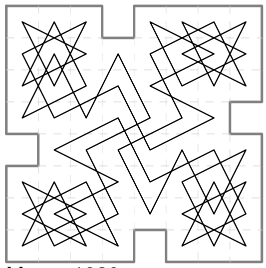


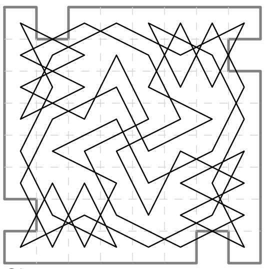
60 cells: There are ten geometrically distinct ways of removing four cells in 90° rotary symmetry from the 8×8 board, but one of these (cell a knight move from corner) does not admit a tour. The nine cases were solved by G. L. Moore (1920). There are four cases with cells removed from the edges: This subject has been independently rediscovered several times since Moore's work (and possibly earlier by Adam 1867). We show some alternative solutions.

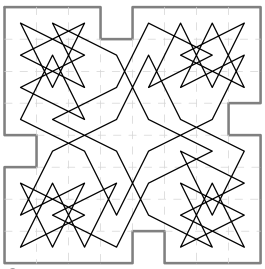

Moore 1920

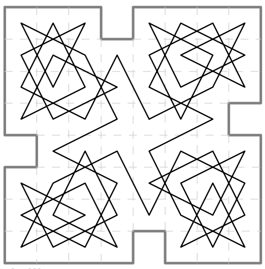

Moore 1920

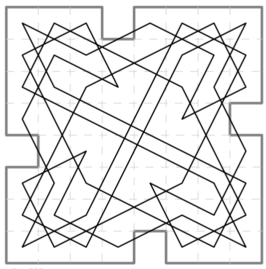

Moore 1920


Moore 1920


Sharp 1925

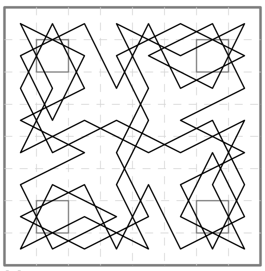

Cozens 1952

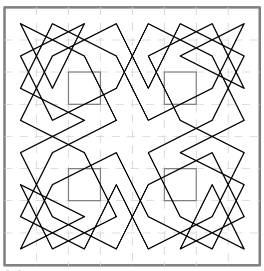

Jelliss 1985

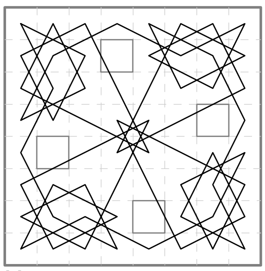

Jelliss 1994

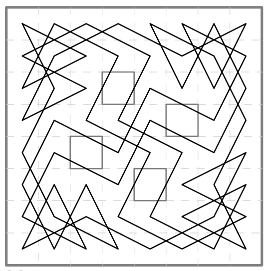
The first of these is from Archibald Sharp *Linaludo* (1925). The second is from W. H. Cozens in *Fairy Chess Review* (vol.8 #3 1952). The third (Jelliss 1985) is formed of four maximal paths covering 15 cells of the 4×4. The fourth (Jelliss *Figured Tours* 1997 p.21) has the multiples of 5 on the diagonals when numbered from g3 = 1 to e4 = 60.

There are four cases with four single-cell holes:

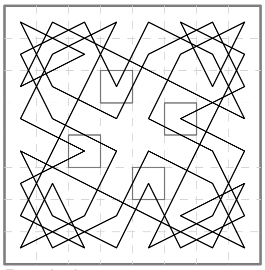

Moore 1920


Moore 1920

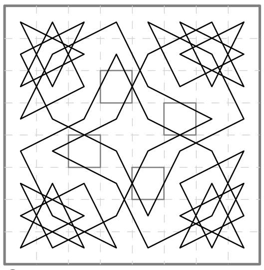

Moore 1920


Moore 1920

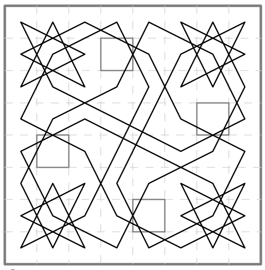
Further by Bergholt (*First Memorandum* 24 Feb 1916) and by W. H. Cozens (*FCR* 1952).



Bergholt 1916

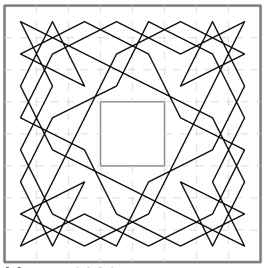


Cozens 1952

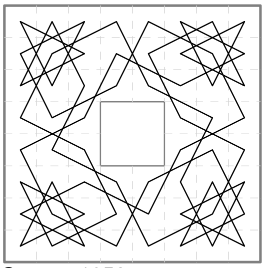


Cozens 1952

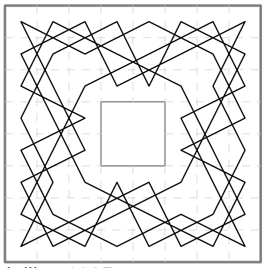
Or the four missing cells can merge to form a single central hole. Extra examples by Cozens 1952 and Jelliss 1985.



Moore 1920



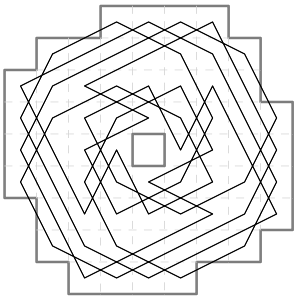
Cozens 1952



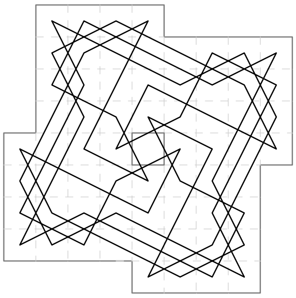
Jelliss 1985

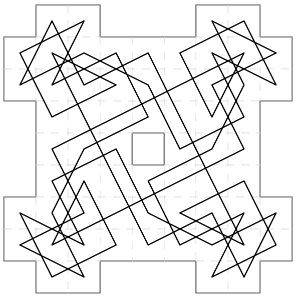
All the tours in this section are examples out of many possible.

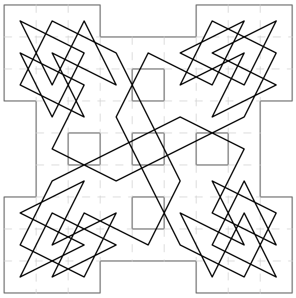
64 cells: A birotary tour is not possible on the normal chessboard, or on any arrangement of 64 squares centred on a point where four cells meet. Solutions are however possible centred on a single cell hole. This first example is from *Zurcher Illustrierte* 27 Oct 1933 collected by Murray.



The next three are my own work (Jelliss 2003).

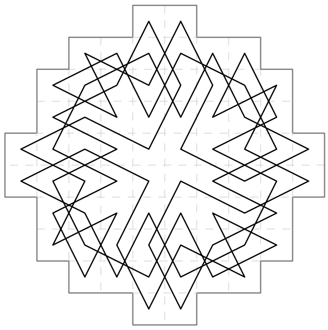
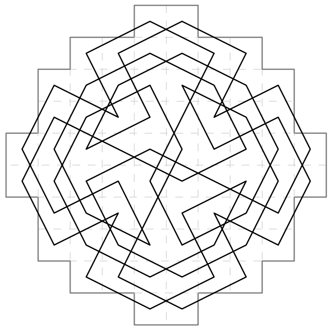
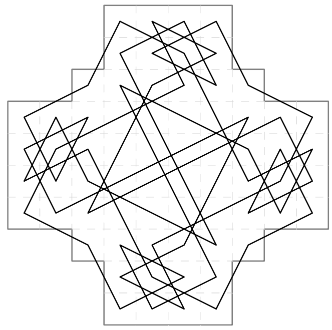
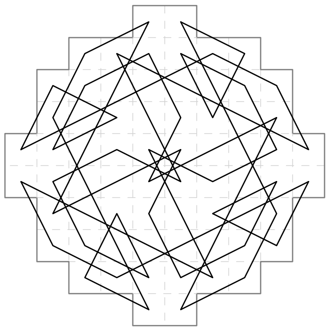
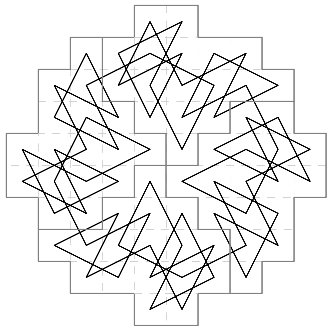
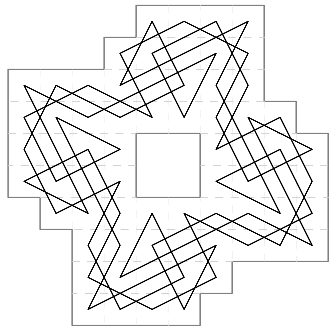






The first here was constructed by starting with four 4×4 ‘squares and diamonds’ arrays, joining the circuits up by Vandermonde's method (deleting parallel pairs of moves) and finally joining the four 16-move circuits by using a linkage polygon of 8 alternating deleted and inserted moves forming a star round the central hole. (See also the 192-cell solution p.122.)

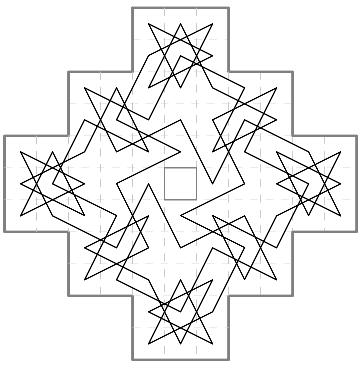
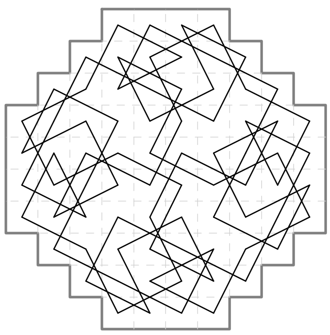
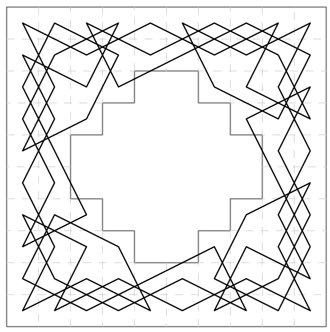
68 cells: A near-circular board formed from the 8×8 chess board by removing corner cells and adding two at the centre of each side, was designed by Paul Byway for ‘Troitzky Chess’ (published in *Variant Chess* 1997) where one of these four birotary tours (Jelliss 1997) was published.

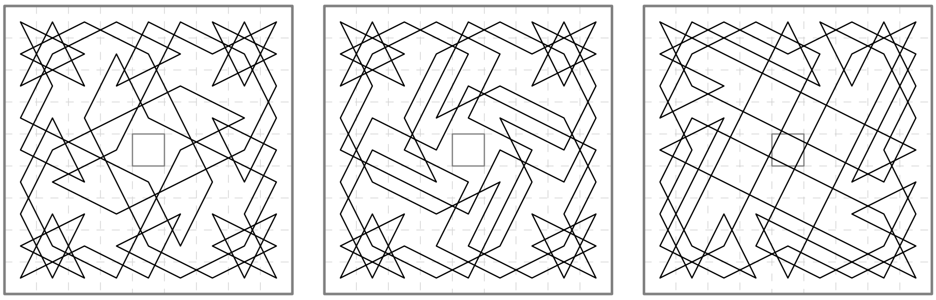
Two more 68-cell boards are shown. An augmented cross-shaped board and a board formed of four 4×4 boards and four single cells arranged around a 2×2 space (Jelliss 2003 and 2013).

72 cells: Birotary tour of 72 cells on a serrated cross with single central hole. Kraitchik 1927.

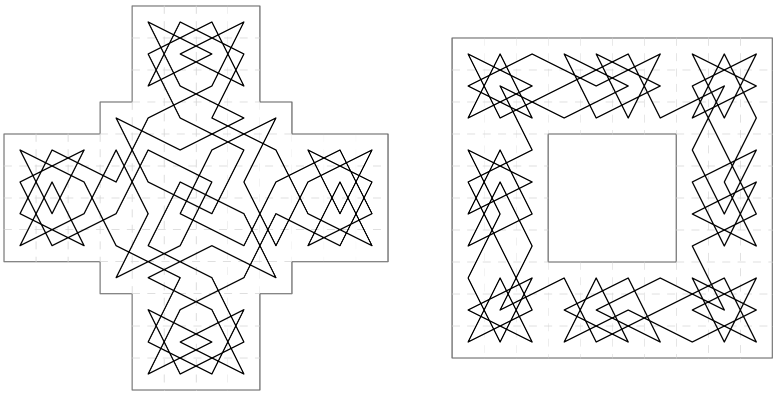
76 cells: Birotary tours of 76 cells (Jelliss 1998 and 2013) that can be regarded as a 10×10 board with four 6-cell triangles removed at the corners or removed internally.

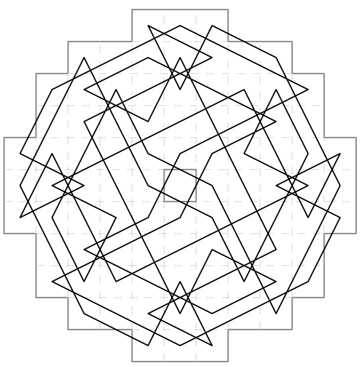
80 cells: Three tours on the centreless 9×9 board with oblique quatenary symmetry. One from Kraitchik 1927, one from *Zurcher Illustrierte* 17 Jun 1932, one by Murray (1942)



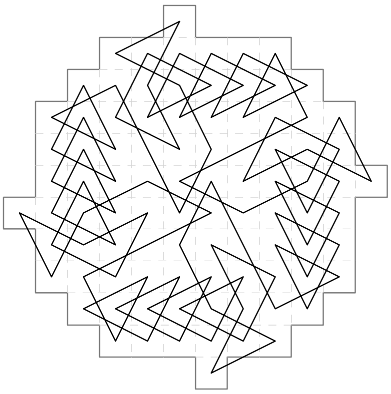
84 cells: Birotary symmetry is not posssible on the 80-cell Greek cross board, but becomes possible when four squares are added to make a type of celtic cross, as was pointed out by Ernest Bergholt in the magazine *Queen* 22 Jan 1916. Also a tour (Jelliss) showing how tours on larger even areas can be constructed by joining together four tours on small boards arranged round a central hole.

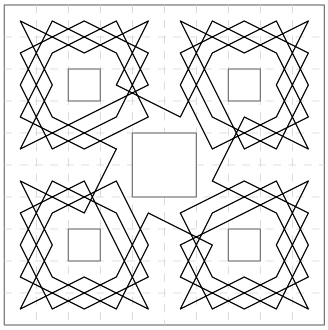


88 cells: Birotary example on a nearly circular board with hole (Jelliss 1998). The distance from the centre cell to the mid-edge cells is the same as to the pairs of corner cells, namely a 5-leap.

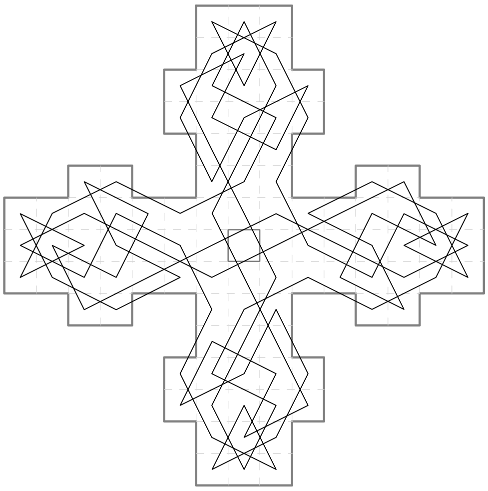


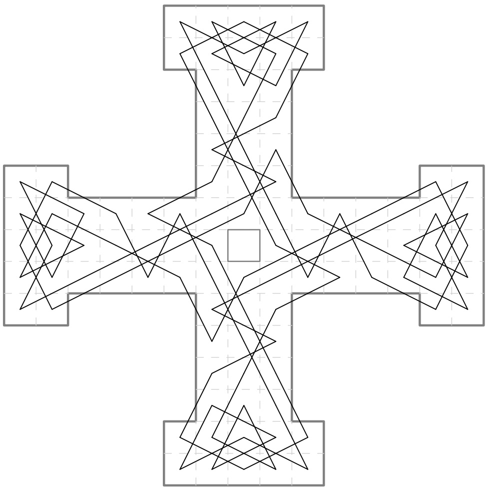
92 cells: Another of Murray's examples in oblique quaternary collected from *Zürcher Illlustrierte* 25 Nov 1932. Another quaternary tour (Jelliss 1996) with five holes in quincunx formation.



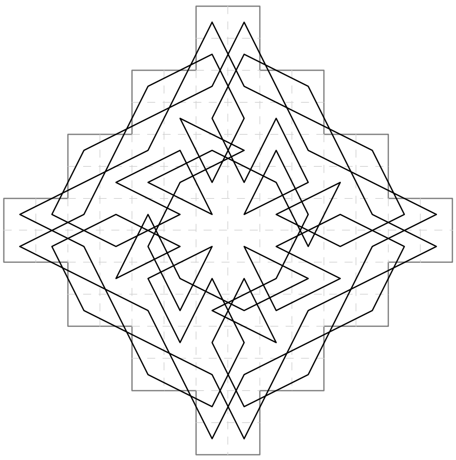


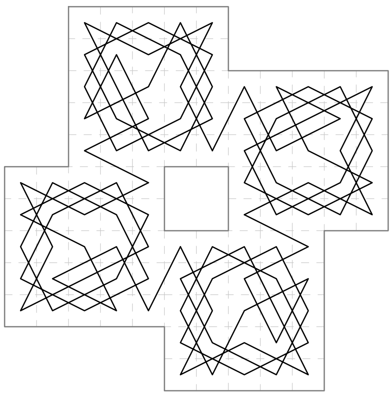
96 cells: Two ornamented crosses with central hole, (Jelliss 1998).





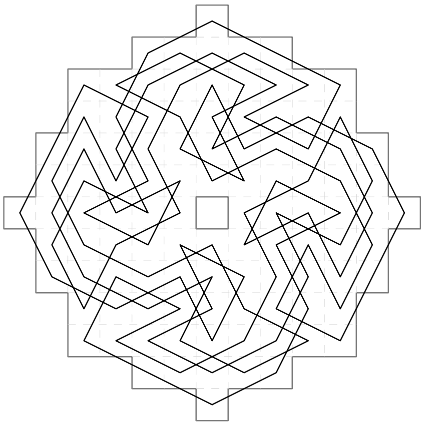
100 cells: Here are two birotary tours (Jelliss 2013) constructed just to show that there are other interesting 100-cell shapes besides squares. A serrated board and a cycle of four 5×5 squares around a 2×2 hole.

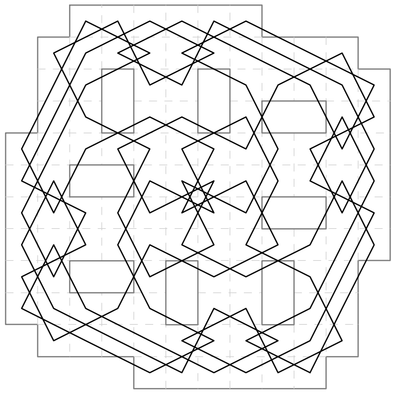




104 cells: On the left an oblique quaternary tour of 104 cells (Jelliss 2003).

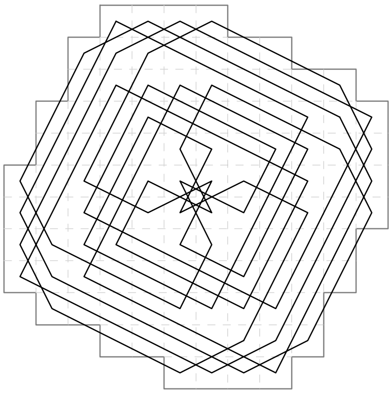
108 cells: On the right tour from *Denken und Raten* 26 Jul 1931, omitting 36 cells from 12×12.

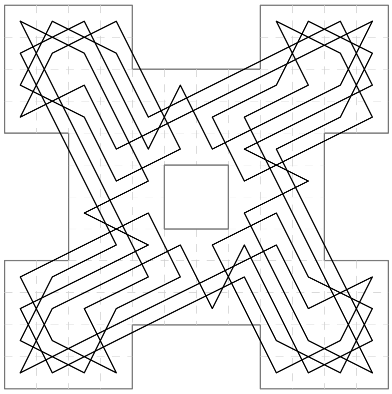




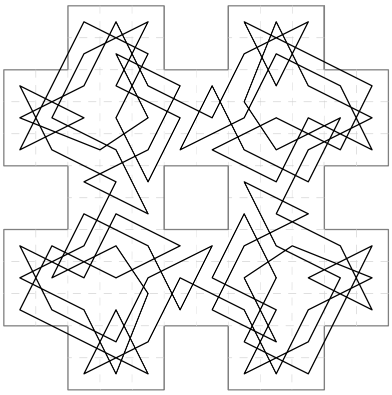
Below another 108-cell tour with oblique quaternary symmetry from *Denken und Raten* 15 March 1931. This is formed by omitting 36 cells from the corners of the 12×12. This tour is another with only four acute angles at the centre (compare the 44 cell examples). These tours from *Denken und Raten* were collected by Murray.

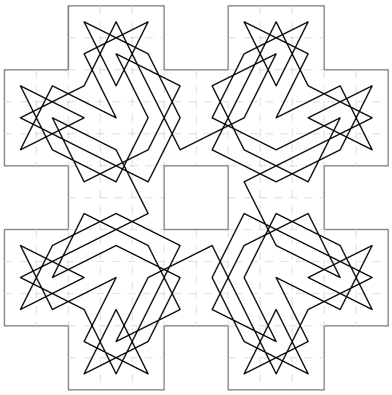
The other example below is a tour constructed by accident when aiming for 112 (Jelliss 2003).



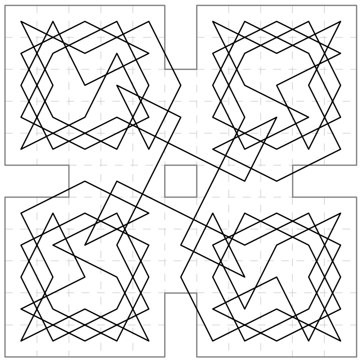


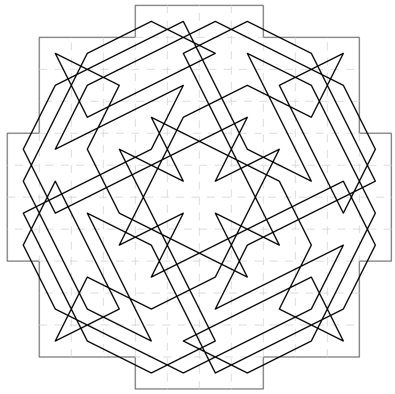
This "hashtag" (#) pattern board is used for an asymmetric tour #17 in Harikrishna 1871, but also allows quaternary tours as in these two examples (Jelliss 2003).





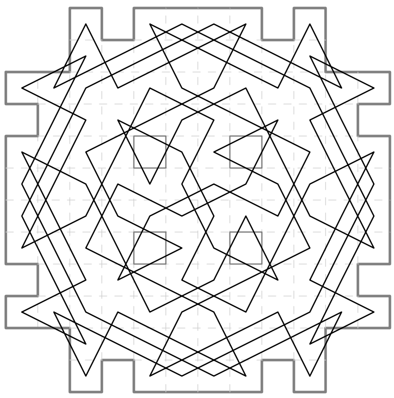
112 cells: A birotary tour based on joining up four 5×5 tours (Jelliss 2003).

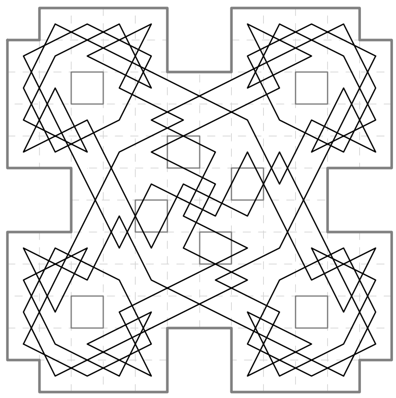




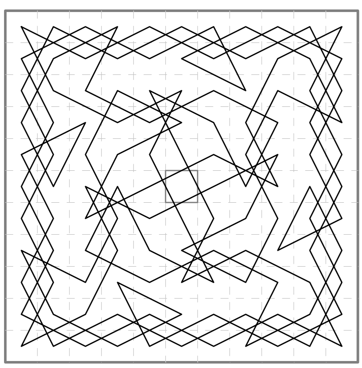
116 cells: The example above right (Jelliss 2013) shows a complete Maltese Cross.

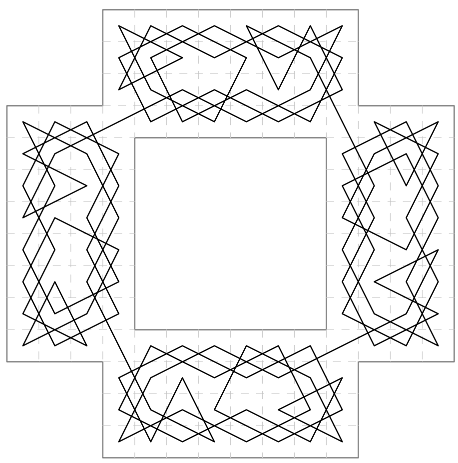
Two more tours collected by Murray from *Denken und Raten* both 25 Aug 1929, omitting 28 cells from the 12×12 in various places.





120 cells: An 11×11 centreless board with quaternary symmetry by Kraitchik 1927.

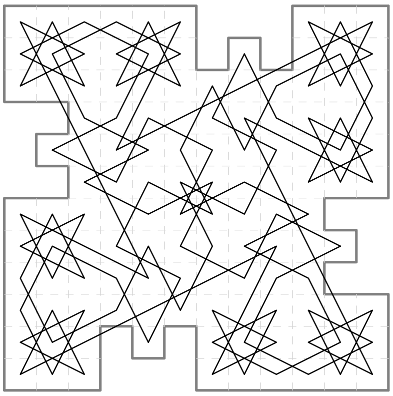


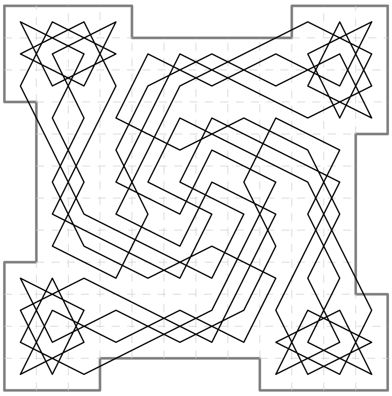


124 cells: Birotary. A board, above right, made up from four 4×8 boards overlapping at corners (Jelliss undated).

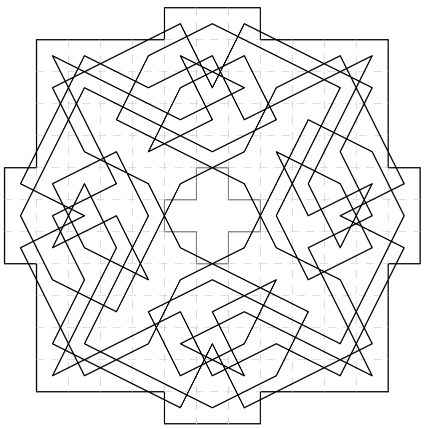
continued

These further 124-cell examples are from *Denken und Raten* 25 Jan 1931 and 14 Jun 1931. They can be regaded as 12×12 minus four groups of five cells in the edges.

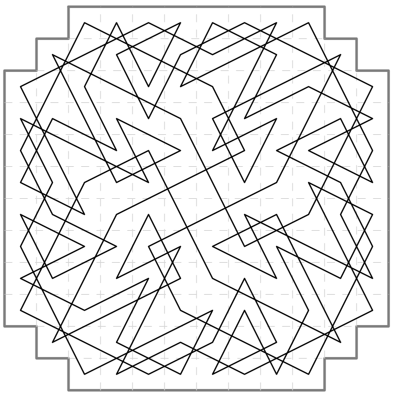


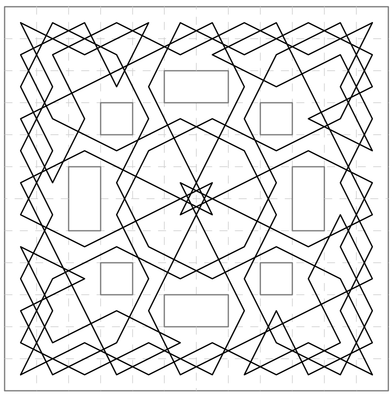


128 cells: It may be worth noting here that in constructing tours like this example the central cross-shaped hole has to be plus-shaped rather than X-shaped to ensure the balance between black and white cells (when the board is chequered) is maintained, as is required in all closed knight tours. Trying to construct a tour on a board where the colours are not in balance is an exercise in futility.

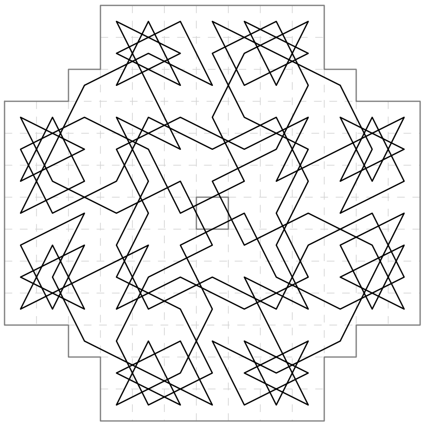


132 cells: Birotary tour (Jelliss 1998) 12×12 board with 12 cells removed at the corners.
Also (Jelliss 2013) 12×12 board with 12 cells removed internally in a circle, forming a clockface.

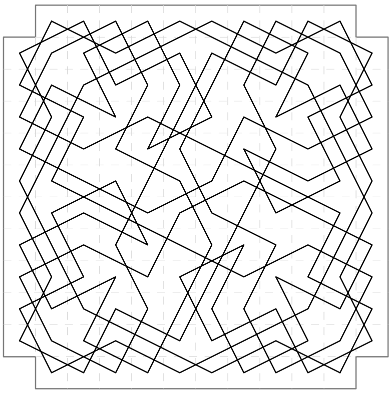


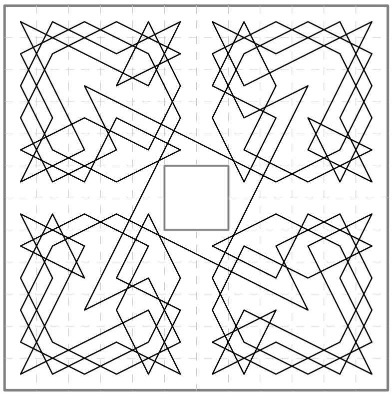


136 cells: Example (Jelliss 2003) formed by joining a 3×7 tour to circuits on the central 7×7.

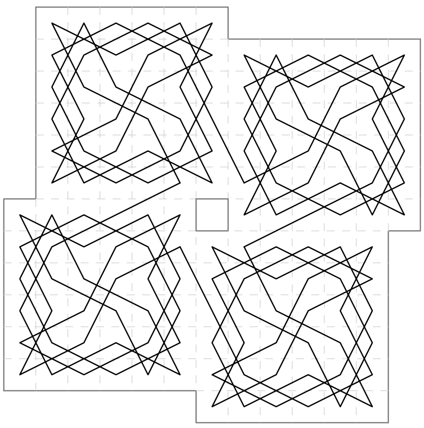


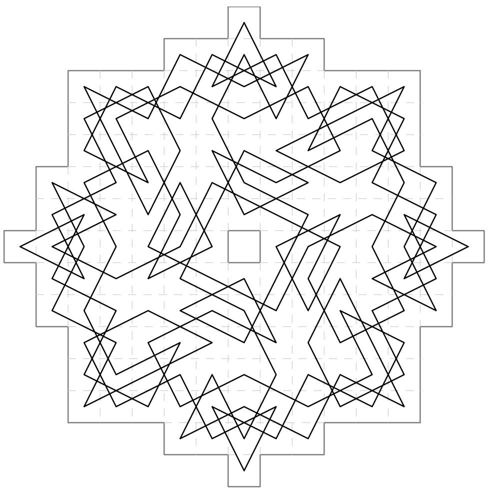
140 cells: Birotary in 12×12 less four cells at corners (Jelliss 2013) or in centre (Kraitchik 1927).



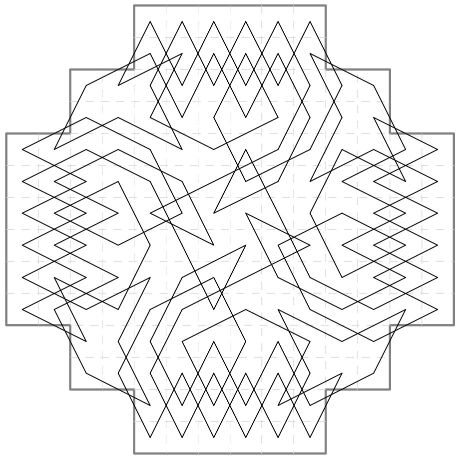


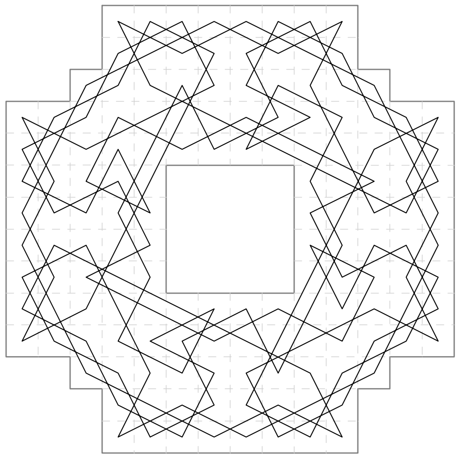
144 cells: Examples of four boards 6×6 and of centreless 11×11 + cells at edges (Jelliss 2003).





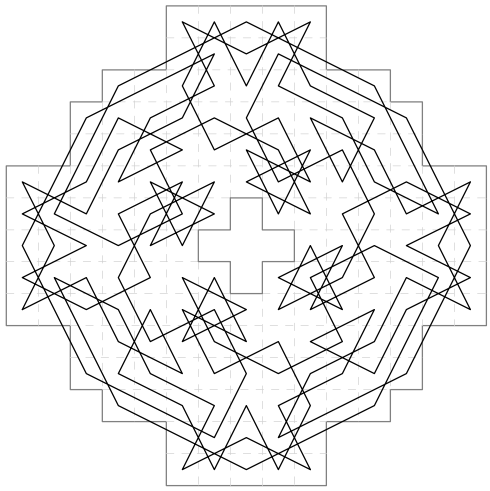
148 cells: Tours on boards 10×10 with 2×6 strips added at edges (Jelliss 1998) or 2×8 strips and central hole (Jelliss 1994). This board was proposed for a 'Superchess' game in *Variant Chess* 1994.

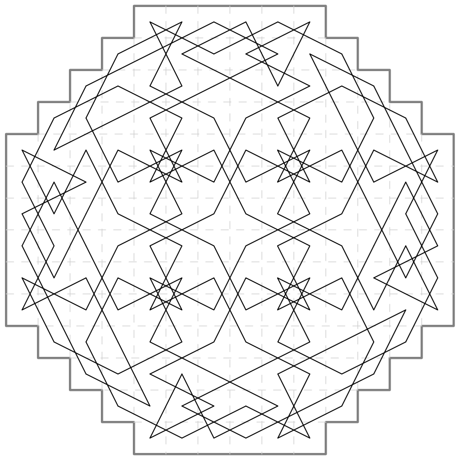




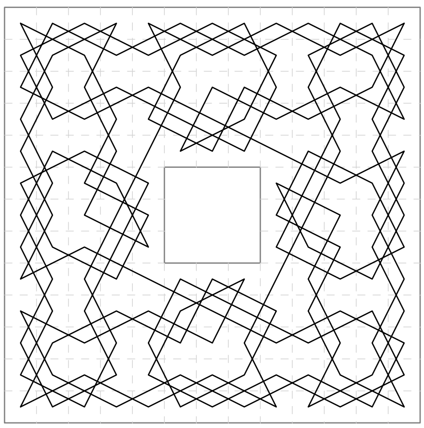
152 cells: A rounded cross board 152 cells with cross-shaped hole (Jelliss 2003).

156 cells: The central pattern in my example on the right is adapted from two designs used by Edward Falkener (1892) in his 160-cell tours (his tours numbers 9 and 15).



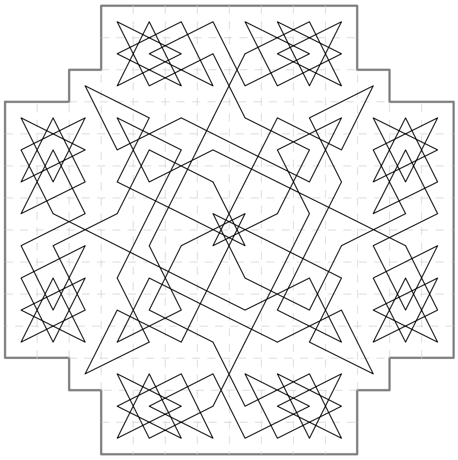


160 cells: The following birotary tour (on a board of size $13^2 - 3^2$) was constructed by first joining four 5×5 corner-to-corner tours together (their corners overlapping), then joining in four 4×4 sets of squares and diamonds in the four corners by Vandermonde's method. (Jelliss 2003).

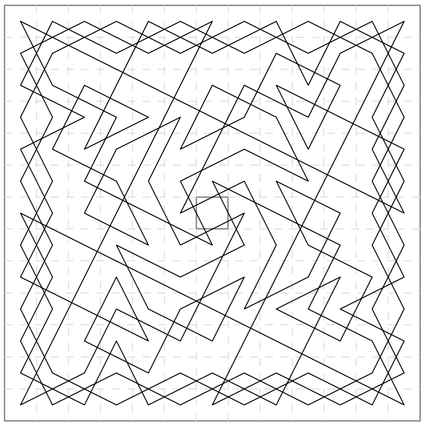


Apart from the final 196-cell board, no further holey examples on boards of size 8n + 4 (i.e. 4 times an odd number) are given since they are easily constructed, and the same frame admits similar tours without holes.

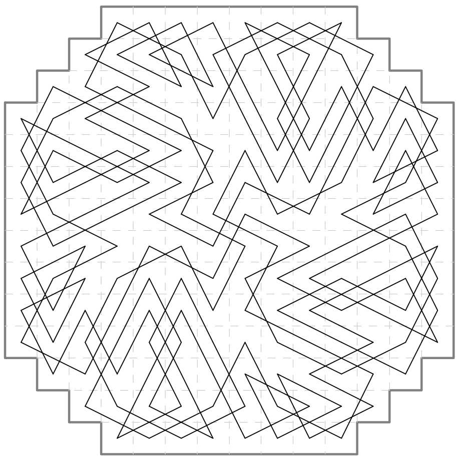
164 cells: This tour is another example by me after Falkener's (1892) near-symmetric design number 16 on a 160-cell 'Four-Handed Chess' board (on which birotary symmetry is impossible). The extra cells in the concave corners of the cross make quaternary symmetry possible.



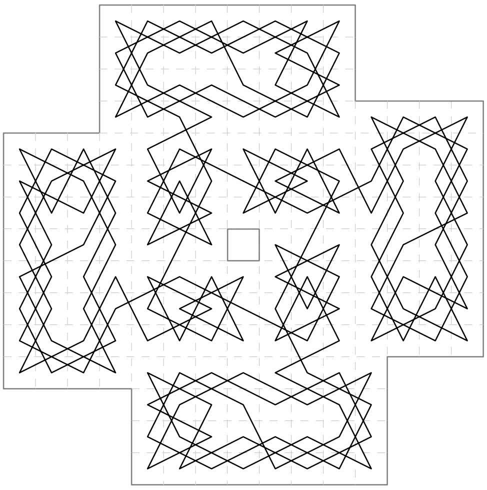
168 cells: This birotary example by Kraitchik (1927) uses the 13×13 with missing centre cell.



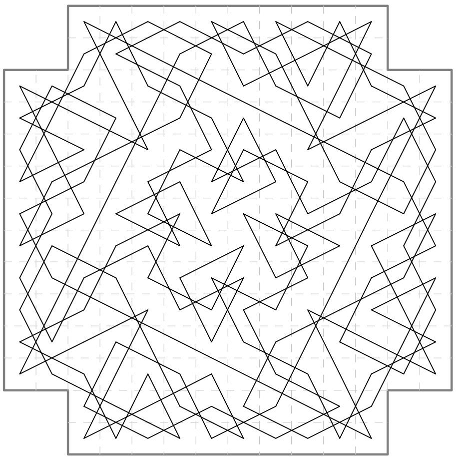
172 cells: This example (Jelliss 1998) is 14×14 minus 24 (six at each corner).



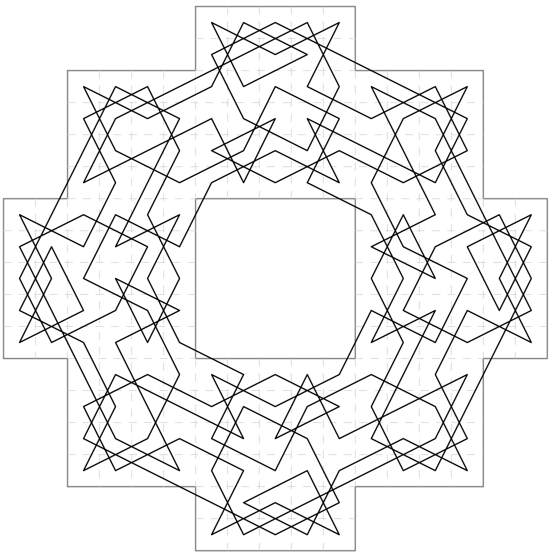
176 cells: This skewed-cross board (Jelliss 2003) makes visual use of the arithmetic decomposition 176 = 4×4×11 = 4×(4×8 + 4×3), being a composite of four tours 4×8 and four tours 4×3 arranged around a central single-cell hole.



180 cells: This tour (Jelliss 1998) uses 180 = 14² – 4×(2²).

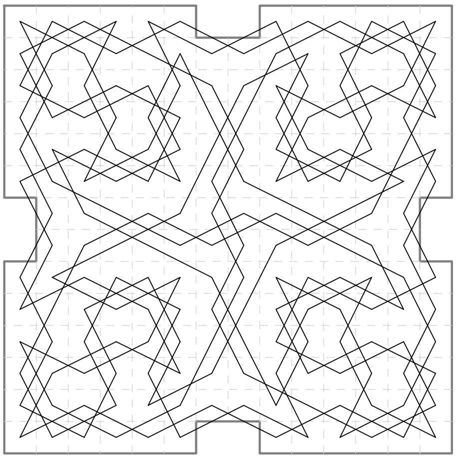


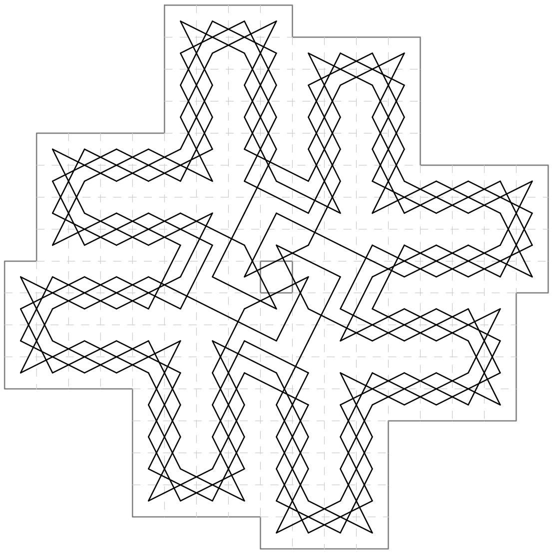
184 cells: This example (Jelliss 2003) uses 184 = 4×46 = 4×(16 + 30) = 4×(4×4 + 5×6).



Thus, similar to the 160 case, four 5×6 open tours were first joined together, then the squares and diamonds in the four corner 4×4s were joined in by Vandermonde's method.

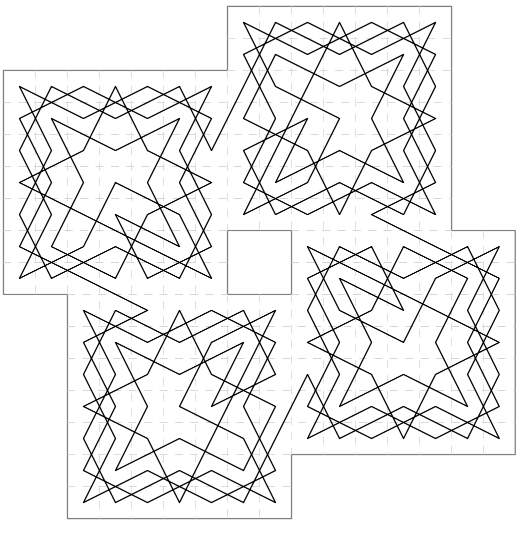
188 cells: The left-hand tour (Jelliss 1998) is 14×14 minus two cells in each edge.

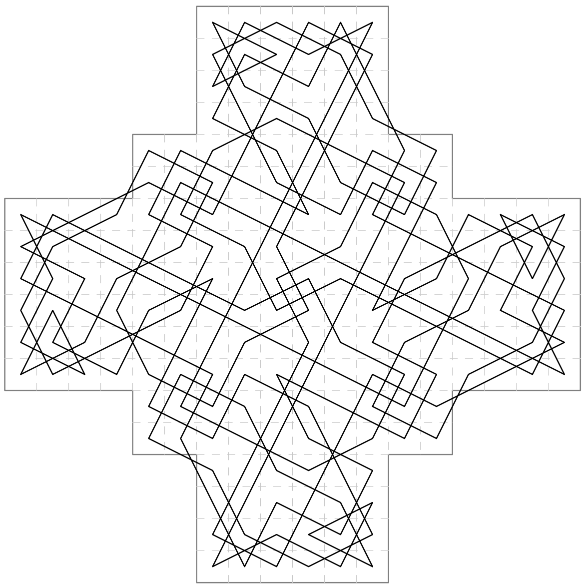




192 cells: The central pattern of the right-hand tour above (Jelliss 2003) is the 64-cell example shown earlier extended to fill 12 more 4×4 areas by the braid method.

196 cells: The 14×14 board of 196 cells admits birotary tours. This tour showing quaternary symmetry in similar fashion to the 144-cell example, uses the fact that 196 = 4×(7^2).





The celtic cross tour (Jelliss 2013) incorporates six complete Greek Crosses formed of knight moves, five of size 1 and one of size 7.

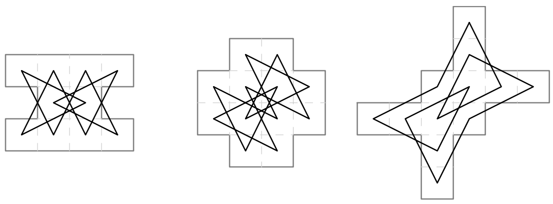
Biaxial

Here we gather examples of biaxial symmetry, also known as ‘direct quaternary symmetry’. This type of symmetry is impossible in knight tours of rectangular boards. Consequently it is less familiar than the birotary type and its study has been neglected. It requires shaped boards having nonrectangular outlines or with holes, or both. The axes can be diagonal or lateral.

It is easy to fall into the error of supposing that a tour with quaternary symmetry can only be possible on a board with a multiple of 4 cells, since it seems obvious that each quarter of the tour must have the same number of moves, m. This is true in the case of biaxial symmetry with diagonal axes, however with lateral axes a tour may be possible using $4 \cdot m + 2$ cells instead of $4 \cdot m$. This is because of the possibility of Bergholtian symmetry, in which the path crosses itself at the centre of the board, so that each of the two central moves contributes half a move to each quarter, thus making the total $4 \cdot m + 2$ possible. The containing rectangle from which the board is cut must be odd by even.

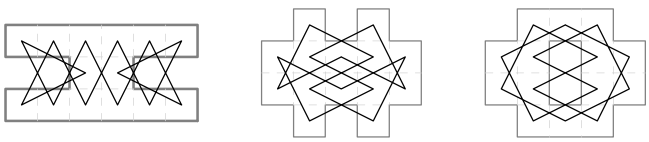
Such tours are in the form of a convoluted figure-of-eight or infinity sign. The mid-point of each loop of the ‘eight’ must be in a cell on the middle line of the odd side, and there can be no other cells on this axis, so the board is of ‘dumb-bell’ or ‘hour-glass’ shape. Apart from the 10-cell tour this type of symmetry does not seem to have been noticed before my study of the subject. All tours of this type with up to 50 cells and omitting cells only in the middle file are included in this collection.

10 cells and 12 cells: See the section on Smallest Tourable Boards in ⌘ 1, where these three tours also appear. Vinje (1949), Euler (1759), Jelliss (1995).

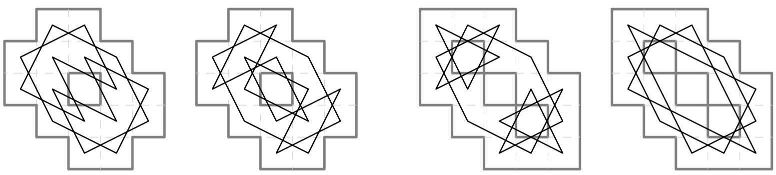


For biaxial symmetry where the axes are diagonal the tour must enter only two cells on each diagonal. If there are to be no holes in the board then the diagonal cells used must be adjacent and the other cells of the board must grow cross-like from this central 2×2 hub. See the entries of 20 and 44 cells for other examples of this type.

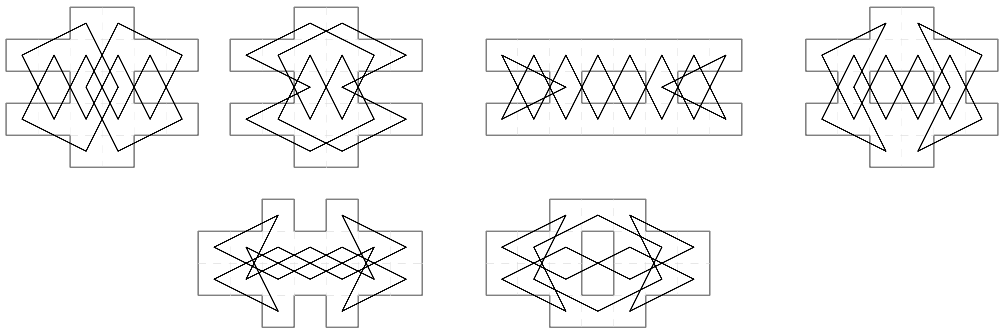
14 cells: Three tours with lateral biaxial symmetry similar to the 10-cell case.



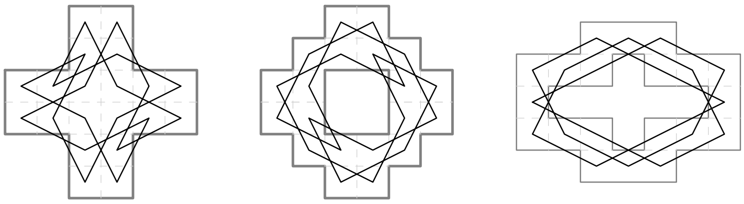
16 cells: Four tours showing biaxial symmetry with diagonal axes on two board shapes with holes. The second diagram is among Murray's notes.



18 cells: More tours with lateral biaxial symmetry (Jelliss 2016). One board has two solutions.

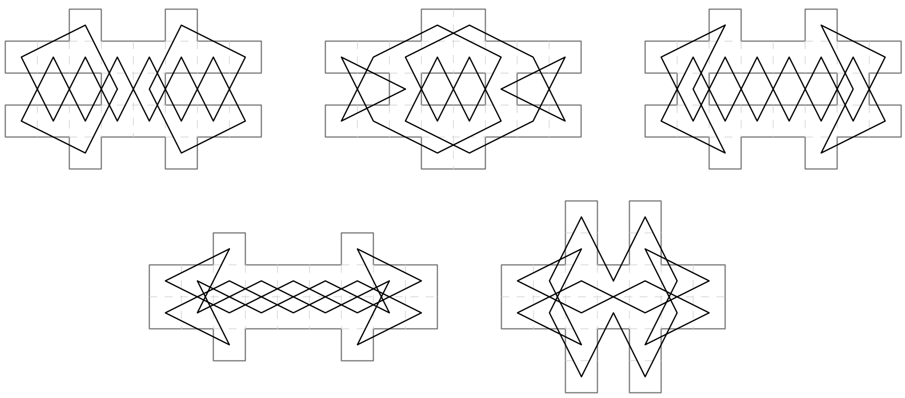


20 cells: The first, on a Greek cross (Euler 1759) and the second, with a 2×2 hole (Jelliss 2013) both have diagonal axes. The third (Jelliss 2017) with cross-shaped hole has lateral axes.

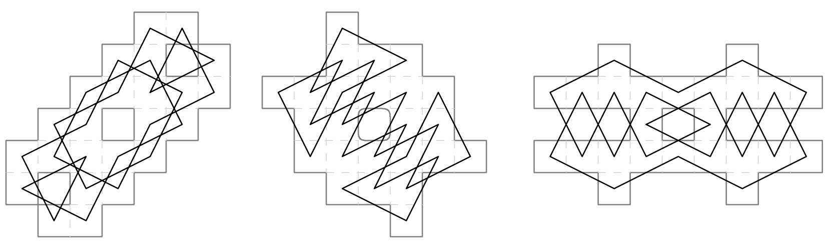


When both sides of the containing rectangle are odd (as in the third example) there is necessarily a central hole. Biaxial tours of 4·m cells with lateral axes are then feasible, the four quarter-paths each joining a cell on the vertical median to a cell on the horizontal median, these being the only cells on the medians that are used. (The 3×3 octonary star is a special case of this.) This 20-cell example is the smallest case.

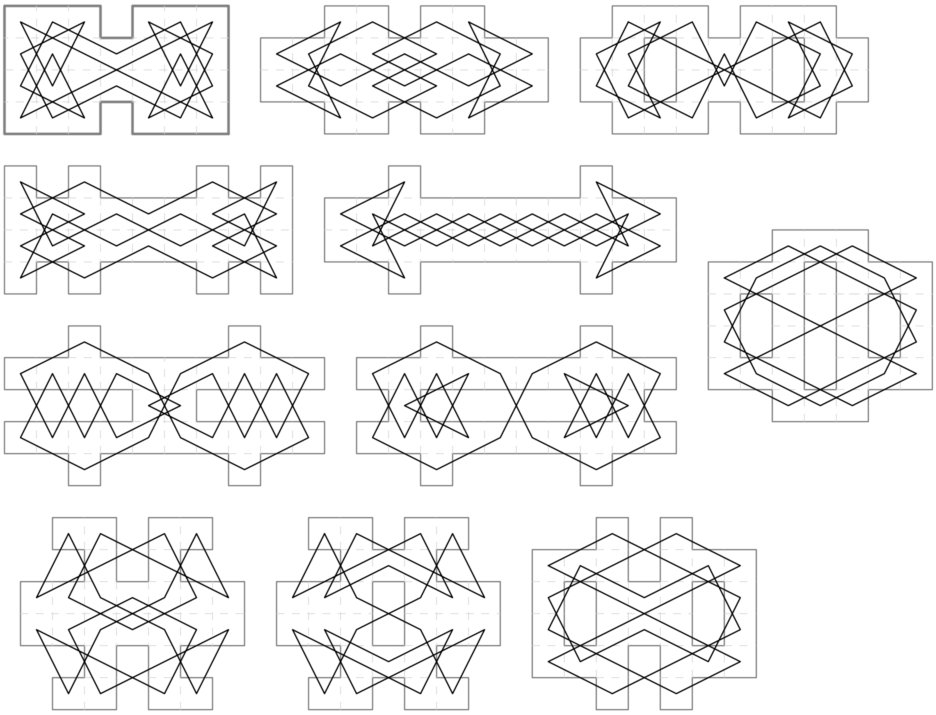
22 cells: More lateral biaxial symmetry (Jelliss 2016).



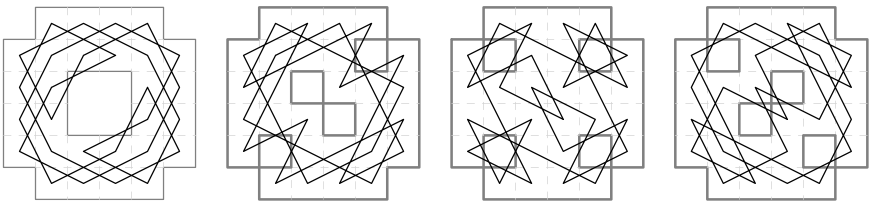
24 cells. Examples with diagonal and with lateral axes (Jelliss 2017). In the second example the pairs of moves at c3 and e5 can be 'folded out' to f6 and b2 respectively, leaving holes at c3 and e5 though the board is still connected. In the third the cells at d3, f3 can similarly be moved to b3, h3, leaving a 5-cell hole.



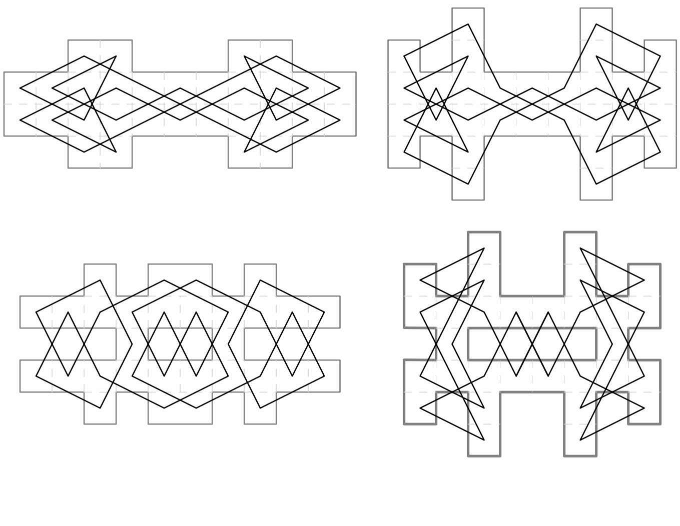
26 cells: Biaxial symmetry, with lateral axes, on assorted board shapes.



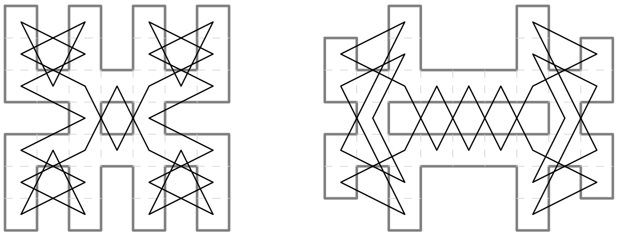
28 cells: Four tours with holes showing diagonal biaxial symmetry. All these boards have the same outline but different hole placements. The last two here are in Murray's notes.



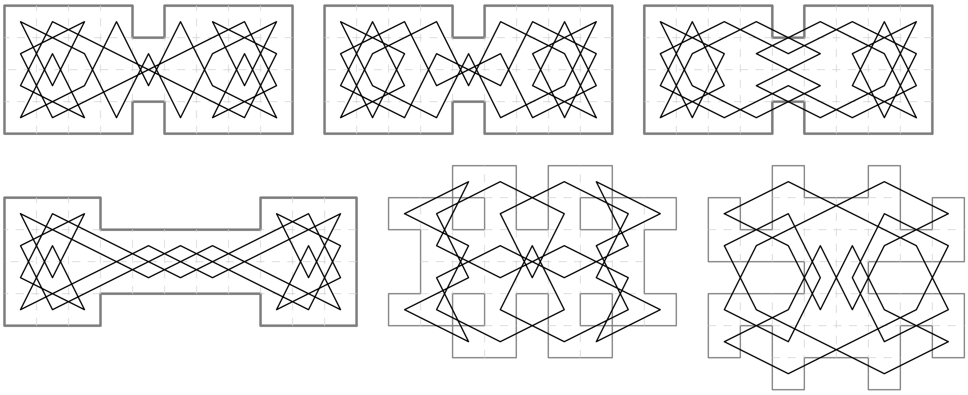
30 cells: Biaxial Bergholtian symmetry with lateral axes.



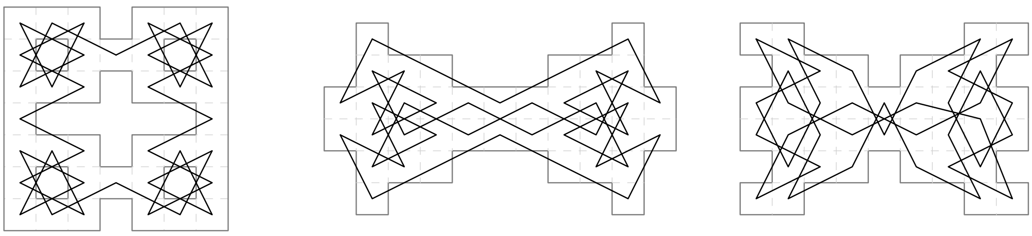
32 cells: Biaxial, lateral. Doubly Murraian. The second is similar to the 30-cell example above.



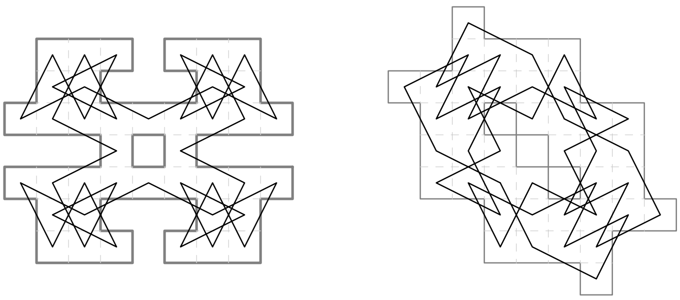
34 cells: Biaxial lateral. One shape has three tours.



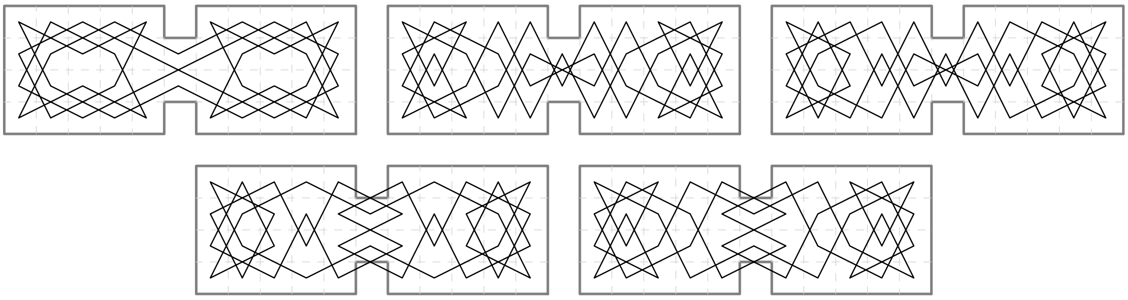
36 cells: Doubly Murraian. **38 cells:** Two Bergjholtian tours.



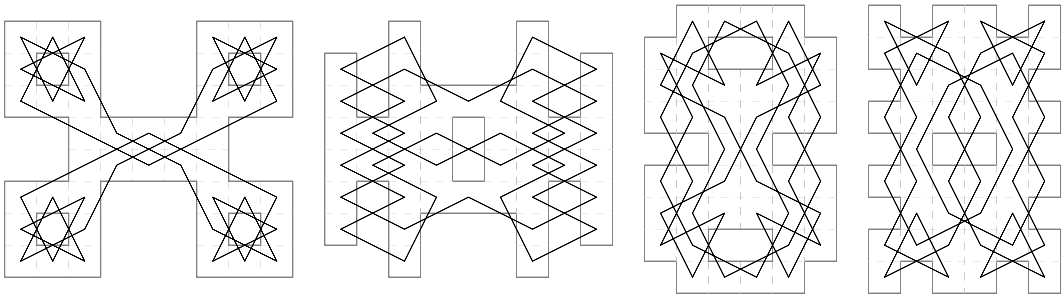
40 cells: Lateral (7×9 area) and Diagonal (Jelliss 2017).



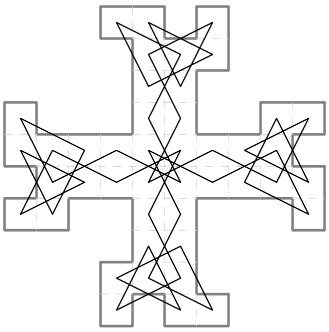
42 cells:. Biaxial, lateral type. There are five tours on the dumbbell board.

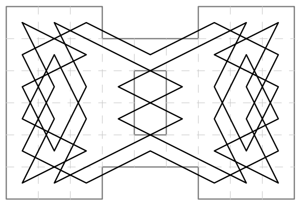


other assorted shapes:

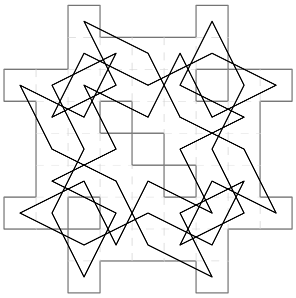


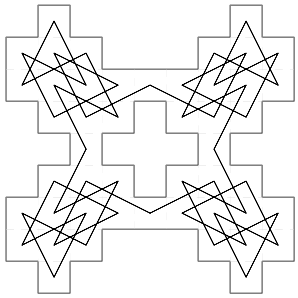
44 cells: Biaxial with diagonal axes (Jelliss 1999). (See earlier for two of 12 and one of 20 cells)
46 cells: Biaxial with lateral axes, Bergholtian.





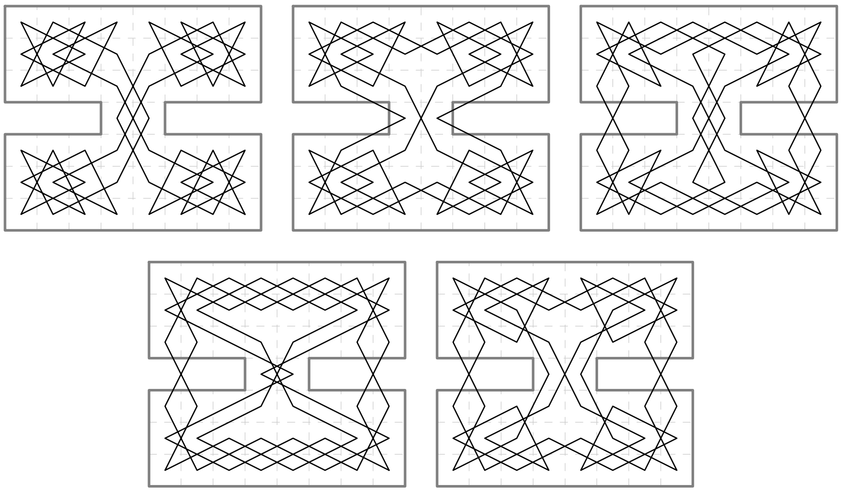
48 cells: Diagonal and Lateral Examples. In the diagonal case the pairs of moves on c4, d3, f7, g6 can be folded out to b1, a2, g9, h8 to leave two three-cell holes, and a new pair of single cell holes at b2, g8, though the board remains connected. (both Jelliss 2017).



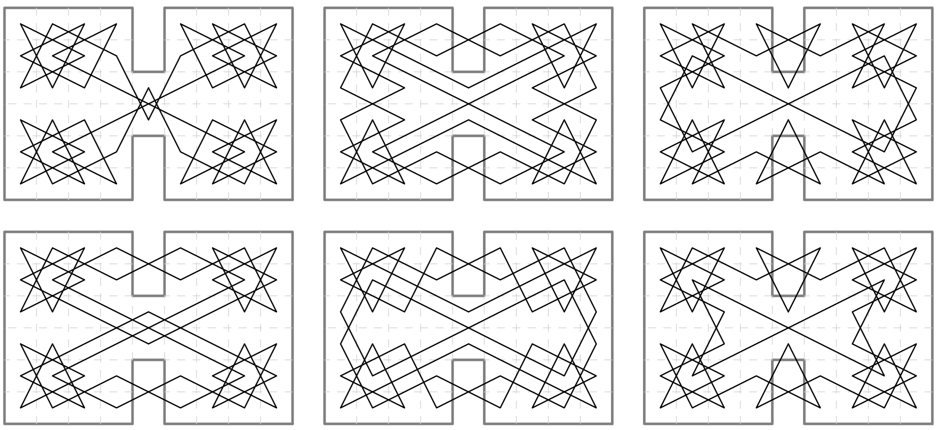


50 cells: This is the largest size of board on which I have constructed tours of the dumbell type with biaxial lateral symmetry. (See also 10, 14, 26, 34, 42). Three differently proportioned dumbells are possible: 7×8 (5 solutions), 9×6 (23 solutions),13×4 (9 solutions).

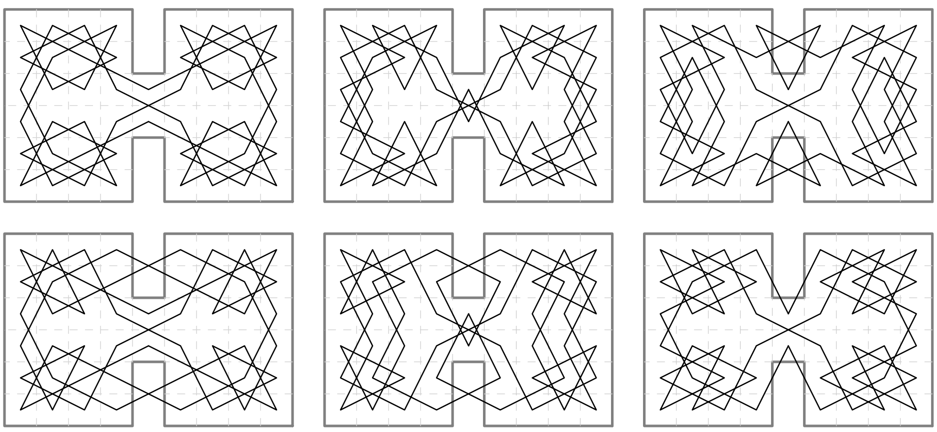
A. 7×8 case: 5 tours. In the striking first tour the knight's moves are all within the boundary.



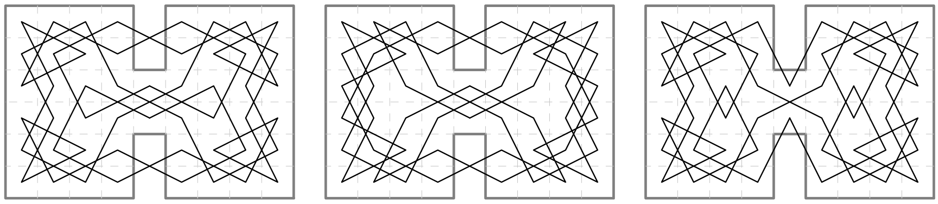
B. 9×6 case: (a) 6 tours with straight cross-centre moves.
Only the first is contained within the border.



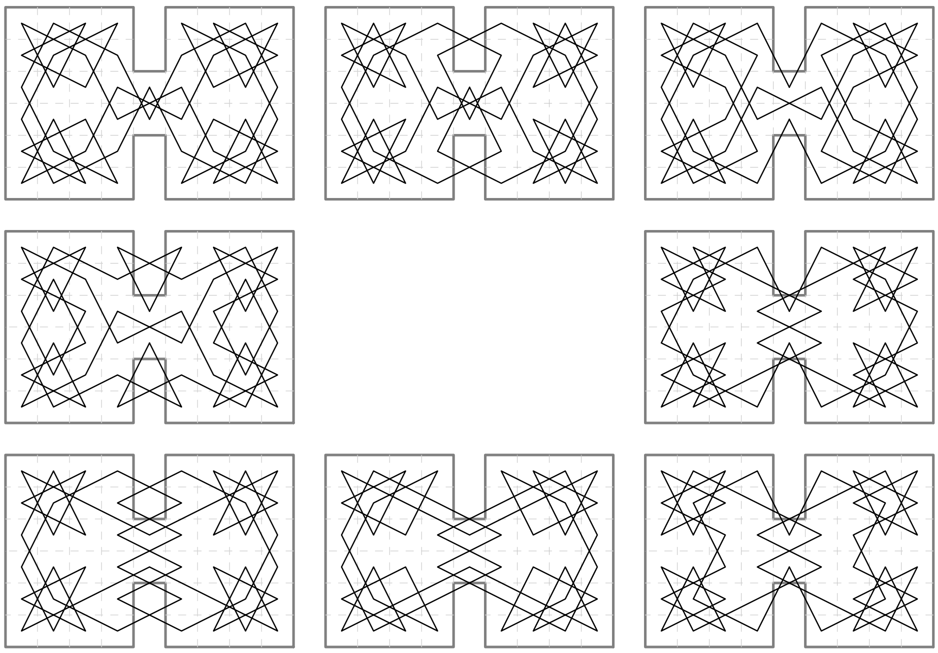
B. 9×6 case: (b) 9 tours with obtuse angled cross-centre moves.
The first two are within the border.



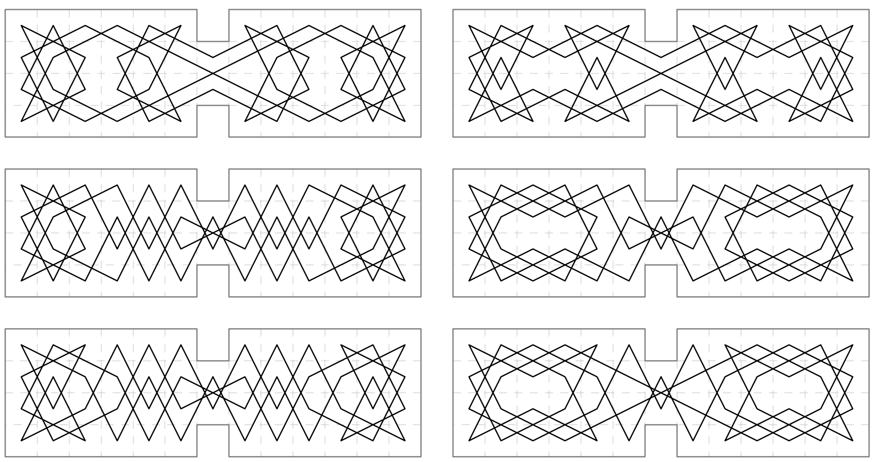
continued



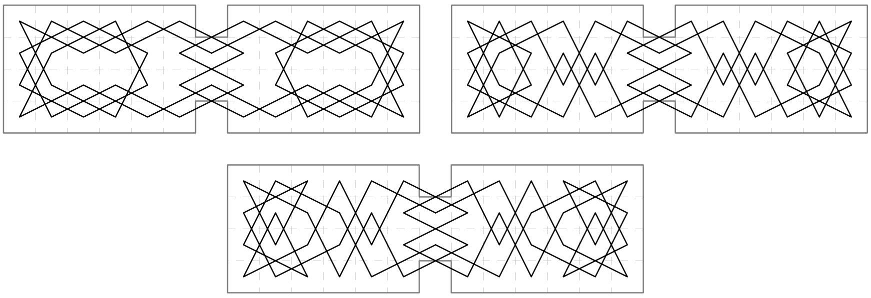
B. 9×6 case: (c) 8 tours with right or acute angled cross-centre. moves
The first one is within the border.



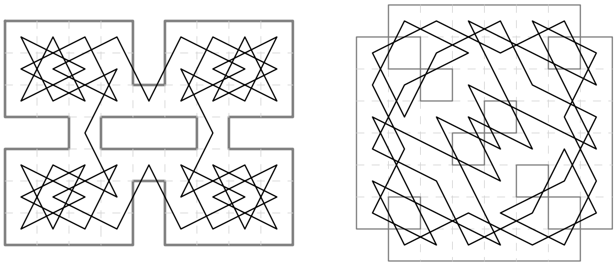
C. 13×4 case: 9 tours. Six are within the border.



and three others:



52 cells: Direct quaternary symmetry. Lateral type (Jelliss) with both sides odd 7×9. See earlier examples of 32, 36, 40 cells. Diagonal type from H. J. R. Murray's unpublished notes.

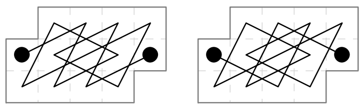


Rotary

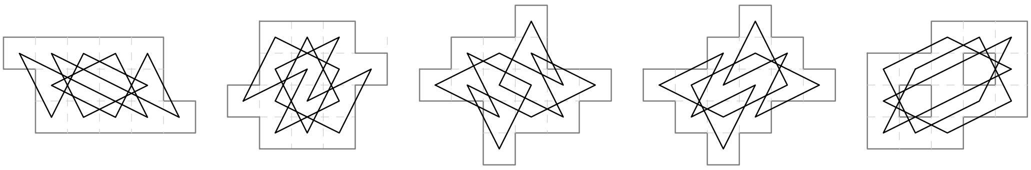
Here we collect tours that show binary rotational symmetry also termed ‘binary oblique’ or ‘diametral’ symmetry, which is invariant to 180° rotation, or a half-turn. It is either Eulerian where the path circles the centre point without passing through it, or Bergholtian where the path passes through the centre, twice. Bergholtian symmetry is closely related to open paths with centrosymmetry. These are of two types: those with an odd number of moves where the middle move crosses over the centre, at the middle of the edge of a cell, and those with an even number of moves in which the two middle moves form a straight line through the centre, at the centre of a cell.

For examples of **8, 9, 10, 11** and **12** cells see ⌘ 1, Smallest Knight-Tourable Boards.

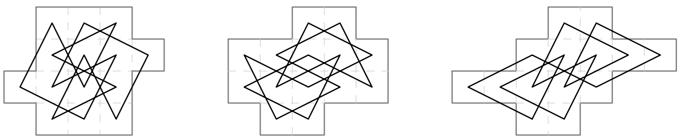
13 cells: Two rotary tours on a board omitting two cells from the untourable 3×5 board.



14 cells: Five Eulerian tours on five shapes, one with two holes.

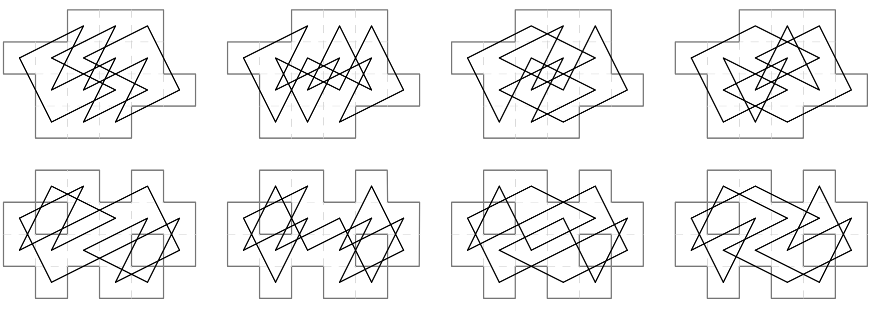


Three Bergholtian tours on three shapes.

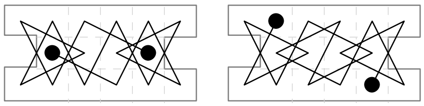


By deleting one of the middle moves these generate six 14-cell centrosymmetric open tours.

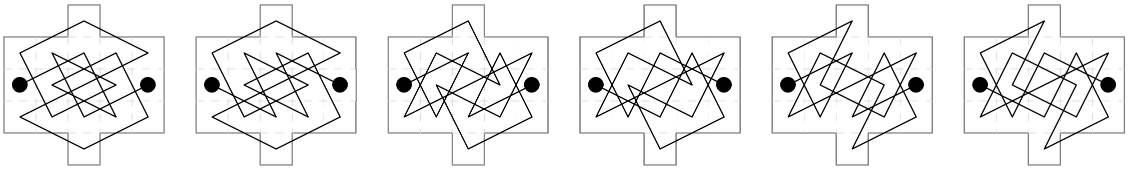
16 cells: The 4×4 board admits no complete tour, but moving two corner cells will produce two centro-symmetric boards with equal numbers of dark and light cells when chequered. One of these has four Eulerian symmetric closed tours (shown below). The other has no closed or symmetric tours but has 16 asymmetric open tours (See the Unary section). I also show the four symmetric tours of a holey board chanced upon while constructing the 20-cell tours shown further below.



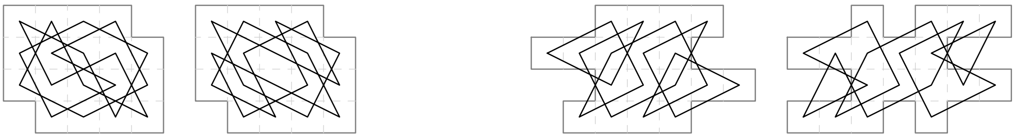
Two rotary tours on a board omitting two cells from the untourable 3×6 board.



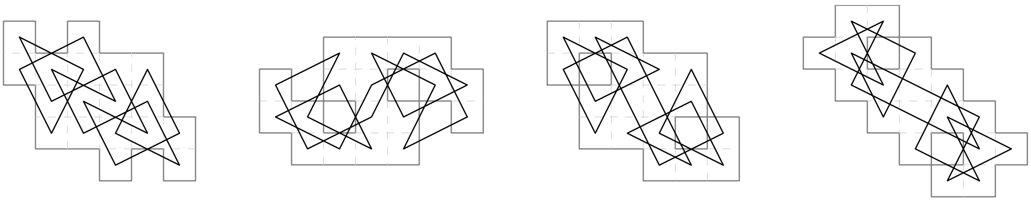
17 cells: The 3×5 with a cell added at the middle of each long edge.



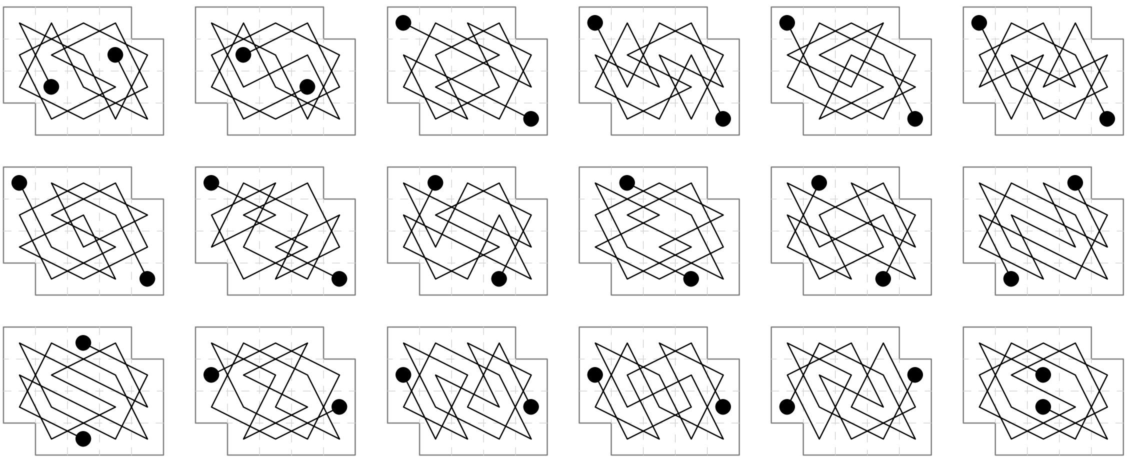
18 cells: There are two closed rotary tours on the 4×5 board with two corner cells omitted, one Bergholtian and one Eulerian. Some impromptu Eulerian examples including three with holes.



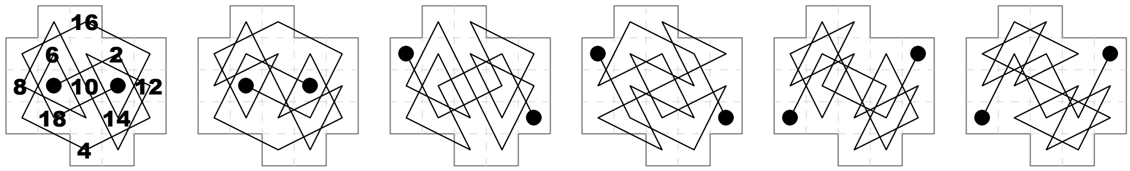
Some other Eulerian examples including three with holes constructed impromptu.



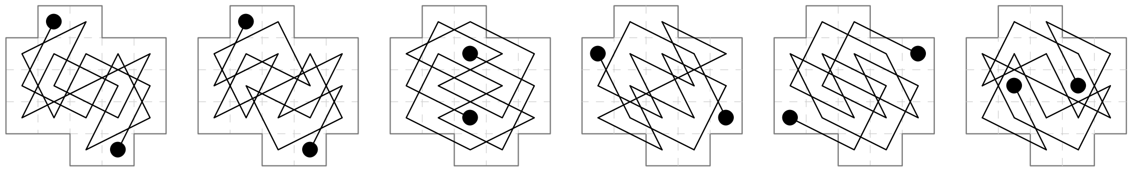
On the 4×5 board with two corner cells omitted, there are 18 open tours with rotary symmery, including two reentrant cases derived from a Bergholtian closed tour.



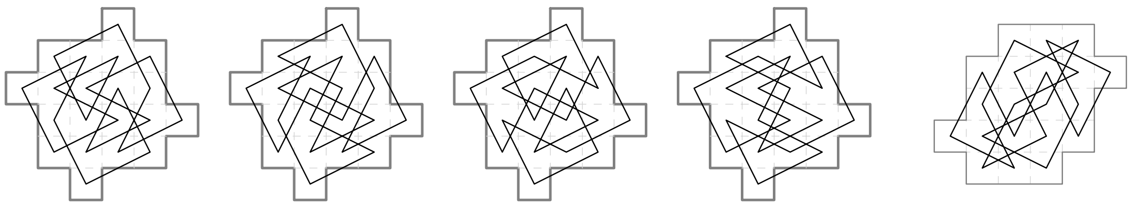
19 cells: The board formed of the 3×5 with a cell added at the middle of each edge has no tours for the same reason as the 3×5 board: formation of a short circuit b2-d2-f2-d4 on the minority colour. However if the end cells are moved next to the side cells to form a board with rotary symmetry many tours become possible, in fact 62, made up of 16 b3-d3, 14 a2-e4, 14 a4-e2, 12 b5-d1, 6 c4-c2 (Jelliss Feb 2018). In these rhe nine even numbers form a diagonal square, and in the six cases shown below this is a magic square! This research was stimulated by an 8×8 figured tour by T. R. Dawson (*Vie Riennaise* 1932) incorporating a magic square, so I wonder if the result is new.



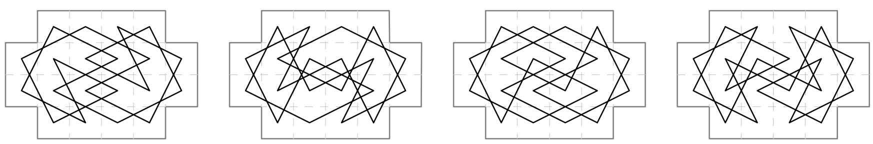
Six further examples from the 62 are diagrammed.



20 cells: The four 20-cell tours here (Jelliss 1996) on a board with birotary symmetry have only rotary symmetry. The first two have four 1-1 moves around the centre in what on a square board would be diagonal symmetry, and the other two have the 1-1 and the 0-2 moves in apparent lateral symmetry. However they are not examples of mixed quaternary symmetry because the board does not have diagonal or lateral axes. The other diagram shows a 20-cell board (4×5 with two corners moved) with a unique Bergholtian tour (Jelliss 2009). It also has 16 asymmetric tours.

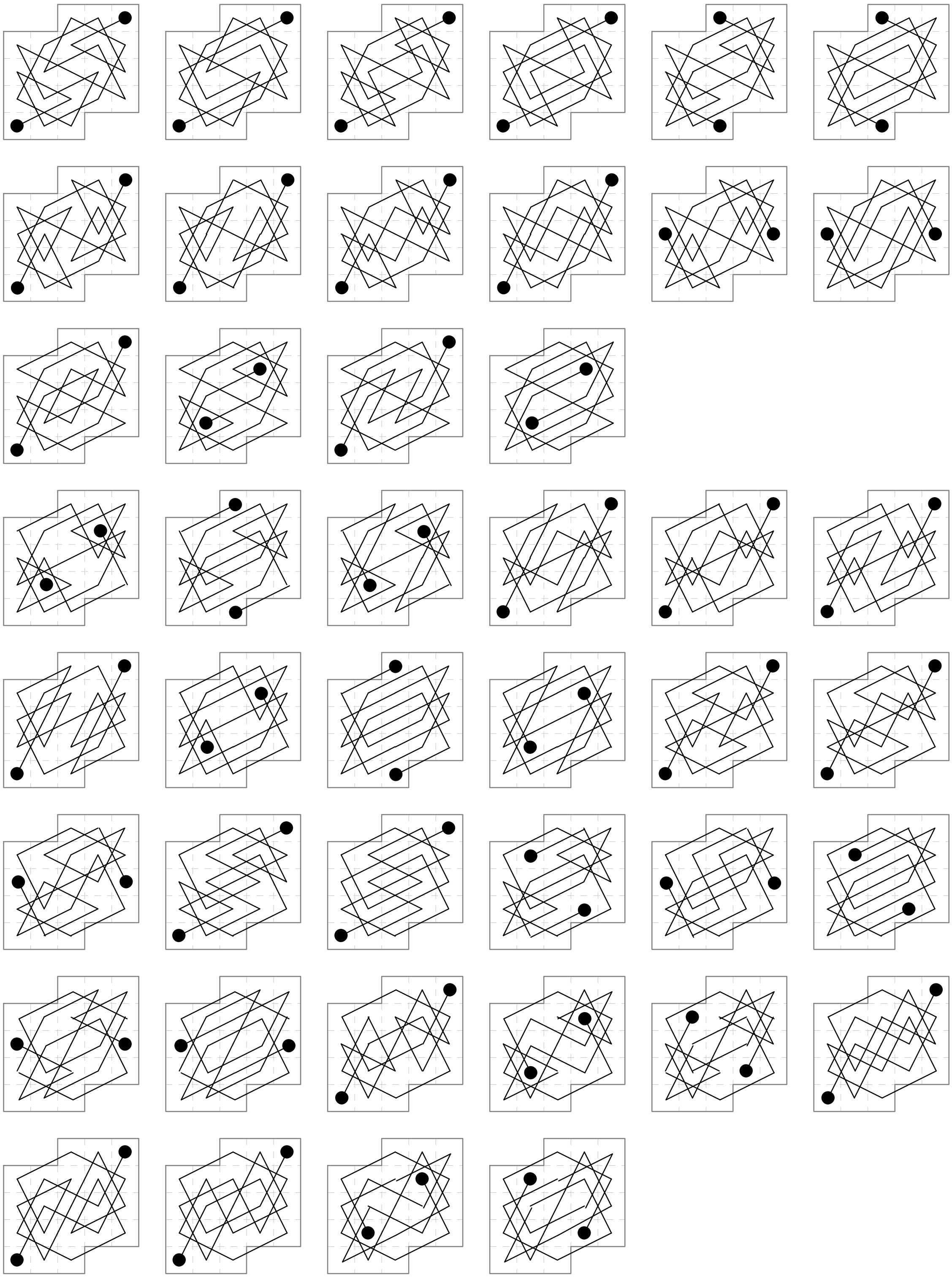


The 4×6 board with corners removed has four symmetric tours (Jelliss 2016) showing a type of ‘pseudo-mixed’ symmetry. Here the main symmetry is biaxial with the 1-1 moves deviating in the first two and the 0-2 moves in the other two.

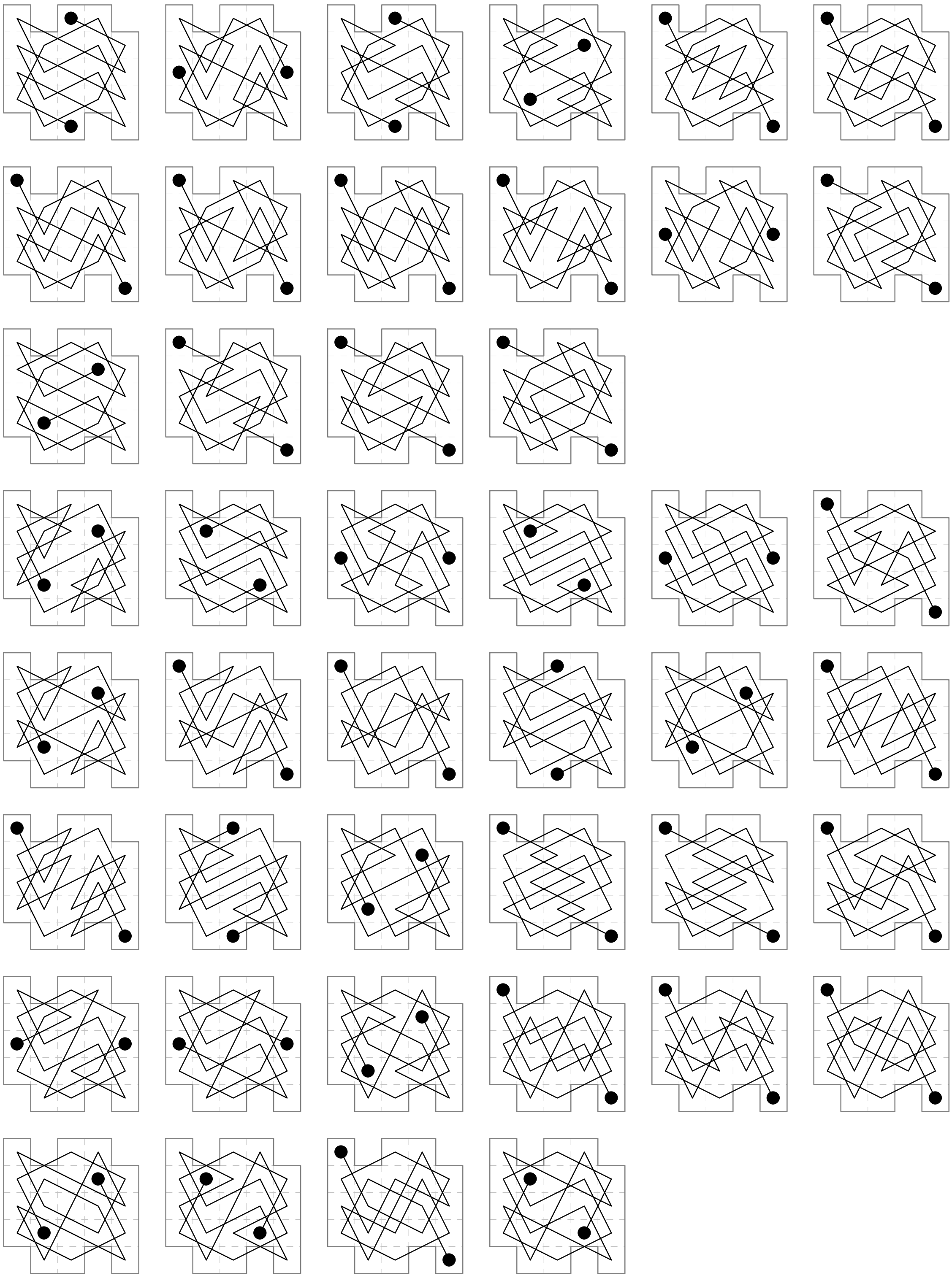


21 cells: Tourable centrosymmetric boards of 21 cells can be formed from the 5×5 with four cells removed at various places, but this does not include the board with the four corners removed, since when chequered it has 12 of one shade and 9 of the other. There are five ways of removing two of each shade that will admit rotary symmetric open tours. I have a note that the number of tours of each shape is: 44, 44, 4, 4, 8. Total 104. Diagrams follow.

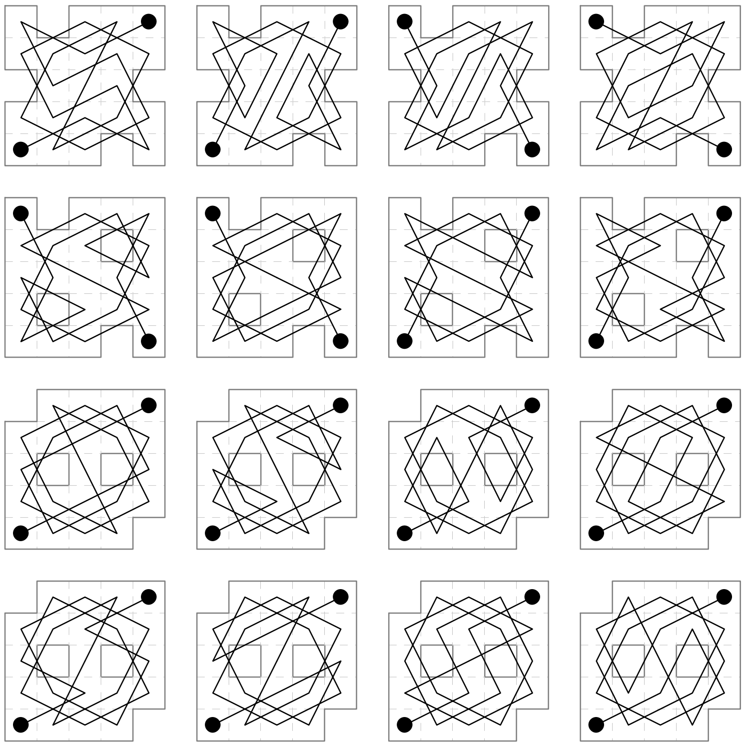
case 1



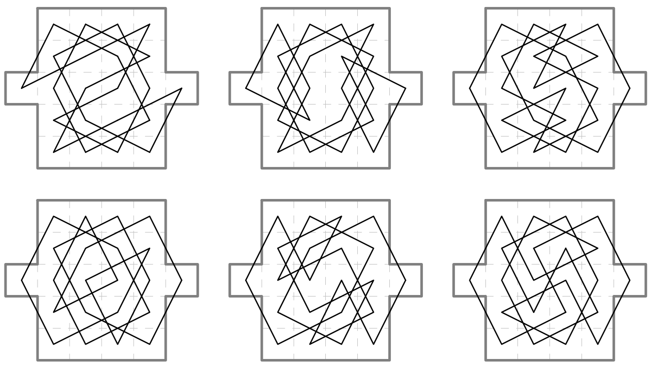
case 2



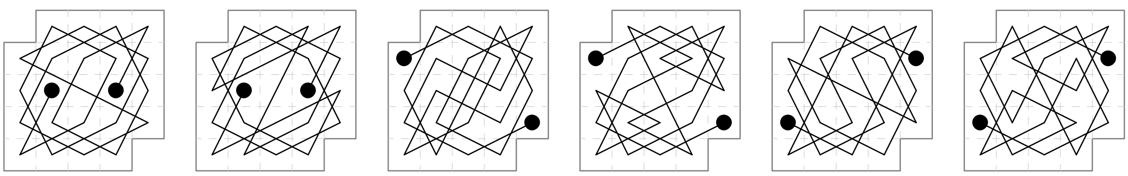
cases 3, 4, 5.



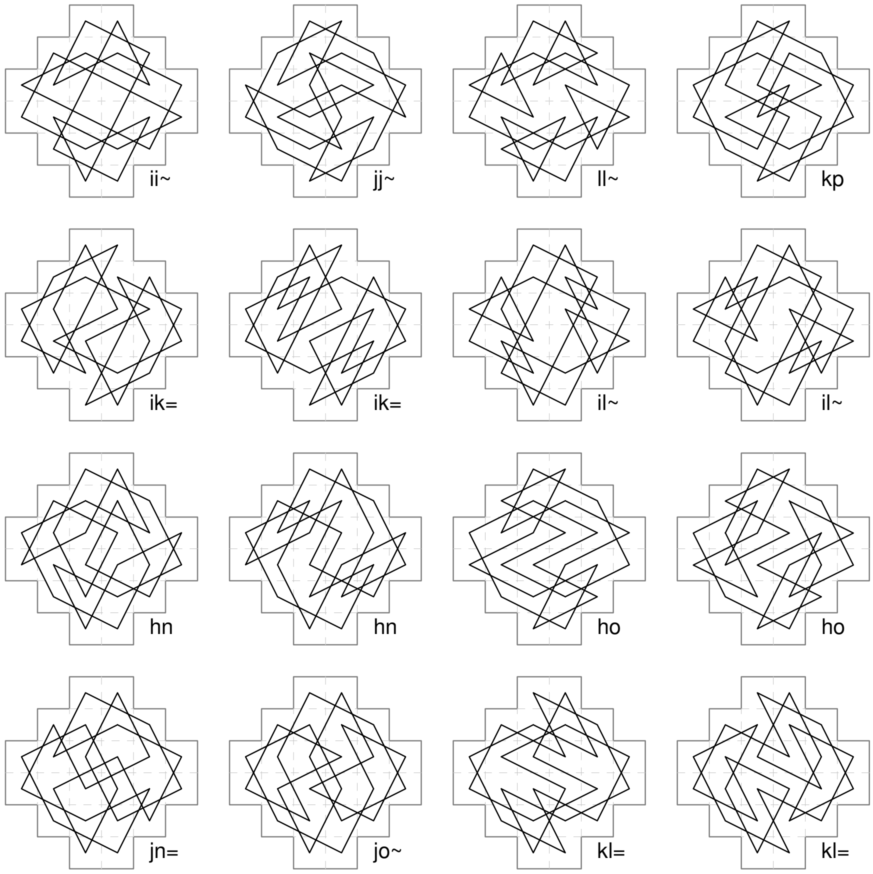
22 cells: The first tour here is given by Godron and Vatriquant in *L'Echiquier* Jan 1929 and has Eulerian rotary symmetry. I find four others of this type and one having Bergholtian symmetry.



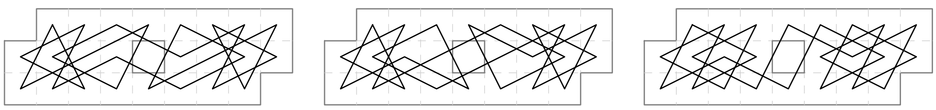
23 cells: Although on the 5×5 board there are only 8 symmetric open tours, all corner to opposite corner, by removing two opposite corner cells we now find 36 symmetric tours. The 8 with ends b3-d3 correspond to the 5×5 cases, but there are also 14 each with ends a4-e2 and a2-e4. We show two of each of these cases (for the rest see the website).



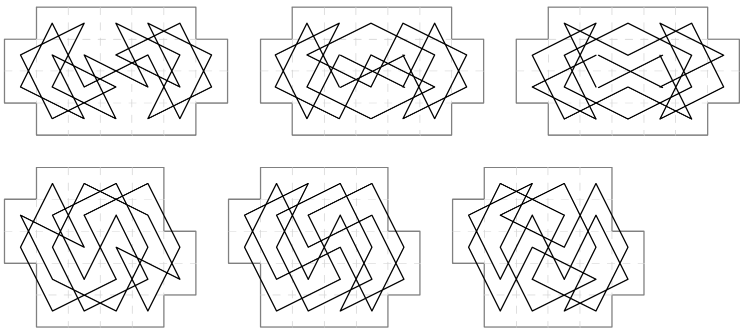
24 cells: This cross-shaped board seems to have been neglected. I found the following 16 symmetric tours Jan 2017 while compiling these pages. The letters indicate the central angles. The pairs with the same lettering differ only by the positioning of the 1-1 links.

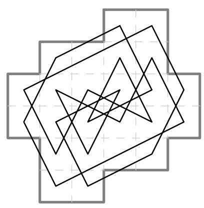


No tours are possible on the centreless 3×9 board due to colour mismatch, but by omitting two corner cells the three Eulerian tours shown become possible (Jelliss 2003).



The first tour here was constructed by Ernest Bergholt (1917) specifically as an example of Bergholtian symmetry on a non-rectangular board. The other two are my own (Jelliss 2009). These three are the only symmetric closed tours possible on this board. Three symmetric Bergholtian closed tours (Jelliss 2009) on a board-shape used for an asymmetric tour in *L'Echiquier* Nov 1928 .

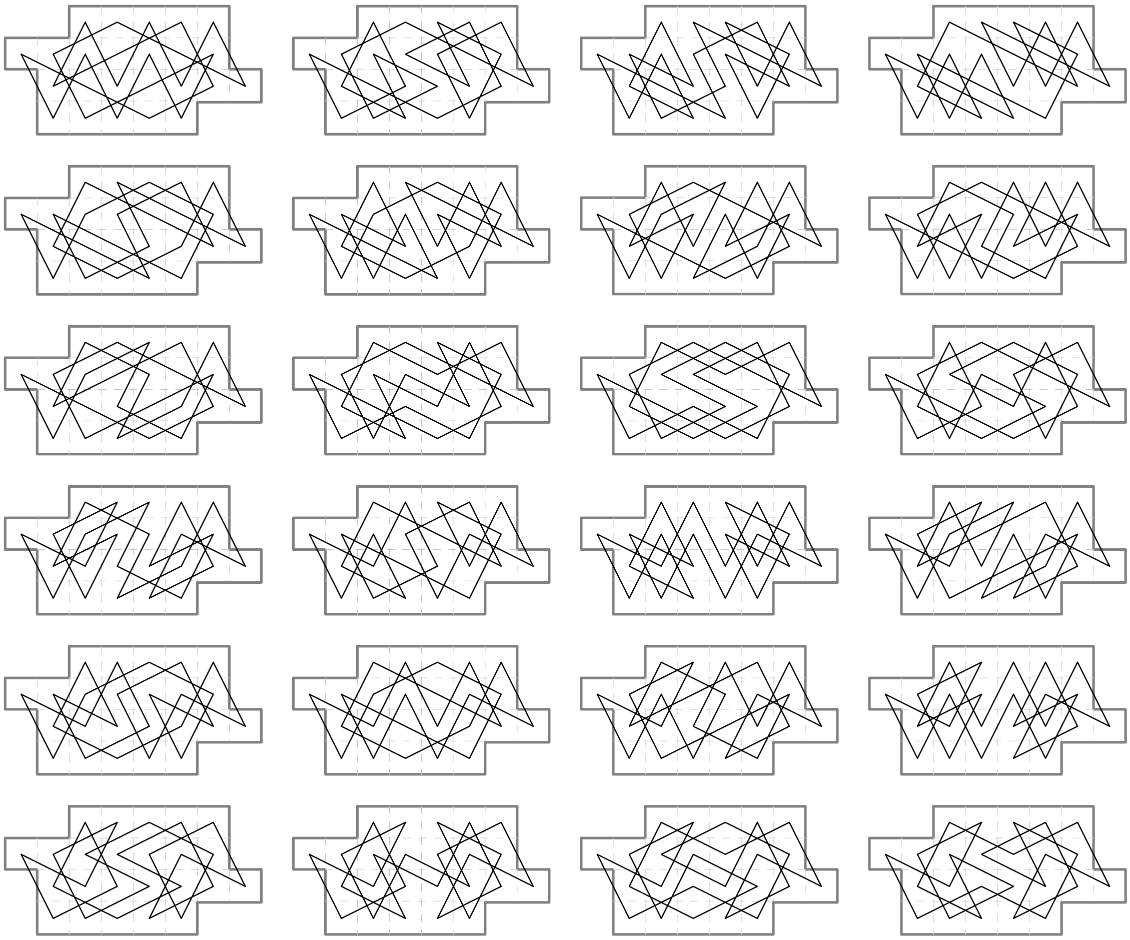




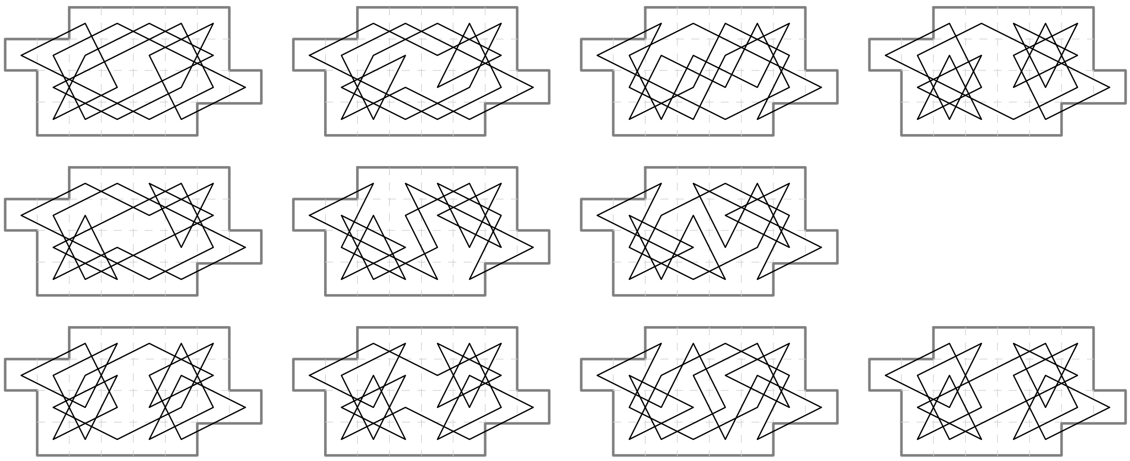
The other shaped Eulerian tour is by Godron and Vatriquant from *L'Echiquier* Jan 1929.

The following symmetric board, used for an asymmetric tour in *L'Echiquier* July 1928, is derived from the 4×6 rectangle with two corner cells moved. It has a large number of tours (total unknown). The following is an enumeration (Jelliss 2015) of the symmetric tours on this board. Total 56.

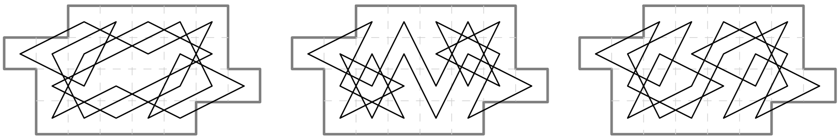
1. Tours with diagonal acute angle at end, total 24.



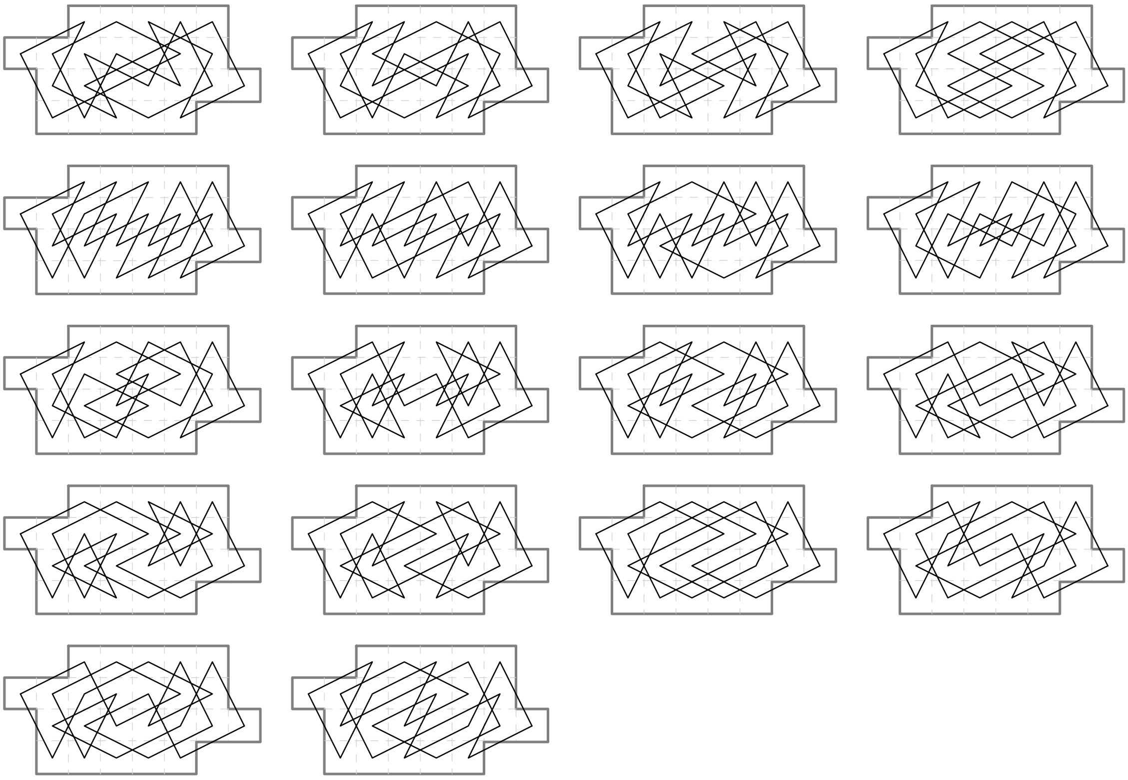
2. Tours with lateral acute angle at end, total 14.



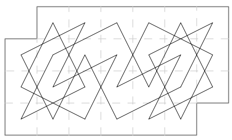
continued



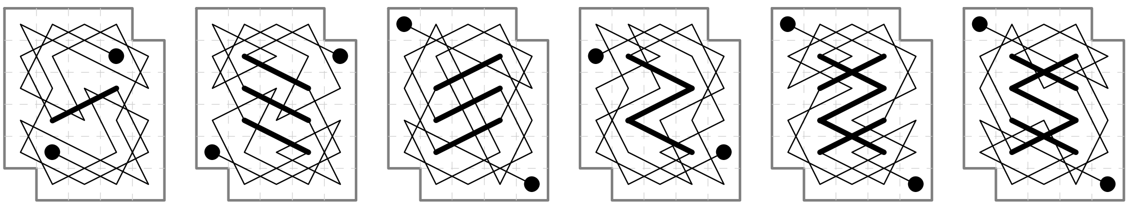
3. Tours with right angle at end total 18.



26 cells: Eulerian tour by Godron and Vatriquant from *L'Echiquier* Jan 1929.

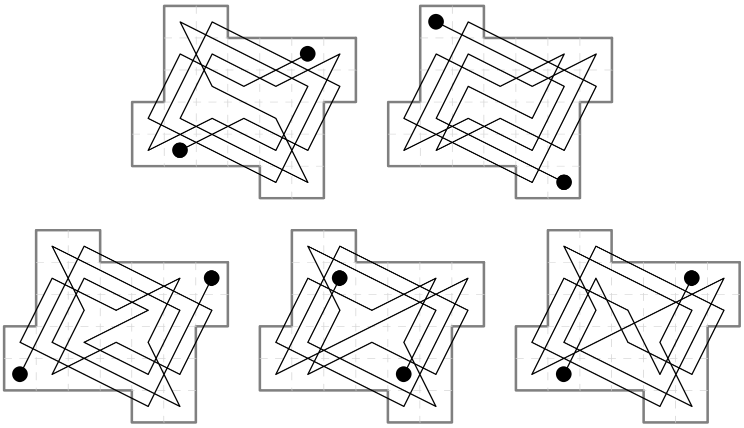


28 cells: Perhaps surprisingly there are no symmetric open tours on the 5×6 board. Tours of this type are however possible by omitting a pair of corner cells. Here are six examples, out of 82 I found, one for each selection of one, three or five ‘inner’ moves.

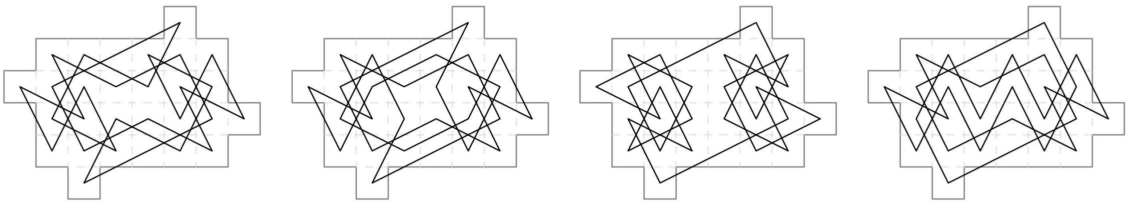


24 20 16 14 6 2

This amusing ‘leotard’ board has rotary symmetry but only has 5 symmetric open tours. The first two are from *L'Echiquier* Feb 1929 where they are attributed to Vatriquant, Post and Deprez, independently. The other three I found (2015).

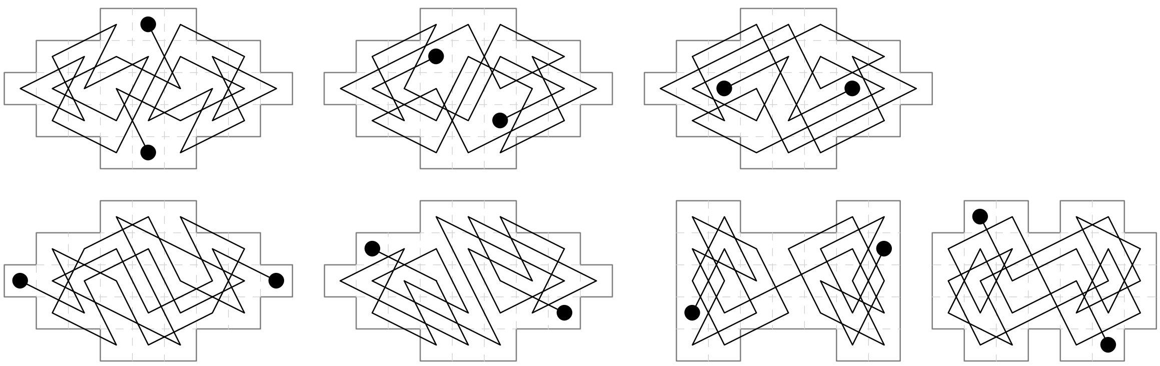


These four Eulerian tours (on a sort of ‘tanned hide’ shape) appeared in *L'Echiquier* Dec 1928 where they are attributed to four composers: Tolmatchoff, Marques, Godron and Vatriquant.

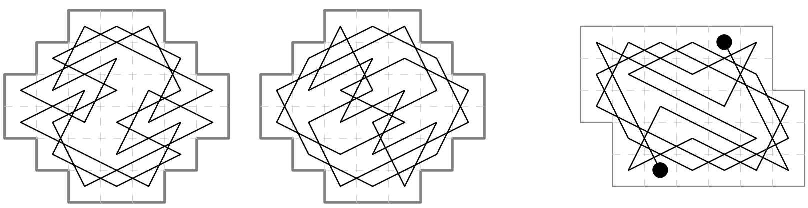


29 cells: Showing the five possible end-positions in an ‘oval’ shaped board.

And rotative open tours of two other board-shapes Suitable boards must have 15 cells of one colour and 14 of the other when chequered.

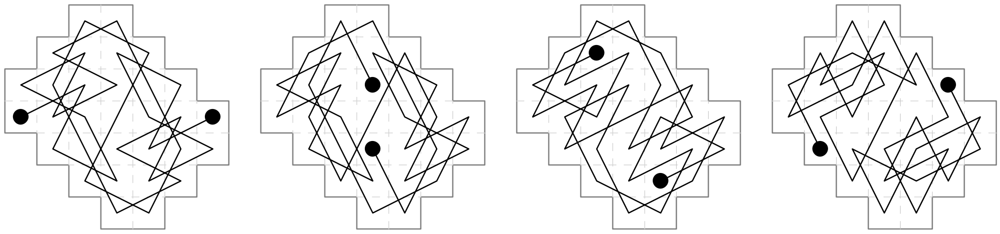


30 cells: Of course no tours are possible on the 4×8 with opposite corner cells removed, since there are 16 cells of one colour and 14 of the other. Here are an Eulerian and a Bergholtian tour by S. Vatriquant, *L'Echiquier* May 1929 on a shaped board. And a 31-cell board tour. See below.

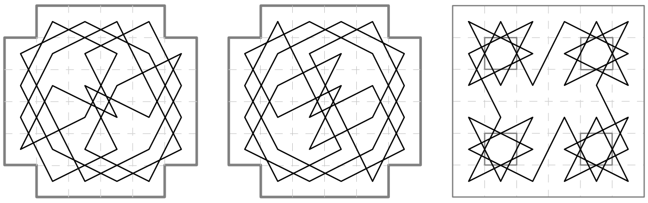


31 cells: The shape of two 4×4s overlapping at a corner has tours with diagonal axis but none centrosymmetric. A centrosymmetric tour of overlapping 3×6 boards is shown inset above.

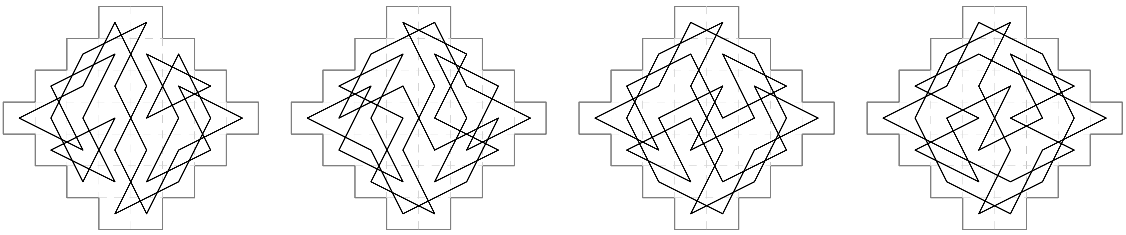
The serrated board below has two diagonal axes of symmetry, and when chequered consists of 3×5 and 4×4 diagonal arrangements intermeshed. The 4 distinct end-positions are shown.



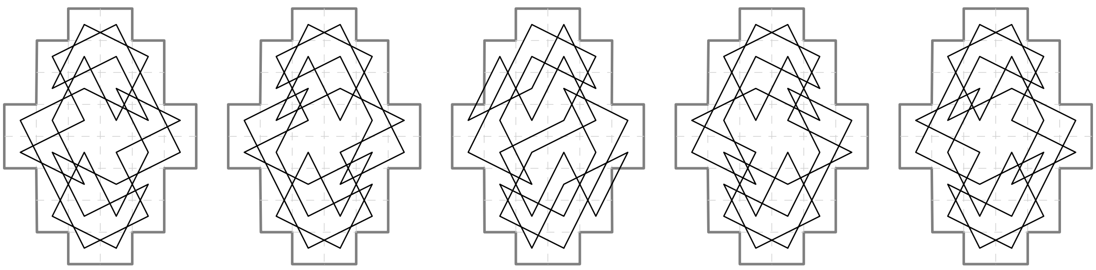
32 cells: Rotary tours on the 6×6 without corners by Euler (1759) and Jelliss (1999). Naidu (1922) gives a symmetric closed tour on the 6×6 board with four holes.



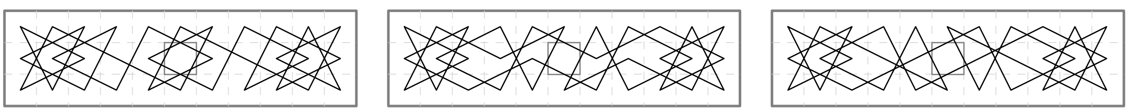
This 32-cell board, can be regarded as a ‘modal transformation’ (a sort of 45 degree rotation) of the white or black cells on an 8×8 chequered chessboard (as was pointed out by T. R. Dawson). It has just four distinct knight tours with Bergholtian symmetry (Jelliss 2009). They correspond exactly to the four {1,3}-leaper (camel) tours on the chessboard.



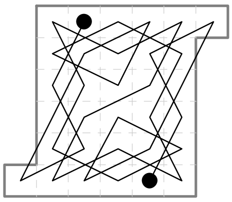
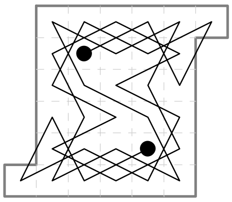
Five tours of a 32-cell board (Jelliss 2018) formed by moving corners of a 4×8 board to the mid-edge. These tours relate to the five magic tours of squares and diamonds type (see ⌘ 9) that have this board shape as their background pattern.



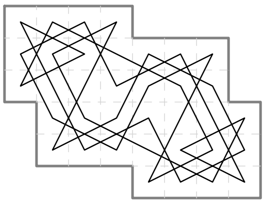
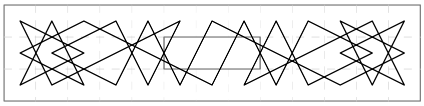
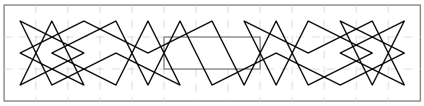
The 3×11, apart from 3×3, is the smallest odd-by-odd centreless board that admits tours.



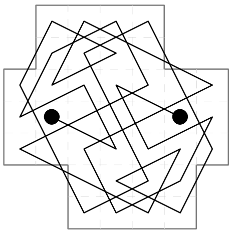
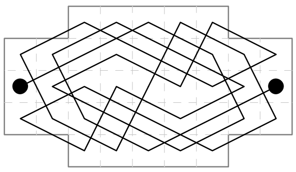
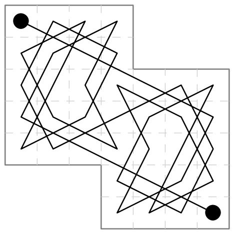
A 5×6 board with two attached cells. The first open tour shown here, by Post (*L'Echiquier* Mar 1929), the second on the same board (Jelliss (1999) has the middle move in a different direction.

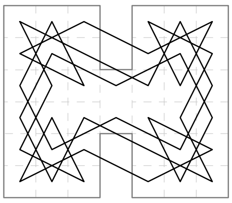
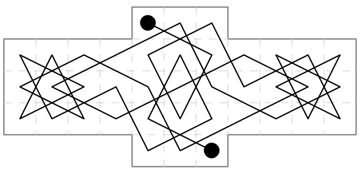
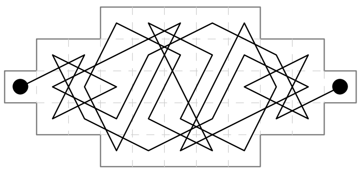
36 cells: This tour is from *L'Echiquier* Aug 1928. Tours are impossible on the centreless 3×13 board due to colour mismatch, but this can be overcome by omitting two more cells.

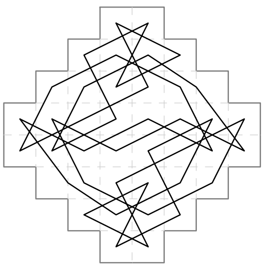
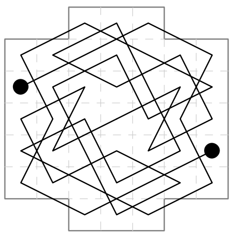
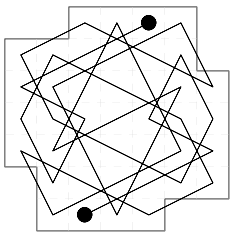
37 cells: The octonary 37-cell solitaire board has no knight tour since when chequered the colours are in the ratio 16 to 21. In these assorted boards the ratio is the required 18 to 19.

38-cells: H-shaped tour from Warnsdorf 1823 (rotated 90 degrees).
39 cells: Two open tours (Jelliss 2017).

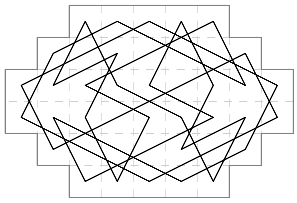




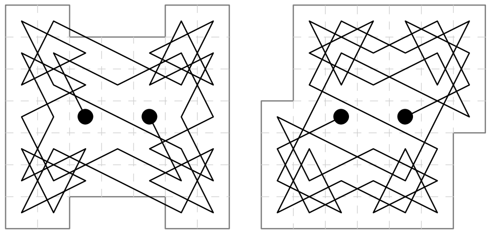
40-cells: A rotary tour of an octonary board (Jelliss 2017) formed by omitting 6 cells from each corner of the 8×8. Birotary is impossible on this board since 40/4 = 10 is even..
41 cells: Two open rotary tours on boards with biaxial and birotary shapes (Jelliss).

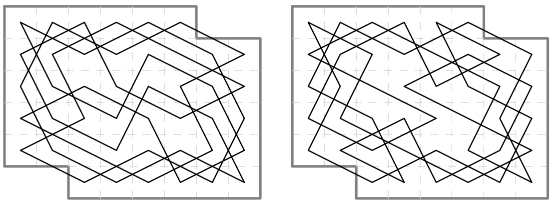
42-cells: A shaped board with Bergholtian tour.

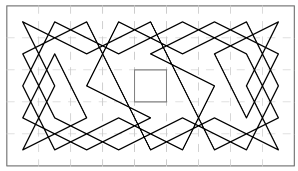
43-cells: Two centrosymmetric examples with different board shapes.



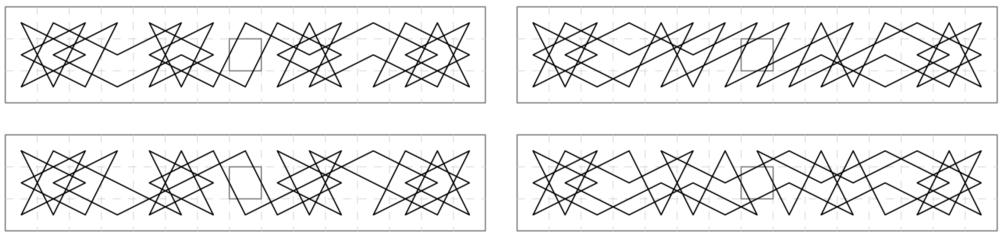


44 cells: Two more rotary examples from *L'Echiquier* Apr 1929. The first by Vatriquant and Post, the second by Godron and Tolmatchoff. Also a tour on centreless 5×9 board (Jelliss 2003).

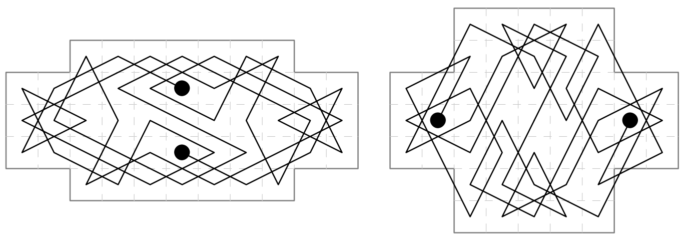




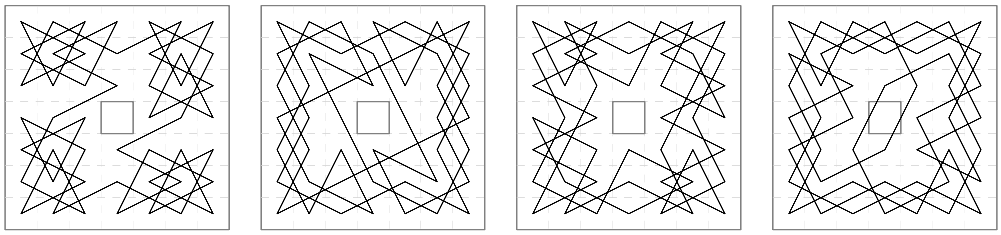
On the centreless 3×15 board there are 14 tours, of which I show four examples.



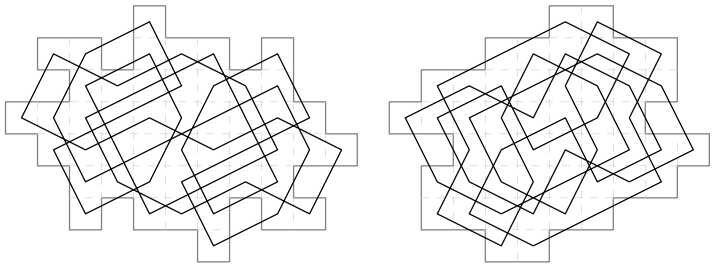
47 cells: Two centrosymmetric examples on cross-like boards.



48 cells: Quaternary symmetry is possible on the centreless 7×7 board but those shown here (Jelliss 2003) have the simpler (or is it more complex?) binary symmetry. See also 120-cells.

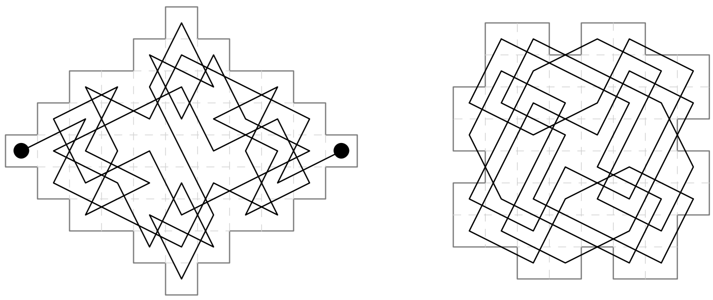


The next two tours on rather amorphous boards (Jelliss 29 May 2013), one Bergholtian and one Eulerian, use no acute angles. See also 52 cells.



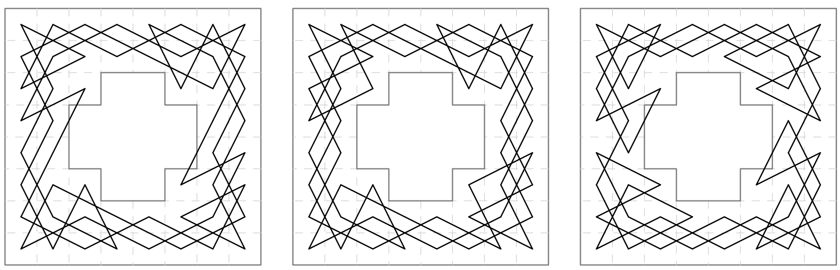
49 cells: A shaped 49-cell board is that consisting of two 5×5 boards overlapping at a corner. A corner to centre tour on each part can join to give both axial and centrosymmetric open tours (unlike the similar 31-cell board where centrosymmetry is impossible) [not diagrammed]

51 cells: A symmetric open tour on a serrated lozenge shape.

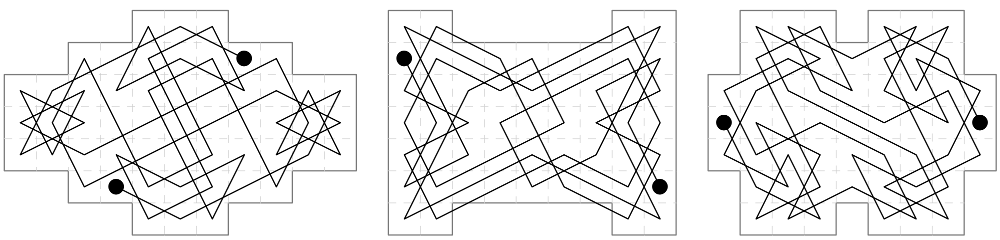


52 cells: Tour above right (Jelliss 30 May 2013) shows 'pseudo-mixed quaternary symmetry' and no acute angles. Only the 1-1 moves deviate from oblique quaternary, but their diagonal axes are not axes of symmetry of the board.

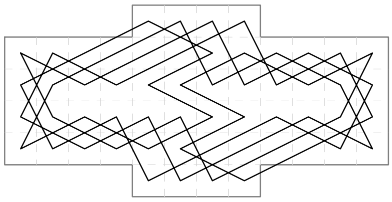
There are just these three geometrically distinct rotative tours (Jelliss 2015) on this 8×8 board with a cross-shaped hole.



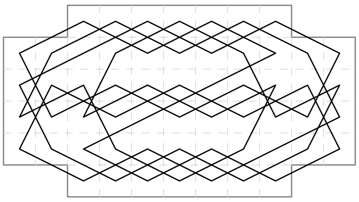
53 cells: Three centrosymmetric open tours on assorted board shapes.

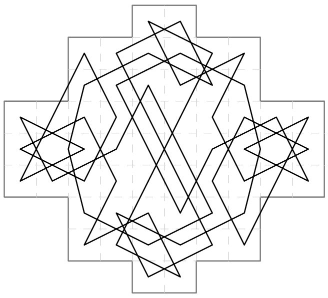


56 cells: Rotary tour by Michael Creighton (2013) on expanded oblong (4×12 + 8).

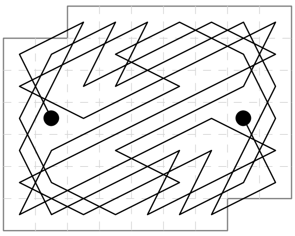


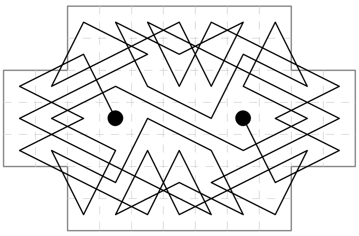
58 cells: Two boards with Bergholtian tours (Jelliss 2014).

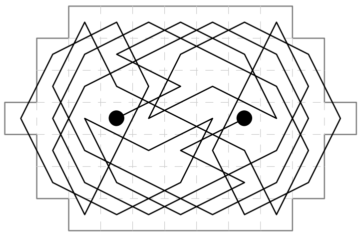




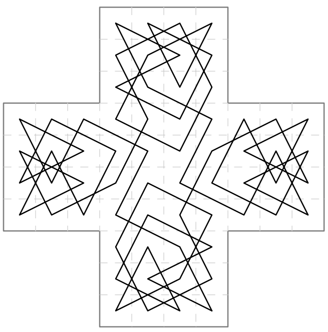
59 cells: Centrosymmetric open tour on a shaped 59-cell board (7×9 − 4).
61 cells: Two centrosymmetric open tours on 61-cell shaped boards (7×7 + 12).

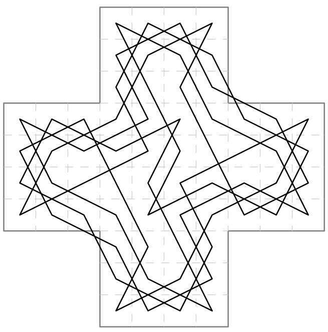






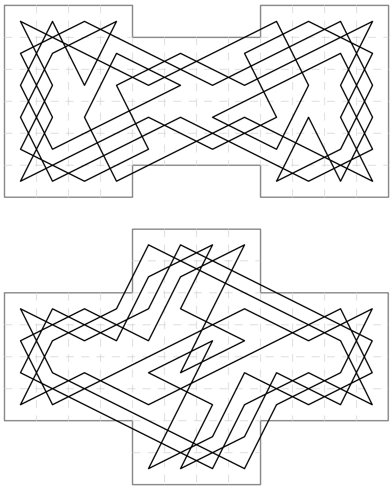
64 cells: Here are two tours with 180 degree rotational symmetry (Jelliss 2014) on an alternative 64-cell board of the truncated Greek Cross-shape used to designate medical services:

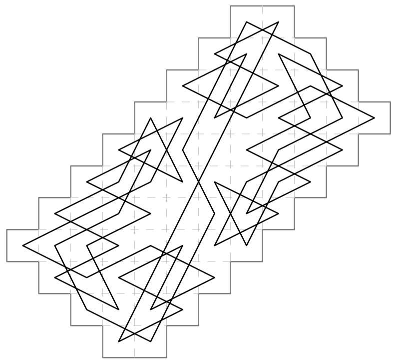




continued

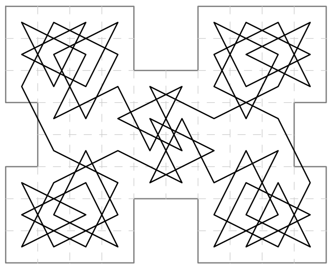
Two rotary tours on expanded oblongs (4×12 + 16) by Michael Creighton (published on the KTN website pages 2013). Besides fitting together with copies of themselves, the first fits with the 56-cell tour shown earlier and the second with the 96-cell board shown later.

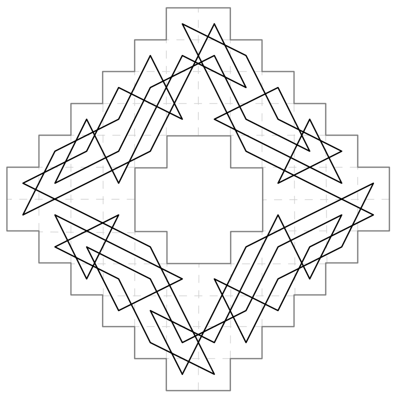




Board with rotary symmetry, the tour (Cognet *L'Echiquier* Oct 1930) being of Bergholtian type.

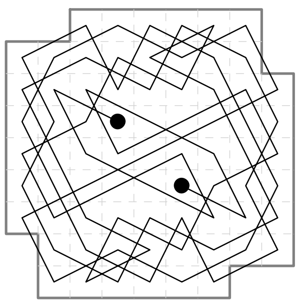
68 cells: A rotary tour by Michael Creighton (2013). This fits with the 92-cell tour shown later.





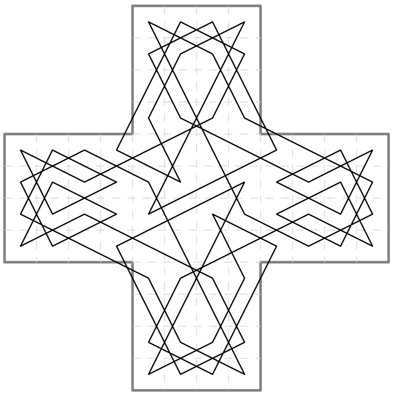
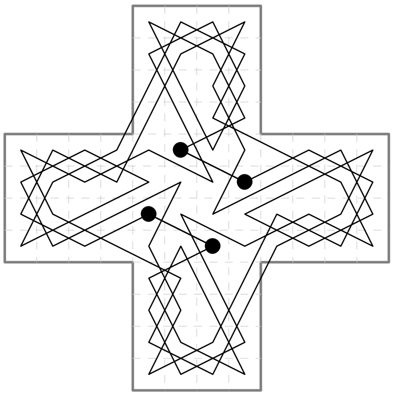
72-cells: A board, above right, with octonary symmetry, depicting a lake with a platform in the centre, used for an asymmetric tour by the Rajah of Mysore, but here showing one with rotary symmetry (Jelliss 2009). This is a board within an even area, 12×12, on which bitotary symmetry is impossible, since 72 = 4×18 and 18 is even. For another 72-cell ring tour see the 120-cell entry.

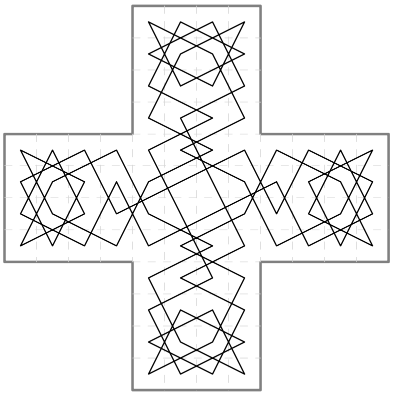
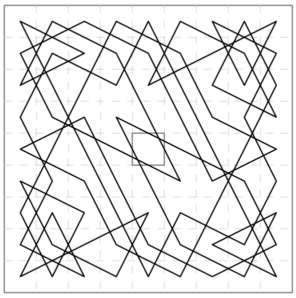
73 cells: An open symmetric 73-cell tour from *L'Echiquier* Jan 1927 (prob.161).



This was used as a cryptotour spelling a quotation from Gauss: "La mathematique est la reine des sciences et l'arithmetique la reine des mathematiques" one letter to a cell.

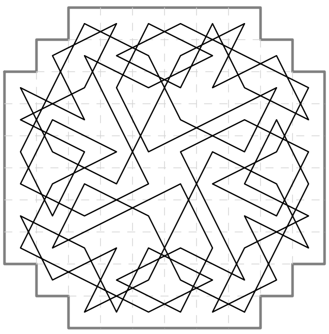
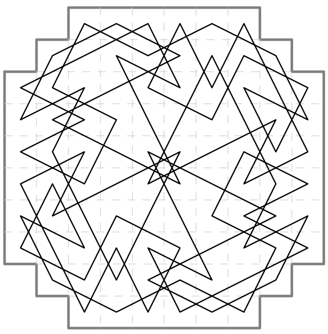
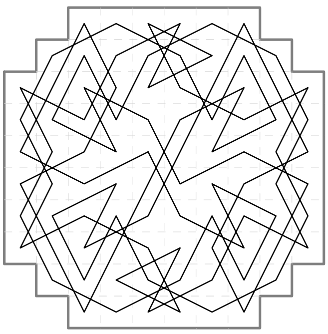
80 cells: The Greek Cross tours below show rotary symmetry. The upper two are by Ernest Bergholt from *Queen* 22 Jan 1916 and from his First Memorandum 24 Feb 1916. A handwritten note by Bergholt alongside the first diagram on the copy of his article in Murray's collection asserts: “I have since constructed several still more remarkable for their elegance” but no others are given there. He may have meant tours with mixed quaternary symmetry, of which Murray composed many, or may refer to his 84-cell tour with birotary symmetry. The lower tour here is by H. J. R. Murray from his 1942 manuscript “selected from some 60 tours on this board in mixed quaternary symmetry composed by the author”.

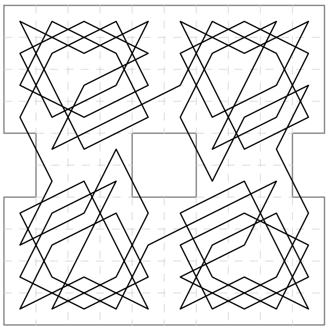



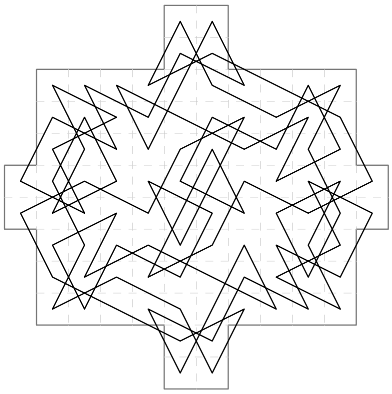
Binary tour on the centreless 9×9 board (Jelliss 2003): See also birotary section.

88 cells: These two tours with oblique binary symmetry by Bergholt are in his notes on symmetry sent to Murray in 1917. They bear the titles ‘Terra Ignota’ and ‘Terra Cognita’. The latter is a tour showing mixed quaternary symmetry, with h:j:k values 3:6:13, the h-moves being a4-c3-b5 and d1-f2 and their reflections in the vertical and horizontal medians, the obvious octonary pattern j-moves being typically b2-d1, d2-e4-f5. The third tour (Jelliss undated) is more obviously binary.

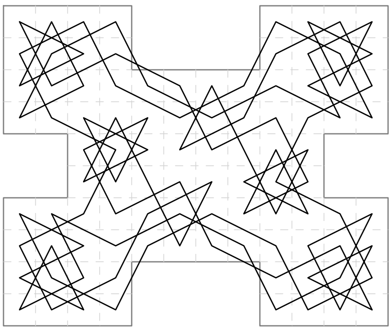
92-cells: Two tours by Michael Creighton (2013). They relate to the 68 cell tour shown earlier.

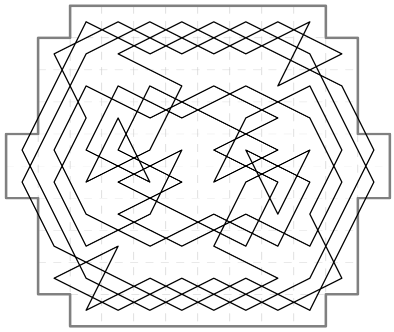




96-cells: An X-cross by Michael Creighton (2013).

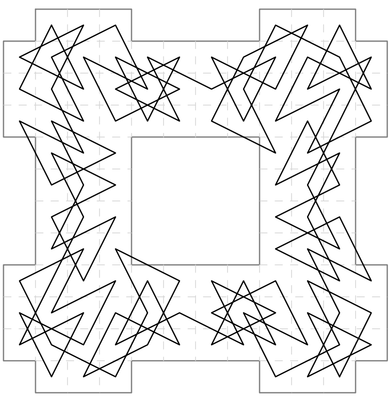
100-cells: A board 10×10 with the four corners moved to the middle of two sides, with closed rotary tour from *L'Echiquier* Nov 1929 by Monsieur Post.

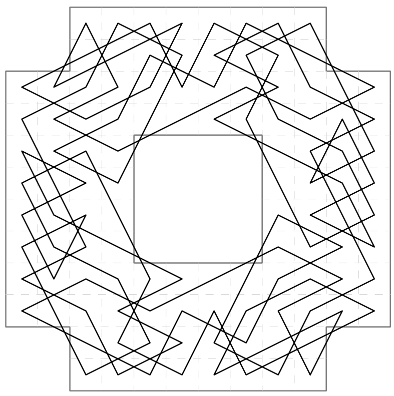




108-cells: A 'hashtag' board with a rotary tour by Michael Creighton (2013).

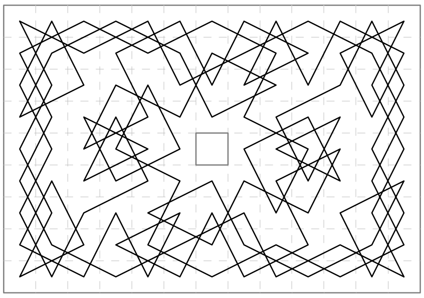
112-cells: A board (144 – 32) used for an asymmetric tour #12 in Harikrishna 1871. Rotary symmetry (Jelliss 2009) is shown on the same octonary board. It does not admit quaternary symmetry. This tour is not in mixed quaternary, as shown for example by the moves of type a4-c5.

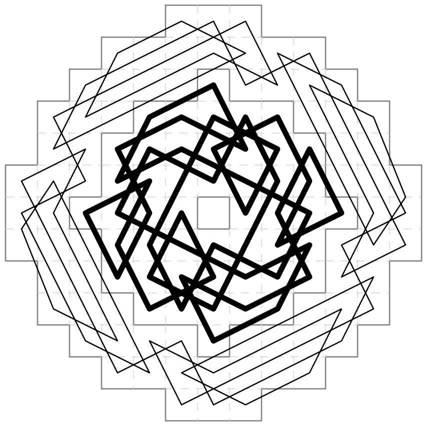




116 cells: Another in my series of centreless odd-by-odd board tours, on the 9×13. (Jelliss 2003)

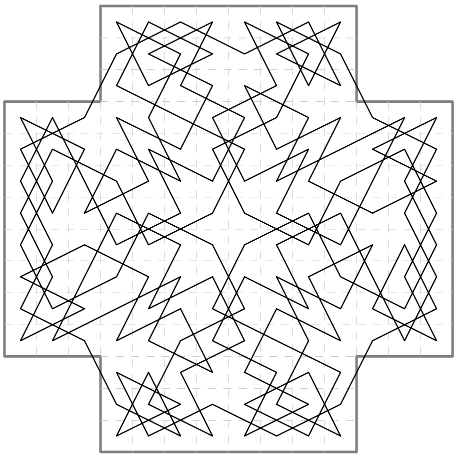
120 cells: A centreless board of 120 cells showing a pseudotour consisting of a rotary 48-cell tour (bold) surrounded by a birotary tour of a 72-cell ring (Jelliss Feb 2017). The board shapes are from my article 'Circling the Squares' in *Games and Puzzles Journal* #29 (online) Sep-Oct 2003.

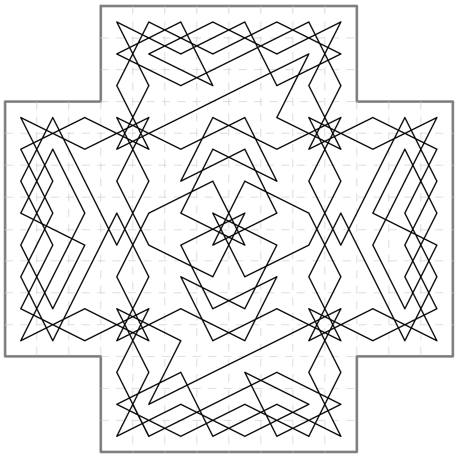




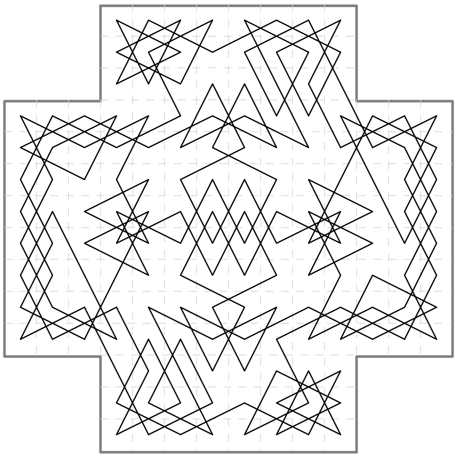
160 cells: Edward Falkener (1892) gave a collection of 18 tours showing approximate quaternary or octonary symmetry on this four-handed chess board. However only #3 has true binary symmetry.

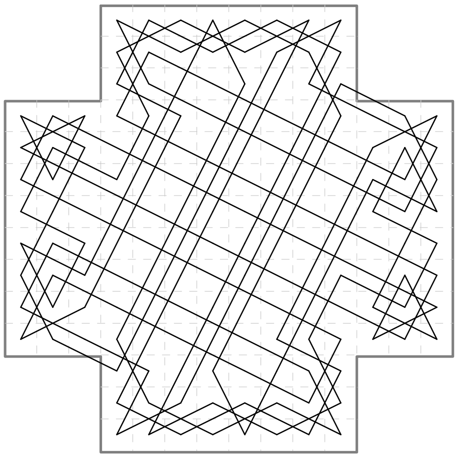
A. Falkener 1892 #3, and a version of his #8.





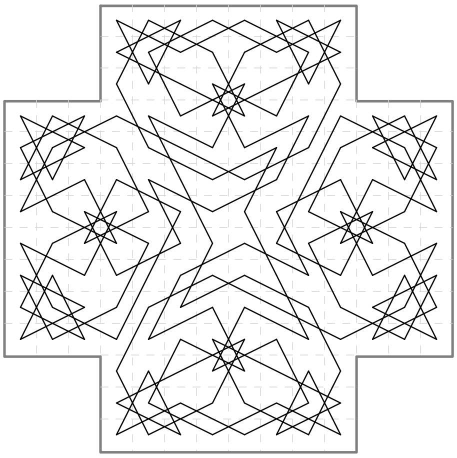
B. after Falkener #6 and #17.

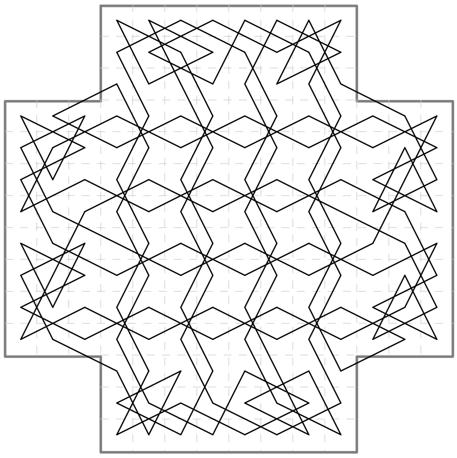




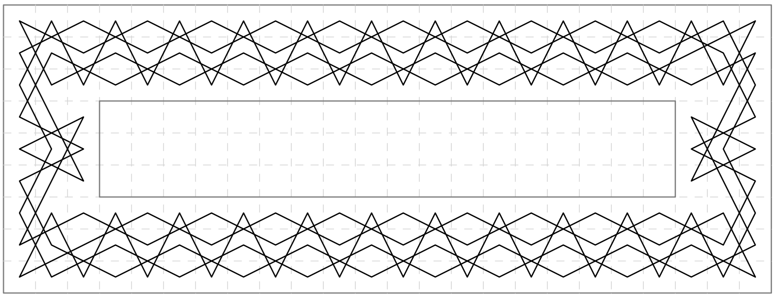
The tours shown here are closely based on his designs and achieve proper binary symmetry.

C. after Falkener #5 and #18.

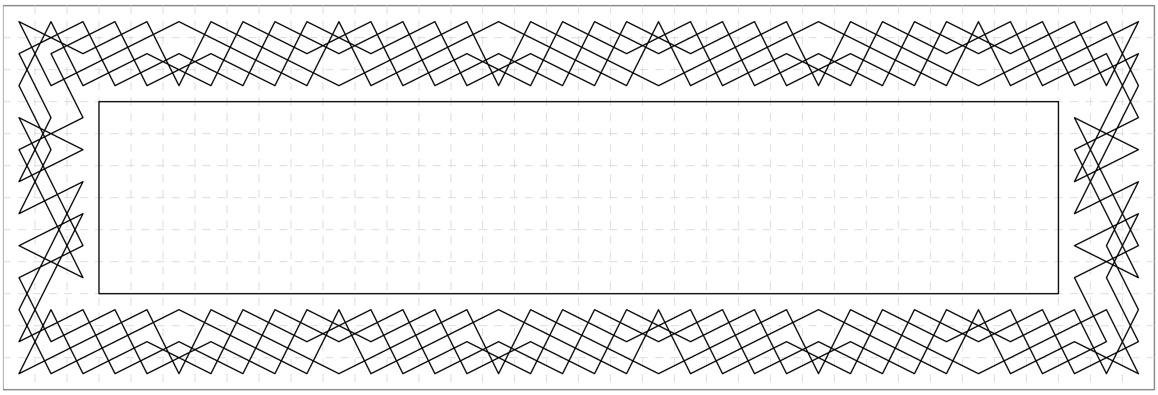




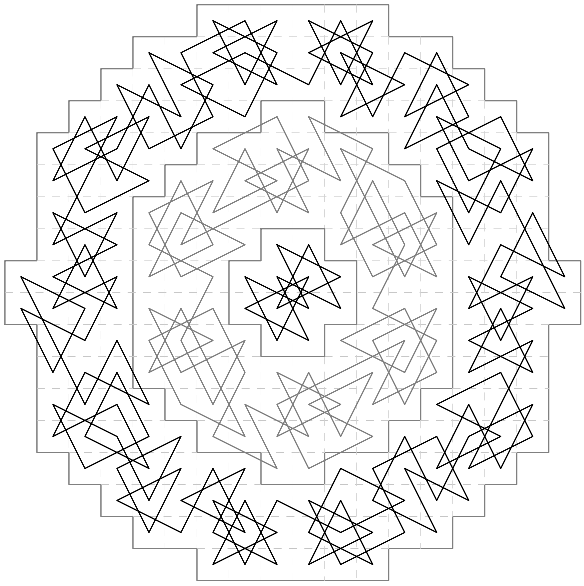
162 cells: A rotary border design for the Knight's Tour Notes title page (24×9 – 18×3).



192 cells: This 36×12 – 30×6 tour was designed as a border for the Bibliography section.



248 cells: Pseudotour by Michael Creighton (10 Feb 2017) the inner part of a much larger pattern which has three further outer bands of “symmetric ripples towards infinity” as he calls them.



The centre is Euler's 12-cell cross. The surrounding rings use 80, 156 cells (then 252, 332, 412 cells). His drawing uses alternate colours and leaves out the border lines. See also 120 cells.

Axial

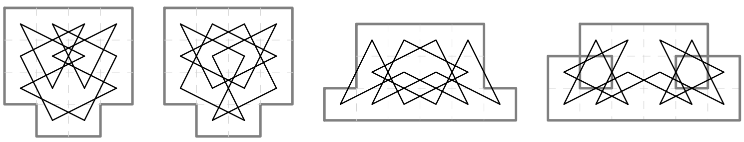
Open tours with a single axis of symmetry must use an even number of moves, since a middle knight move would have to cross the axis of symmetry at right angles, and this is impossible because the axes are diagonal or lateral and knight moves are skew. Thus the number of cells used must be odd and there can only be one cell, the middle cell of the tour, on the axis of symmetry.

Closed tours with a single axis of symmetry and cells on the axis I call **Murraian** since H. J. R. Murray made a special study of this type in his 1942 manuscript. The axis must contain just two cells and can be lateral or diagonal. A Murraian tour can be regarded (in two ways) as an axially symmetric open tour with two extra moves connecting the ends to a second point on the axis. I include both open and closed tours under this description.

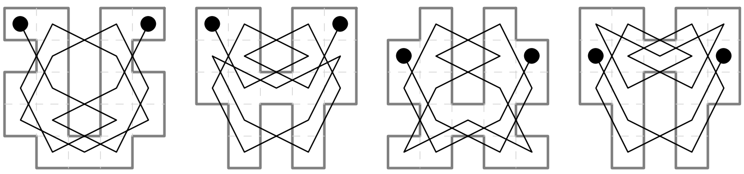
Closed tours with a single axis of symmetry and no cells on the axis I call **Sulian** (after As-Suli who c.900 constructed an 8×8 tour incorporating a section with this type of symmetry). In Sulian symmetry the axis can only be lateral, not diagonal. This type of symmetry seems to me particularly fascinating, but very little work has been done on it until my own reported here. Corresponding cells on opposite sides of the axis are of opposite colours if the board is chequered, so the paths of knight moves leading from one to the other must be of an odd number of moves, $2 \cdot h - 1$. The whole tour must therefore cover a board of $4 \cdot h - 2$ cells, twice an odd number. This sequence runs 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50 ... Sulian tours cannot exist on boards where the number of cells is a multiple of 4, nor on any square boards without holes. (See also Oblong Boards ⌘ 4)

For axial tours on boards of 7, 9, 10, 11, 12 cells see ⌘ 1.

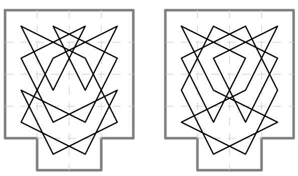
14 cells: Four tours with binary axial symmetry, all of Sulian type, including two on the same board shape. and one with two holes. The first is in Murray's notes.



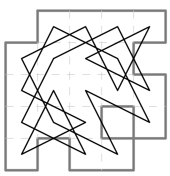
17 cells: Here are some shaped boards with open tours showing Murraian axial symmetry.



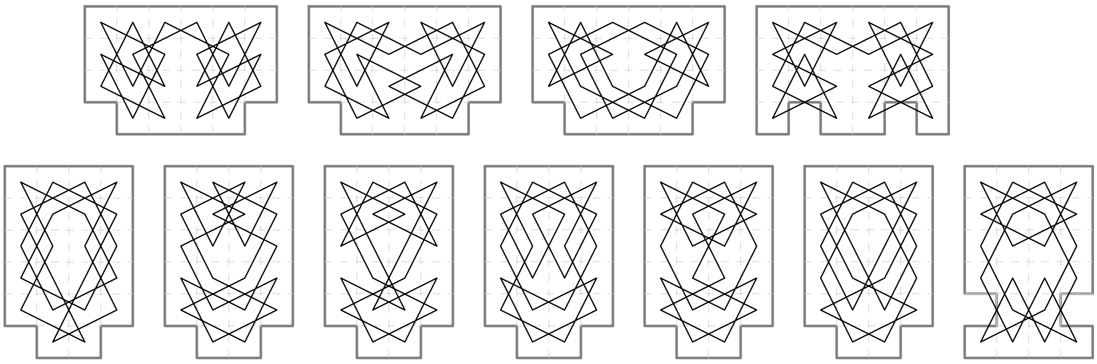
18 cells: There are two with axial symmetry of Sulian type.



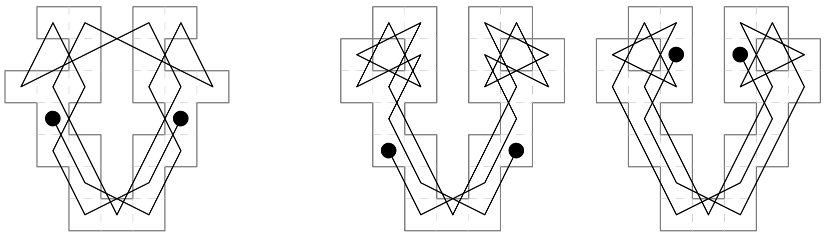
20 cells: A board with hole showing Murraian symmetry with diagonal axis (Murray 1942).



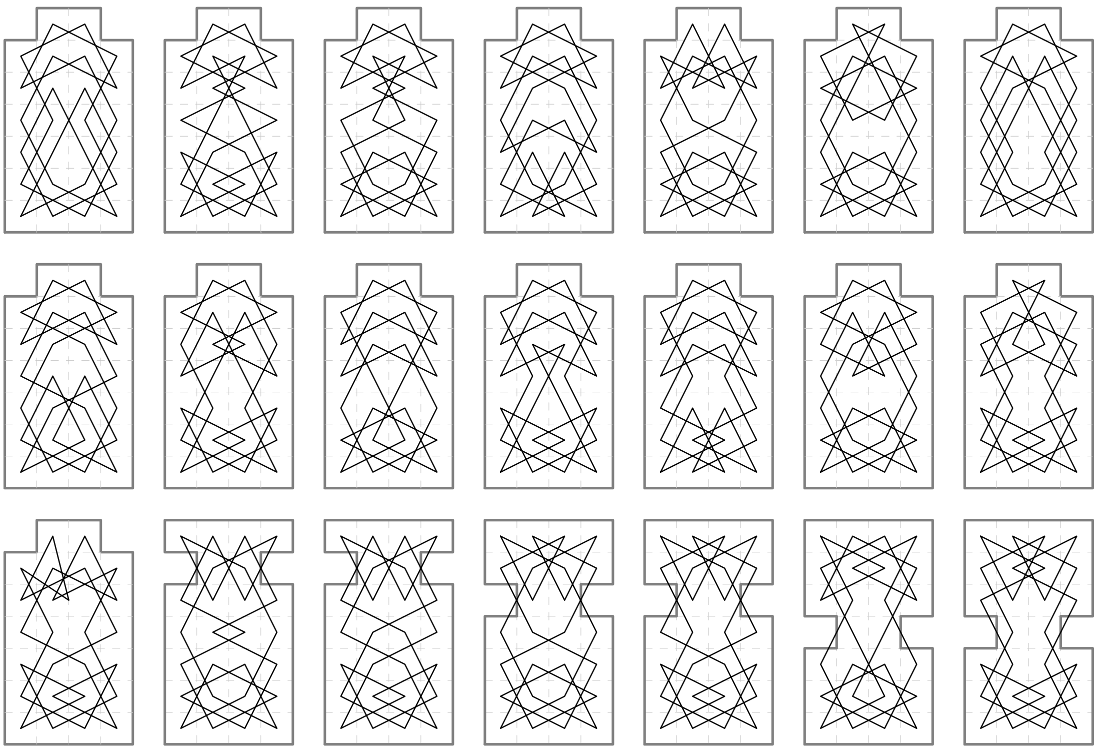
22 cells: There are 11 tours with Sulian symmetry within a 4×6 area. The third is in Murray.



23 and 25 cells: Some ‘just connected’ open tours with lateral axis; one of 22 moves on 23 cells, two of 24 moves on a board of 25 cells. Other moves can be added, e.g. in the second b3-d2-f3 making a Murraian closed tour on 26 cells.

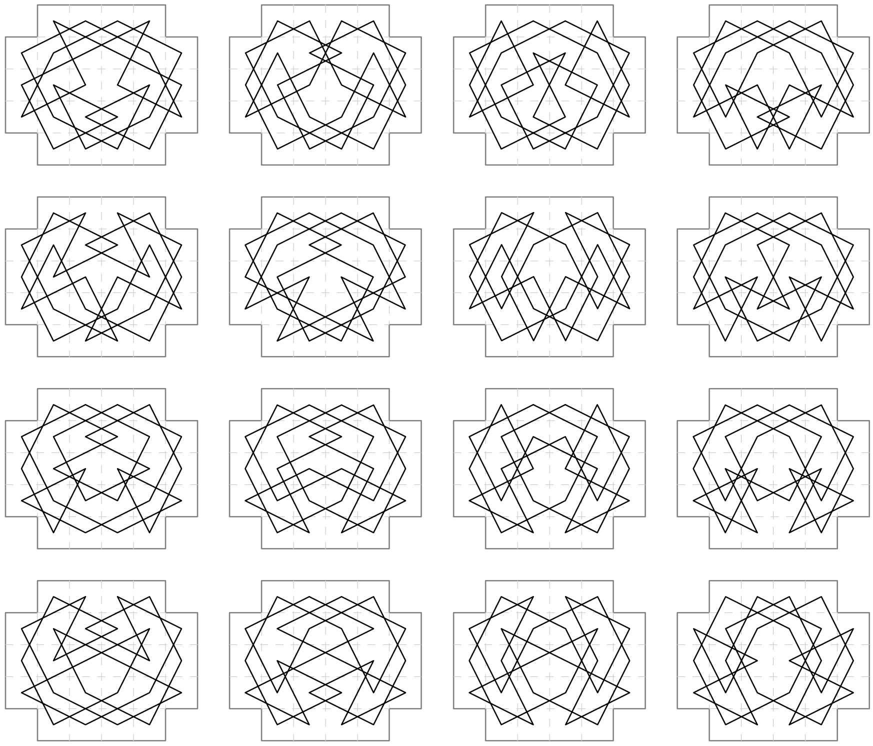


26 cells: Here are 21 tours omitting two cells from the 4×7 frame.



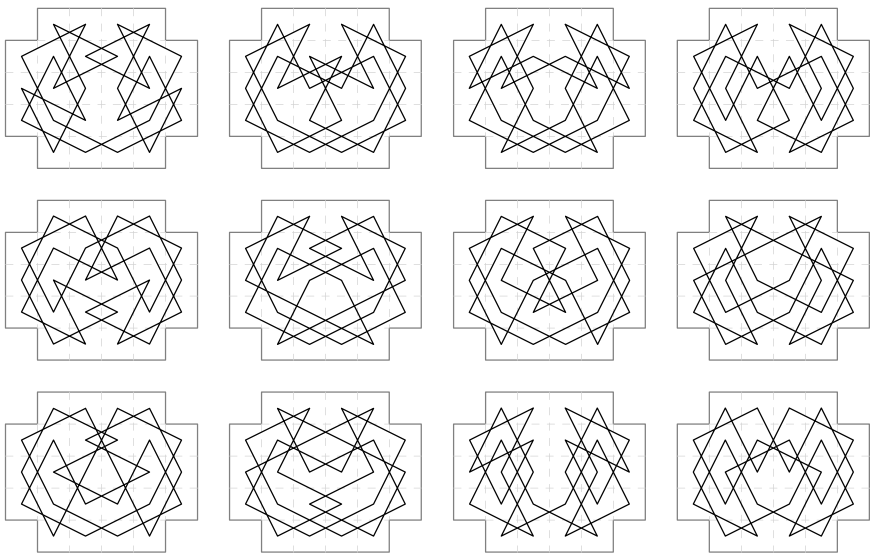
And here are 28 tours on a cross-shaped board of 26 cells (Jelliss 2009) in batches of 16 and 12.

A. 16 with acute angle at a2.

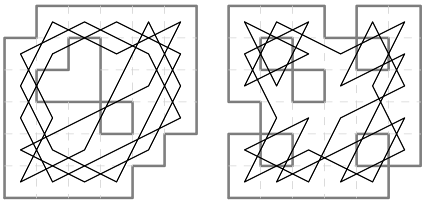


The final tour is the only case with acute angles at both a2 and a4.

B. The 12 with right angles at a2 and a4:

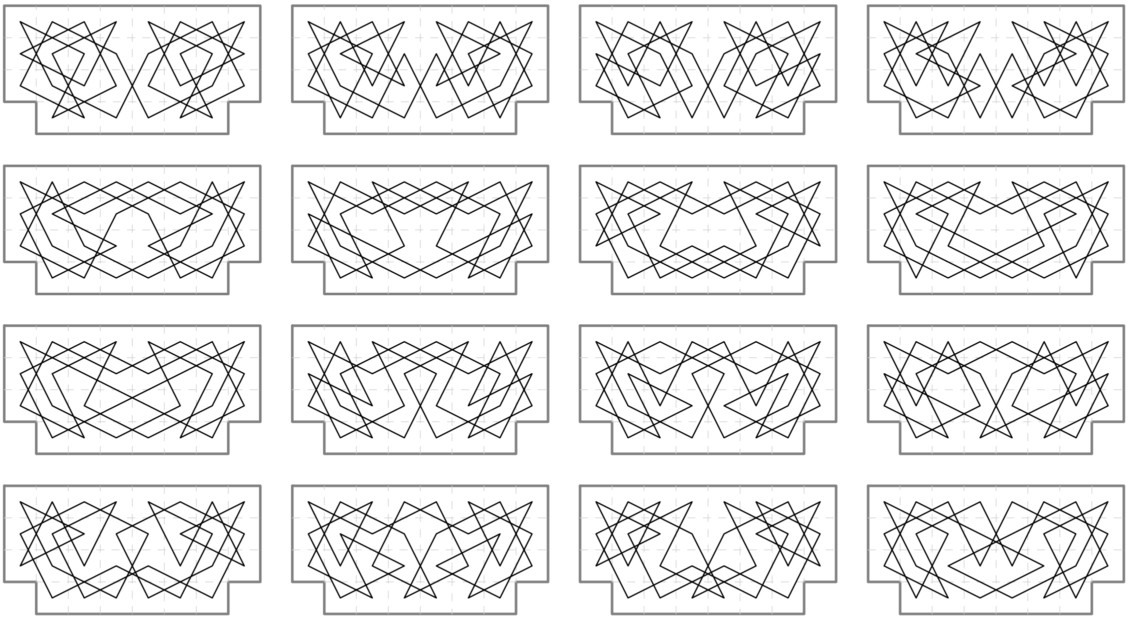


28 cells: Murraian symmetry with diagonal axis (Murray 1942) two of 28 cells

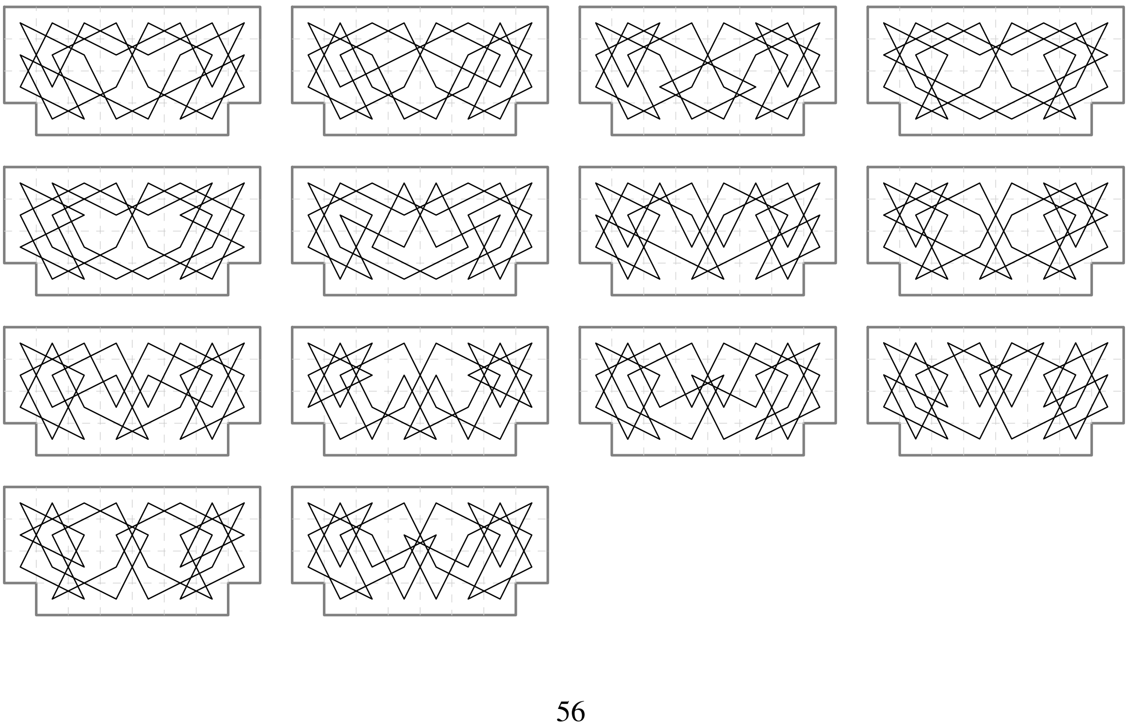


30 cells: Suli's original axisymmetric tour was on the 4×8 board omitting two cells. The following charts show all possible tours of this type. The axis of symmetry can be the short or long median. There are 45 with axis along the short median.

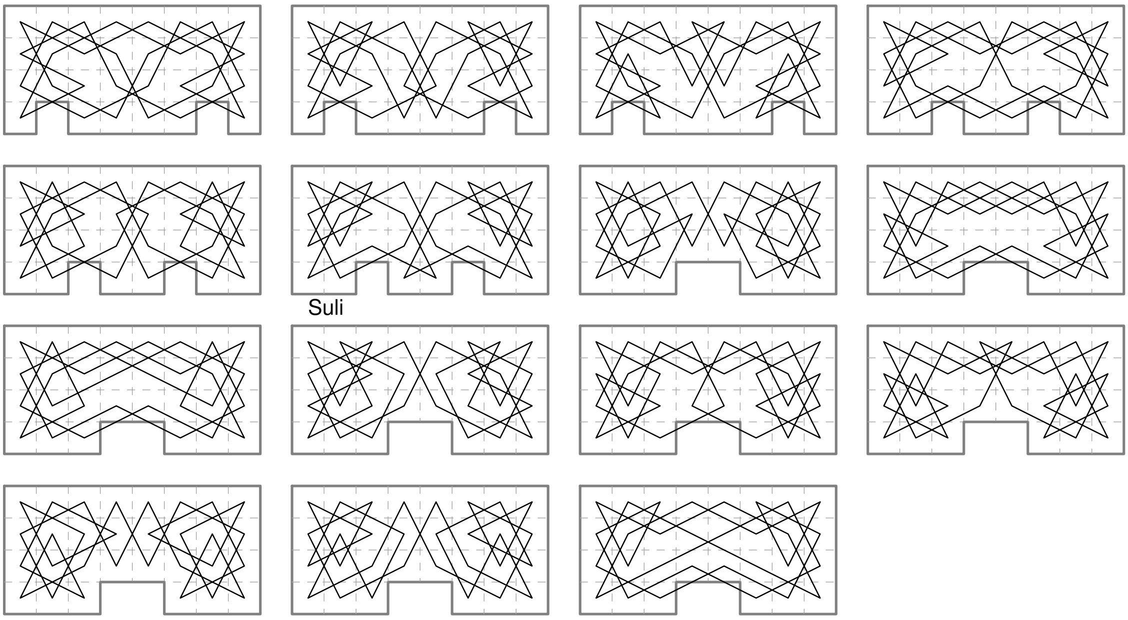
Here are the 30 cases that omit cells at the corners (ah1): First 16:



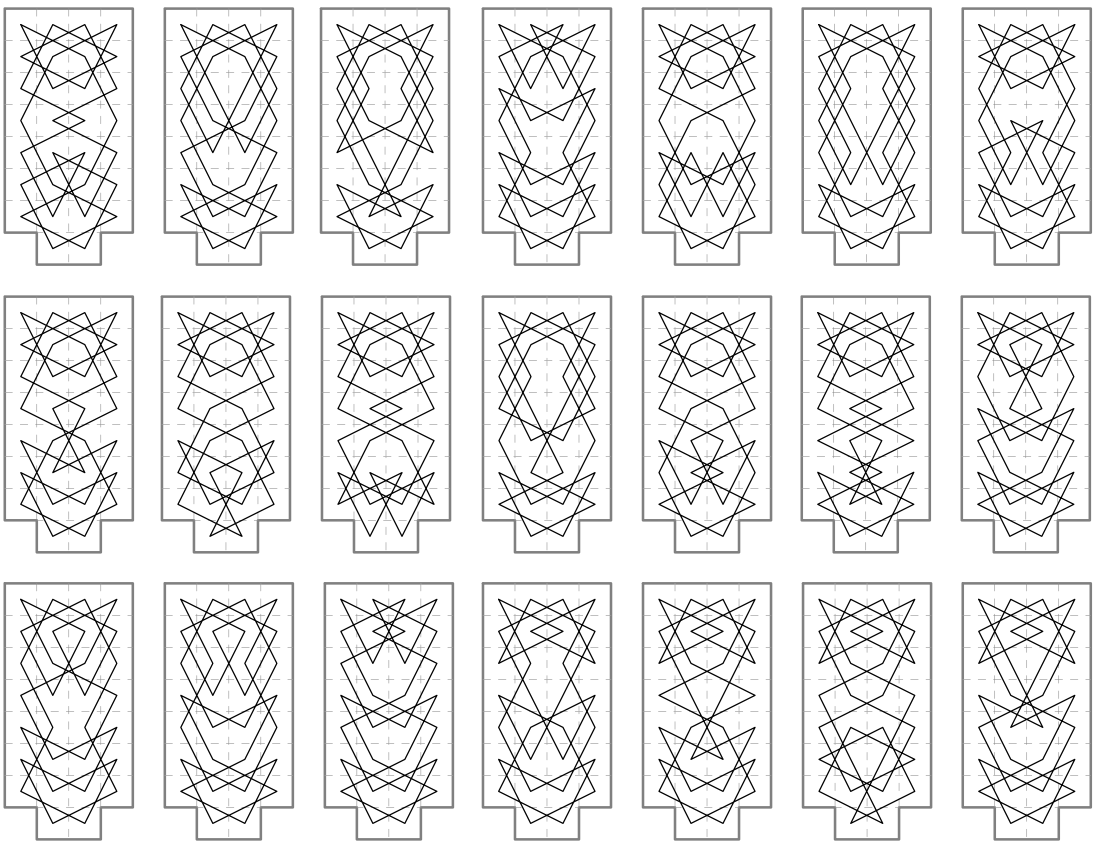
remaining 14:



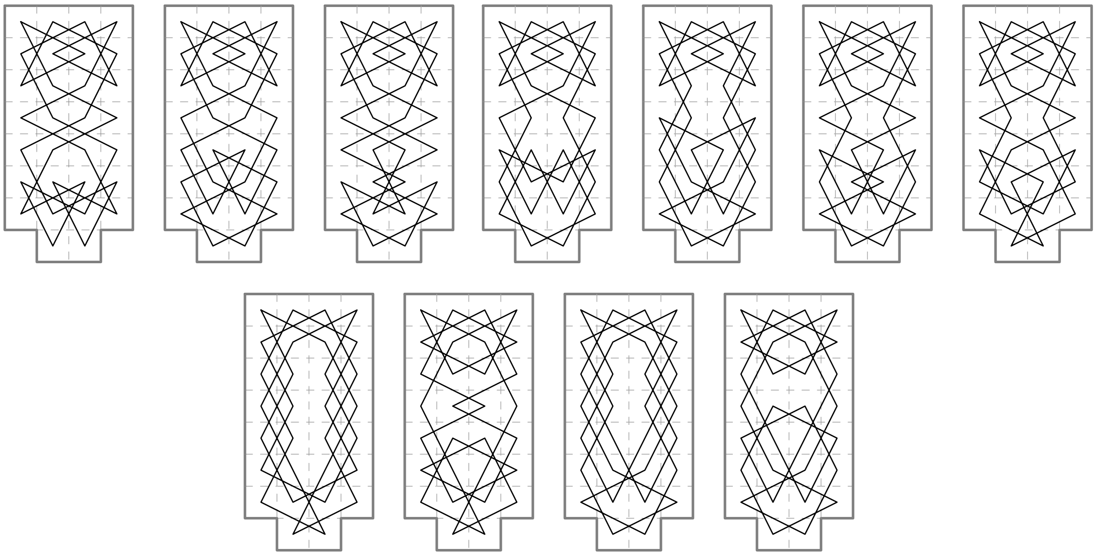
Here are the 15 with missing cells at bg1 (3 cases), cf1 (3 cases), the last of which is Suli's original example and de1 (9 cases).



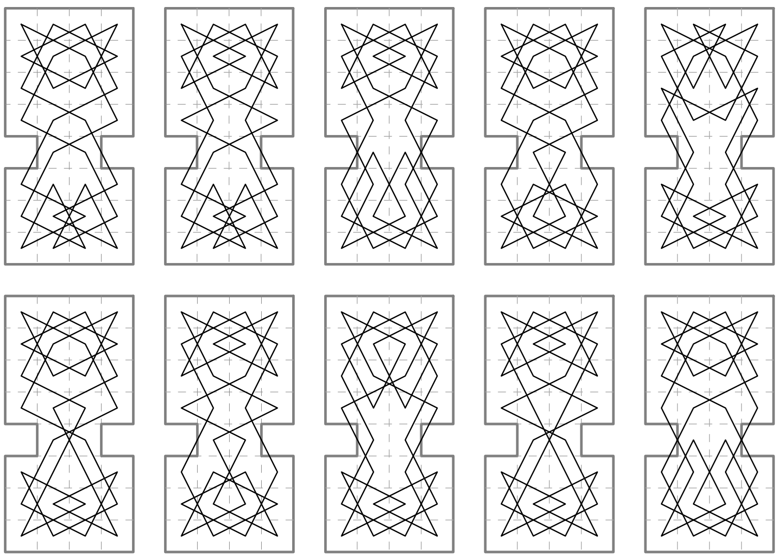
There are 47 tours with axis along the long median: 32 tours with corner cells (ad1) missing.



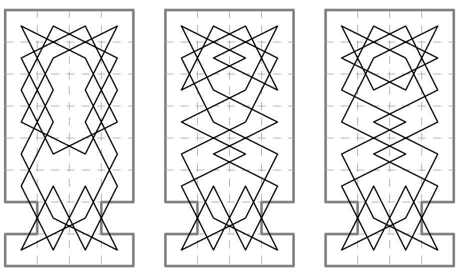
continued

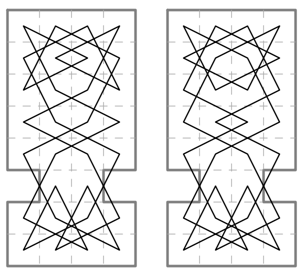


plus 10 with missing cells in the edges at ad4.

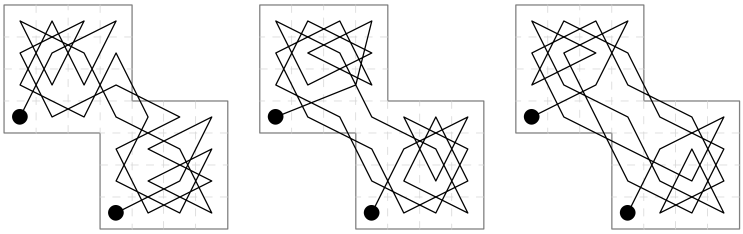


plus ad2 (3 tours) and ad3 (2 tours).

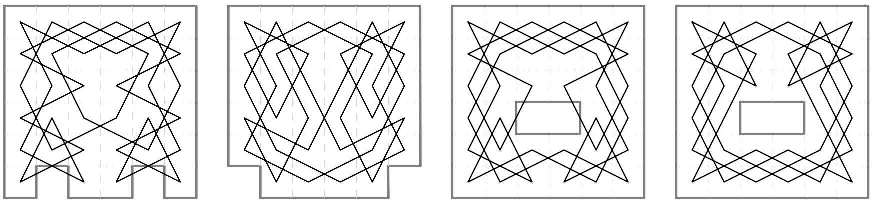




31 cells: The shape of two 4×4s overlapping at a corner, has tours with diagonal axis (three examples shown) but none centrosymmetric. There are also asymmetric tours (not shown).

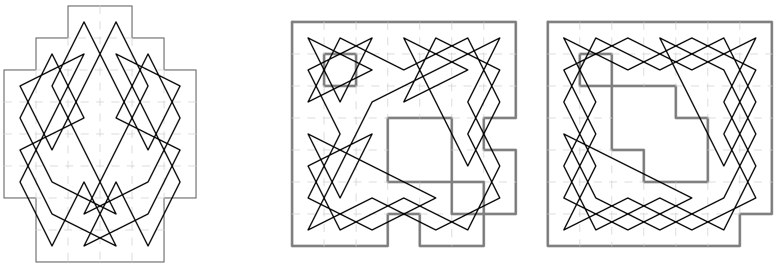


34 cells: Sulian symmetry on the 6×6 with 2 cells missing, from *Zurcher Illustrierte* 23 Jan 1931, collected by H. J. R. Murray, the second is also in Murray's notes. Two with a 2-cell hole (Jelliss).



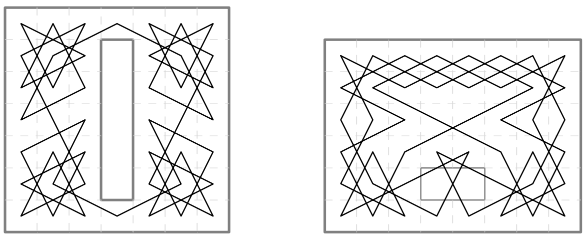
38 cells: Tour with direct binary symmetry, Sulian type, from *Zurcher Illustrierte* 8 June 1934.

40 cells: Two binary tours of Murraian type with diagonal axis (Murray (1942).



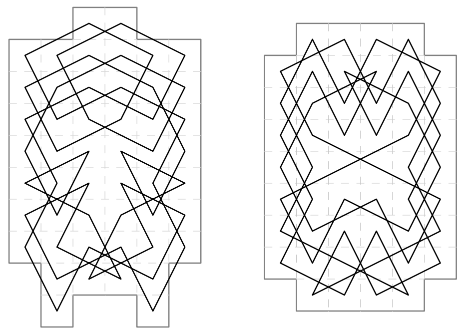
44 cells: A tour with axial symmetry of Murraian type with lateral axis (Murray 1942).

46 cells: Sulian symmetry on a 6×8 board with 2-cell hole (46 cells) by Bergholt First Memorandum 24 Feb 1916.



49 cells: A shaped 49-cell board is that consisting of two 5×5 boards overlapping at a corner. A corner to centre tour on each part can join to give both axial and centrosymmetric open tours (unlike the similar 31-cell board where centrosymmetry is impossible).

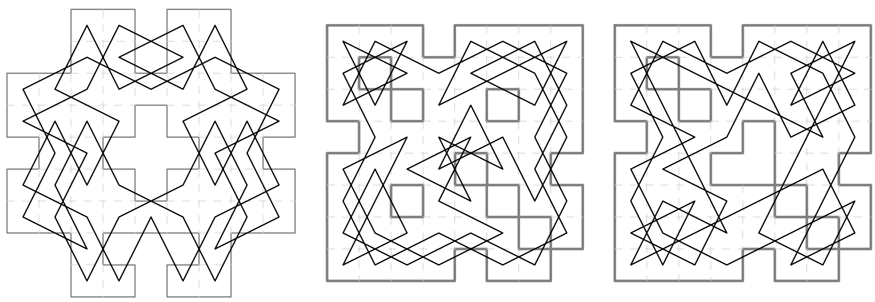
50 cells: Sulian symmetry on a 50-cell shape from *Zurcher Illustrierte* 13 July 1934. The tour looks as if it may have been intended to resemble a human skeleton (the 38-cell tour above may be its head). Psychologists use patterns with this type of bilateral symmetry in the Rorschach inkblot test, because of its biological significance, which the human eye has evolved to recognise quickly.



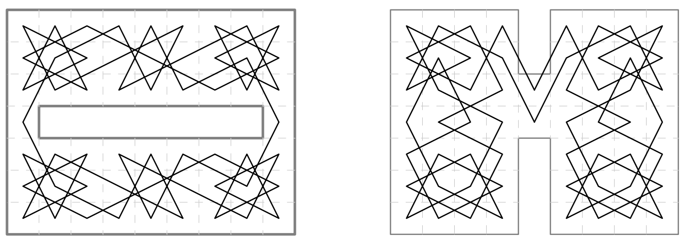
Sulian tour on board 6×9 minus corners (Jelliss 2018).

52 cells: Direct binary symmetry. Murraian type with lateral axis. This 52-cell cross example was collected by Murray from *Zurcher Illustrierte* 27 Jun 1930.

Murraian type with diagonal axis (Murray 1942) two tours. (He also gave two other diagonal examples of 52 cells on the 8 by 8 board, but they omit cells that result in boards that are not wazir-connected, so are not included here.)

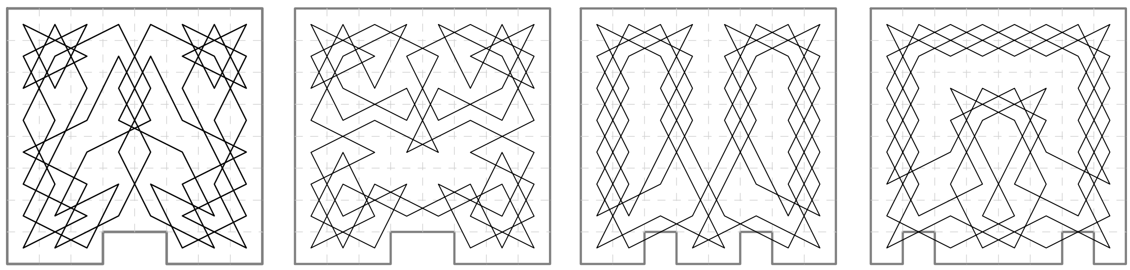


56 and 58 cells: Examples by Murray. First diagram rotated here.

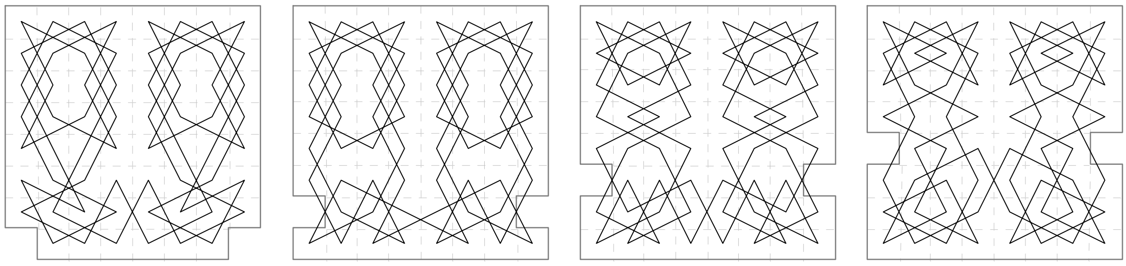


62 cells: Here we consider boards formed by omitting two cells from the standard chessboard.

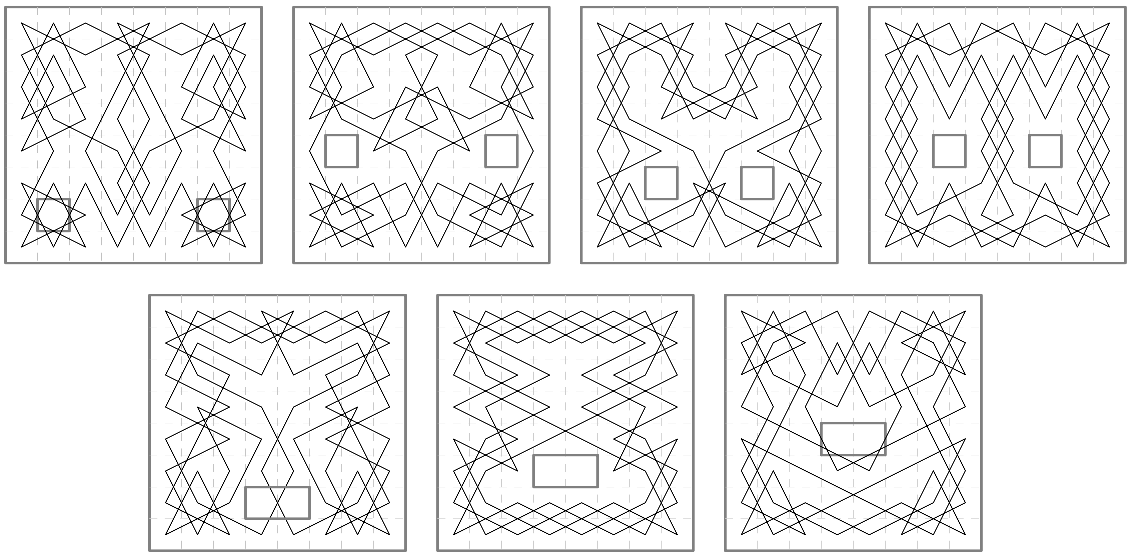
We show examples with direct binary symmetry (Sulian type) for each of the 7 ways of omitting two cells from the board edge. The first two examples are from Kratchik (1927) and Murray (1942).



The others are by Jelliss (constructed 1996) the first below being after Naidu (1922).

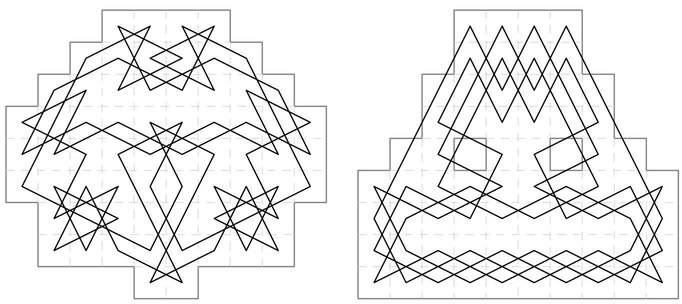


We also show one Sulian example for each of the 7 positions of the two voids forming one or two holes in the 8×8 board. (The cases with holes a knight move from the corner cells do not of course admit a tour.) The last is from Murray (1942) diagram inverted, the rest are Jelliss (1996), the third being after Hoffmann (1893), the fifth after a German example (c.1860), the sixth after Babu (1901).

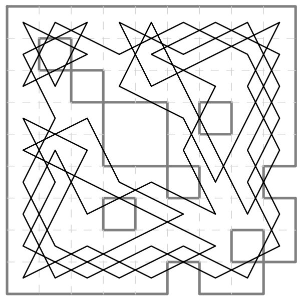


The tours described as ‘after’ earlier composers were originally published with the path diverted through the void cells to give either a near-symmetric open tour, or a pseudotour.

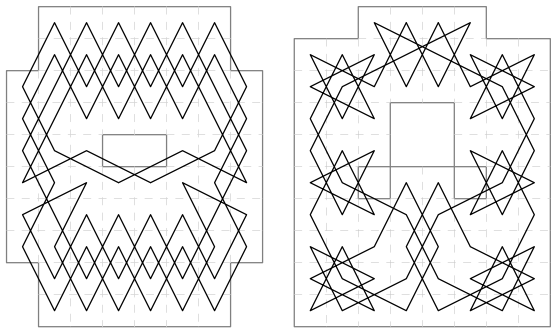
66 cells: Sulian binary direct symmetry with vertical axis, collected by H. J. R. Murray from *Zurcher Illustrierte* 28 Aug 1931 and from *Denken und Raten* 25 Aug 1929 with two holes.



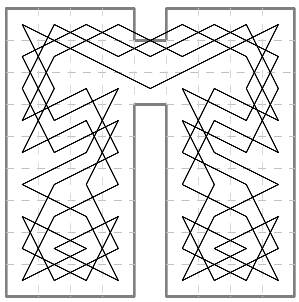
68 cells: A holey board with diagonal axial tour of Murraian type (Murray 1942).

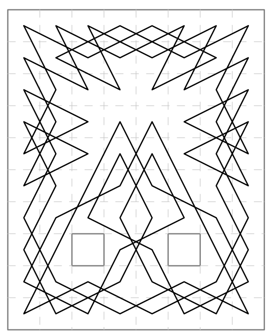


70 cells: Sulian direct binary tours on shaped boards with holes collected by Murray from *Zurcher Illustrierte* 22 Dec 1933, and from *Denken und Raten* 25 Aug 1929.

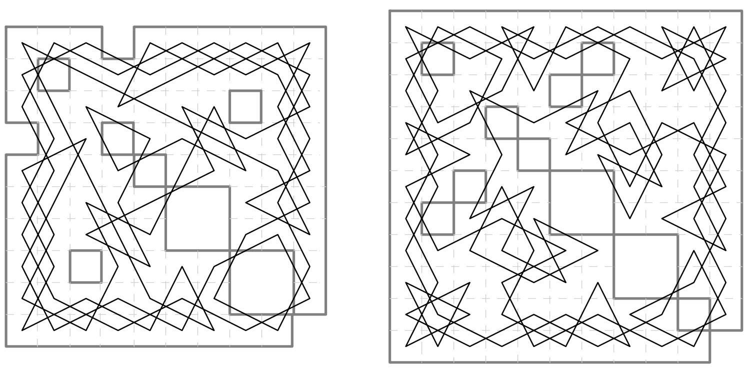


74 cells: Murraian type with lateral axis (Murray 1942).
78 cells: Sulian type on 8×10 board with two holes from *Denken und Raten* 25 Jan 1931.

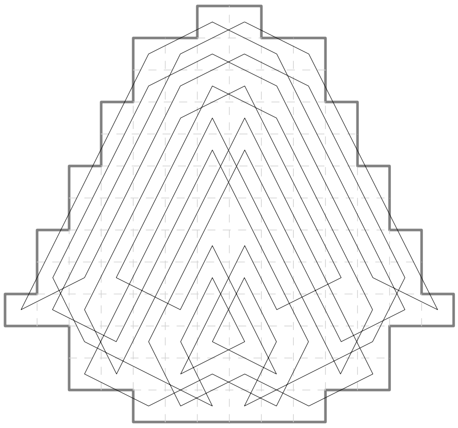




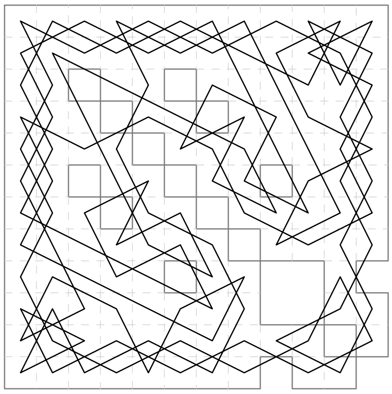
84 and 104 cells: Two of Murraian type with diagonal axis (Murray 1942).



114 cells: This is our final Sulian example with vertical axis, another collected by Murray from *Zurcher Illustrierte* (7 Jul 1933), composer unknown.



124 cells: A final Murraian example with diagonal axis (Murray 1942).



Unary

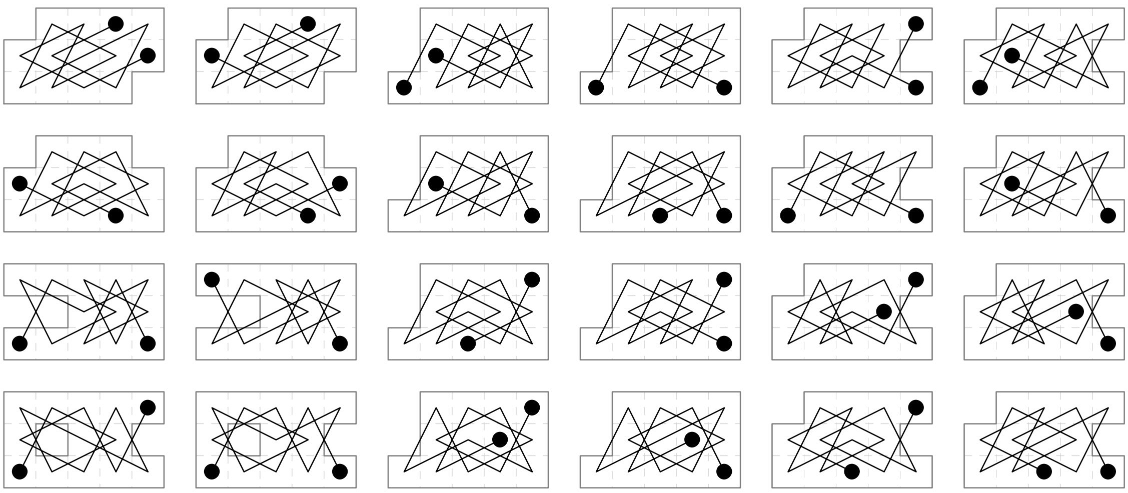
A unary tour is one that is all of one piece and cannot be split up into two or more congruent components. In negative terms it is an asymmetric tour, even if it has a high degree of symmetry. Usually most tours on a board are asymmetric, so their study should certainly not be neglected.

We include a selection of Indian ‘pictorial’ examples some of which use irregularly shaped boards. In Harikrishna (1871) as reproduced by Iyer (1982) these tours are presented on unchequered chessboards, the tour being given by numbered squares within a line drawing depicting the subject. In his review of Naidu (1922), and in regard to these pictorial tours, Murray (1930) expressed the view that “These appear to me to be of no importance or interest”. However, I find them a welcome antidote to too much of a diet of mathematically perfect symmetry. In particular the Indian tours include a study of tours on boards of 32 cells that retain the biaxal symmetry of the 4×8 board by cutting out some squares and affixing them in new positions so that the revised board admits a closed tour, which is impossible on the rectangle.

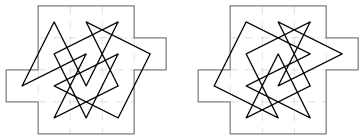
Another related question that leads to asymmetry concerns what board shapes admit unique tours of given types. It is a paradox that some symmetric boards do not admit symmetric tours. Of course if the board is asymmetric then so must the tour be. A series of tours of these paradoxical types were given by various contributors to the chess magazine *L'Echiquier* in the 1920s and 30s. Some boards do not admit any tour, open or closed, in particular the 3×5, 4×4 and 3×6 rectangles, so it is natural on such boards to find the largest possible tours within the rectangle, omitting fewest cells. Some boards allow open tours but not closed tours. Obvious cases are rectangles odd×odd and boards 4×n.

For asymmetric tours on boards of 8, 9, 10, 11, 12 cells see ⌘ 1.

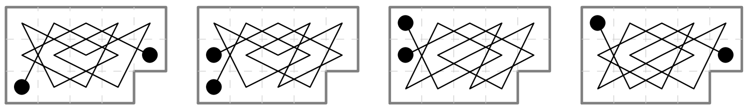
13 cells: These 24 asymmetric tours investigate boards formed by omitting two cells from the untourable 3×5 board. Two centrosymmetric open tours are also possible. 8 asymmetric open tours on 4 symmetric boards (one with a hole) and two asymmetric board shapes, each tourable in 8 ways.



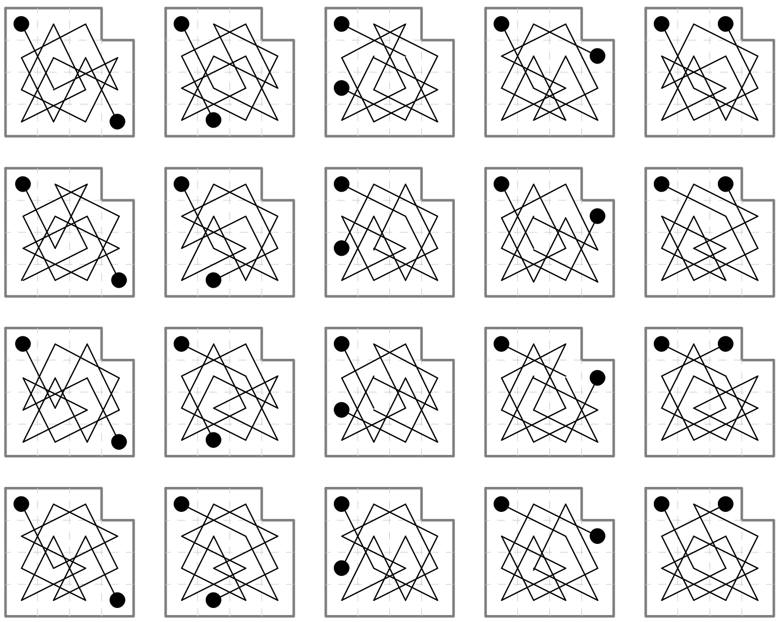
14 cells: Two asymmetric tours on a board that also has symmetric tours (one Eulerian, one Bergholtian).



The four asymmetric open tours here solve the problem of a maximum length knight path on the 3×5 by omitting one cell, which has to be a corner cell.



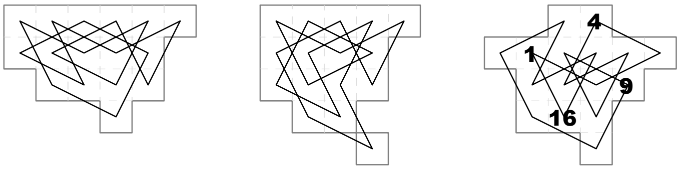
15 cells: No complete tour is possible on the 4×4 board. The diagrams shown here solve the problem of a maximum length open knight path on this board. There are 20 distinct 15-cell open paths on the 4×4 board, omitting one corner cell. They all have at least one end in a corner



16 cells: The impossibility of a knight tour on the 4×4 board was known in mediaeval times, although no proof was published. Murray (1930) reports: “A Dresden manuscript of the end of the fourteenth century gives a half-tour without solution and sets as a wager game a tour over a board of 4×4 squares” and: “The sixteenth century Persian manuscript on chess in the library of the Royal Asiatic Society makes some remarks on the tour, and promises to give tours on the whole board and on boards of 4×8 and 4×4 squares, which are lost owing to the fragmentary condition of the manuscript. The author boasts a little, for the tour on the 4×4 board is an impossibility.”

. KNIGHT'S TOUR NOTES

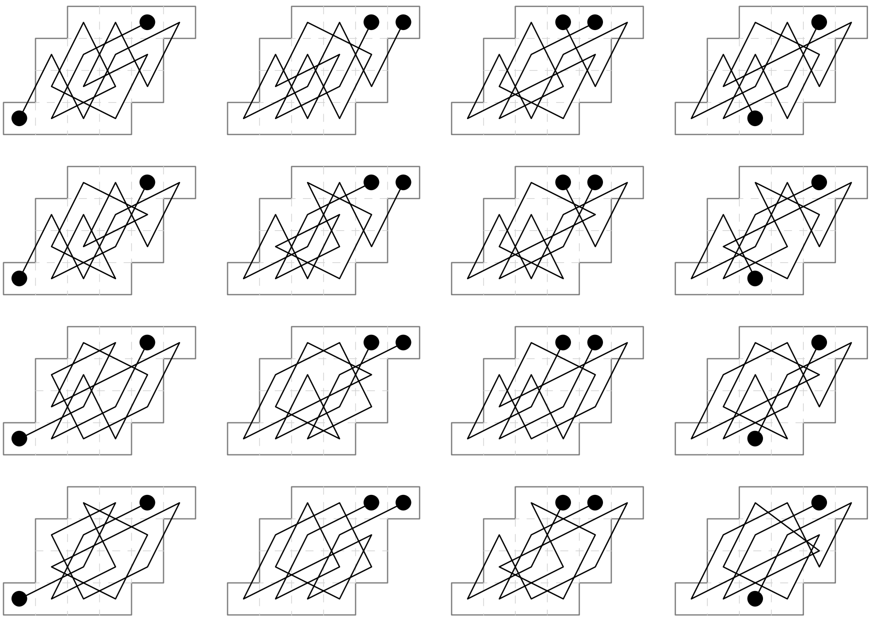
In his *History* (1913 p.171, 177, 335 footnote) Murray gives reasons to believe that the author of the work reproduced in this ms may have been Ala'addin Tabrizi (also known as Ali ash Shatranji, i.e. Ali the chessplayer) who is known to have written a work on chess. He was the leading player at the court of Timur (also known as Tamerlane, 1336-1405). A translation of this ms by Forbes (1860) is subtly different, refering to “one quarter of the board” rather than to a 4×4 board. The wording is critical, since a closed tour is possible on one particular non-rectangular quarter-board. There is also a solution on a disconnected quarterboard.



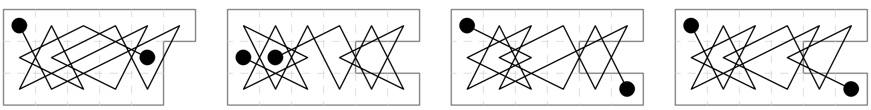
I call the quarter-board tour puzzle Aladdin's Conundrum. However, there is no firm evidence that these quarter-board tours were known before being listed in Paul de Hijo's 1882 catalogue of all the possible 16-move knight circuits in quaternary symmetry. He makes special note of them but does not give a diagram. (I have since found the disconnected case earlier as ¶118 in *Le Siècle* 16 Mar 1877.) They were rediscovered by H. E. Dudeney (1917) who made them the basis of several puzzles.

Another unique 16 cell tour on a shaped board solves a problem in my booklet on *Figured Tours* (1997): find a 16-cell knight's tour, on a connected board, showing the first four square numbers 1, 4, 9, 16, forming a square. This can be regarded as a 4×4 board with three corner cells moved.

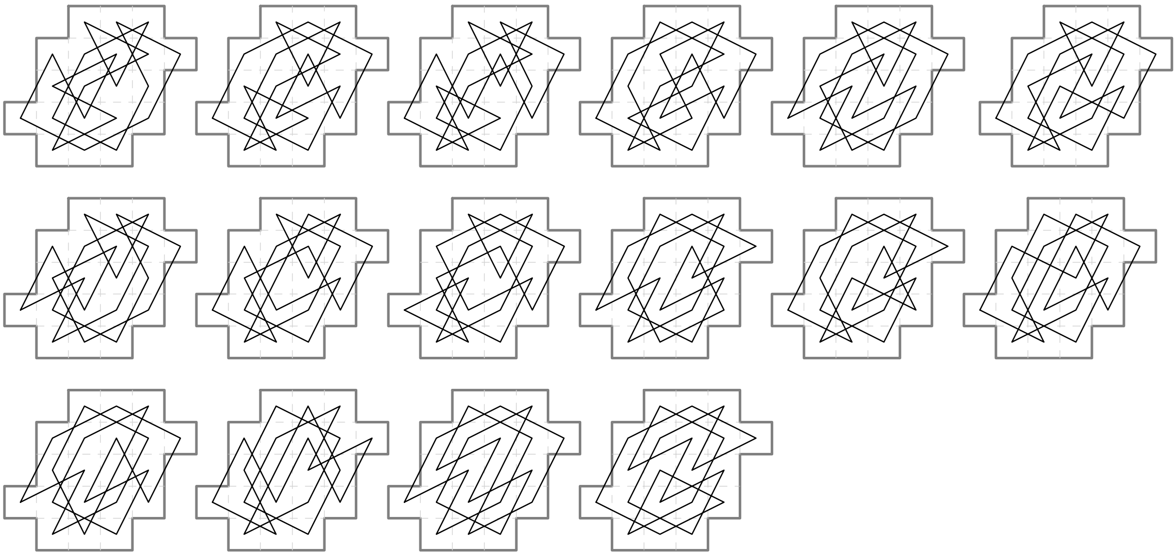
The 4×4 board admits no complete tour, open or closed. However, moving two corner cells will produce two centro-symmetric boards with equal numbers of dark and light cells when chequered. One of these has four rotary closed tours (see earlier section). The other has no closed or symmetric tours but has 16 asymmetric open tours, all shown here.



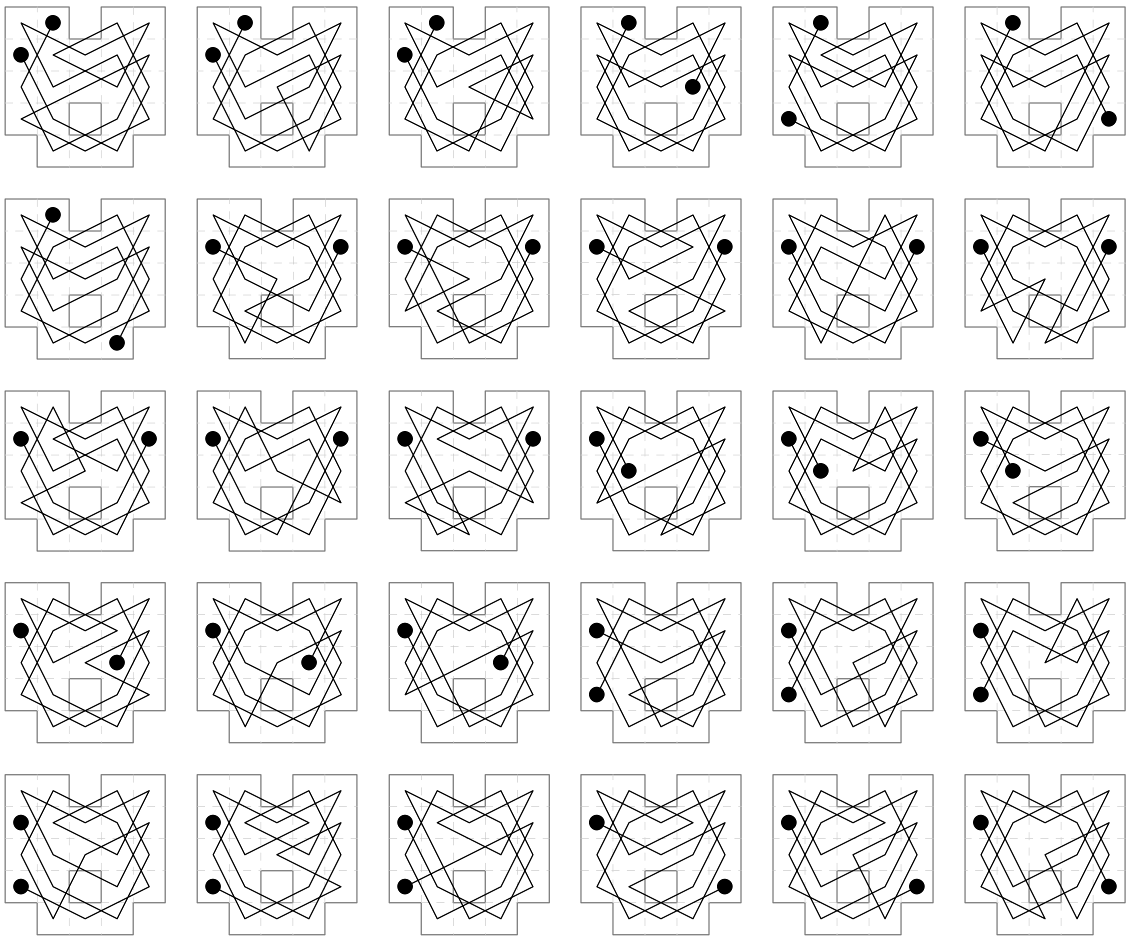
The problem of a maximum length knight path on the 3×6 board is solved by open tours, omitting two cells, four asymmetric (three on a symmetric board), also two symmetric solutions (p.134).



20 cells: The following board, formed from 4×5 with two corners moved has a unique symmetric Bergholtian tour and 16 asymmetric closed tours (Jelliss 2009). One such asymmetric tour was shown by Godron and Vatriquant in *L'Echiquier* January 1929 (the last one shown here).



21 cells: Here is an axially symmetric board formed by removing four cells from the 5×5 board. It has no axially symmetric tours, but 51 asymmetric tours.



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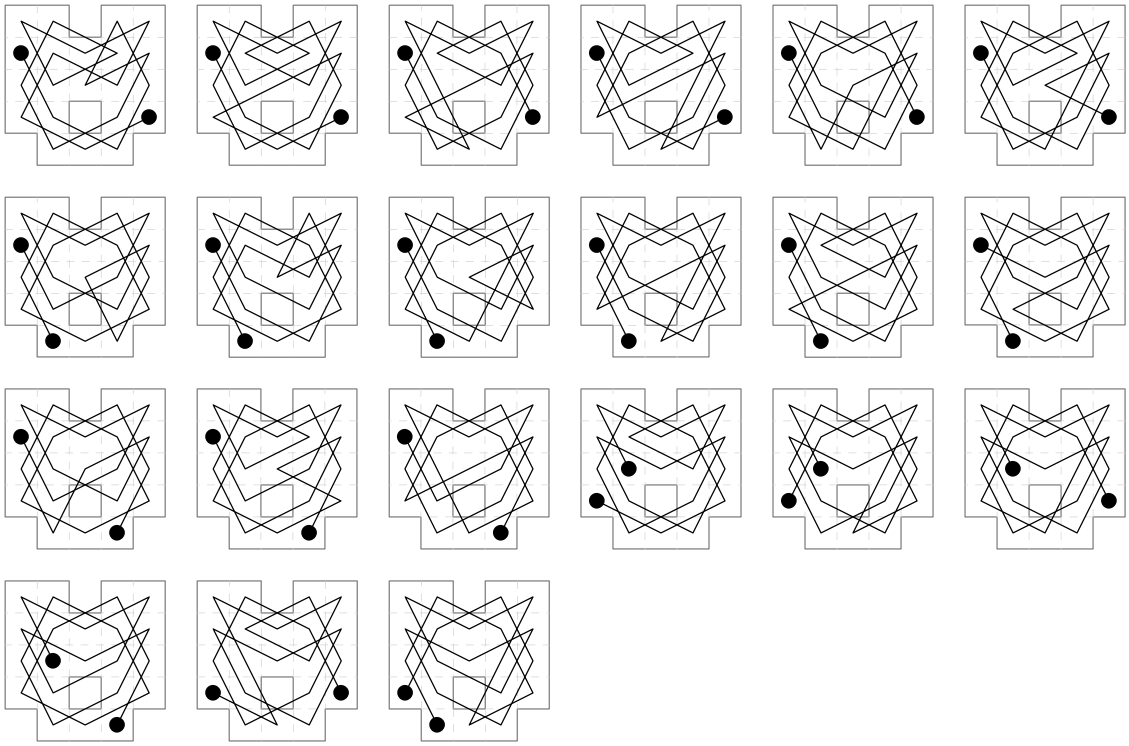
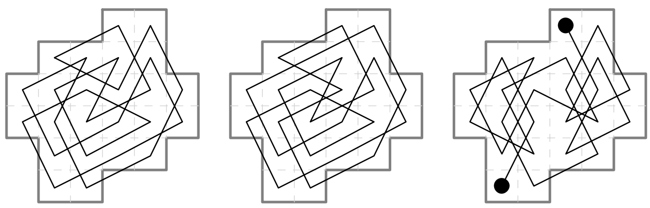
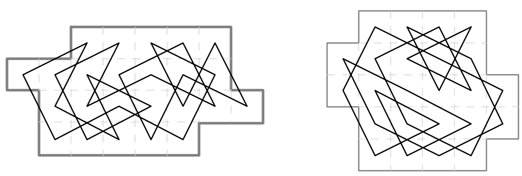


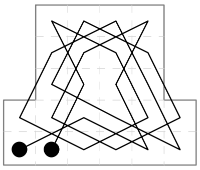
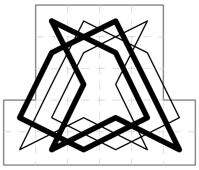
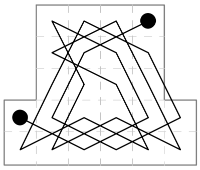
24 cells: Here is a shaped board with two asymmetric closed tours and an asymmetric open tour by Godron and Vatriquant from *L'Echiquier* Jun 1928 and Jan 1929. A symmetric open tour is impossible since the centre point is not centre of a cell or mid-point of an edge, but this board also has a symmetric closed tour (see rotary section).



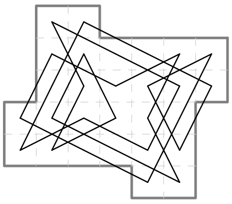
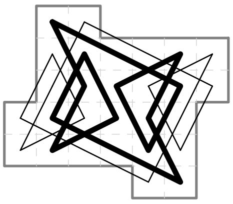
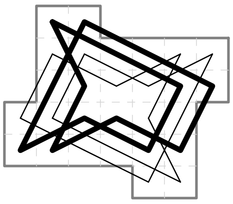
We show two other symmetric boards with asymmetric tours (from *L'Echiquier* Jul and Nov 1928). See the Rotary section (p.140) for an enumeration of 56 symmetric tours on the first board. Three symmetric closed tours are possible on the other board, all Bergholtian.



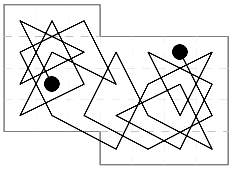
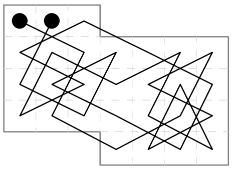
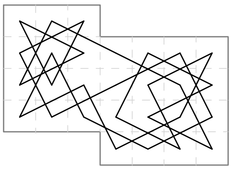
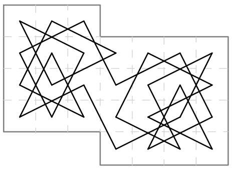
On the 24-cell ‘top-hat’ shaped board a closed tour is impossible, but there is a unique closed pseudotour of two 12-move circuits which can be linked in two ways to form open tours (given by Tolmatchoff and Vatriquant in *L'Echiquier* July 1929).

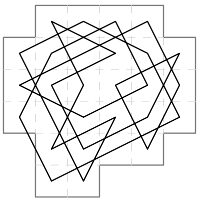
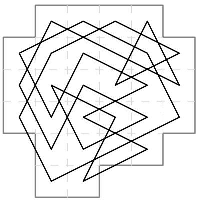
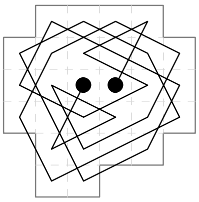
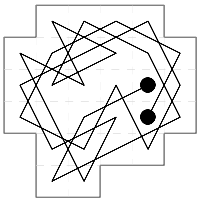
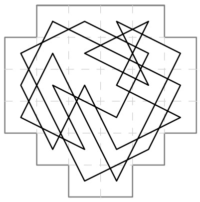
28 cells: This amusing ‘leotard’ board has rotary symmetry but only has the one asymmetric closed tour. There are however two symmetric pseudotours of two types: one in which both components are symmetric, and one in which the two components are the same and are symmetrically related. (There are also 5 symmetric open tours. See p.142)

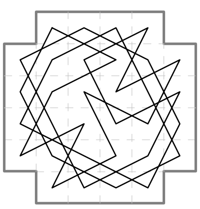
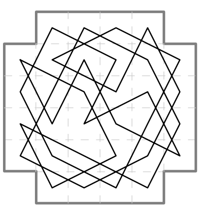
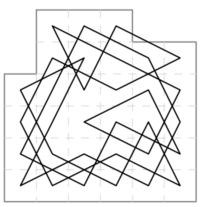
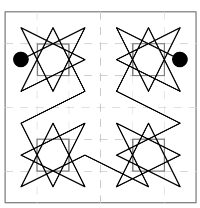
We show two tours of a 28-cell asymmetric board (formed as 3×4 + 4×4) by Vatriquant and Godron (*L'Echiquier* Oct 1927). Closed tours are also possible on this board, two examples shown.

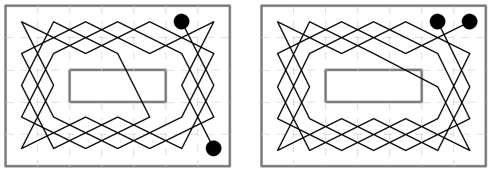
An asymmetric board with two closed and two open tours. The other tour on a related symmetric board is from Naidu 1922, representing a peacock.

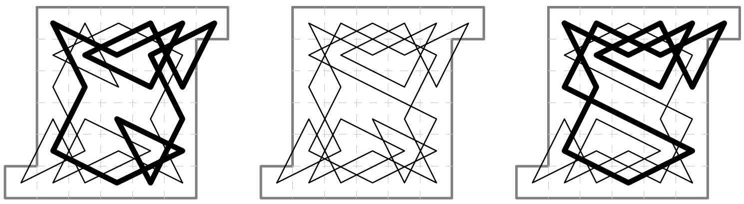
32 cells: Tours on the 6×6 with four cells cut away. The first is by Euler (1759) second and third by Naidu (1922, #S26). The fouth is an open approximate axial tour by Ambikadatta Vyasa *Chaturanga Chaturi* (1884), rotated here so the axis is vertical. Naidu (1922) also gives a symmetric closed tour on the 6×6 board with four holes (see rotary section).

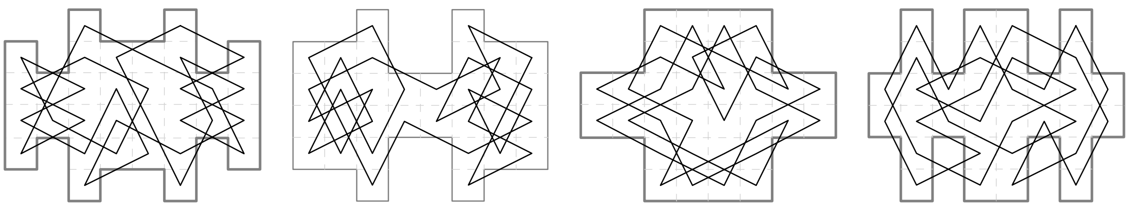
On this ‘letterbox’ board (5×7 with 1×3 hole) the border moves form two circuits, of 8 and 24 moves. Two open tours can be formed as shown by simple linking.



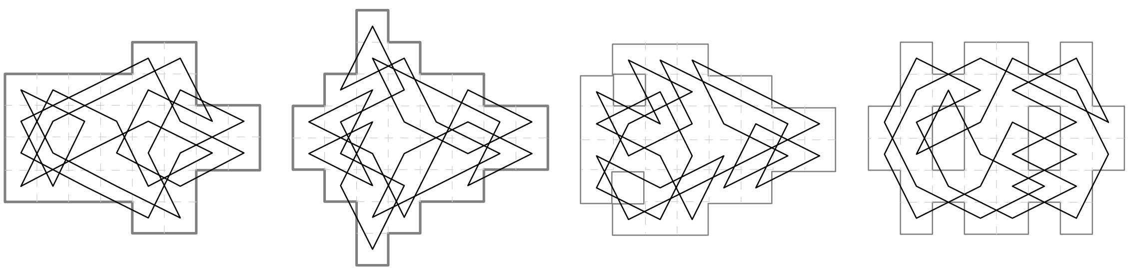
This 5×6 board with two attached cells, is another example that has symmetric open tours and pseudotours (shown here) but on which a closed tour must be asymmetric. The tour shown is by Tolmatchoff (*L'Echiquier* March 1929).



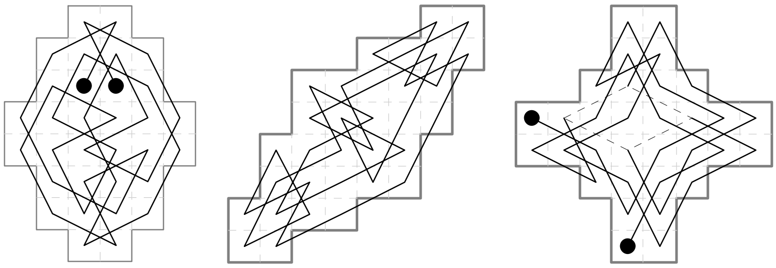
The following tours form part of a systematic study of boards derived from the 4×8 rectangle with some edge cells cut out and reattached elsewhere.



This is a natural subject to study since closed tours on the 4×8 are impossible. The first four boards abovee are biaxially symmetric and the asymmetric tours are #21, #30, #45 and #64 from Naidu (1922). The next three boards only have one axis of symmetry, and the tours are #20, #S51, #S22 from Naidu (the third having holes). The last example, another from Naidu, is biaxial with holes. The diagrams have been rotated 90 degrees to save space.

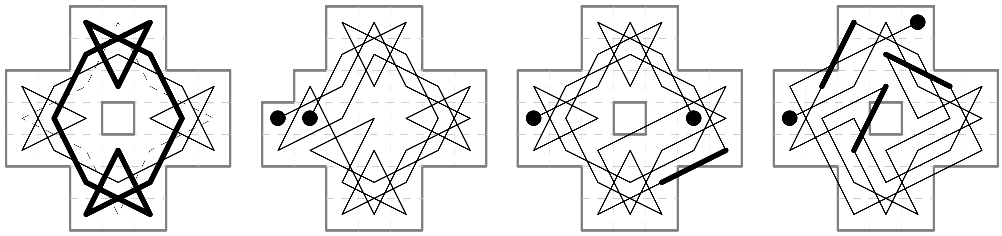


Two more tours from Naidu. The first, like those above, is derived from the 4×8 rectangle with corner cells cut out and reattached to the sides, to make a board with biaxial symmetry, but with an open tour The second (#S13) is a nearly symmetric tour of a board with diagonal axes of symmetry.



The celtic cross board with octonary symmetry can be covered by a pseudotour of four 8-move circuits, but has no closed tours, and only two geometrically distinct open tours, differing only by which sides of the dashed lozenge are taken (Jelliss 2003).

A 32-cell board that makes an interesting case for study is the problem of a tour on the cross-shaped 33-cell solitaire board, where one cell has to be omitted. On the solitaire board of 33 cells there are 17 white and 16 black (assuming the corners are coloured white). A closed tour is not possible because of the odd number of cells. An open tour if possible must start and end on the majority colour white cells, but moves through the black cells a4, d1, d7, g4 form an 8-move short circuit (dotted moves in A), so no open tour is possible.



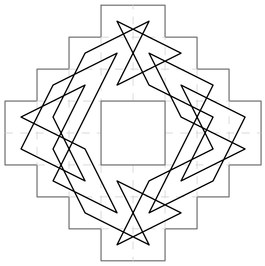
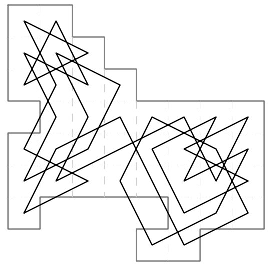
A B C D

A tour omitting one cell must omit a white cell and must include 7 moves of the 8-move circuit. Because of the octonary symmetry of the circuit we can take a4-c5 as the first move without loss of generality. Under the restriction that moves must be within the area of the cross, the end cell must be on one of b4, c1, c7, e1, e7, f4, and the unused cell at one of a3, a5, d2, d6, g3, g5, since otherwise one or both of the 12-move short circuits shown (in A) are forced. A shaped board solution with start, finish and omitted cells adjacent is shown in (B).

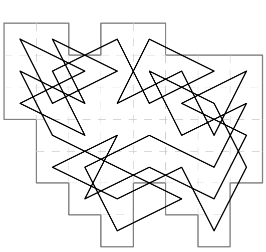
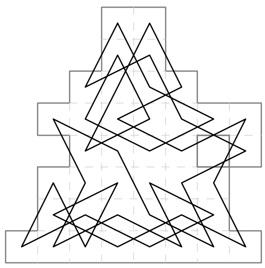
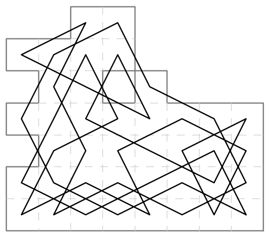
If the unused cell or the end cell is at the centre then the tour must go outside the border at least once, as in (C). The tour in (D) is piece-wise symmetric: when numbered the diametrally opposite cells differ by the constant value 4. [These notes were inspired by a puzzle proposed by Angus Lavery in *Games & Puzzles* No.2, May 1994, page 34.]

36 cells: The Rajah of Mysore's #45 from Harikrishna (1871) uses this octonary board. Despite appearances the tour is asymmetric.

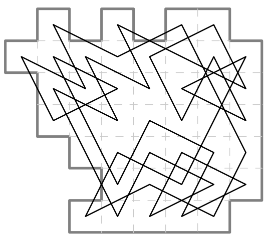
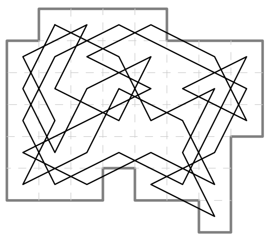
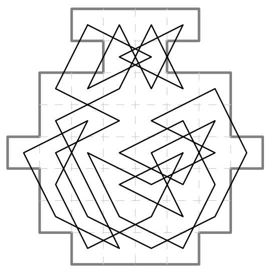
38 cells: Tour #57 of the Rajah of Mysore in Harikrishna (1871) representing a prancing horse also occurs, reflected, as #33 in Naidu (and #61 is similar).

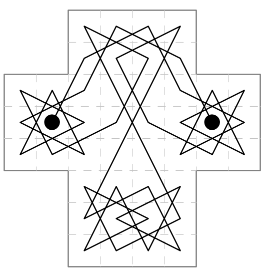
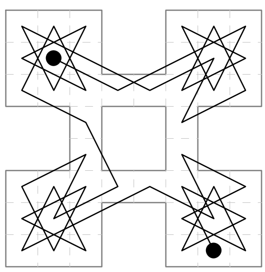
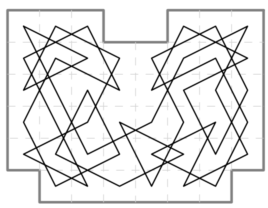
40 cells: More typical Indian tours on irregular boards. The first is #32 in Naidu and represents a yak. The second is #54 in Harikrishna (1871) and #S31 in Naidu (1922) and represents an ‘installed Shiva-linga’. The other is #55 in Harikrishna and represents a bull. These two have a single hole.

42 cells: Three more asymmetric pictorial tours. Naidu #15 and #25 represent camel and tiger, and #60 in Harikrisha (1871) representing a vase.

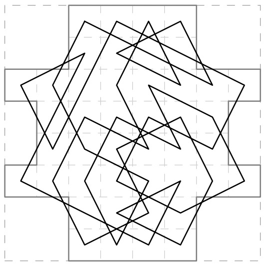




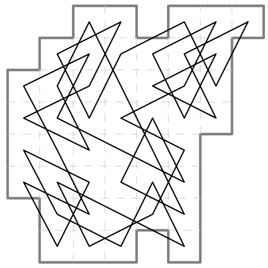
44 cells: First an open tour by Haldeman (1864) with approximate axial symmetry on a cross-shaped board. Second an open asymmetric tour of 8×8 board with five 2×2 voids by Ambikadatta Vyasa *Chaturanga Chaturi* (1884). Third an asymmetric tour of a symmetric board that cannot be toured symmetrically, by S. Vatriquant *L'Echiquier* October 1928.

The next tour is the solution of an ingenious puzzle titled ‘A Personally Conducted Tour’ in *Pittsburgh Gazette Times* (‘Some Puzzles’ XVI, 19 Aug 1917) composed by T. R. Dawson and C. D. Locock. On the 8×8 board begin with a Black Knight g8, White King e1. “The knight moves only to check, and may not play into any of the 20 squares cut off round the edges (where it cannot be forced to check). White giving Black no choice of check, makes the knight tour the other 44 squares.”

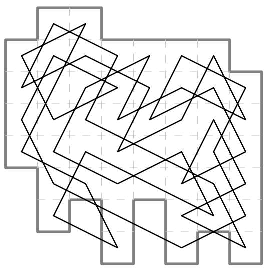
To force this unusual knight's tour the King has to make a much longer journey: 5 c6, 8 f8, 10 h7, 13 f6, 15 e8, 17 g7, 18 f7, 21 c5, 23 a6, 24 a5, 27 a2, 30 c3, 33 c6, 38 h6, 39 h5, 42 h2, 43 f3, 47 e1, 49 g2, 52 e3, 55 b1, 58 c3, 61 a6, 63 b8, 66 c6, 68 a5, 70 b7, 73 d6, 76 g8, 79 f6, 82 c6, 87 c1, 88 d1, 91 g1, 94 f3, 98 g7, 100 h5, 103 e3, 106 b1, 107 b2, 108 c2, 111 a5, 114 a8, 115 a7, and with 122 King g6, Knight e7+ the tour is made reentrant.

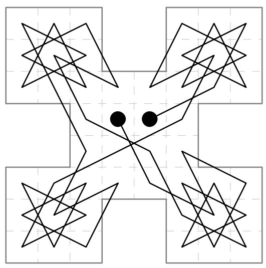


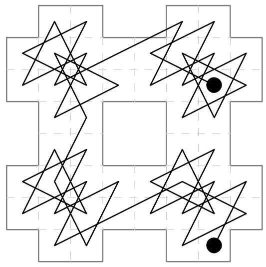


46 cells: Diagram above right. Naidu (1922) #43 depicting a camel.

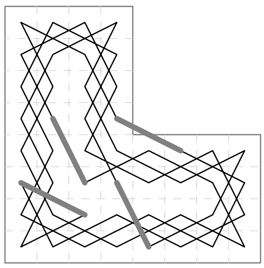
48 cells: Closed tour of asymmetric board by Naidu (1922) #46 representing an elephant, and two open asymmetric tours from Indian sources. The first is from Lala Raja Babu Sahib *Mo'allim ul Shatranj* 1901. The second, on a hatch-shaped board, appears in Ambikadatta Vyasa *Chaturanga Chaturi* 1884. It joins together four copies of Euler's biaxial 12-cell tour. These tours were collected by H. J. R. Murray (1930).

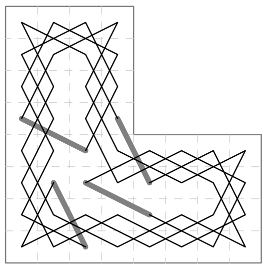


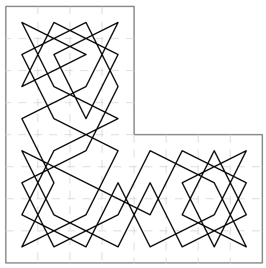




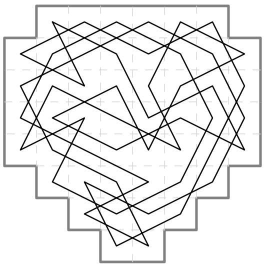
The first L-shaped tour here is from Eggleton and Eid (1984), formed by simple linking of four edge-hugging circuits; their linkage polygon is noncrossing. My solution alongside keeps within the boundary but the linkage includes a crossover. The third is a squares and diamonds solution (both Jelliss 2016).

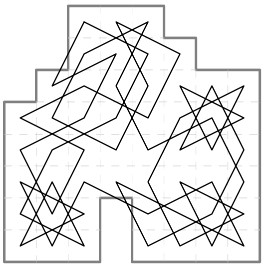




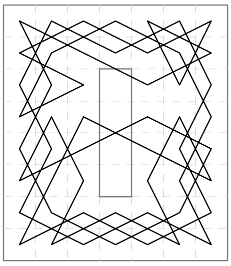


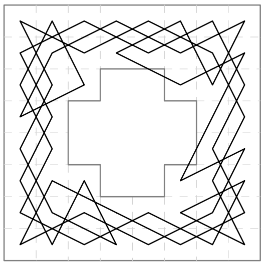
50 and 52 cells: #8(S) and #10 from Naidu (1922) representing peacock and elephant.

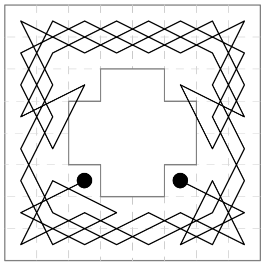




52 cells: Closed tours by H. C. von Warnsdorf (1823) on 7×8 – 4 and 8×8 – 12 boards. Open tour on the same board from Lala Raja Babu Sahib *Mo'allim ul Shatranj* 1901 (rotated) with approximate axial symmetry. (Closed tours with rotative symmetry are also possible on this board. See p.146.)

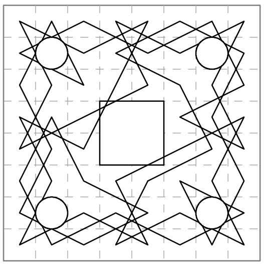


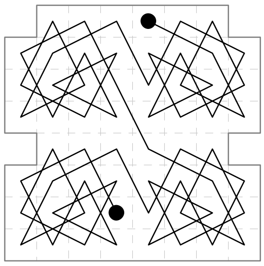




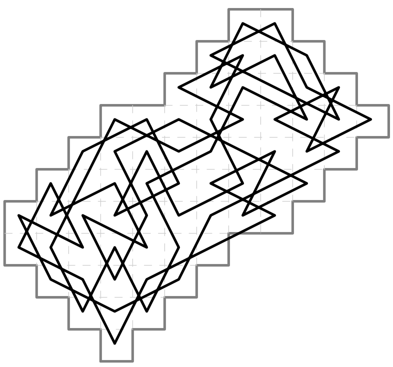
56 cells: Rajah of Mysore #64 board (8×8 – 8).

60 cells: Figure 85 in Wenzelides *Schachzeitung* (1850), joins four 15 cell tours of the 4×4.

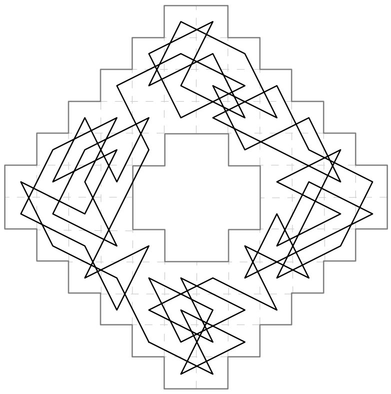


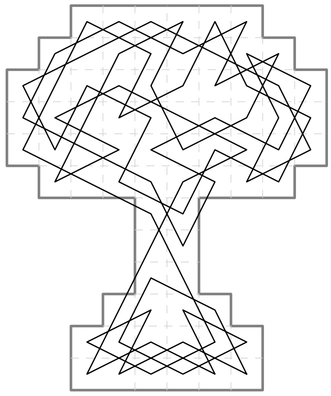


64 cells: This board is formed by combining two identical 32-cell biaxially symmetric boards, but the boards combine asymmetrically, and the tour (Kraitchik 1926 §16) is of course asymmetric. Curiously the board can be dissected into three parts to form an orthodox 8×8 board (Sam Loyd?).



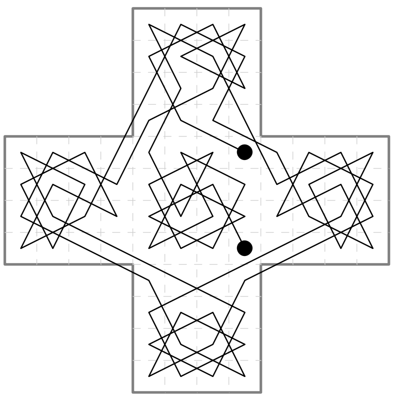
72 cells: A board with octonary symmetry, said to represent a lake with a platform in the centre. Showing an asymmetric tour, #15 by the Rajah of Mysore in Harikrishna 1871. Rotary symmetry is also possible. This is a board within an even frame, 12×12, on which Birotary symmetry is impossible, since 72 = 4×18 and 18 is even.

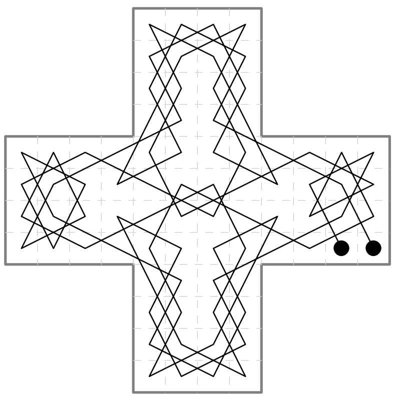




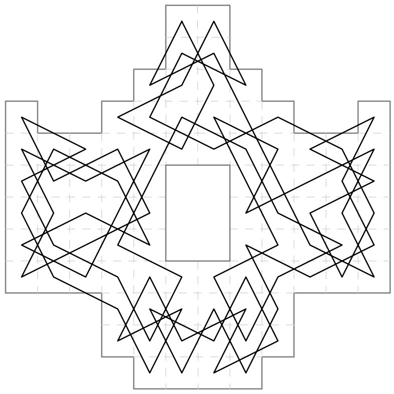
74 cells: This example above (#25 from the Rajah of Mysore in Harikrishna 1871) was designed to represent a tree, but nowadays invokes the more ominous mushroom cloud of a nuclear explosion. A similar tour appears as #35 in Naidu (1922). Harikrishna's #28 representing a prancing horse, not shown here, also uses 74 cells, but on an irregular board with two single-cell holes.

80 cells: These two open tour diagrams represent successive attempts at improved symmetry on the Greek Cross board of five 4×4 components. The first by H. E. Dudeney in *Queen* 12 Nov 1910: “We select ... one of the more elegant ways of solving this puzzle. Perfect symmetry is not possible, but our path approaches it.” It may be noted that the central 4×4 less two cells is based on a closed path with Sulian symmetry, one of two possible. [Cited by Bergholt 1916 who responded with a discussion on symmetry, and more symmetrical examples, open and closed.] This led to the much more symmetrical solution by E. Bergholt in *Queen* 22 Jan 1916. The linkage k5-l7-j6-l5 of inserted and deleted moves converts it into a pseudotour with biaxial symmetry.

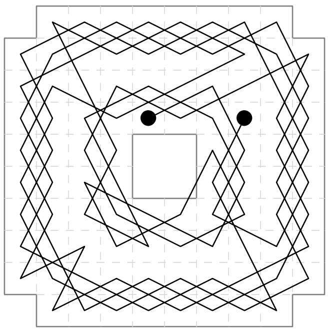




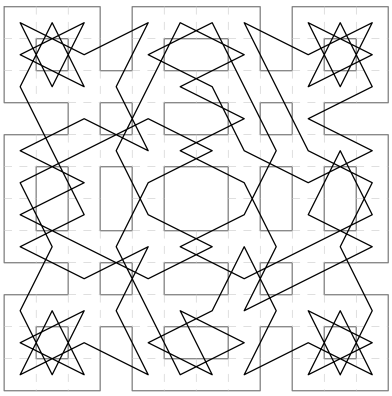
86 cells: This tour by the Rajah of M|ysore #20 from Harikrishna (1871) is said to represent an aerial chariot. The board has axial symmetry but the tour is asymmetric.

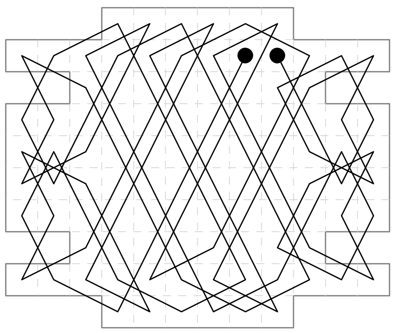


92 cells: Open asymmetric tour on a board 10×10 with 8 cells removed collected by Murray from *Zurcher Illustrierte* 20 Mar 1931. Has both rotative and diagonal reflective elements.



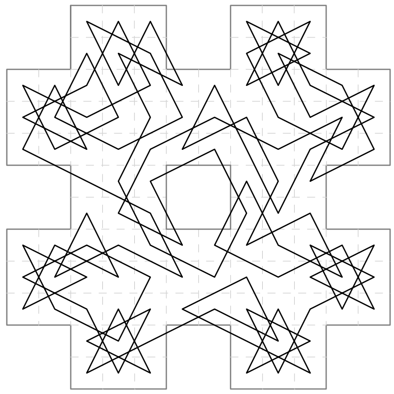
100 cells: Asymmetric closed tour, #23 from Harikrishna 1871, said to represent intertwined serpents. The board is 12×12 minus 44 cells, nearly biaxial. Asymmetric open tour from *Zurcher Illustrierte* 26 Jun 1931on a board 10×12 = 120 minus 20 cells. Also nearly biaxial.



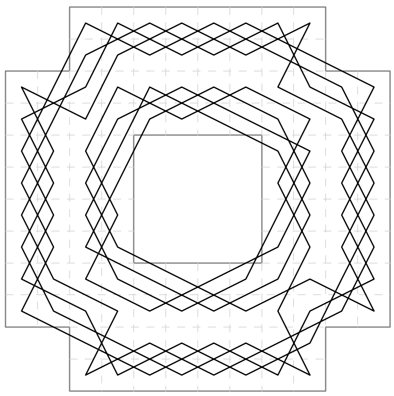


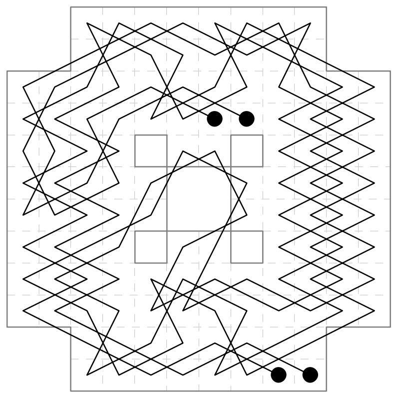
On a 10×10 board with the four corner cells moved to the middle of two sides, D. Fiomare in *L'Echiquier* Nov 1929 gave a simple asymmetric open tour, connecting tours of the four quarters.

108 cells: This 'hash' diagram is an asymmetric tour #17 from Harikrishna 1871, the nine areas removed are described as ponds. (#21 is also on 108 cells but uses an irregular board with a single cell hole, it is said to represent ‘an installed Shiva-linga’ not reproduced here). Oblique quaternary symmetry is possible on this board (se p.115).



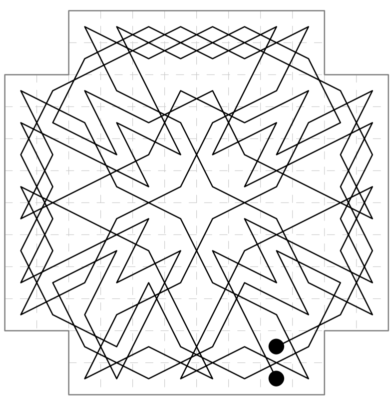
112 and 120 cells: Asymmetric tour (144 – 32), #12 from Harikrishna 1871. Oblique binary symmetry is possible on the same board. It does not admit quaternary symmetry.

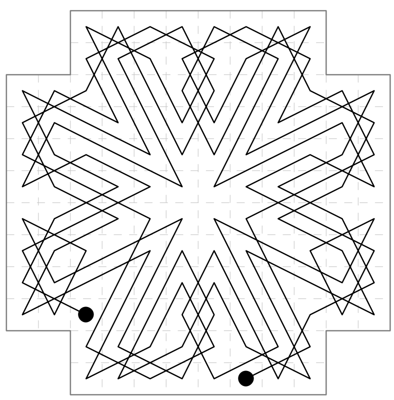




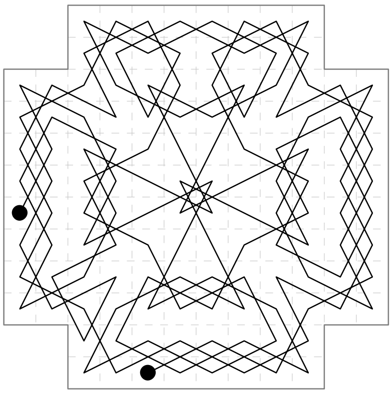
120 cells (above). Rajah of Mysore: Tour #19 is a ‘two-horse movement’ each tour covering 60 cells. When numbered the pairs of same numbers occur in adjacent cells, forming a division of the board into dominoes; one pair of dominoes being vertical (30 at h67 and 32 at e67) the others all horizontal. This can be converted into a tour by the linkage: h9-f8-e6-g5-h7-g9.

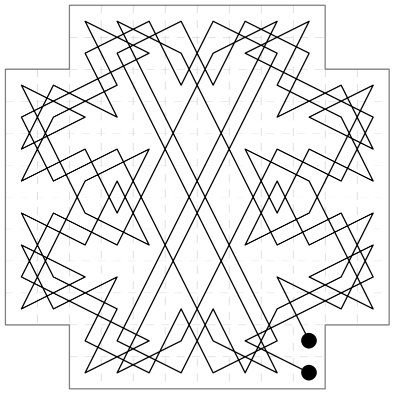
128 cells: The next four open tours exhibit approximate direct quaternary symmetry. The first is from *Essener Anzeiger* 12 Feb 1933, the others are from the H. Staeker manuscript (1849).



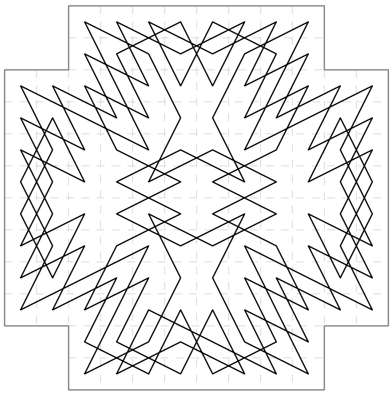


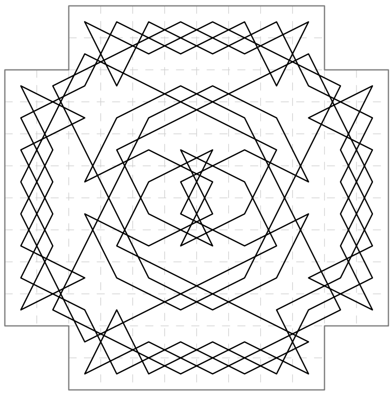
Two others from the Staeker ms:



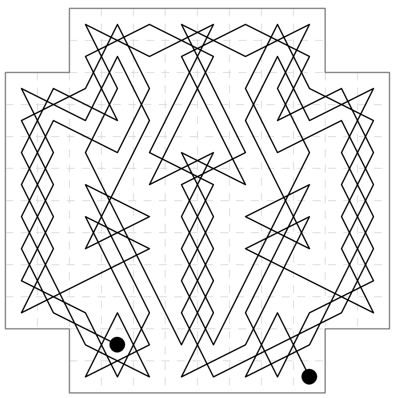


These closed tours from the Staeker ms are very nearly symmetric, appearing to show direct quaternary symmetry, but this of course is impossible on a board within an even by even frame and with an even number of cells in each quarter, and on close examination small flaws in the symmetry can be found. These flaws can be rectified but the result will be a pseudotour. This board is of a type used for four-handed chess.

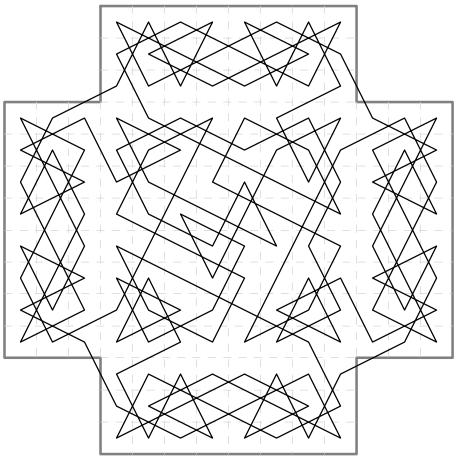
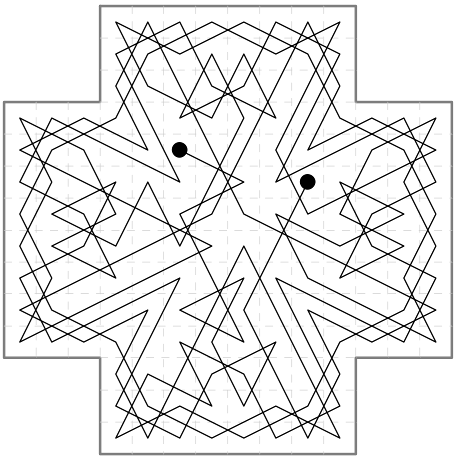




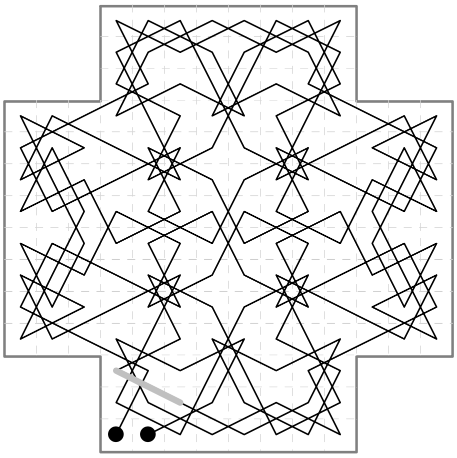
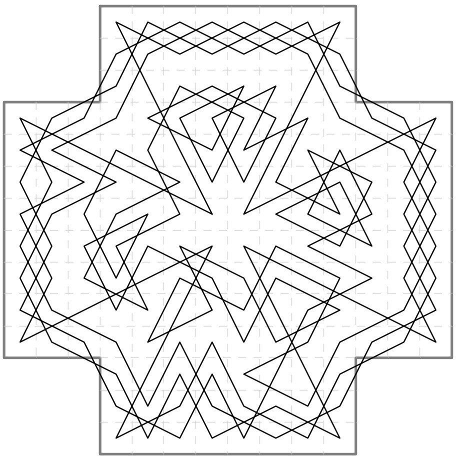
This tour is from G Mann jr (1859). At the end of his collection of chessboard tours he gives this near-axial tour on a four-handed chess board. He also shows a tour on a three-handed board which is similar but with the two ranks at the bottom omitted.



160 cells: The following diagrams are of asymmetric tours on the board for four-handed chess that was popular in the late ninreteenth century. The first two were decoded from hand-written copies of cryptotours taken from *Illustrierte Zeitung* for the period 1852-63:

The next is ‘The Knight's Tour on the Board Used in Chess for Four Players’ Frontispiece to Vol 3 of *Westminster Papers* 1871.

The above frontispiece tour is described as ‘a figure of the greatest beauty and regularity’ and the author as Victor Gorgias ‘of Oëdenberg’. It shows the maximum amount of direct quaternary symmetry possible in an open tour (and also a high degree of octonary symmetry). If the move marked in grey is deleted and the two loose ends of the move are connected to the ends of the tour the result is a pseudotour.

The closed partially symmetric tour is by A. H. Frost (1877). He also gives a tour by the compartmetal method, joining up ten tours 3×4 and two 4×5 on this board.