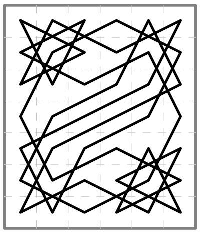
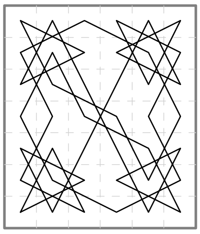
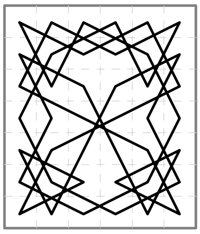
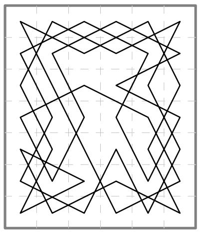
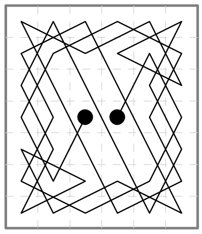
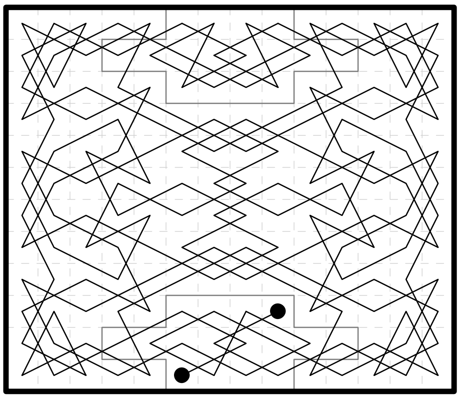
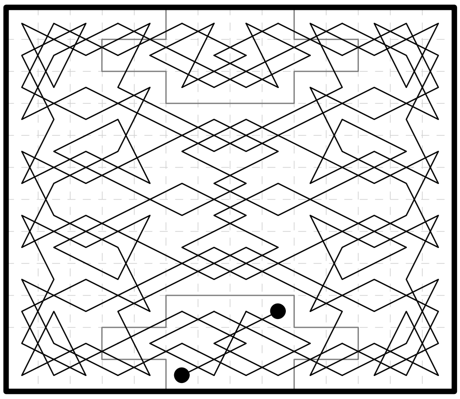
. KNIGHT'S TOUR NOTES

⌘ 4

Oblong Boards

by G. P. Jelliss

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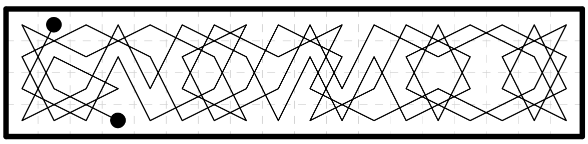

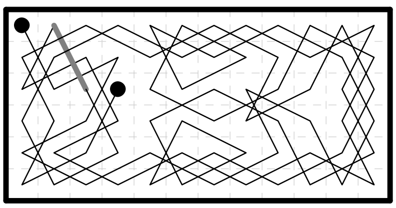 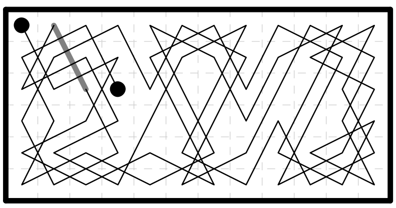

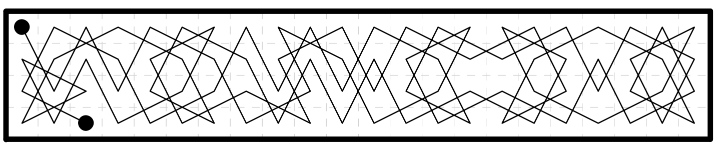

Title Page Illustrations:
Tours with assorted types of symmetry on 6×7 board.
Two magic tours on the 12×14 rectangle (Jelliss 2011).
Magic Tours 4×18, 6×12, 4×22, found by Awani Kumar (2018)

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http://www.mayhematics.com/
Knight's Tour Notes, Volume 4, Oblong Boards.

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Knight Tours of Oblong Boards

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Which Oblong Boards Have Knight Tours?

One of the attractions of knight tours on oblong boards is that they admit a greater variety of overall symmetry than on square boards. In particular, on boards with one side even and the other odd, closed tours may be possible with 180 degree rotational symmetry that cross at the centre point (termed Bergholtian symmetry) or with reflective symmetry about the median joining the mid-points of the even sides (termed Sulian symmetry). On small boards complete enumerations are possible.

THEOREM (Schwenk): An m×n board with m ≤ n has a closed knight tour unless one or more of these three conditions holds: (a) m and n are both odd, (b) m = 1, 2 or 4, (c) m = 3 and n = 4, 6 or 8.

This theorem for rectangular boards was rigorously proved by Allen J. Schwenk in the MAA *Mathematics Magazine* 1991. However, most of these facts were known to earlier authors (e.g. Frost 1876, Kraitchik 1927) though not expressed as a single theorem. The proof given depends on mathematical induction, using the Vandermonde method (see ⌘ 6) to extend a tour of side n to n+4.

Figure 9 in the paper shows “The nine Hamiltonian cycles [i.e. closed tours] that form the base of the inductive construction.” These are tours 3×10, 3×12, 5×6, 5×8, 6×6, 6×7, 6×8, 7×8, 8×8. The 3×10 and 3×12 examples are among those known to Bergholt and Moore. The 5×6 example is the Haldeman solution. The 6×6 example is the quaternary one with a Greek cross. Almost any tour with considerable border braids would do for the others.

Open Tours of course exist on all boards that have a closed tour, but are also possible on boards 3×4 and 3×8 (but still not on 3×6) and on all boards 4×n, except 4×4, and on all boards u×v where u and v are odd and greater than 1, except for 3×3 and 3×5.

Symmetric Closed Tours exist on all boards with closed tours except for 3×12.

Proofs of these special cases are given in the following sections.

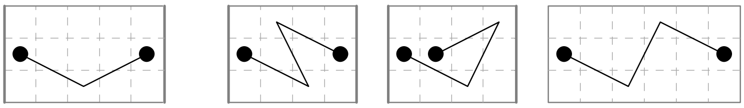
Magic Knight Tours are discussed ⌘ 1 p.37. They require a board even×even, and are possible on all boards (4·h)×(4·k) except 4×4, 4×8, 4×12, 4×16. They are also possible on certain boards where one side is not divisible by 4, that is (4·h)×(4·k + 2). Cases known to be impossible are 4×6, 4×10, 4×14 and 8×6. The 8×10 and 8×14 cases are undecided. The 8×18 and larger cases are probably possible (since 4×18 and larger are). Awani Kumar has now shown (2018) that magic knight tours are possible on all boards 4×2k from 4×18 upwards and has constructed two examples on the 12×6. One of these I have found can be extended to 12×10 by using a 12×4 braid. A few years earlier (2011) I found two magic tours on 12×14, the first of this type.

For results on square boards see ⌘ 9. The larger the board of course the easier it is to construct magic tours and the more there are, provided they are of the dimensions indicated above.

Three-Rank Boards

General Principles

The enumerations of 3×n knight tours described in the following pages have mainly been obtained by the method of breaking up a tour into **components**, which are groups of cells symmetrically arranged, with the property that the pattern of knight's moves connecting the cells of the component can be reflected or rotated without breaking its links with the other components of the tour. The cells and moves within a component are usually interconnected, but this is not essential. A diagram with k asymmetric components represents a family of 2^(k–1) tours, formed by 'twiddling' the components, keeping one fixed (i.e. 2 components give 2 tours, 3 give 4 tours, and so on). If a component is symmetric however the number is reduced due to duplication.



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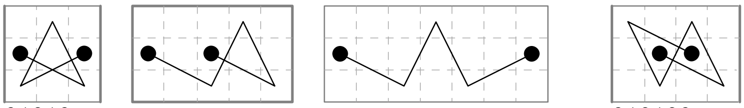

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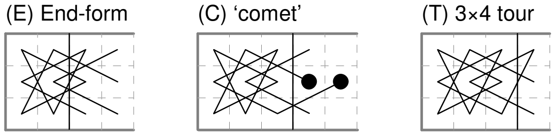
By numbering the ranks 1, 2, 3 we see that only moves 1-2, 1-3, and 2-3, and their reversals, are possible, so that a knight's tour will consist of a sequence of 2s separated, or at the ends preceded or followed, by sequences of alternating 1s and 3s. The sequences of 1s and 3s represent zigzag paths of vertical knight moves, while the moves to and from the centre rank are horizontal knight moves linking these zigzags. The following are diagrams of some of the shorter patterns of 2…2 linkage. Components are made up of such linkages.

THEOREM (End-Form): All 3-rank tours without end-points in the two end files have the same end-form (E) in those two files. If there are no end-points in three end files, the moves through them form one of two patterns: a 10-cell 'comet' component (C) or a 12-cell open 3×4 tour (T).

(E) End-form (C) 'comet' (T) 3×4 tour



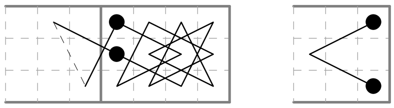
Proof: In a tour without end-points on the first two files, the moves through a1, a2, a3 and b2 on any three-rank board are fixed, and they determine also the path through c2. Now we cannot have b1-d2 and b3-d2 because a 6-move short circuit is formed, and we cannot have b1-c3 and b3-c1 since an 8-move short circuit is formed. We must therefore have either b1-d2 and b3-c1 or its reflection b3-d2 and b1-c3. A tour with no end-points in two end files can thus always be oriented to show the pattern (E) of the first diagram. From c1 we must now connect either to e2, which produces the **comet** pattern (C) or to d3 which completes the 3×4 tour (T).

Closed tours can thus be classified by their end-formations as of types CC, TT or CT. Those of type CT are all asymmetric. Denoting 180° rotation by ~ and left-to-right reflection by ^ the CC and TT types can be subclassified as C~C, T~T, including tours with rotary symmetry, and C^C, T^T, including tours with axial symmetry.

The 'comet' is an example of a component (as defined above); its end-points are on the middle rank and when reflected top to bottom it still tours the same 10 cells, and its end-points are unmoved.

THEOREM (Extension): Any 3×n knight's tour can be extended by four files.

Proof: The asymmetric 3×4 tour can be attached to one end of the tour by deleting a vertical move in the last two files and joining the loose ends to the ends of the 3×4 tour. In the diagram the deleted move is shown as a dashed line. Such a vertical end move always exists since if none existed the only possible end formation would have end points in both corners, entered horizontally, and two such moves join to give a two-move path, not a tour.



Centre-File Formations

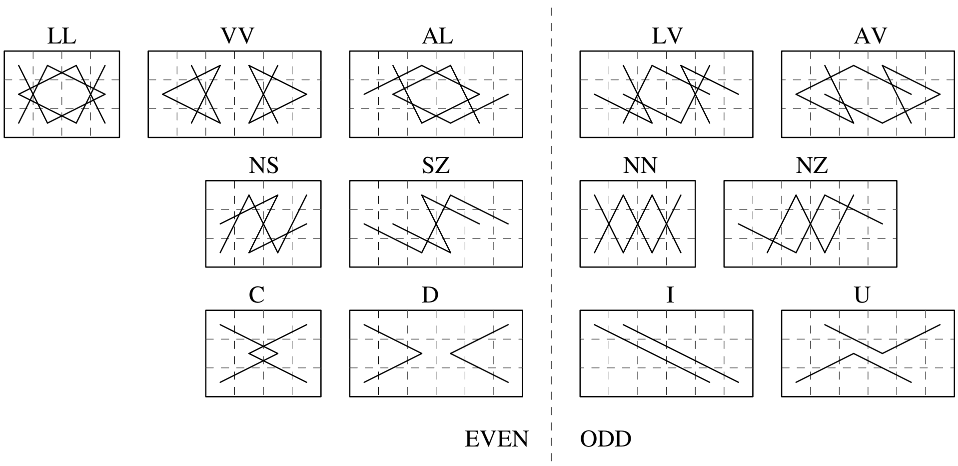
THEOREM (Rotary Centres): There are 26 formations that can occur on the two centre files in rotary closed tours 3×n, consisting of 18 on boards 3×(4k+2) and another 8 on boards 3×4k.

Proof: (a) If the symmetry is Eulerian (circling the centre) the moves through the upper and lower cells of the two middle files can be taken in five ways: LL, VV, AL, LV, AV (not AA as it forms two circuits). If the symmetry is Bergholtian the moves through these cells can be taken in four different ways: NN, NS, NZ, SZ. (not SS, ZZ which form closed circuits). In either type of symmetry there are four ways in which the centre cells can be taken: I, U, C and D:

(b) Each formation on the two central files can thus be represented by a three-letter code, e.g. ALU or NNC. The codes that include one or more of the reflectively symmetric components LL, VV, NN, C or D each represent one pattern, but in the other cases there are two ways of combining the formations. Those in which the centre-cell formation is reflected can be distinguished by an asterisk, e.g. NZI and NZI*. This gives 48 formations to consider.

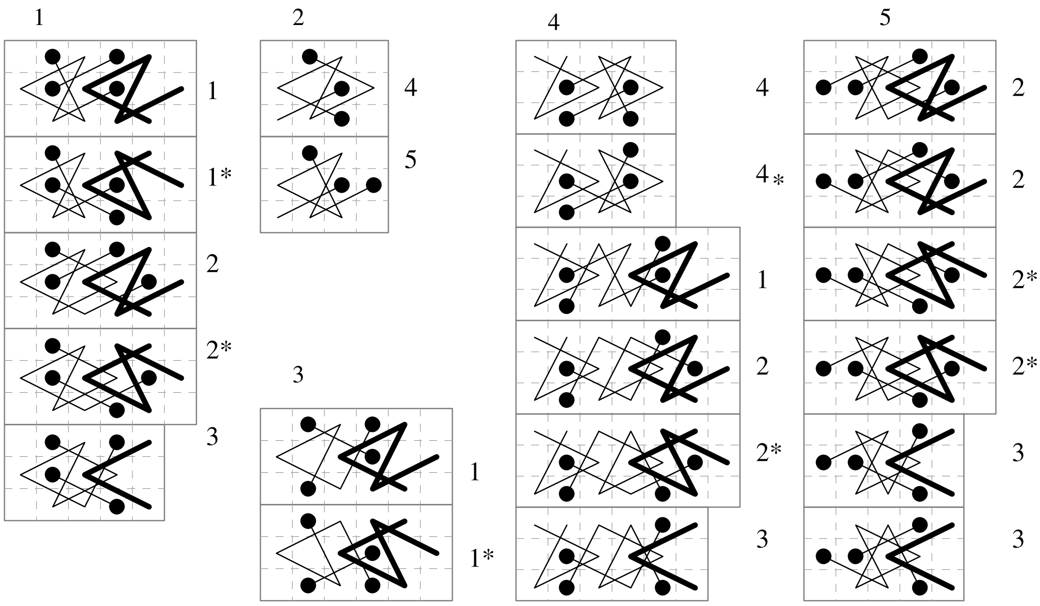
(c) The 14 formations LLC, LLD, VVC, VVD, ALC, ALD, AVI, AVI*, AVU, AVU*, LVI, LVI*, LVU, LVU* cannot occur in any symmetric tour since joining the loose ends in the same way on each side results in two circuits instead of a single tour (LLC and VVC in fact already form two circuits). This reduces us to 34 formations.

(d) The case NNC forms a single circuit but not using all the cells of the 3 by 4 rectangle it occupies. In the case SZD a six-move short circuit is forced so no tour is possible. In the cases NSD, NZC, VVI, VVU, LVC, AVC iterative construction shows that these central formations lead to end-formations which, when extended further, lead only to the same end-formations (to present all this would take up too much space and be tedious, but an example case is shown below). This means that these paths can be extended to any length, but the ends of the tour can never be **squared-off** to complete a rectangle, so no (rectangular) tours derive from these cases. This reduces us to the 26 formations stated in the theorem.

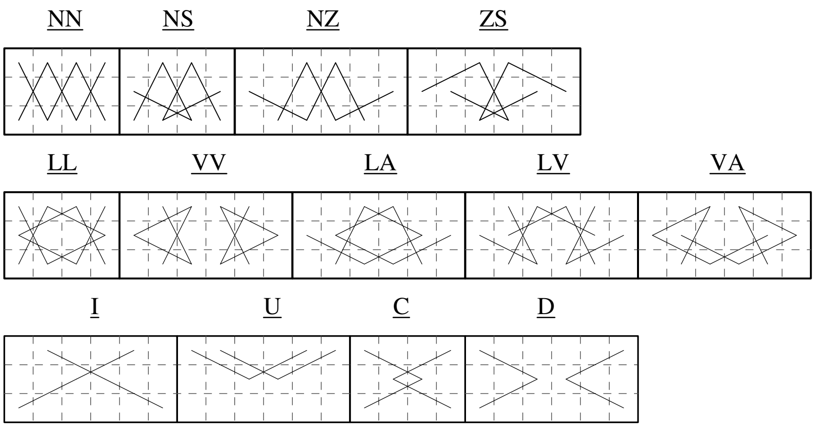


(e) A necessary condition for a tour to be possible is that the number of cells needed to join up the loose ends on one side of the central formation, plus the number of cells used in that half of the formation, must be of the same parity (even or odd) as that of half the board, i.e. odd on 3×(4k+2) and even on 3×4k (the partial components are classified as odd or even according to this criterion in the figure above). The 18 cases possible in tours on 4k+2 boards are: NSI, NSU (occurring on 3×10), SZI, SZI*, ALU, ALU*, LVD, AVD (on 3×14) and NND, NZD, NSI*, NSU*, SZU, SZU*, LLI, LLU, ALI, ALI* (3×18). The 8 cases possible in tours on 3×4k boards are: NZI, NZI*, NZU, NZU*, SZC (possible on the 3×16 board) and NNI, NNU, NSC (which occur in 3×20 tours).

(f) Here is an example of the iteration process. In the even case NSD the diagrams below show that this central formation leads to other formations, numbered 1 to 5, and that these lead only to the same formations (1 to 1, 2 or 3; 2 to 4 or 5; 3 to 1; 4 to 1, 2, 3 or 4; 5 to 2 or 3) never to a squared-off end. Only the right-hand half of the board is shown.



THEOREM (Equality): The numbers of tours in reflective and rotational symmetry on boards 3×(4k+2) are equal. *Proof*: Consider the possible formations on the two central files.

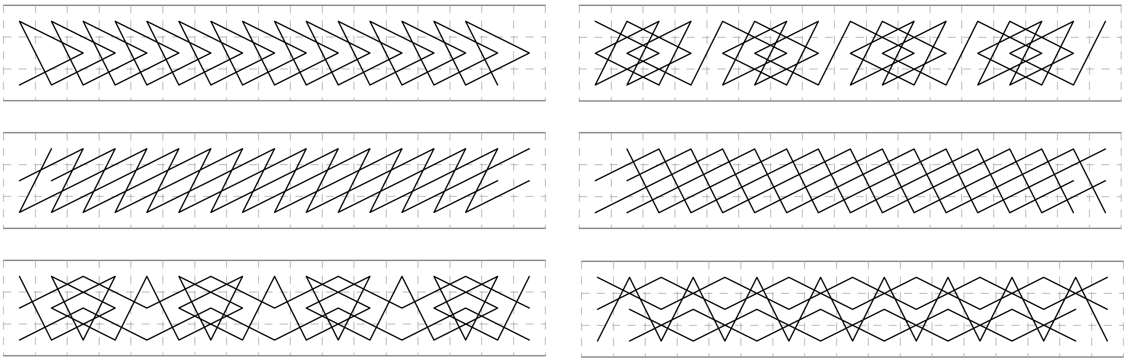


These formations are found to occur in reflective and rotational related pairs. The reflective forms, illustrated above, are denoted by the same letters as for the rotational forms but underlined. (NN, LL, VV, C, D, are in fact the same as the rotational cases).

The same arguments as in the previous theorem show whether tours are possible; in fact each reflective tour can be formed by reflecting half of the corresponding rotational tour. It will be found that in the eight cases that apply on 4k boards the transformation from rotational to reflective alters a tour to a pseudotour (a superposition of circuits). This result is thus compatible with the fact that tours with reflective symmetry are not possible on 3×4k boards. This equality was conjectured by me in *Chessics* (#22 1985 p.65) but a more complete proof is given here.

Repeating Patterns

More convoluted repeating patterns, analogous to the 2-rank braids are possible on 3-rank boards. Here are some designs. Note that the number of strands in these patterns varies from one to five. Using these in tours of course requires the strands to be linked to one another on other parts of the board. These patterns can be coded by the angle numbers (1 and 2 indicating the acute angles, 3 the right angle, 4 and 5 the obtuse angles and 6 the straight). In the first the angles follow the repeated sequence 213. The second 21321123123. The third 116. The fourth 336. The fifth 4 in two strands and 232121232 in the other. The sixth 4 in four strands and 2 in one.



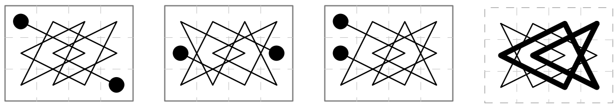
Catalogue of Three-Rank Tours

We now catalogue tours on 3×n oblong boards of increasing length. The number of geomerically distinct tours is denoted by G, the number symmetric by S and the number of tour diagrams (oriented with long side horizontal) by T. In the case of binary symmetry these are related by T = 4·G – 2·S.

3×4

The 3×4 twelve-cell board is the smallest knight-tourable rectangle. No closed tour is possible since the passages through a1, a3 and d2 are fixed and form a six-move short circuit. Two such circuits however combine to form a crosspatch pseudotour. There are three geometrically distinct tours, all found by Euler (1759), two symmetric, one asymmetric.

3	**6**	**9**	**12**
8	11	2	5
1	**4**	**7**	**10**

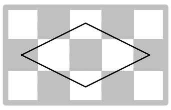


The corner to corner tour is 'tridirectional' using moves in only three of the four directions, and can be joined to copies of itself to form tridirectional open tours 3×4n. For other examples see boards 5×11, 6×9 and 7×7. When numbered, as in Euler's paper, there are four tours, since the asymmetric one gives two distinct numerical forms. The top and bottom rows of the corner to corner tour form arithmetic progressions 1, 4, 7, 10 and 3, 6, 9, 12 making it a Figured Tour.

Summary 3×4: G = 3, S = 2, A = 1, T = 8.

3×5

THEOREM 3×5: No knight tour is possible on the 3×5 board. *Proof*: A tour on an odd×odd board must be open and have its ends on cells of the majority colour when the board is chequered. The cells a2 and e2 on the 3×5 are of minority colour and moves there form a 4-move short circuit.

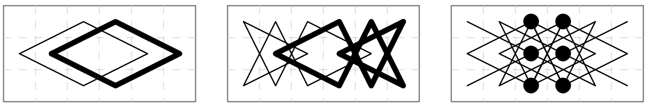


3×6

THEOREM 3×6: No knight tour is possible on the 3×6 board.

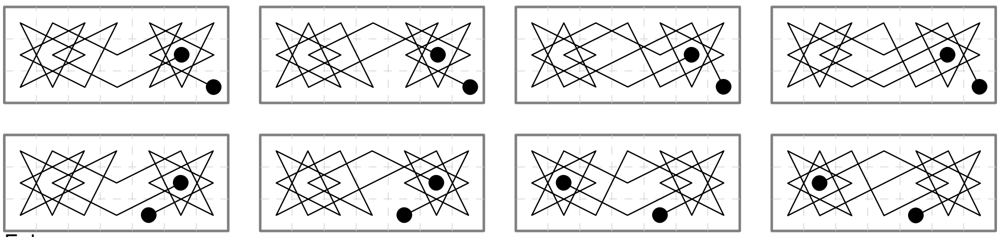
Proof: The knight paths through the cells a2 and e2 form a four-move short circuit, which shows a closed tour is impossible. The passages through b2, f2 also form a four-move circuit, so if an open tour is possible it must begin in one of these circuits and end in the other. The corner cells are thus not end cells of a tour, and the paths through them are fixed. These paths also fix the paths through c2 and d2. So now the paths through b1, b3, e1, e3 are fixed, to c3, c1, d3, d1 respectively. Finally a pair of moves through any of a2, e2 and b2, f2 completes an 8-move short circuit.

Alternative Proof: There are 22 moves incident with the two middle files, and only 4 other moves. An open tour would use 17 moves, at most two on each cell of the middle files, making 12, leaving 5 others required, but only 4 are available.

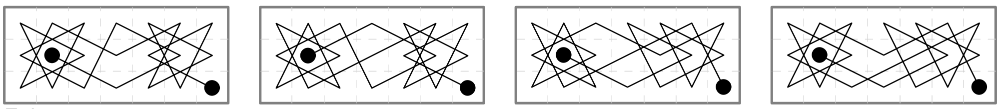


3×7

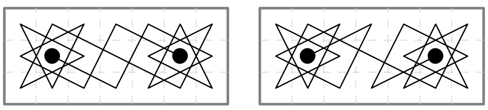
On the 3×7 board the 14 geometrically distinct were first counted by U. Papa (1920). Two were shown by Euler (1759). By end-separations there are: 6 {1,1}, 2 {1,3}, 4 {1,5}, 2 {0,4}.



Euler



Euler



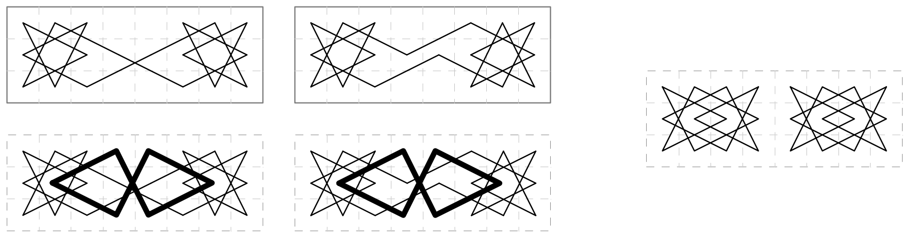
The last two are symmetric. The tours are related by reflection of components. All twelve of the asymmetric tours have a central lozenge which can be reflected left-right. Four of the asymmetric tours include a 3×4 tour at one end, and in four others the comet component can be reflected up-down. Eight of the asymmetric tours have a star in the last three files which can reflect in the f file (pivot about f3), or when extended by one move reflect in the middle rank (pivot about d2).

Summary of open tours 3×7: G = 14, S = 2, A = 12, T = 2×2 + 4×12 = 52.

3×8

THEOREM 3×8: No closed tour is possible on the 3×8 board. *Proof*: The passages through the six end cells are fixed and we may also fix b1-c3 and b3-d2 without loss of generality (since the alternative choice b3-c1 and c1-d2 is merely a reflection of this case). We now have two choices: if g1-f3 and g3-e2 then e2-c1 and d2-f1 are forced, while if g3-f1 and g1-e2 then e2-c1 and d2-g3 are forced; these forced moves complete 18-move short circuits.

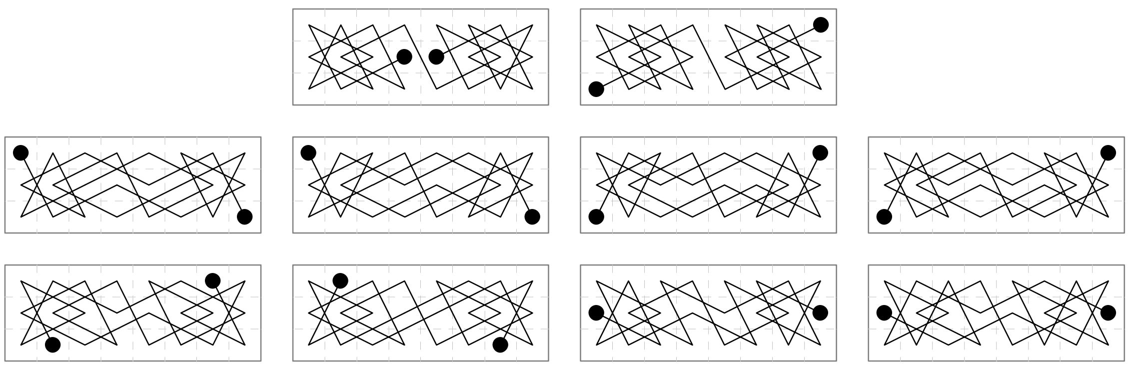
These circuits can be combined with a 6-move circuit on the remaining cells to produce two pseudotours (Schuh 1968 p.350). There is also a crosspatch pseudotour of two 3×4 components.



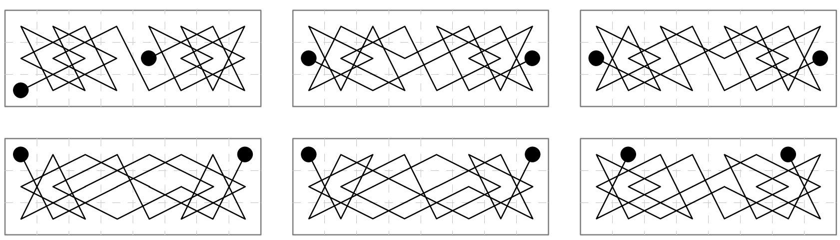
The construction of open tours on this board has been especially popular, since they can be used together with 5×8 tours to form compartmental tours of the 8×8 chessboard. However, the correct total of 104 geometrically distinct tours does not seem to have been given until my article in *Chessics* (1985). My method of counting the tours was to classify them by separation of end-points. There are nine classes: 29 {0,1}, 13 {0,3}, 9 {2,3}, 7 {1,4}, 3 {0,5}, 4 {2,5}, 18 {1,6}, 11 {0,7}, 10 {2,7}.

Summary of open tours 3×8: G = 104, S = 10, A = 94, T = 2×10 + 4×94 = 396.

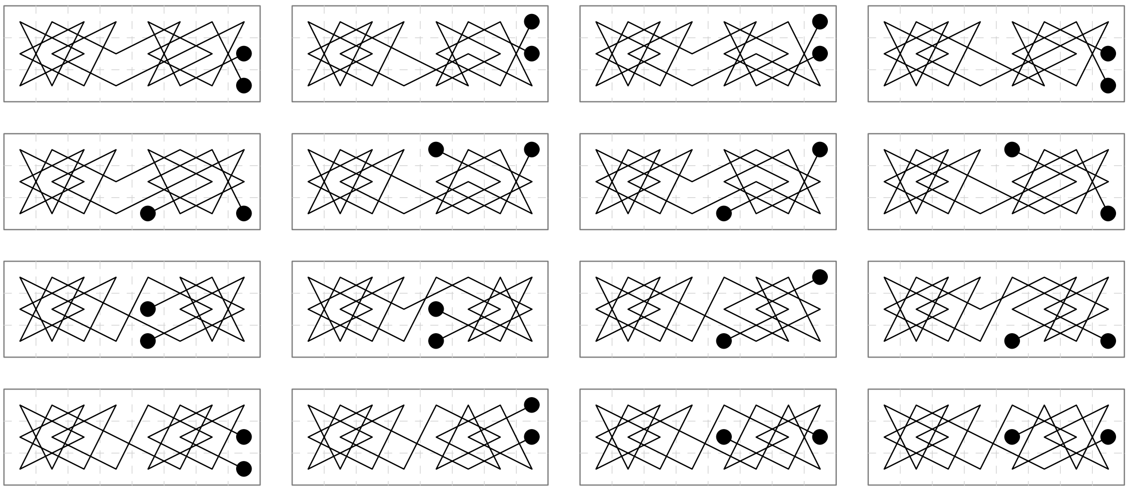
The 10 symmetic tours wete counted by U. Papa (1920). These comprise two compartmental tours, formed by joining two 3×4 tours, one from the {0,1} class and one {2.7} corner to corner. There are six other corner to corner, and two end to end {0,7}.



Diagrams of the 94 asymmetric tours follow, classified according to their end-formations.
The first six are related to the symmetric tours by twiddlng of components.

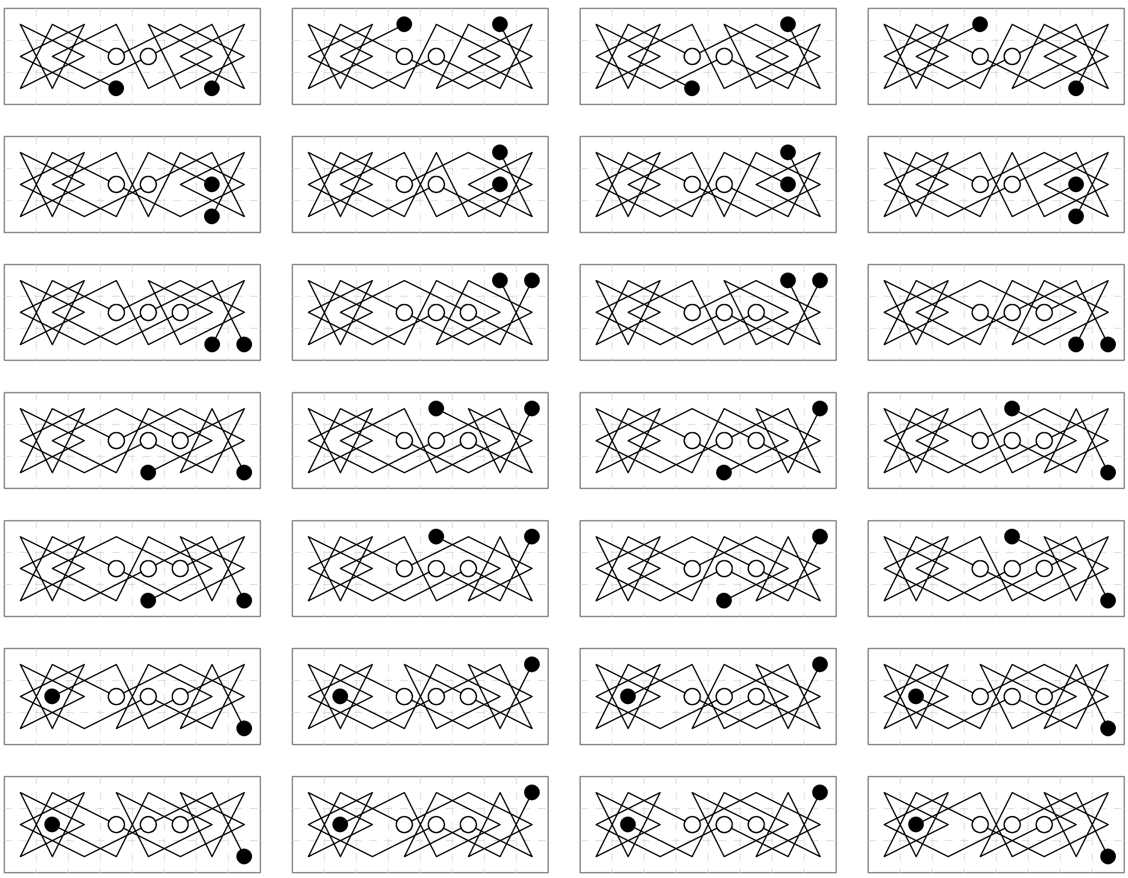


There are 16 ‘semi-compartmental’ tours in which the asymmetric 3×4 tour occurs at one end, extending two ‘tendrils’ to cover the other half of the board.

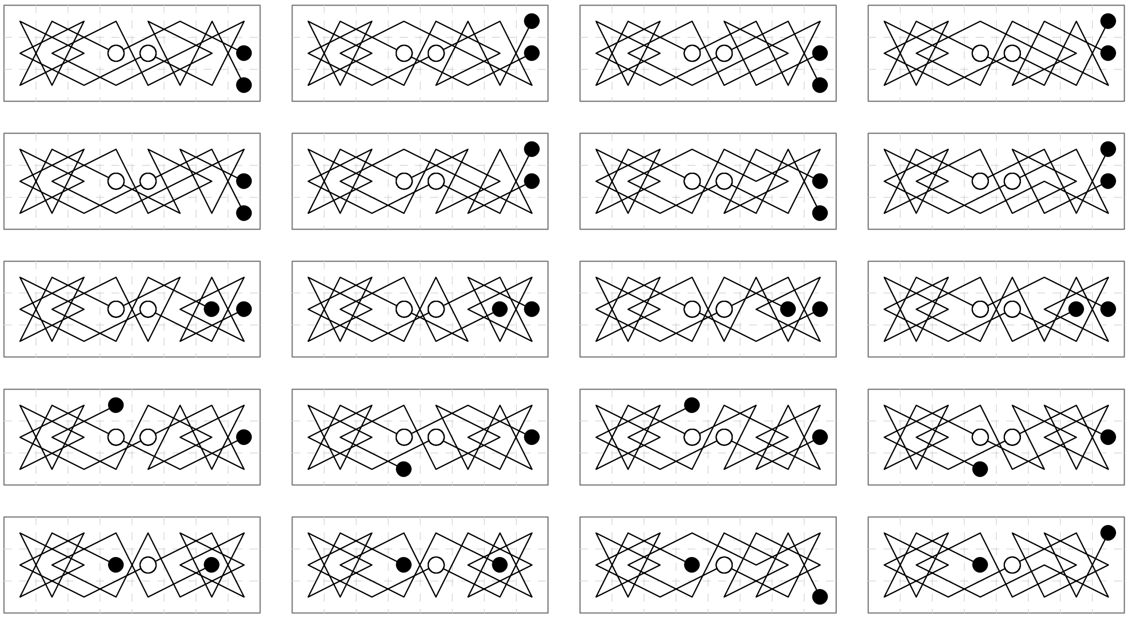


There are 48 tours that have the 9-move ‘comet’ at one end. They are shown with the comet on the left and in the same orientation throughout, the ends of the comet being marked by white dots. They occur in pairs, the right-hand part reflected top to bottom.

Of these 48 there are 28 where the right-hand part is formed of two separate components. These occur in sets of four formed by ‘twiddling’the components:

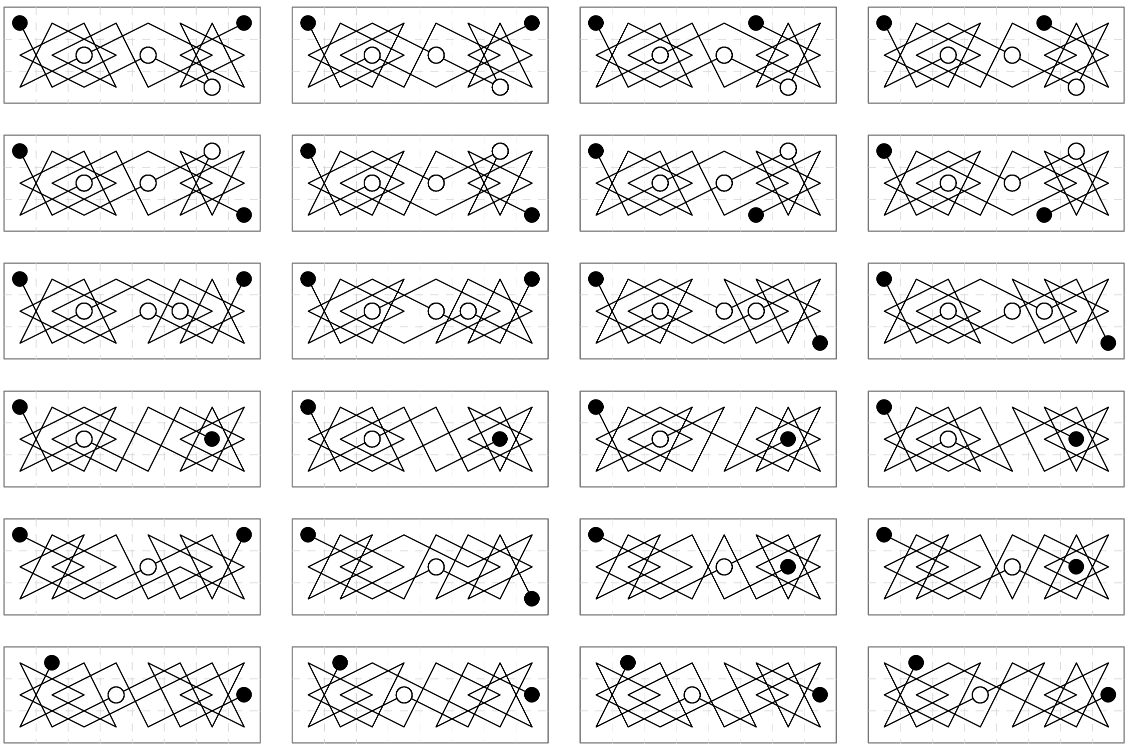


Of these 48 the other are 20 two-component comet tours:



In the last four cases one end of the comet is an end-point of the tour.

There is another set of 8 tours formed by twiddling four components,



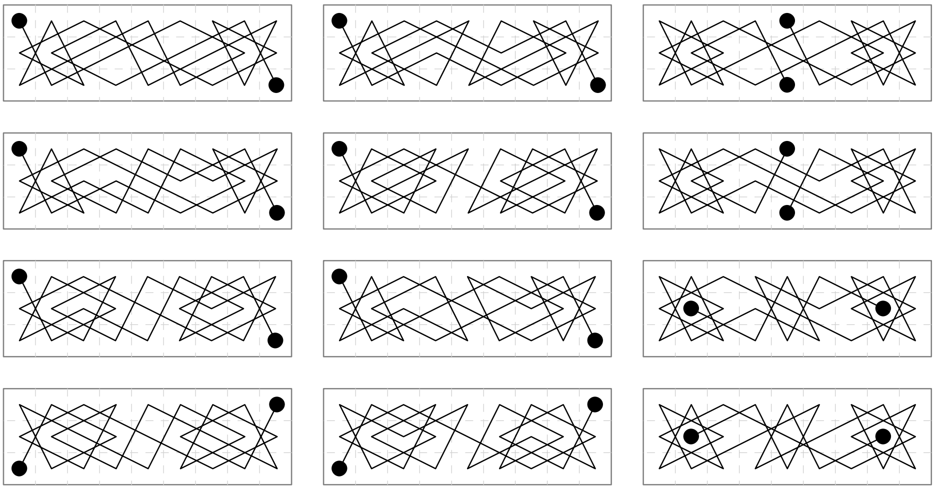
And 16 other 3×8 tours (last four rows above). Pivot points marked with white dots.

3×9

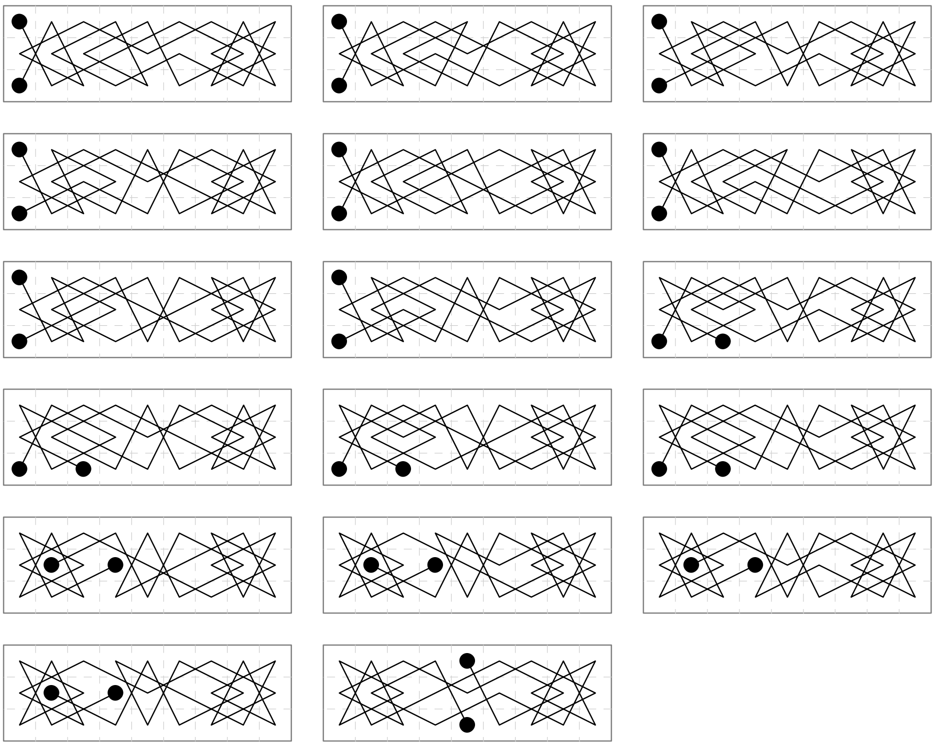
In terms of separation of ends the numbers are: 19 {0,2}, 10 {0,4}, 3 {0,6}, 20 {0,8}, 28 {1,1}, 8 {1,3}, 8 {1,5}, 12 {1,7}, 4 {2,2}, 10 {2,4}, 24 {2,8}. Total 146. Of these 12 are symmetric.

Summary of open tours 3×9: G = 146, S = 12, A = 134, T = 2×12 + 4×134 = 560.

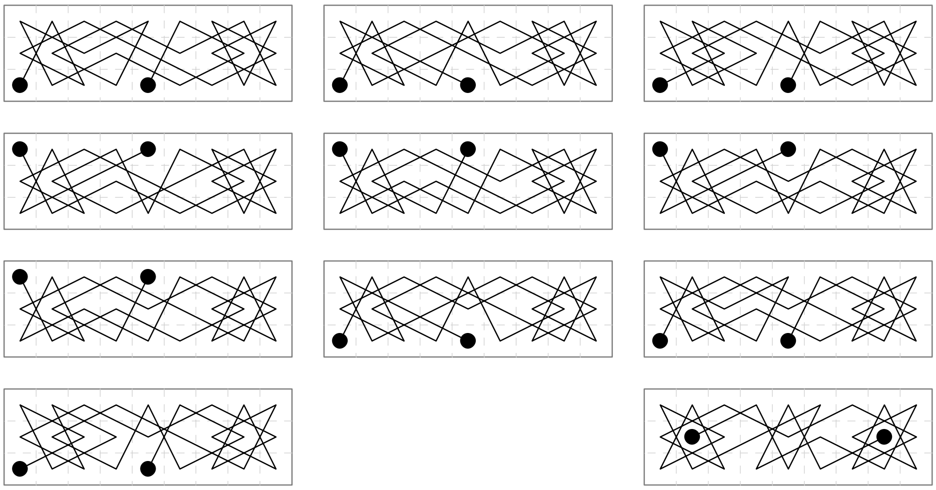
The 12 symmetric open tours were given by Kraitchik (1927), though he counted 16. They are oriented here so the two moves through the centre are c3-e2-g1.



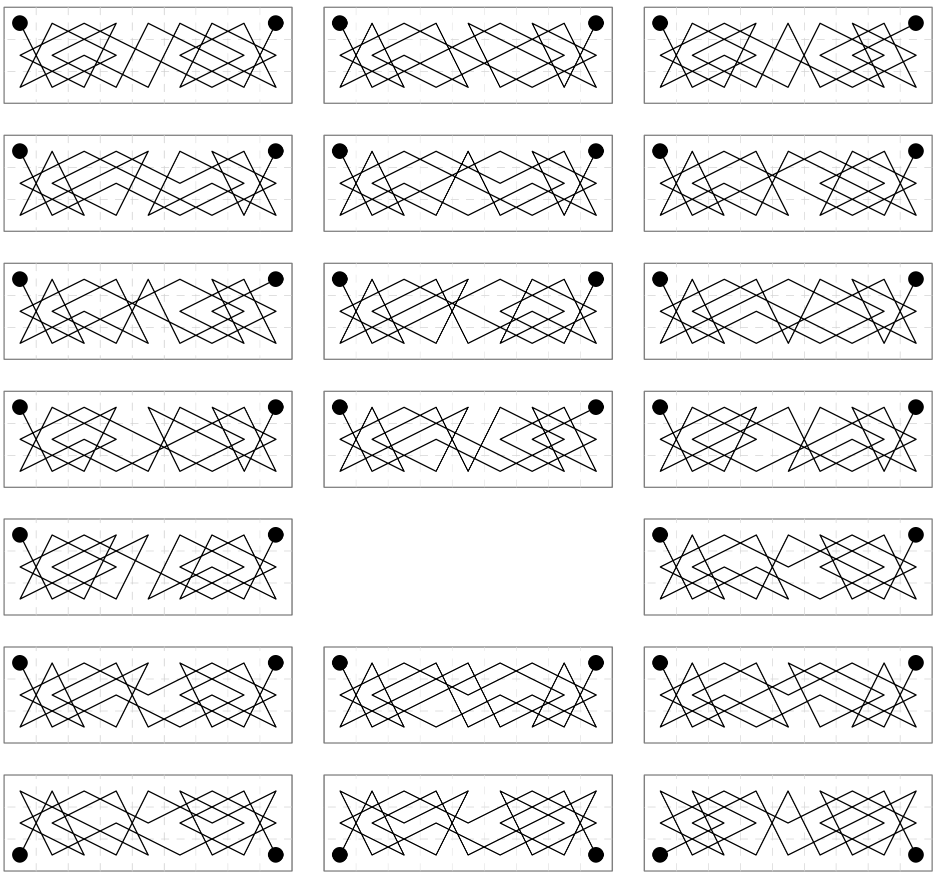
There are 17 asymmetric {0,2} tours:



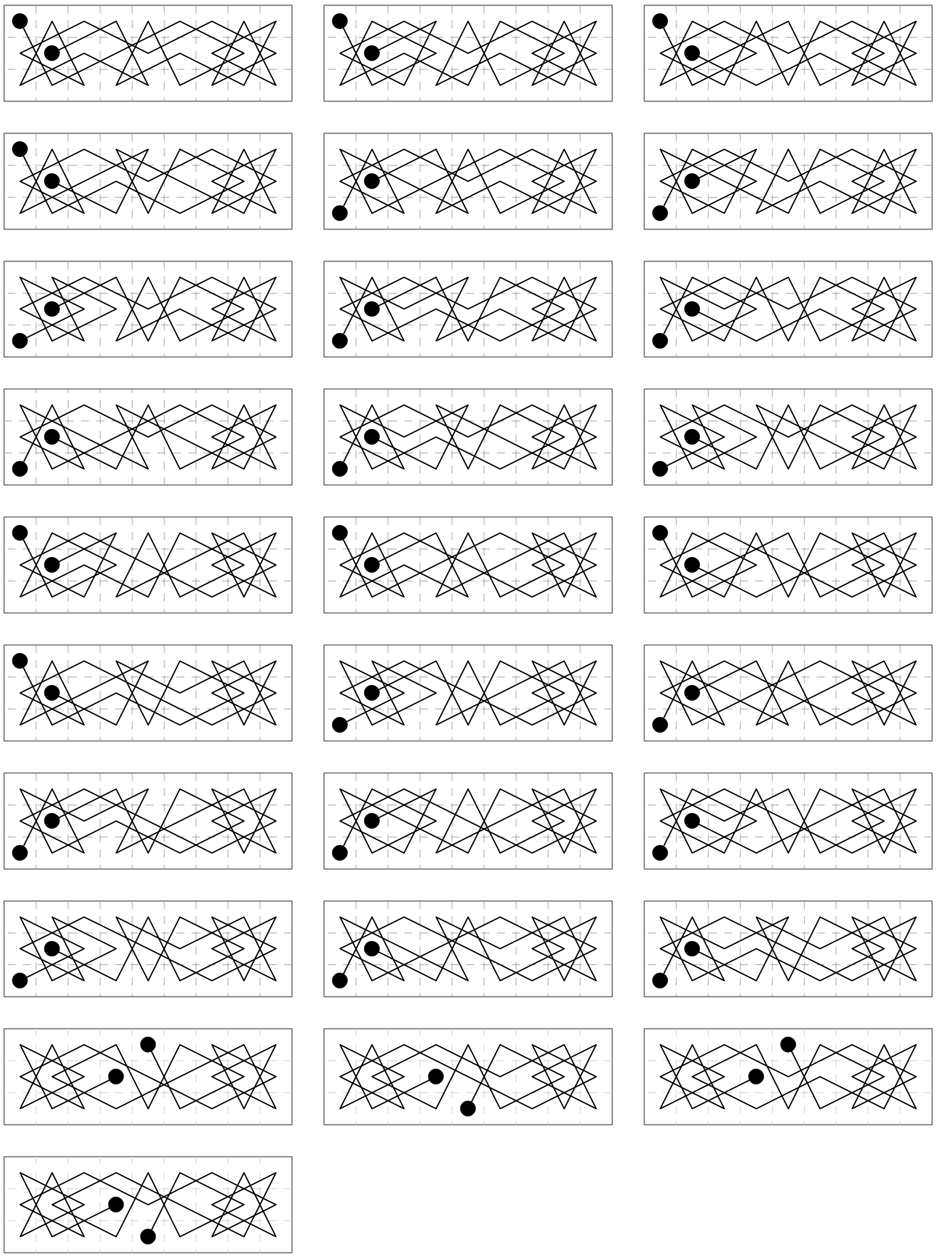
There are 10 {0,4} tours all asymmetric, but only one {0,6} to add to the two symmetric.



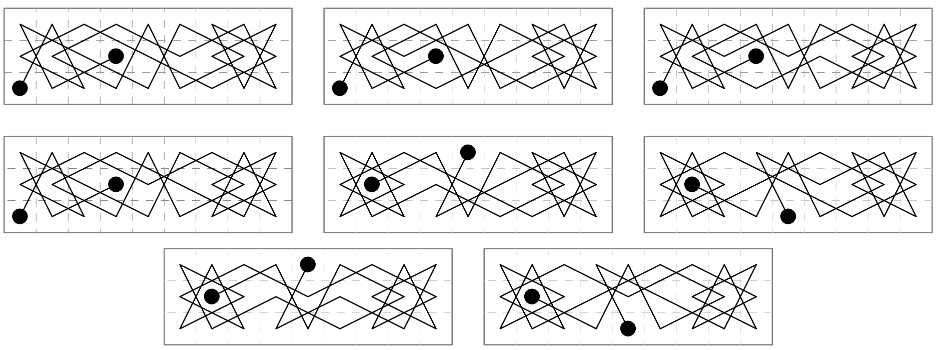
There are 20 {0,8} tours, all asymmetric. The middle moves of the asymmetric tours are also oriented c3-e2-g1 or in the vee c3-e2-g3. A complication in counting these tours is that the seven with the vee-shaped central move can be reflected left to right without altering the relative position of the endpoints, whereas the 13 with the middle two-move line, oriented c3-e2-g1, can always be rotated so that the end cells are in the top rank.



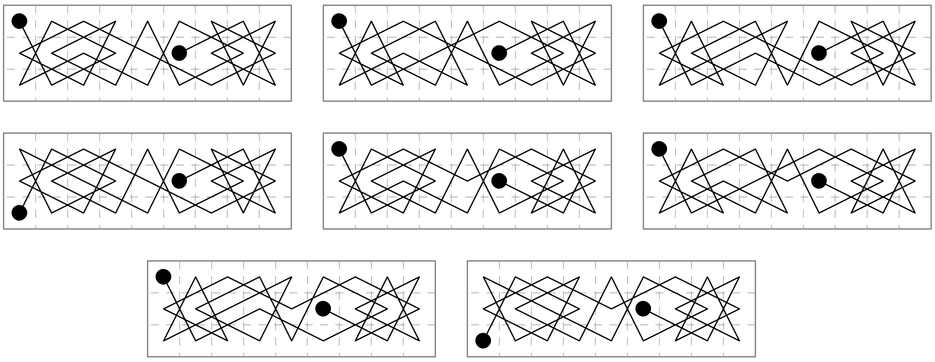
There are 28 tours of {1,1} type all asymmetric:



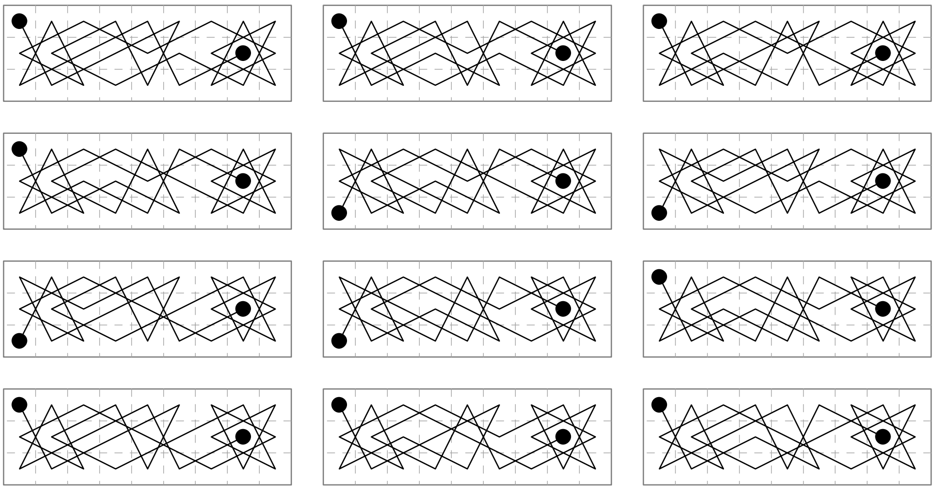
There are 8 {1,3}



There are 8 {1,5}

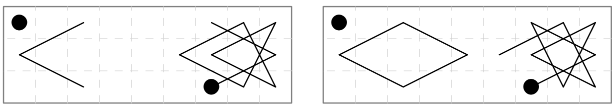


There are 12 {1,7}:

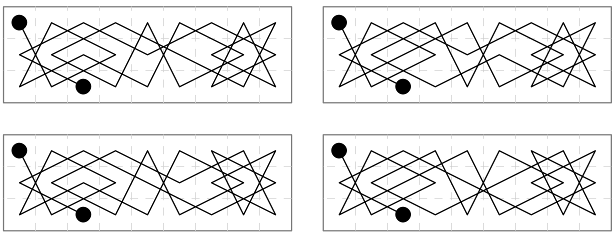


We now look at those with minimum coordinate 2.

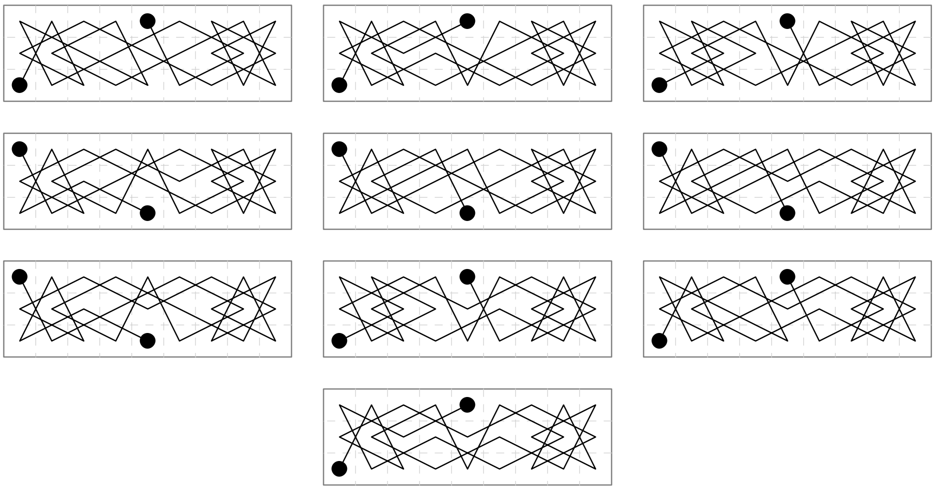
There are no {2,6} tours since if we take the ends to be at say a3, g1 the moves through a2, i1, i2, i3 are fixed, and since g1 is an end-point h3 must connect to f2, and to avoid a 6-move circuit h1 must connect to g3, but now e2 can only go to c1, c3, forming a 4-move circuit.



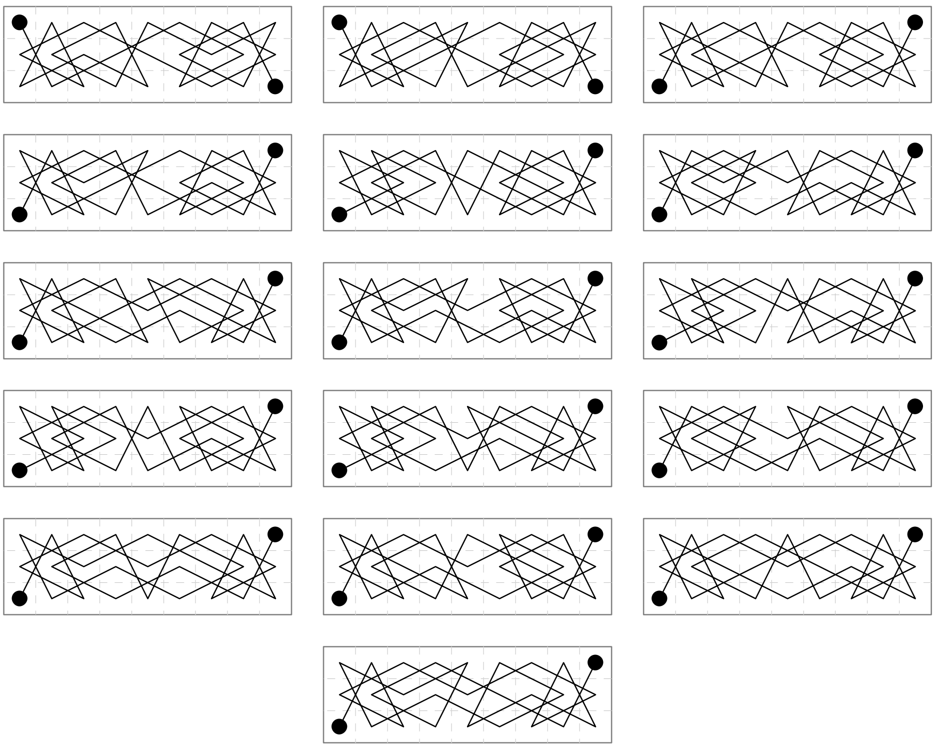
There are four {2,2} tours, all asymmetric:



There are also 10 {2,4} tours all asymmetric:



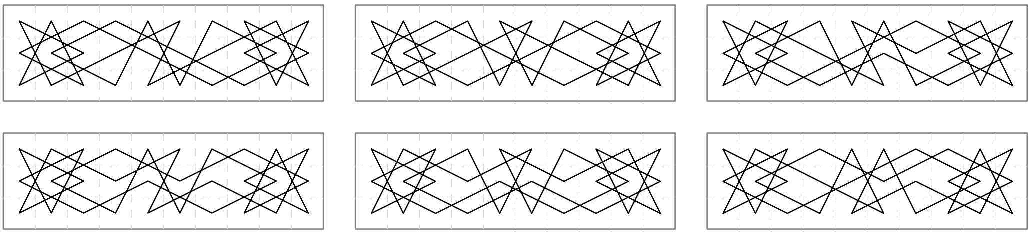
The remaining 16 {2,8} corner to corner tours, to add to the 8 symmetric cases:



Most of the tours have an end point in a corner, there are only 18 without (4 symmetric and 14 asymmetric). The two of types {0,2} and {0,6} types with end-points symmetrically placed are formed of the same components as the corresponding symmetric tours but with the end components (d2-e2, e2-f2) being reflectively instead of rotationally related.

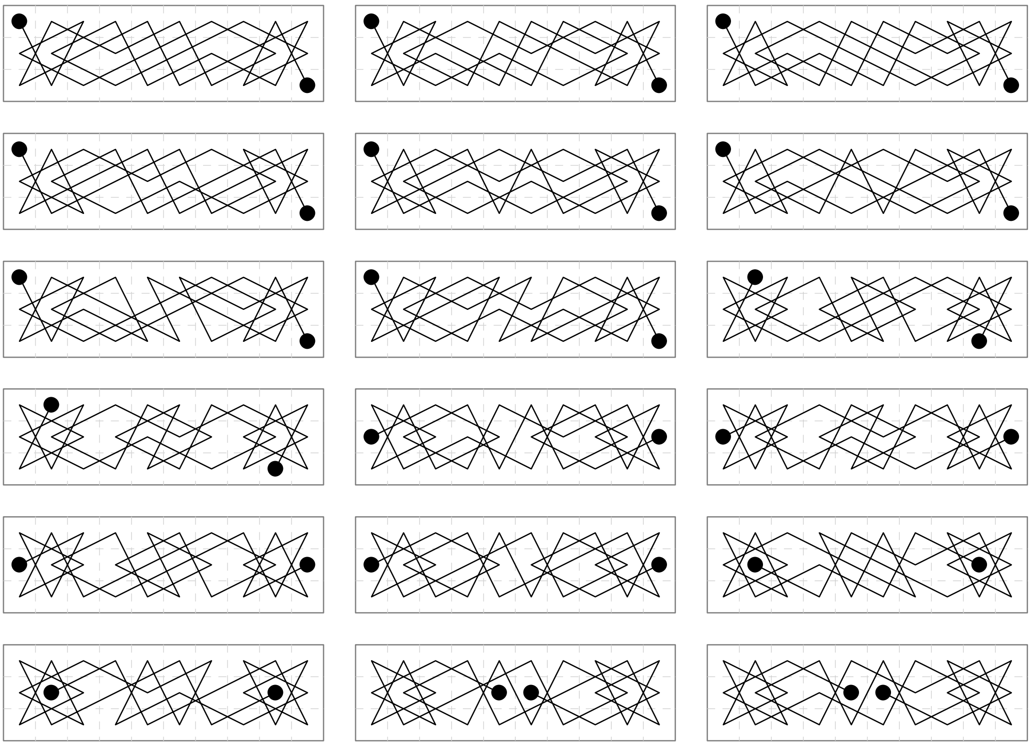
3×10

The existence of closed tours on 3-rank boards was first shown by Ernest Bergholt in 1917 by constructing 3×10 and 3×12 closed tours. He and G. L. Moore (1920) found all six 3×10 solutions, although Kraitchik (1927) was the first to publish all six.



Two are centrosymmetric tours of the Bergholtian type in which two moves cross at the centre, two are axisymmetric of the Sulian type in which the axis does not pass through any cell-centre, and two are asymmetric.

There are 22 symmetric open tours (4 of which are reentrant, derived from the Bergholtian closed tours by deleting one of the central crossing moves). The 18 non-rentrant are shown.



Summary for closed tours: S = 4, A = 2, G = 6, T = 16 = 4A + 2S.
Summary for open tours: S = 22, A = 751, G = 773, T = 3048 = 2S + 4A.

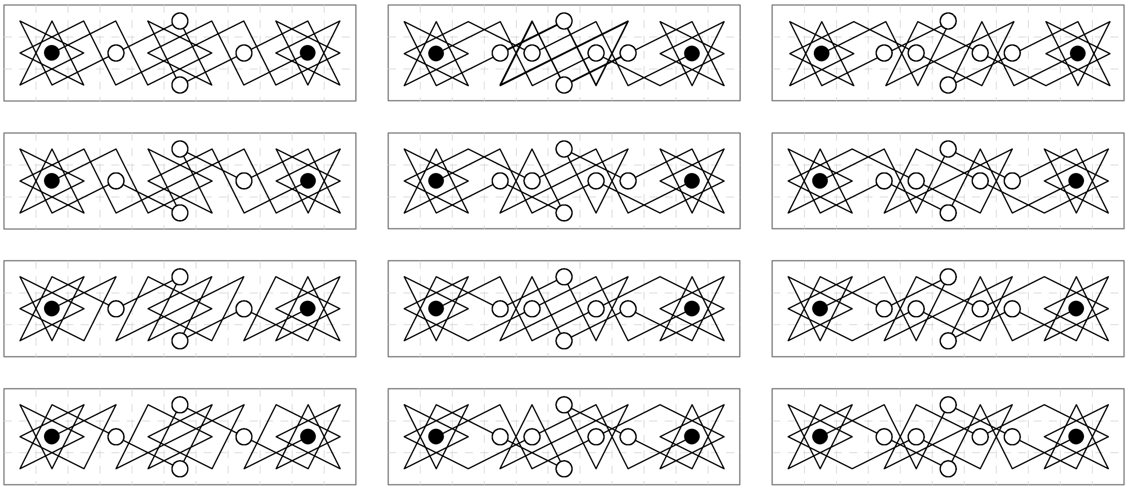
3×11

The enumeration of the symmetric open tours on the 3×11 board has proved to be another difficult case. The correct total is 60. (Kraitchik 1927 reported 58, Murray 1942 counted only 56, computer work by D. E. Knuth reported to me in 1994 confirms the total 60). The numbers arranged by end-point separation are: 2 {0,4}, 12 {0,8}, 12 {2,2}, 4 {2,6}, 30 {2,10}.

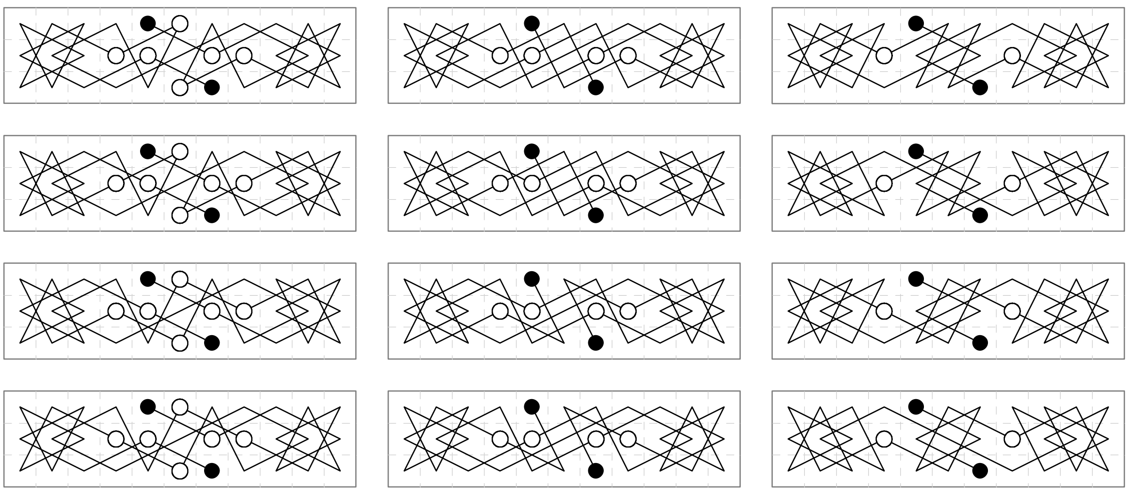
Summary for open tours 3×11: S = 60, A = 2638, G = 2698, T = 2×60 + 4×2638 = 10672

Diagrams of all 60 tours follow [The two {0,4} cases are at the end.]

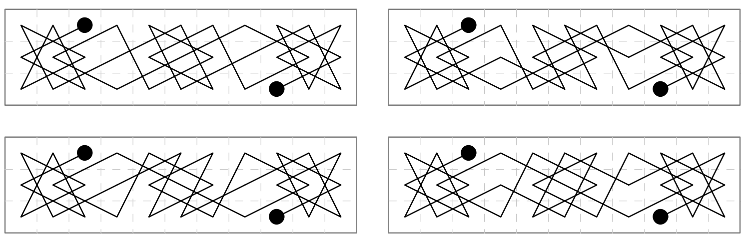
{0,8} 12: Where two moves of the central lozenge d2,f1,f3,h2 are used the other two can be taken instead (this is a case of a disconnected component).



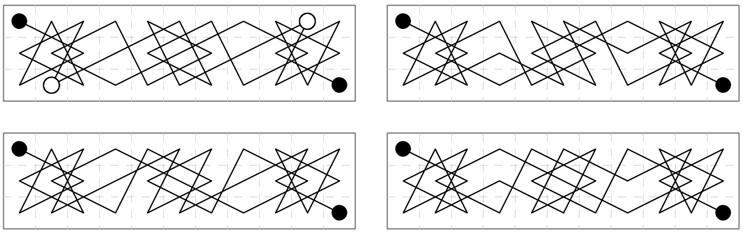
{2,2} 12



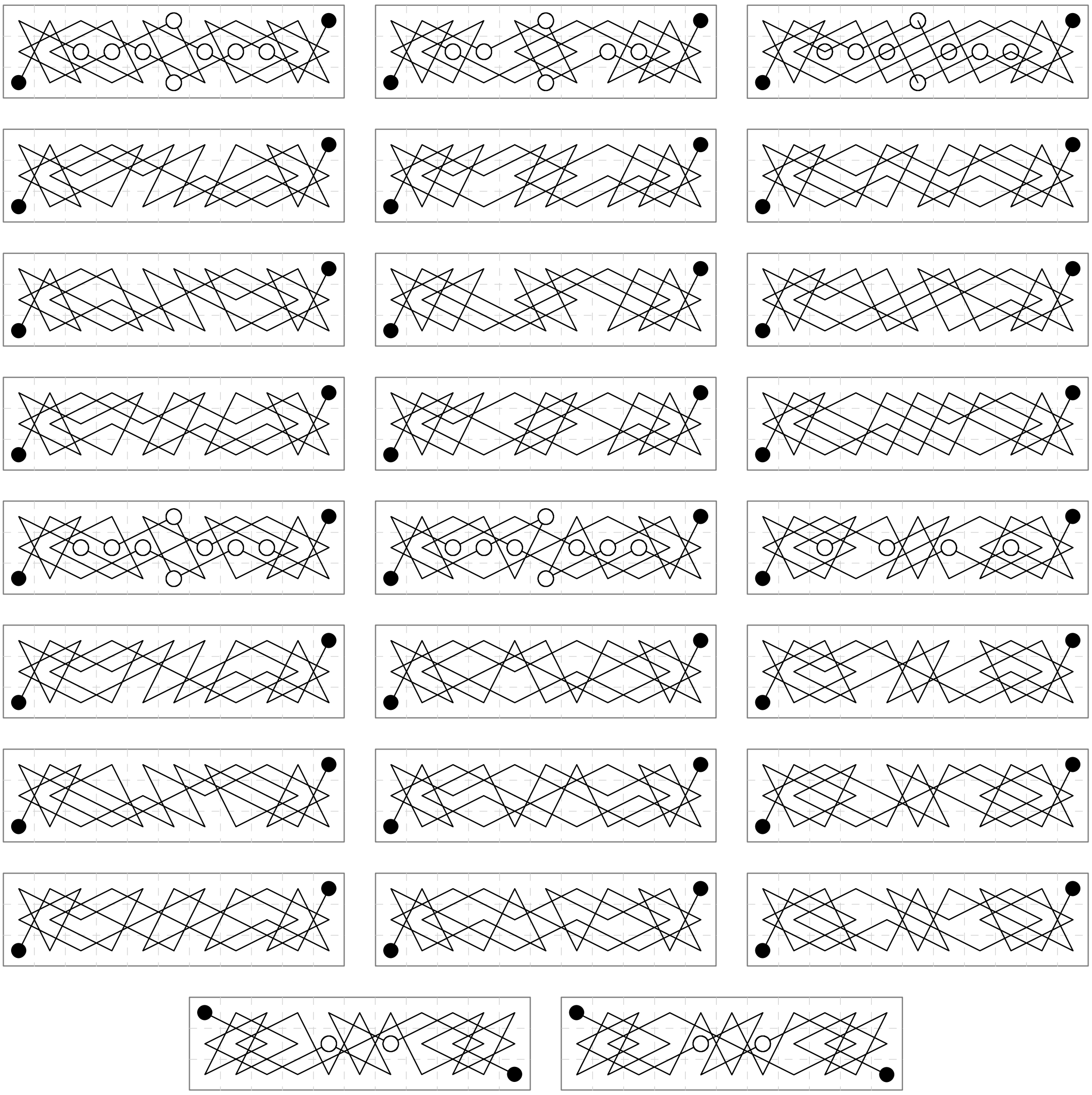
{2,6} 4



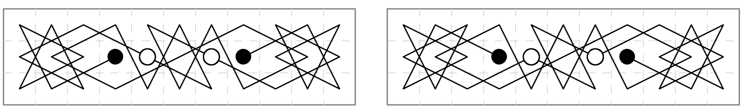
{2,10} 30: The above four diagrams also give 4 of {2,10} type, since the 7-move stars at the ends can be twiddled about b1 and j3.



There are a further 26 of {2,10} type:



Finally {0,4} 2:



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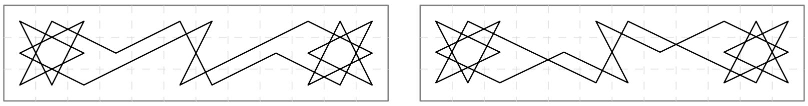
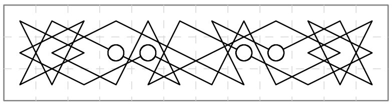
3 ×12

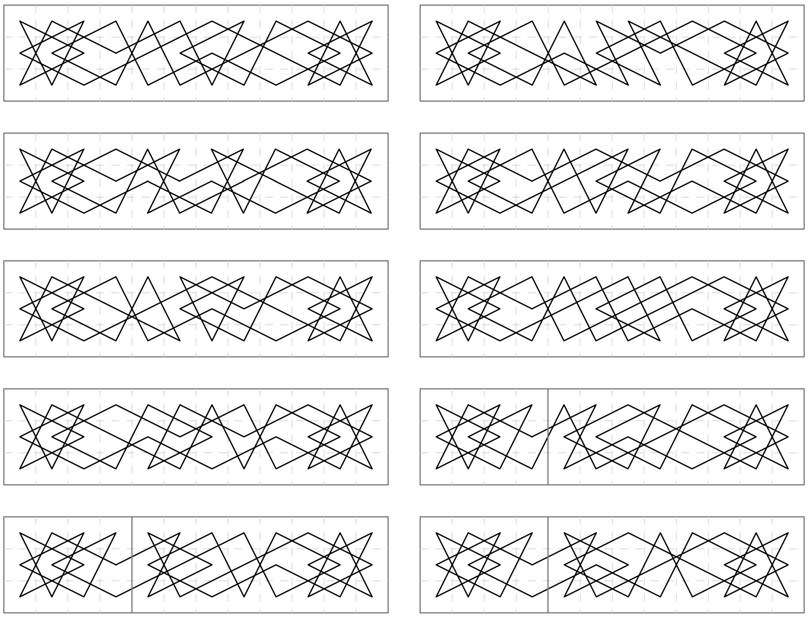
Theorem: No symmetric closed tours are possible on the 3×12 board. *Proof*: On the 3×12 board a closed tour with reflective symmetry is not possible since such a tour requires an odd×singly-even or an even×even board. If a tour with rotational symmetry exists it must pass twice through the centre, i.e. contain the moves f1-g3 and f3-g1. The moves through a1, a2, a3, b2 and l1, l2, l3, k2 are fixed, and we can take b1-c3, j1-k3, b3-d2, i2-k1. We must then have either (1) d2-f1, g3-i2 in which case c1-e2-g1 and f3-h2-j3 are forced, or (2) d2-f3, g1-i2 in which case c1-e2-g3 and f1-h2-j3 are forced. In each case a 24-move short circuit is completed.



A closed tour of the 3×12 board was first found by Bergholt (1917), as shown in the first diagram below. By twiddling the end formations it generates three more, all having four two-unit lines.

There are 44 closed tours in all, none symmetric. Each of these diagrams generates 4 tours by twiddling the end components. Twelve of these tours are compartmental (3×4 + 3×8).

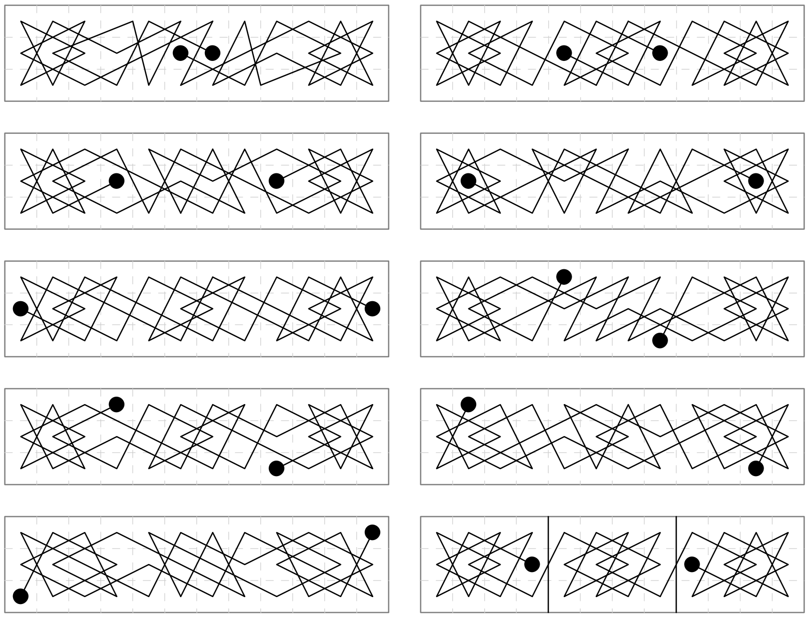




Each tour, being asymmetric, can be viewed in four different orientations.
Summary for 3×12 closed tours: S = 0, A = 44, G = 44, T = 176 = 4×44.

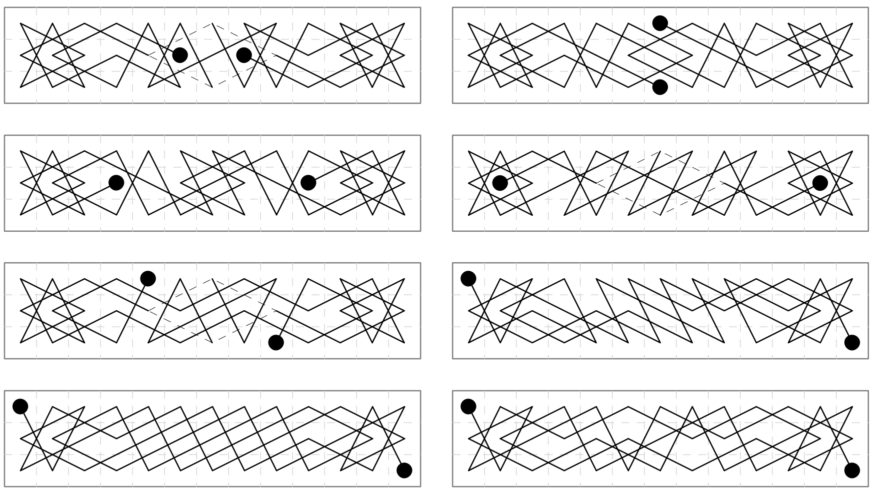
There are 76 symmetric open tours 3×12. By end-point separation: 4 {0,1}, 6 {0,3}, 5 {0,5}, 6 {0,9}, 10 {0,11}, 10 {2,3}, 4 {2,5}, 4 {2,9}, 27 {2,11}. Two are compartmental tours 3×(3×4).
Summary for 3×12 open tours: S = 76, A = 14274, G = 14350, T = 57248 = 4A + 2S.

We show one example for each position of the end-points, and one compartmental case.

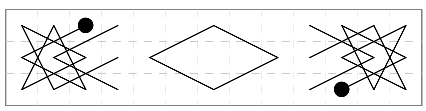


3×13

There are 160 symmetric open knight tours. Classified by end separation: 24 {0,2}, 8 {0,6}, 16 {0,10}, 16 {2,4}, 96 {2,12}. Thus 60% are corner to corner tours. Many of the tours occur in pairs according to which way the moves of the central lozenge are taken. We have space only for a few examples, including each possible position of the end-points.



Theorem: A 3×13 symmetric open tour with end-point separation {2,8} is impossible. *Proof*: If the ends are taken as c3, k1, then moves through a1, a2, a3, b2, l2, m1, m2 m3 are fixed; now b1 must go to d2 and l3 to to j2 because of the ends; now we must connect b3-c1 and l1-k3, since b3-d2 or l1-j2 form 6-move circuits; but now the paths through e2, i2 form a 4-move circuit.



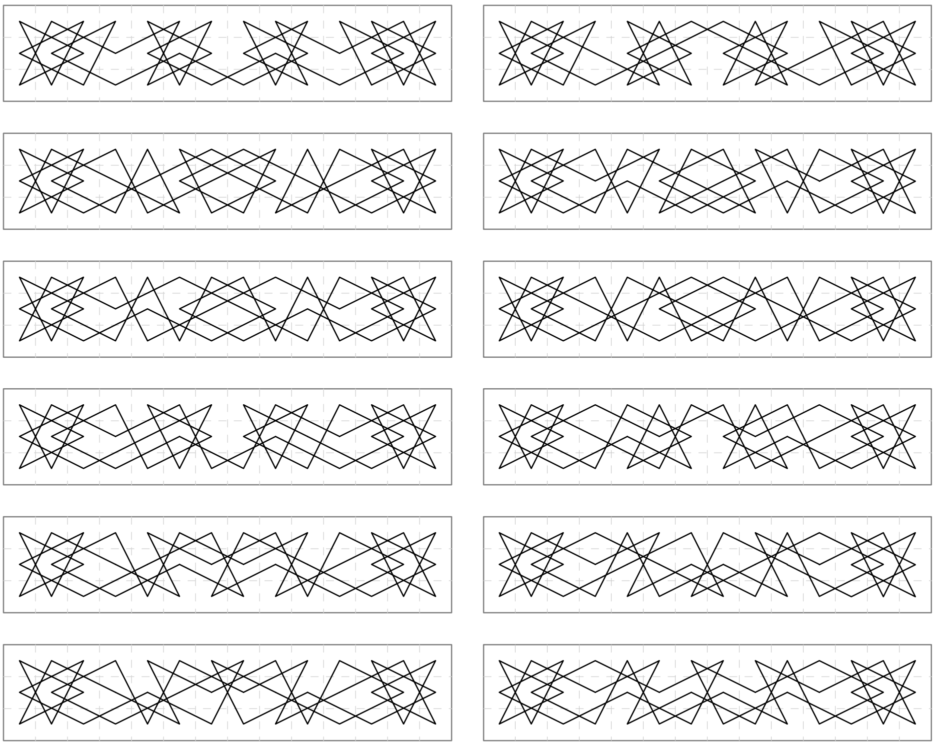
Summary of open tours: S = 160, A = 32136, G = 32296, T = 128864 = 2S + 4A.

3×14

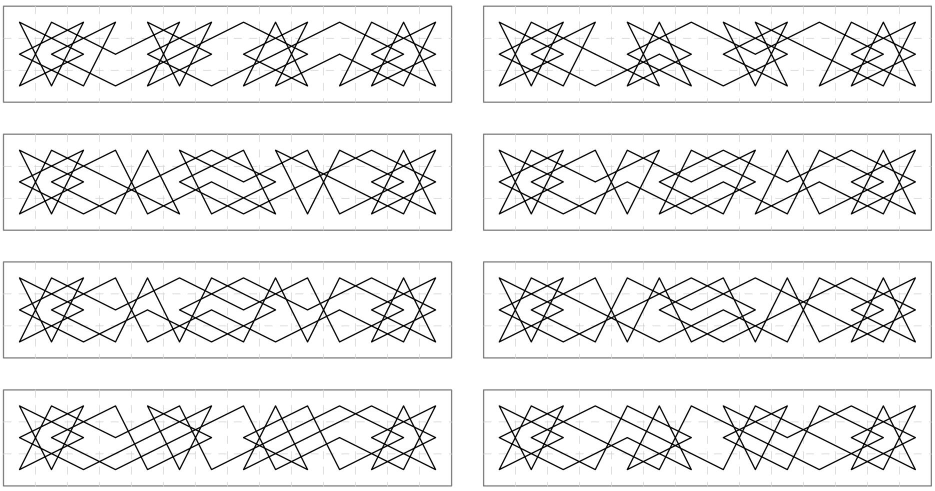
Summary for closed tours: S = 24, G = 396, T = 1536.

There are 396 closed tours of which 24 are symmetric, of which

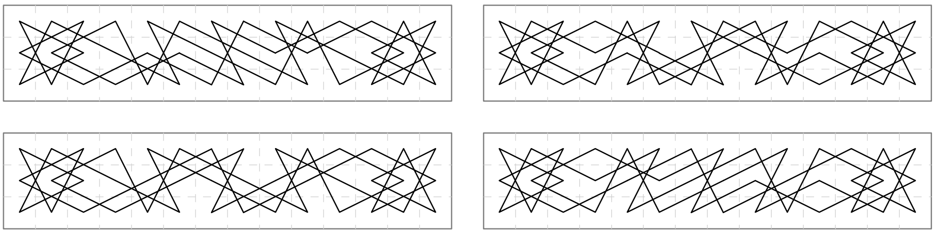
12 are reflective (Sulian):



8 are Eulerian:



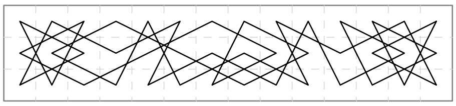
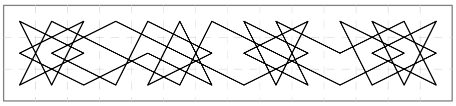
and 4 are Bergholtian (two of these were diagrammed by Murray):

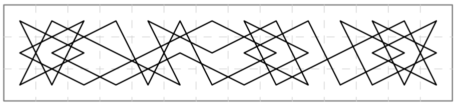


The tours can be classified as 318 of type CC (20 symmetric), 6 of type TT (4 symmetric) and 72 of type CT, where C denotes the comet end formation and T the 3×4 tour formation.

By twiddling the components of these symmetric tours we can form 20 asymmetric tours, 18 of CC type and 2 of TT type.

The eight diagrams marked 8 and 16 below generate the 72 CT tours.


8

8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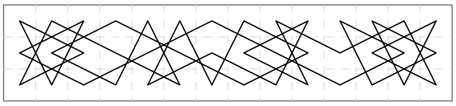
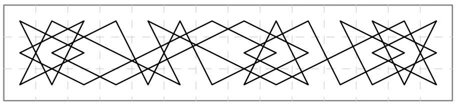

8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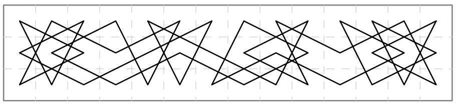
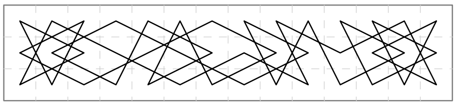
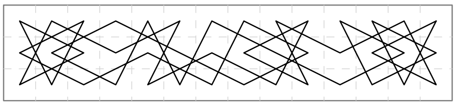

8

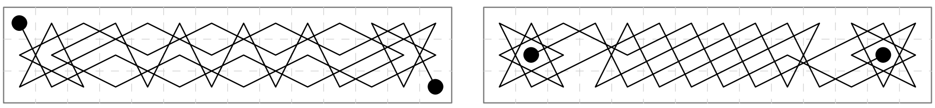

8

16

In a tour of CC type, each C uses 10 cells. The remaining 22 cells are covered by two paths, linking the Cs together, and the tours can be classified according to the way these 22 cells are divided between the two paths.

In the symmetric CC tours, and the asymmetric tours derived from them, the 22 cells are shared out equally 11:11, and there are another 8 of this type.

The 318 CC tours are distributed: 11:11 (46), 10:12 (12), 9:13 (28), 8:14 (24), 7:15 (40), 6:16 (20), 5:17 (32), 4:18 (80), 3:19 (4) and 2:20 (32).

D. E. Knuth reported 292 symmetric open tours 3×14. These two examples illustrate general designs I found, which I call ‘barbed wire’ and ‘brickwork’, that can provide symmetric open tours on most 3-rank boards of greater length by extending the repetitive scheme.



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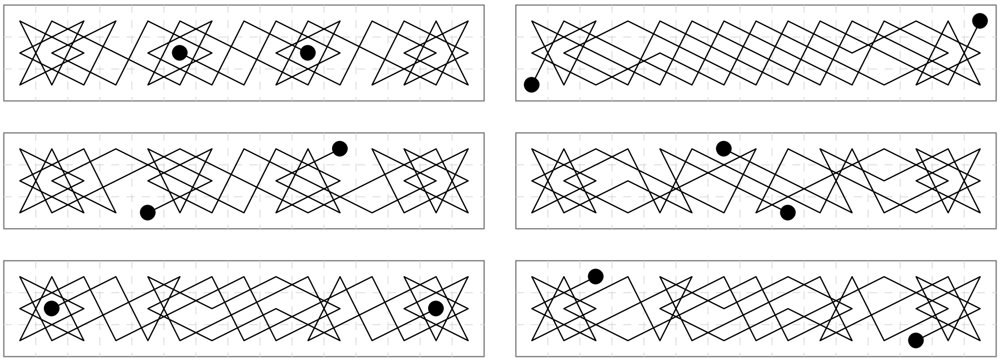
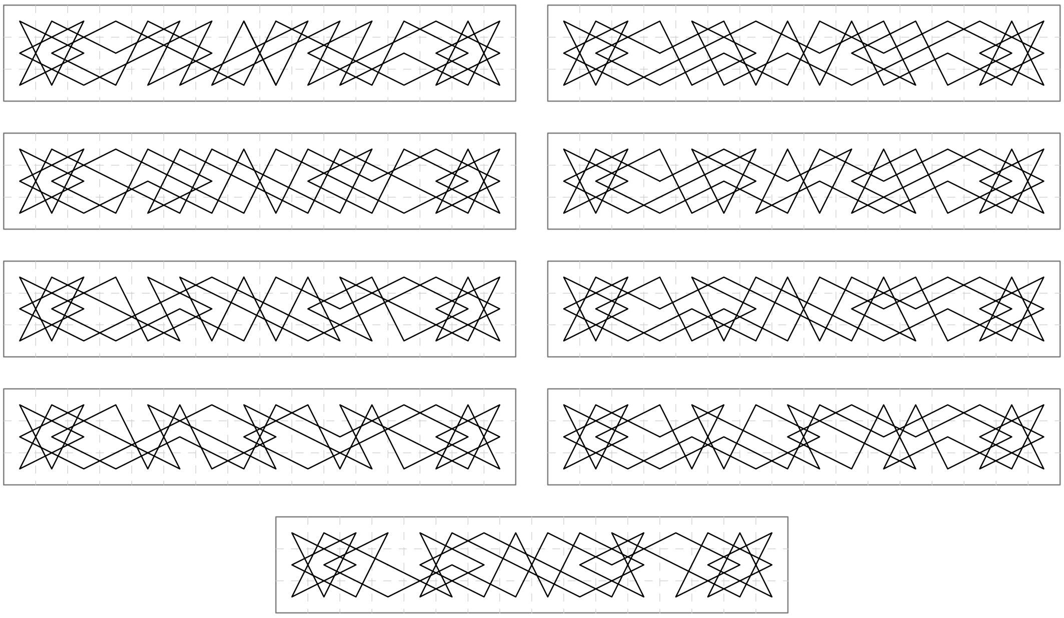
3 ×15

There are 652 symmetric open 3×15 tours reported by D. E. Knuth.
These examples are of my own construction:



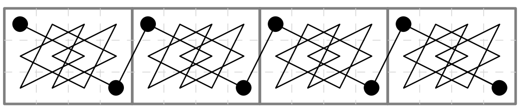
3×16

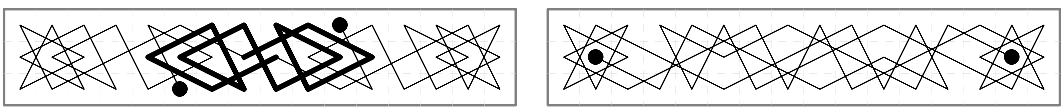
There are 24 symmetric closed tours, all Bergholtian. 20 have ends CC and 4 TT. The centre patterns are NZI, NZI*, NZU, NZU*, SZC. The first six diagrams generate 2 and the others 4.



Summary for closed tours 3×16: S = 24, A = 3844, G = 3868, T= 15424 = 2S + 4A.

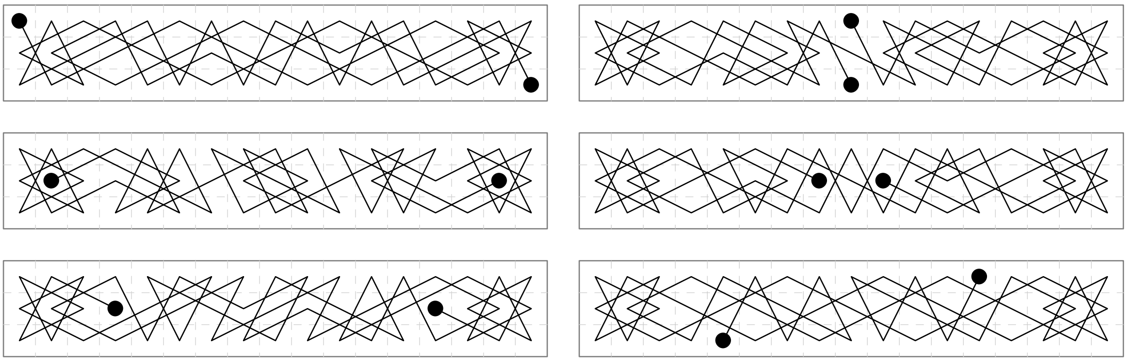
D. E. Knuth reports 1148 symmetric open tours 3×16, e.g. by the compartmental method:





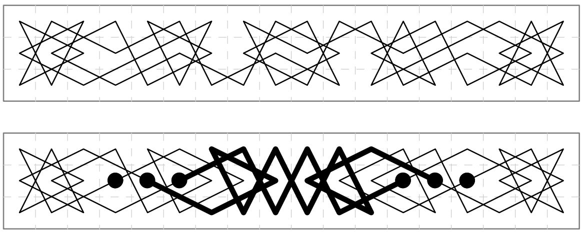
3×17

D. E. Knuth reports 2600 symmetric open tours (which seems a strangely round figure for a prime-dimensioned board!). Six examples shown.



3×18

On the 3×18 board there are 292 symmetric closed tours. These consist of 146 direct (Sulian) and 146 oblique (62 Eulerian, 84 Bergholtian). Second diagram shows NND centre.



Summary for closed tours: S = 292, A = 36786, G = 37078, = T = 147728.

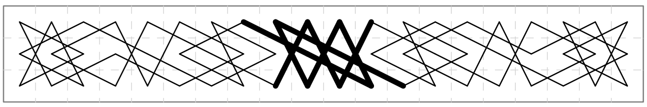
Besides the closed tours, D. E. Knuth reports 3870 symmetric open tours.

3×19

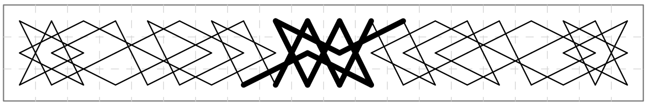
D. E. Knuth reports 9152 symmetric open tours.

3×20

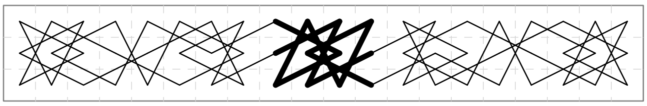
There are 176 symmetric closed tours, all Bergholtian. The following three centres are those not possible on boards shorter than length 20.



NNI



NNU



NSC

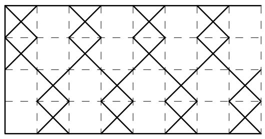
Summary for closed tours 3×20: S = 176, A = 362016, G =362192, T = 1448416.

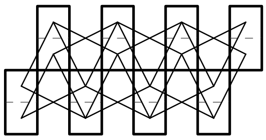
Four-Rank Boards

Sainte-Marie's Theorem

There are indications in the mediaeval manuscripts of an understanding of the two-part structure of knight's tours on the 4×8 half chessboard. It was realised that closed tours on that board are impossible, and several open tours were constructed. Euler (1759) was aware that a closed knight tour is impossible on any 4-rank board of any length and that the tour must start and finish in outer ranks. Jaenisch (1862) also wrote on this subject. A proof of these points and a full and sound analysis of the problem was presented to the Société Mathématique de France by C. Flye Sainte-Marie at their meeting of 18 April 1877.

Sainte-Marie divides the cells into two sets: the white cells on the outer ranks combined with the black cells on the inner ranks (which I call WOBI) and the white cells on the inner ranks combined with the black cells on the outer ranks (WIBO). A knight move from an outer cell can only reach an inner cell of opposite colour, which is in the same set, so a passage from one set to the other is only possible by a move on the inner ranks.





THEOREM (Sainte-Marie 1877). A knight's tour on a 4×n board must be open, with its ends in the outer ranks, and of two equal parts using $2\cdot n$ cells, linked by a single move on the inner ranks. *Proof*: An open tour of a 4×n board consists of $4\cdot n - 1$ moves, and at least one of these must join inner cells, so we cannot have more than $4\cdot n - 2$ moves that link to an outer cell, but on the other hand it is not possible to have less than this number, since there are $2\cdot n$ outer cells, and of these at least $2\cdot n - 2$ must have two knight's moves linked to them, there being only two cells (the ends of the tour) linked to one knight's move, and $(2\cdot n - 2)\cdot 2 + (2)\cdot 1 = 4\cdot n - 2$. QED

An alternative argument by contradiction, ascribed to Louis Pósa, to prove the impossibility of a closed tour, says that the numbers of inner and outer cells are equal and outer cells link only to inner cells, therefore in a closed tour inner and outer cells must alternate, but we know that white and black cells must also alternate, so the outer cells must all be of one colour and the inner cells the other colour, which we know is untrue (Honsberger 1973).

This shows that to enumerate tours on a 4×n board we need to count the tours on the two half-sets of cells and calculate the number of ways they can join. The two sets are the same shape, one the reflection of the other in the horizontal, so we only need to enumerate the partial tours in one set.

Denoting the number of half-tours with inner terminal on file f on the n-file board by $H_n(f)$, we have $H_n(f) = H_n(n+1-f)$ from symmetry. Knowing these values we can calculate the number of complete tours with internal connection f to f+2. This is normally $H_n(f)\cdot H_n(f+2)$, but there are two special cases on odd-length boards.

First, if the f-file is the centre file then the required total is $2\cdot H_n(f)\cdot H_n(f+2)$ since $H_n(f)$ is the number of geometrically distinct half-tours and each can be taken to left or right.

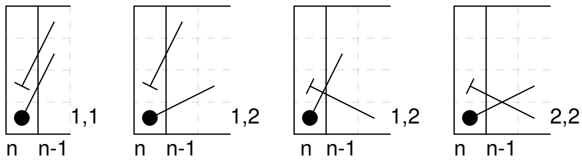
Second, if the (f+1) file is the centre file the total is $H_n(f)\cdot[H_n(f) + 1]/2$, since $H_n(f+2) = H_n(f)$ in this case and the tours consist of $H_n(f)$ symmetric tours where the two half-tours are the same plus $H_n(f)\cdot[H_n(f) - 1]/2$ asymmetric tours where the two half-tours are different.

Recurrences for Counting Half-Tours

Finding the $H_n(f)$ values still requires much work. Recurrrence relations have been devised by Kraitchik and Murray that go part way to solving this. Denote by $H_n(f,g)$ the number of half-tours on the 4×n board that begin on the f file and end in the g file. This number is the same no matter on which cell in the f file the half-tour starts. This is due to the regularity of the net of knight moves. We also have $H_n(f,g) = H_n(g,f)$ since either end can be taken as the start, and $H_n(f,g) = H_n(n+1-f, n+1-g)$ since the board can be reflected (when n is odd) or rotated (when n is even). If we know these numbers $H_n(f)$ can be calculated as the sum of the values $H_n(f,g)$ over all values of g.

Half-tours with ends on the first file follow the recurrence:

(1) $H_n(1,1) = H_{n-1}(1,1) + 2 \cdot H_{n-1}(1,2) + H_{n-1}(2,2)$.

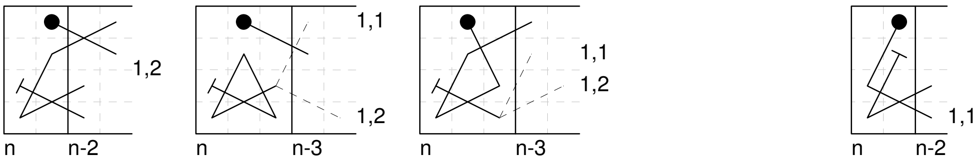


Half-tours with ends on the first two files follow the recurrences:

$H_n(1,2)$ is also the number of closed half-tours

(2a) $H_n(1,2) = H_{n-2}(1,2) + 2 \cdot H_{n-3}(1,1) + 2 \cdot H_{n-3}(1,2)$

(2b) $H_n(2,2) = H_{n-2}(1,1)$



Half-tours with ends on the first three files follow the recurrences:

(3a) $H_n(1,3) = H_n(1,2)$

(3b) $H_n(2,3) = H_{n-3}(1,1) + H_{n-3}(1,2)$

(3c) $H_n(3,3) = H_{n-3}(1,1)$

Half-tours with ends on the first four files follow the recurrences:

(4a) $H_n(1,4) = H_{n-2}(1,4) + 2 \cdot H_{n-3}(1,1) + 4 \cdot H_{n-4}(1,1) + 3 \cdot H_{n-4}(1,2) + 2 \cdot H_{n-5}(1,2)$

(4b) $H_n(2,4) = H_n(1,2) - H_n(2,3)$

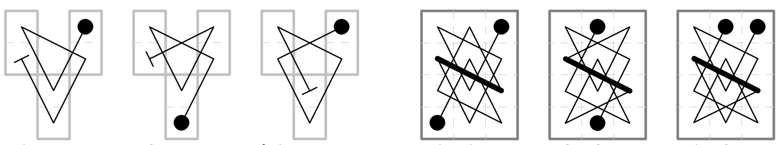
(4c) $H_n(3,4) = H_{n-3}(1,1)$

(4d) $H_n(4,4) = 3 \cdot H_{n-4}(1,1) + 2 \cdot H_{n-4}(1,2)$

I'm not sure of the correctness of (4a) though. Can it be simplified?

4×3

We can illustrate Saint-Marie's analysis on the 4×3 board, which of course is the 3×4 board turned through a right angle. Two half-tours have the inner end on the a-file, and one on the b file. We label these a1, a2 and b1. The half-tour b1 cannot be used to form a tour since its inner end has nothing to link to. The three complete tours can be represented as a1-a1, a2-a2 and a1-a2, the first two being symmetric. This confirms the three tours found in the 3×4 section earlier.



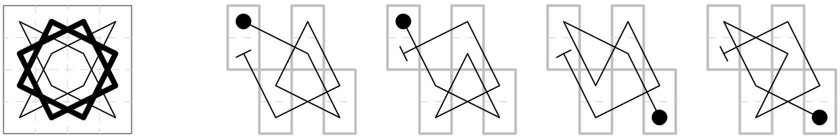
a1 a2 b1 a1-a1 a2-a2 a1-a2

We have $H_3(i, j) = 0$ when $i = j$ and $= 1$ when $i \neq j$. Results: G = 3, S = 2, A = 1, T = 8.

4×4

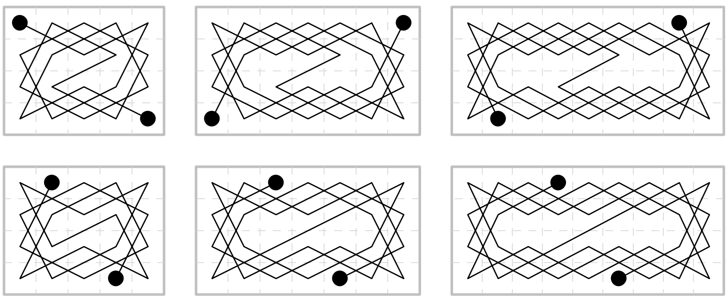
On the 4×4 knight moves through the corners form two circuits of 4 so there is no closed tour. The moves through the remaining cells also form two circuits of 4, making a pseudotour, known as the ‘squares and diamonds’ pattern which features in construction of tours on larger boards.

There are four half-tours, $H_4(1) = 4$, but no two can join to give a full tour since their inner ends are on the 1st file (or the 4th when rotated) which have no horizontal knight-move link: $H_4(1,1) = 2$ and $H_4(1,4) = 2$. Each is composed of one ‘square’ and one ‘diamond’.



Edge-Hugging Tours on 4-Rank Boards

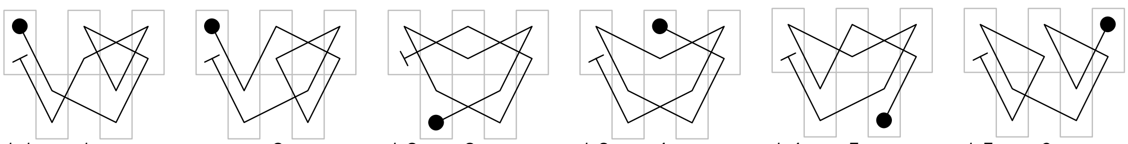
On the 4×n board (n > 4) if we draw a barrier along the horizontal centre-line, apart from the first two and last two files, then the knight's moves that do not cross this barrier form an edge hugging, or centre circling, braid, making a pseudotour. On an odd board the braid consists only of two circuits, so the single middle move of the tour is the only link needed to convert the pseudotour to a tour. Each position of the middle move gives four geometrically distinct tours, except when the move is in the middle of the board in which case the number reduces to three. The number of tours 4×n with n odd is $2 \cdot n - 3$, two of which are symmetric. The following diagrams begin the two series of symmetric centre-circlng tours on odd-length boards:



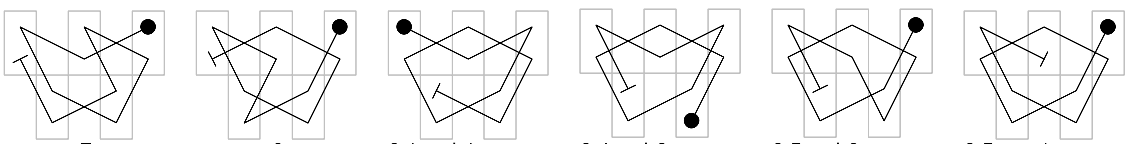
On an even board the braid is of four circuits so three links are needed. We give examples in the sections below. There are 12 tours 4×6 of this type, 24 on the 4×8, 40 on the 4×10, and the numbers on larger boards add 16 at each step. The number on a 4×n board with n even and > 6 is $8 \cdot (n - 5)$.

4×5

On the 4×5 board there are 12 knight half-tours. From which: $H_5(1) = 8$, $H_5(2) = 3$, $H_5(3) = 1$. According to the end-pont positions. $H_5(1,1) = 2$, $H_5(1,2) = 1$, $H_5(1,3) = 1$, $H_5(1,4) = 1$, $H_5(1,5) = 3$, $H_5(2,1) = 1$, $H_5(2,4) = 1$, $H_5(2,5) = 1$, $H_5(3,5) = 1$, other values being 0. Five are edge-hugging.



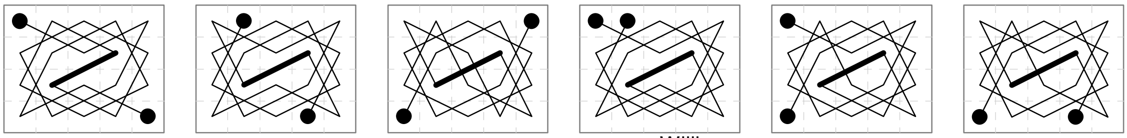
1,1 a1   a2   1,2 a3 E   1,3 a4 E   1,4 a5   1,5 a6



a7   a8   2,1 b1 E   2,4 b2 E   2,5 b3   3,5 c1 E

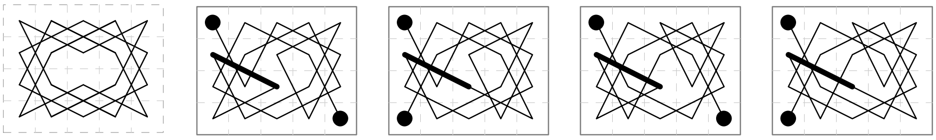
Both of the special cases mentioned in the note on Sainte-Marie's Theorem above apply to the calculation of the number of tours in this case. File 3 (c) being central, with inner link 1-3 (a-c or c-e) the total is $H_5(1)\cdot 2\cdot H_5(3) = 8\cdot 2\cdot 1 = 16$, and with inner link 2-4 (b-d or d-b) the total is $H_5(2)\cdot[H_5(2)+1]/2 = 3\cdot 4/2 = 6$. If we label the half-tours a1-a8, b1-b3 and c1 as shown above then the complete tours can be named by the two half-tours from which they are formed.

There are six tours using the b-file half-tours, three being symmetric, and three edge-hugging (E). These have the horizontal link central, b-d. One is in Willis (1821).

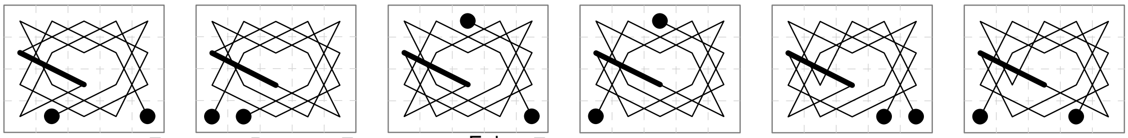


b1-b1 E | b2-b2 E | b3-b3 | b1-b2 Willis E | b1-b3 | b2-b3

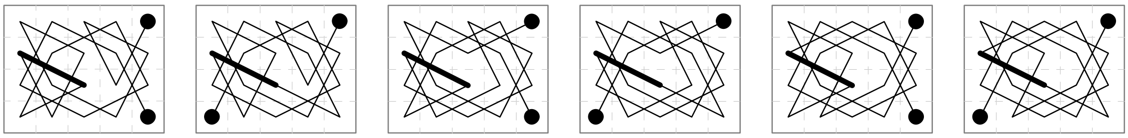
The other 16 asymmetric tours consist of the eight a-file half-tours linked to the two positions of the c-file hallf-tour. Four are edge-hugging. One was given by Euler (1759).



a1-c1 | a1-c1R | a2-c1 | a2-c1R



a3-c1 E | a3-c1R E | a4-c2 Euler E | a4-c1R E | a5-c1 | a5-c1R

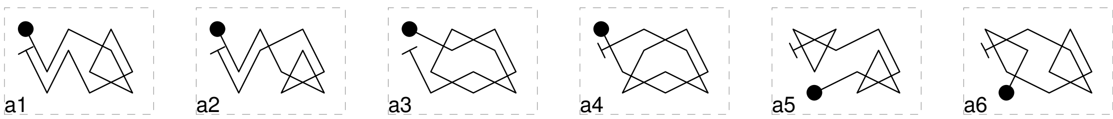


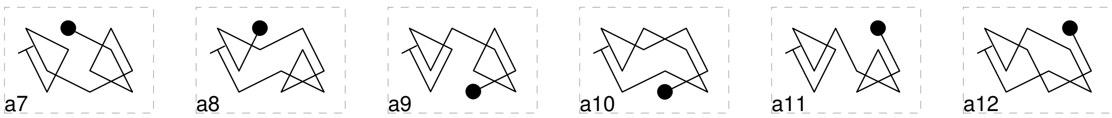
a6-c1 | a6-c1R | a7-c1 | a7-c1R | a8-c1 | a8-c1R

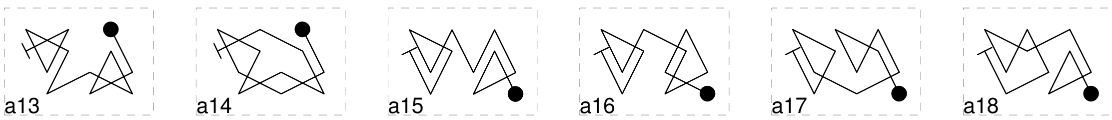
All the tours except the second symmetric case have one end at a corner. None are semi-magic.
G = 16 + 6 = 22, S = 3, A = 19, T = 82.

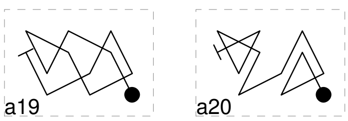
4×6

There are 37 partial tours. (Same as number of wazir tours - this is probably just a coincidence.)
a-file:

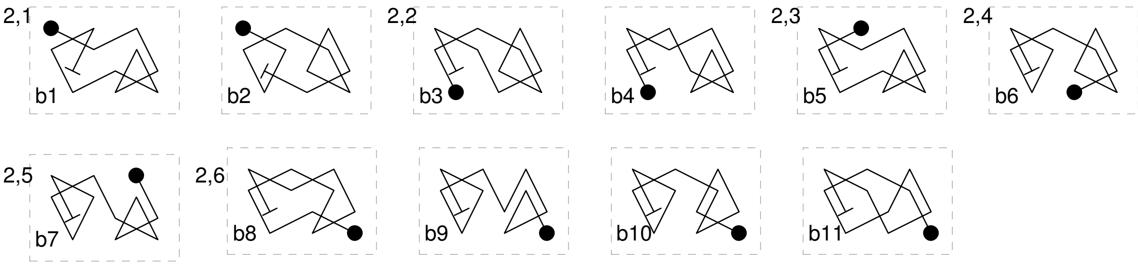




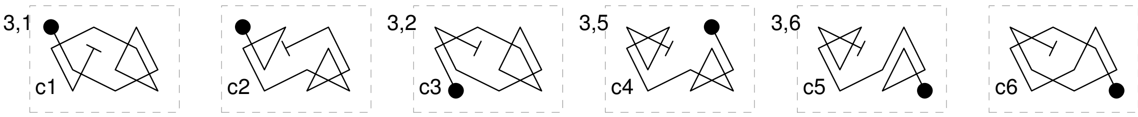




b-file:



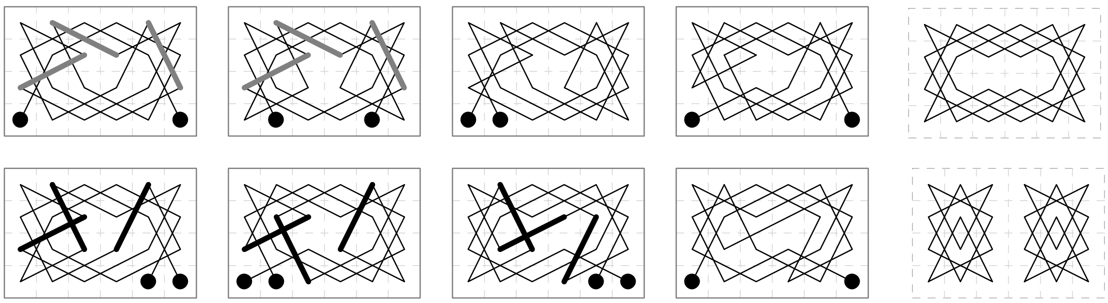
c-file:



H(1) = 20: H(1,1) = 4, H(1,2) = 2, H(1,3) = 2, H(1,4) = 2, H(1,5) = 4, H(1,6) = 6.
H(2) = 11: H(2,1) = 2, H(2,2) = 2, H(2,3) = 1, H(2,4) = 1, H(2,5) = 1, H(2,6) = 4.
H(3) = 6: H(3,1) = 2, H(3,2) = 1, H(3,3) = 0, H(3,4) = 0, H(3,5) = 1, H(3,6) = 2.

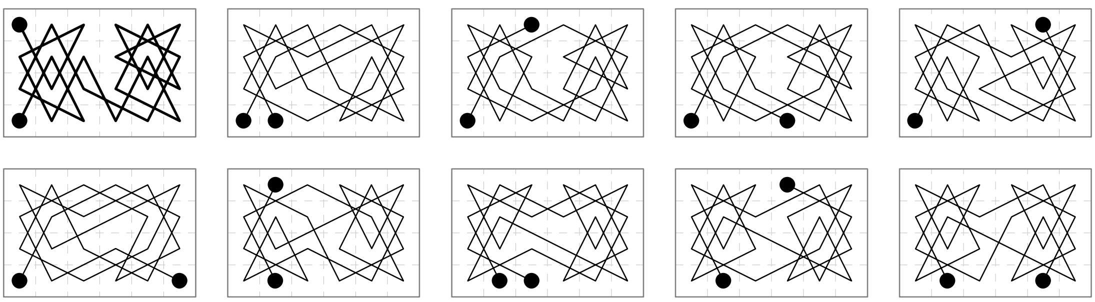
From these values we get 20·6 = 120 tours with middle move 1-3 (a-c or d-f), and 11·6 = 66 with middle move 2-4 (b-d or c-e). Symmetry is impossible, so: G = 120 + 66 = 186 = A, S = 0, T = 744.

Edge-hugging 4×6 tours. There are 12 tours 4×6 of edge-hugging type. Two are of the near-symmetric type of the next section. Four are semi-magic, shown in the section that follows. The remaining six, shown here, include historical examples. The links form three patterns, shown by the thick black lines in the bottom row, each yielding four tours.

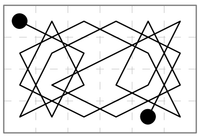


Euler 1759 Willis 1821

Miscellaneous examples. With ends in all relative positions (c1-d1 or c1-c4 are impossible).

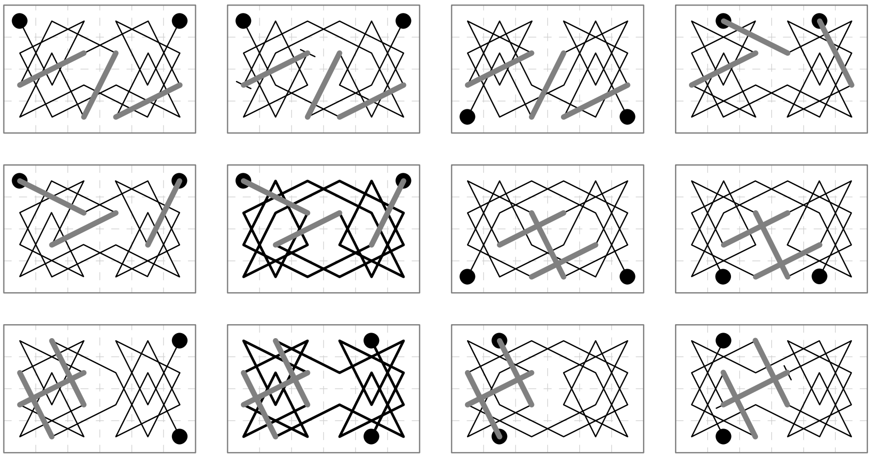


This piece-wise symmetric tour 4×6 with numbers in opposite cells differing by 6 is by S. Vatriquant (*L'Echiquier* Sep 1929 problem 363).



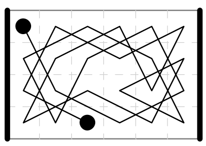
1	18	5	22	9	14
4	21	2	13	6	23
17	12	19	8	15	10
20	3	16	11	24	7

Near-symmetric 4×6 tours. I have found 14 tours with the end-points symmetrically placed and with the minimum of three moves having no symmetric counterpart. These moves, shown by thick grey lines, are alternate moves of a six-move circuit. One is always the middle link. The other two are shown in the next section, since they are also edge-hugging tours

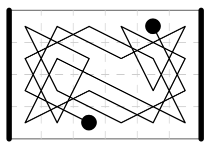


Semi-magic 4×6 tours. Here are the four edge-hugging tours that are semi-magic, summing to 75 in the 6-cell lines, shown here in arithmetical and geometrical forms, oriented by the Frénicle rule.

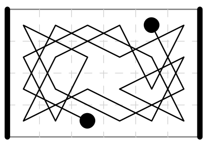
1	22	9	18	5	20
8	17	6	21	10	13
23	2	15	12	19	4
16	7	24	3	14	11



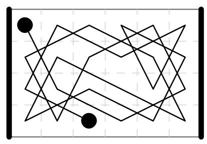
5	22	9	18	1	20
8	13	6	21	10	17
23	4	15	12	19	2
14	7	24	3	16	11



5	22	9	18	1	20
8	17	6	21	10	13
23	4	15	12	19	2
16	7	24	3	14	11

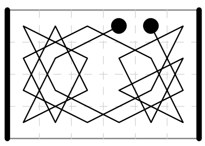


1	22	9	18	5	20
8	13	6	21	10	17
23	2	15	12	19	4
14	7	24	3	16	11

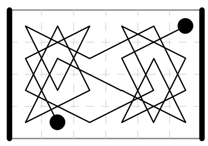


There are also four others. There are no 4×6 semi-magic tours with the 4-cell lines magic.

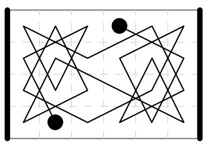
5	20	9	24	1	16
8	23	6	15	10	13
19	4	21	12	17	2
22	7	18	3	14	11



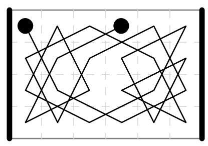
5	16	3	20	7	24
2	13	6	23	10	21
17	4	15	12	19	8
14	1	18	9	22	11



5	16	3	24	7	20
2	13	6	21	10	23
17	4	15	12	19	8
14	1	18	9	22	11



1	20	9	24	5	16
8	23	6	15	10	13
19	2	21	12	17	4
22	7	18	3	14	11

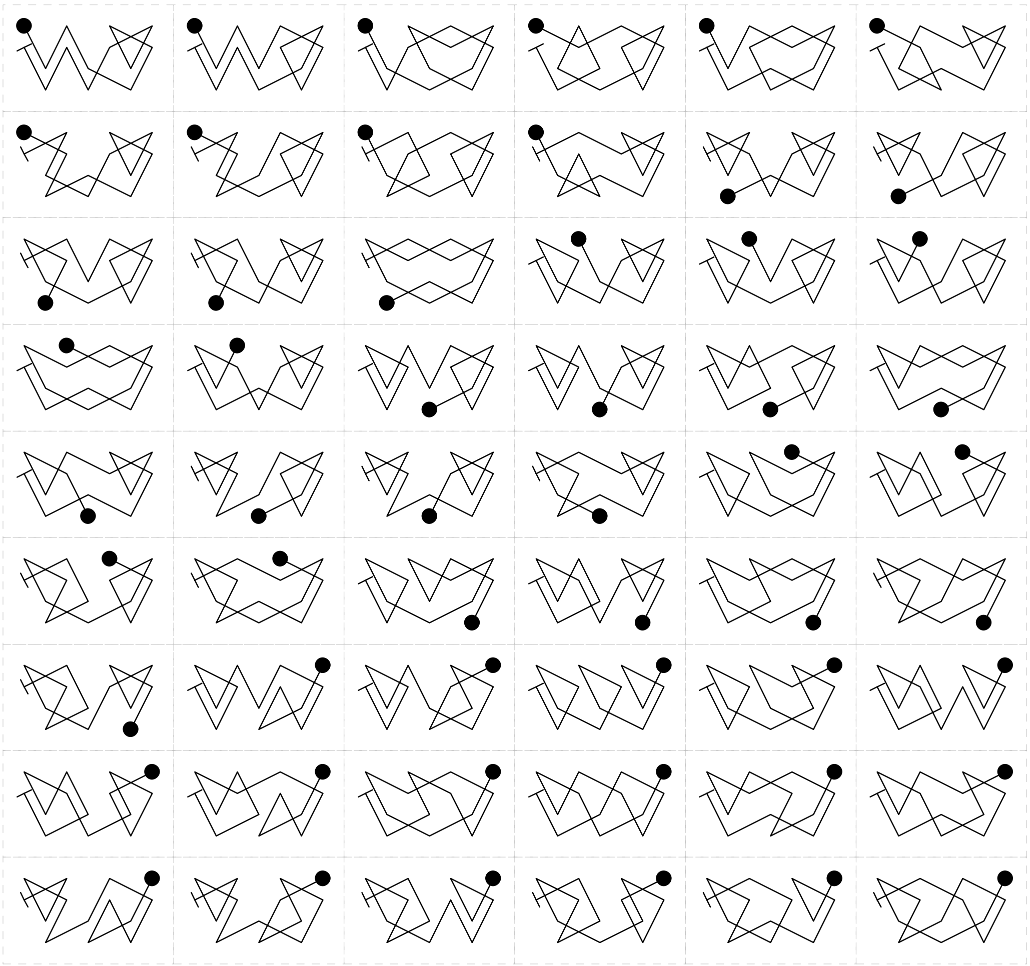


The first tour in each batch is ‘quasi-magic’, having just two different file sums. (a) 4 of 48, 2 of 54, (b) 4 of 54, 2 of 42 (or 100 minus these values when numbered in reverse). The others have more than two file sums. (Source: *The Games and Puzzles Journal*, #26 April 2003).

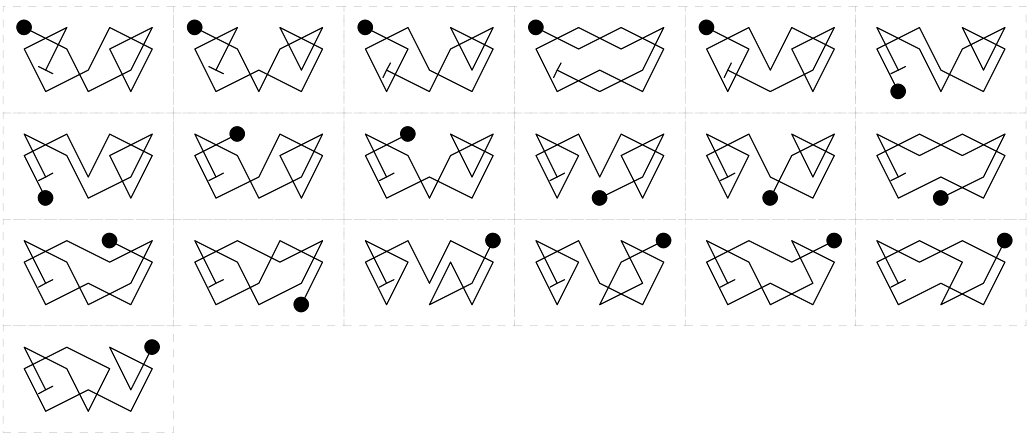
4×7

Diagrams of the 104 half-tours, arranged according to (a) the initial inner and terminal outer files, and (b) the sequence of files visited, the sequences being arranged in numerical order.

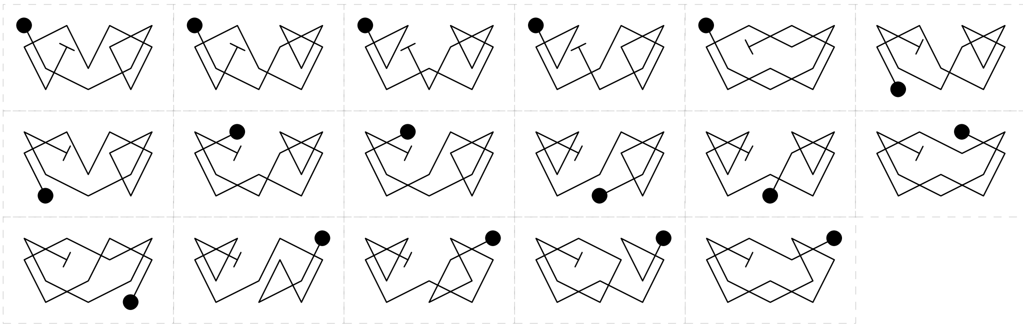
a-file: a1-a54



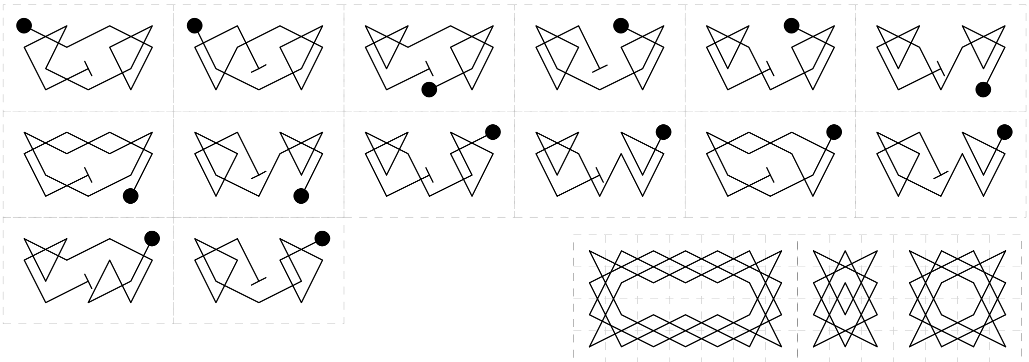
b-file: b1-b19



c-file: c1-c17. This group H(3) are those that generate the symmetric tours.



d-file: d1-d14.



The 104 half-tours of the 4×7 are made up as follows:

H(1) = 54: H(1,1) = 10, H(1,2) = 5, H(1,3) = 5, H(1,4) = 8, H(1,5) = 4, H(1,6) = 5, H(1,7) = 17

H(2) = 19: H(2,1) = 5, H(2,2) = 2, H(2,3) = 2, H(2,4) = 3, H(2,5) = 1, H(2,6) = 1, H(2,7) = 5.

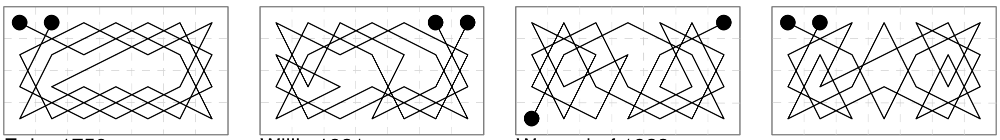
H(3) = 17: H(3,1) = 5, H(3,2) = 2, H(3,3) = 2, H(3,4) = 2, H(3,5) = 1, H(3,6) = 1, H(3,7) = 4.

H(4) = 14: H(4,1) = 2, H(4,2) = 0, H(4,3) = 0, H(4,4) = 1, H(4,5) = 2, H(4,6) = 3, H(4,7) = 6.

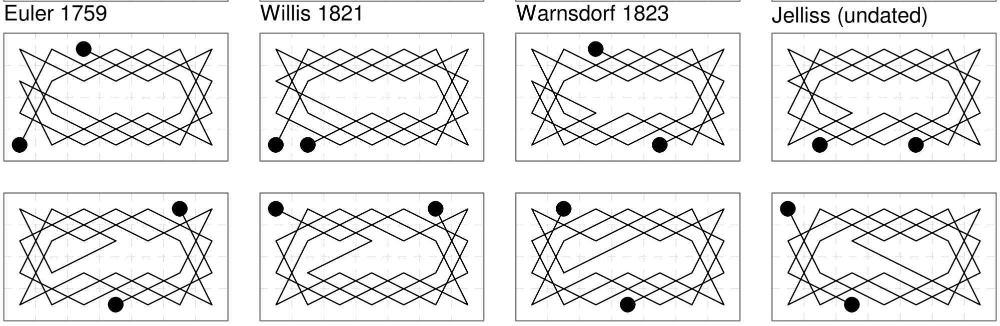
From these values we can calculate there are 54·17 = 918 tours with middle move 1-3 (a-c or e-g), 19·14·2 = 532 of type 2-4 (b-d or d-f), and 17·(17 + 1)/2 = 153 of type 3-5 (c-e).

Totals: G = 918 + 532 + 153 = 1603. S = 17, A = 1586, T = 6378.

We show three asymmetric tours of historical interest, by Euler, Willis and Warnsdorf. My example is piece-wise symmetric, numbers in opposite cells differing by 5 or 19. The Euler tour is of edge-hugging type. We show eight other asymmetric edge-hugging tours.

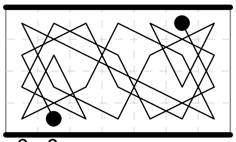


Euler 1759 Willis 1821 Warnsdorf 1823 Jelliss (undated)

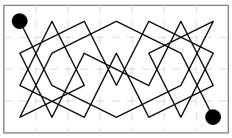


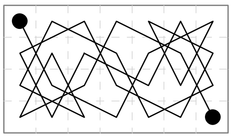
This unique 4×7 symmetric and semi-magic tour was given in *L'Echiquier* Oct 1929. Files add to 58. Rank sums, alternately even and odd of course, are all different (102, 95, 108, 101).

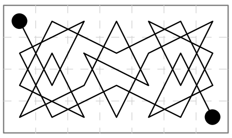
13	24	3	18	7	28	9
2	21	14	25	10	17	6
23	12	19	4	15	8	27
20	1	22	11	26	5	16

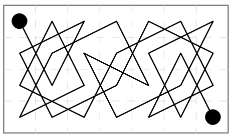

c6-c6

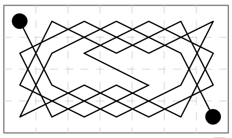
In symmetric open tours on this board of course the numbers in opposite cells add to 29. There are a further 16 symmetric tours, including two edge-hugging tours, marked E.

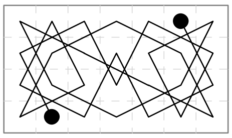

c1-c1

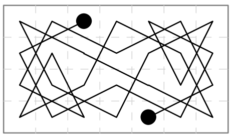

c2-c2

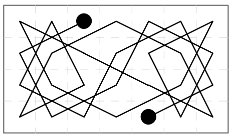

c3-c3

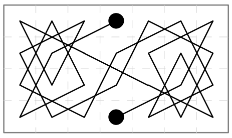

c4-c4

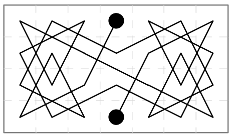

c5-c5 E

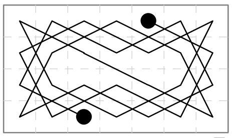

c7-c7

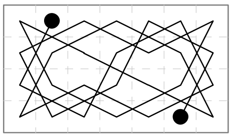

c8-c8

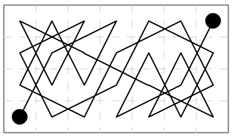

c9-c9

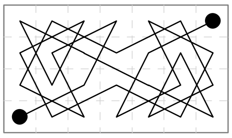

c10-c10

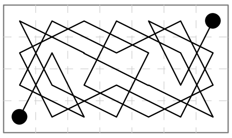

c11-c11

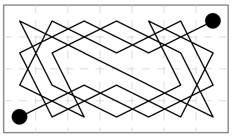

c12-c12 E


c13-c13


c14-c14

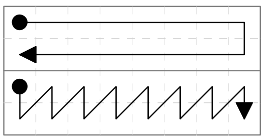

c15-c15

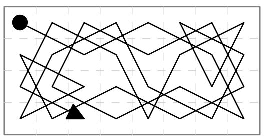

c16-c16

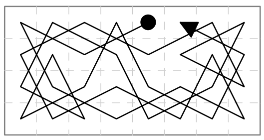

c17-c17

4×8

The 4×8 board is half the standard chessboard, and examples of tours on this board have been constructed dating back to the Middle Ages, from 900 to 1500, sometimes combining two such half-board tours to cover the whole chessboard. We summarise here the results reported by the chess historians H. J. R. Murray (*British Chess Magazine* article 1902, *A History of Chess* 1913, unpublished ms on *The Early History of the Knight's Tour* written about 1930, now in the Bodleian Library, Oxford University) and Antonius van der Linde (*Geschichte und Litteratur des Schachspiels* 1874 and *Quellenstudien zur Geschichte des Schachspiels* 1881) For much fuller details of the mediaeval manuscripts the reader should consult these more scholarly works. The earliest known examples date back to around the year 900 in Kashmir and Baghdad. They survive in manuscripts written by later commentators and in the collections of chess problems associated with 'Bonus Socius' {Good Companion} and later 'Civis Bononiae' {Citizen of Bologna}.







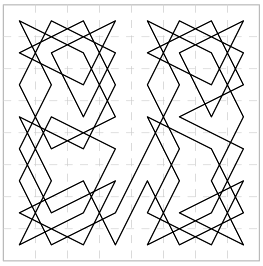
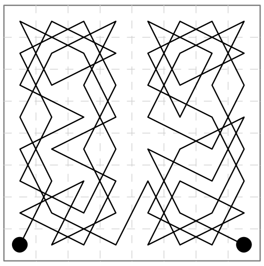
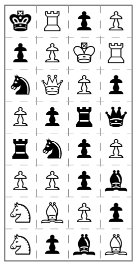
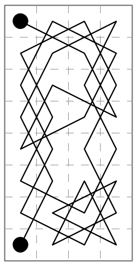
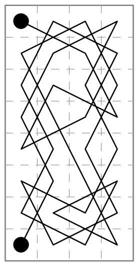
One of the earliest is in the *Kavyalankara* by **Rudrata** which is a verse work believed by Murray (1913) to have been written in Kashmir during the reign of Sankaravarman 884-903. This includes a simple boustrophedonal tour by *rat-ha* (chariot, i.e. rook) and a sawtooth tour by *gaja* (elephant) on two ranks, as well as a tour by *turaga* (horse, i.e. knight) on the 4×8 half chessboard.

More recent research places the work earlier. [Commentary by Nami of Guzerat 1069. Analysis by H. Jacobi 1896, Murray 1913, p.53–55, and G. Sastri 1943 (cited in Wikipedia)].

These tours are said to be presented by syllables on the cells which when read normally or in the sequence of the tour give the same verse. The elephant, which is a piece still in use in Burmese and Thai forms of chess and was described by al Beruni as used in India, is a {1,1}-mover (i.e. one step diagonally) with the extra power of a single pawn-like forward step (the five moves thus representing the elephant's four legs and trunk). Rudrata's knight tour can be repeated so as to give an 8×8 open tour. A diagram showing it in this form, reflected left to right, appears in “a Persian manuscript of the early 19th century probably compiled in Northern India”, described by Walker (1844).

Another version, in which the lower half-board tour is rotated 180° and modified to join to the upper half appears in the *Manasollasa* of Somesvara III (c.1150), described by F. Bernhauer (*Der Rösselsprung im Manasollasa*, *Aus dem wissenschaftlichen Leben des Südasien-Instituts* 1997).

Among full-board tours constructed in Baghdad about the same period is a tour by **Abu-Bakr Muhammad ben Yahya as Suli** (c.880-946) that joins together two half-board tours to cover the whole board (first diagram below). Murray describes this as “given in two poems, one by **Tahir al Basri**, the other by **ibn Duraid** (died 934). Each poem consists of 64 lines, and the first two letters in each line give the name of the square in the literal notation used by the Muslim chessplayers.”

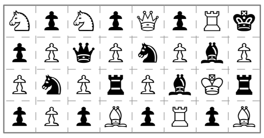






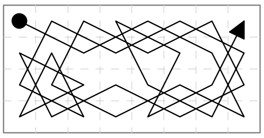
A much later work known as the **King's Library** manuscript, described by Murray, is written in Anglo-Norman, and dated as of “the last quarter of the thirteenth century” held in The King’s Library (now part of the British Library). This is one of the Bonus Socius group of chess problem collections.

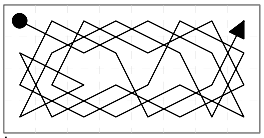
This contains the only mediaeval European tour on the whole chessboard, but formed of two half-board tours with a single link between them (diagram above). This has the description ‘Guy de Chivaler’; where ‘guy’ has the archaic meaning of joke or puzzle. It may be noted that the route in the top right quarter, g5 to g6 is as near as it is possible to get to a 4×4 tour, using only the cell g4 outside the 4×4 instead of the corner h5. This ms uses a,b,c,d,e,f,g,h as coordinates for the files and i,k,l,m,n,o,p,q for the ranks (from top to bottom).

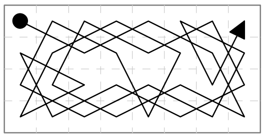
A half-board tour occurs in the same manuscript, presented in the form of the 32 chessmen occupying the left half of a board. The knight in the corner is to take the white pawns first then the black pawns, “travelling twice round the board”, then the bishops, knights, rooks, queens and kings in that order. Without the “twice round the board” condition there is a second solution, as shown alongside, incorporating a three-move line.

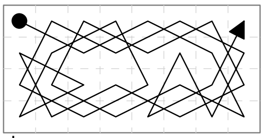
A Latin manuscript of the first half of the 14th century (i.e. c.1325) in the Bibliotheque de Paris is a version of the Bonus Socius chess problem collection. Unusually, the scribe of this manuscript is identified, as **Nicolas de Nicolai**, a scholar from Picardy who studied and lectured at the Lombard universities. He presents a knight's tour problem in the form of an arrangement of the 32 chessmen in half of a chessboard. The white knight in one corner is to capture all the other officers except the kings, and then all the pawns and finally the two kings. The first half of the tour is determinate, but the sequence of capture of the remaining pawns can be varied, giving six solutions, (a) to (f) as shown below. Note that the cells used in the first and second halves of the tour form a domino pattern. This is typical of all tours on 4-rank boards and this was evidently known to Nicolai, though a strict proof was not given until Sainte-Marie (1877).

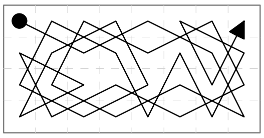


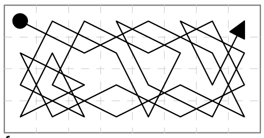

a


b


c

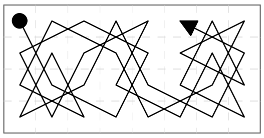

d

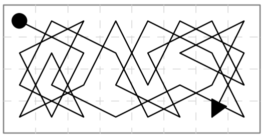

e


f

The arrangement of the chessmen on the board in the King's Library ms is different from that of the Bonus Socius manuscript, but the solution is the same as in diagram (a). The 4×8 tour given in the later Civis Bononiae manuscripts is identical to solution (f) of the puzzle in the Nikolai ms. [Linde 1874, 1881, Murray 1913. See also Guarini 1512.] So this ms can be seen as a transitional one.

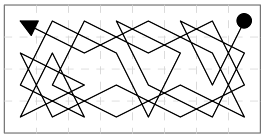
Other half-board tours noted from the Early Modern period make no advance on previous work. One in Italian written in a **Florence** ms c.1490. [Lasa 1850 p.164, Linde 1874, 1881, Murray 1902.] One by **Johannes Chachi**, of Terni (1511) in his ms collection of chess problems [Murray 1913, p.727, 730]. And one by **Paulo Guarini di Forli** (or Paulus Guarinus d.1520) in a 1512 ms in the J. G. White collection, Cleveland Library, Ohio, USA, contains the Civis Bononiae tour. His diagram shows the Nicolai tour (f) traversed in reverse sequence, with a knight at the top right corner (shown here by X) and the successive squares lettered a to p (no j) and A to Q (no J) in two type styles (represented here as lower and upper case), showing the two-part structure of the tour.


Florentine


Chachi

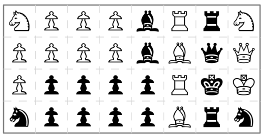
Q	k	O	b	K	h	F	X
N	c	I	i	G	a	C	p
l	P	e	L	n	A	g	E
d	M	m	H	f	D	0	B

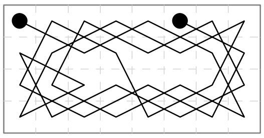
Guarini



The same tour as Civis Bononiae and Guarini is the first tour to appear in a printed work, produced by **Denis Janot** printed in Paris between 1530 and 1540. See Bibliography for title.

Our final early half-board example is provided by **Orazio Gianutio della Mantia** (known as Gianutio) in an Italian work published in 1597. It is puzzling that the arrangement of the chess pieces shown bears little relation to the tour, which is in fact shown by numbers printed below the chess symbols. It cannot be deduced from the sequence of capture of the pieces as in the case of the mediaeval problems. Diagram 246 in van der Linde 1874 has a black knight at f7 instead of a white bishop, also in the original it is difficult to distinguish between the king and queen symbols.



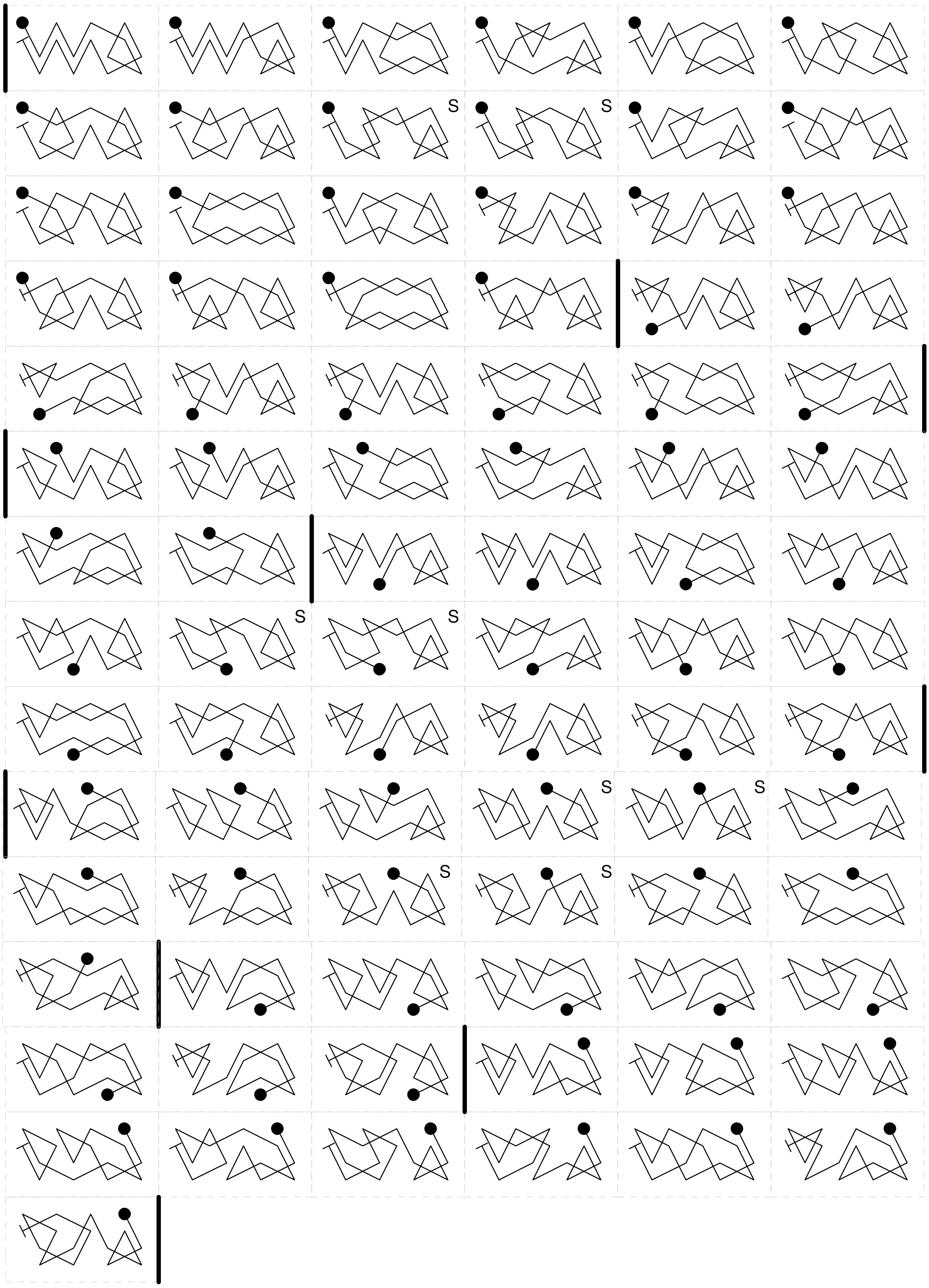


Several writers, following L. Perenyi (1842), namely Lasa (1850), Haldeman (1864) and Lucas (1872), have noted that the Gianutio tour can be combined with a copy of itself to form a closed 8×8 symmetric tour. The same is true of one half of the Suli tour, the Somesvara variant of Rudrata, and the Florentine tour. However D. E. Knuth points out that the Gianutio tour is in fact diagrammed at the top of an 8×8 board so the possibility of duplicating it to form a symmetric compartmental tour was evidently missed. We have to await Euler (1759) for tours deliberately constructed in this way.

For more on the early history of tours on the full chessboard see ⌘ 6.
We now return to the enumeration of half-board tours.

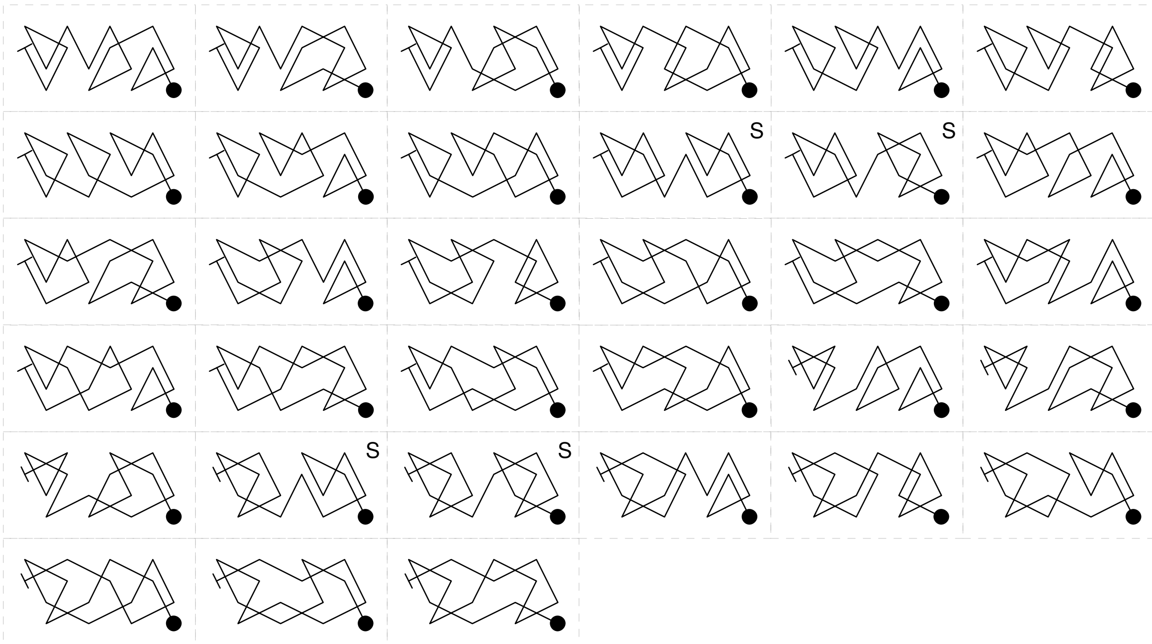
Diagrams of all the 4×8 Partal Tours

a-file: a1-a118 (S = squares and diamonds)

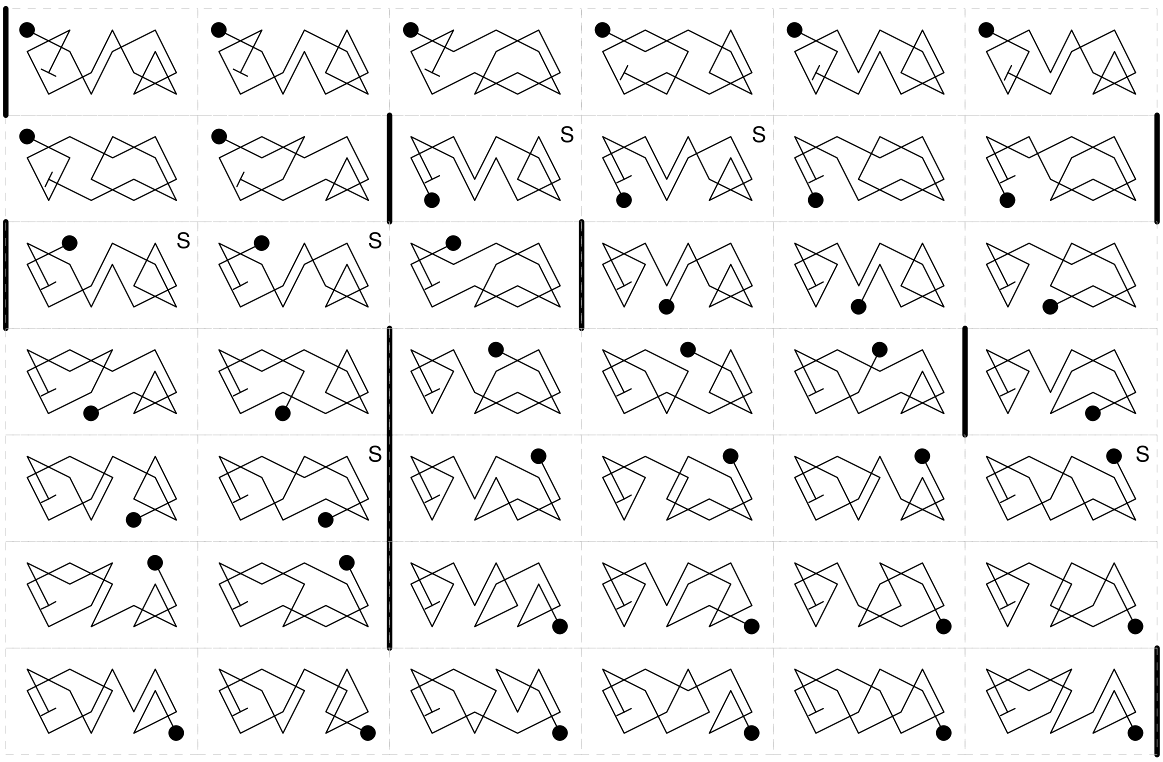


The simple a14 edge-hugging half-tour is the first component in many of the mediaeval tours.

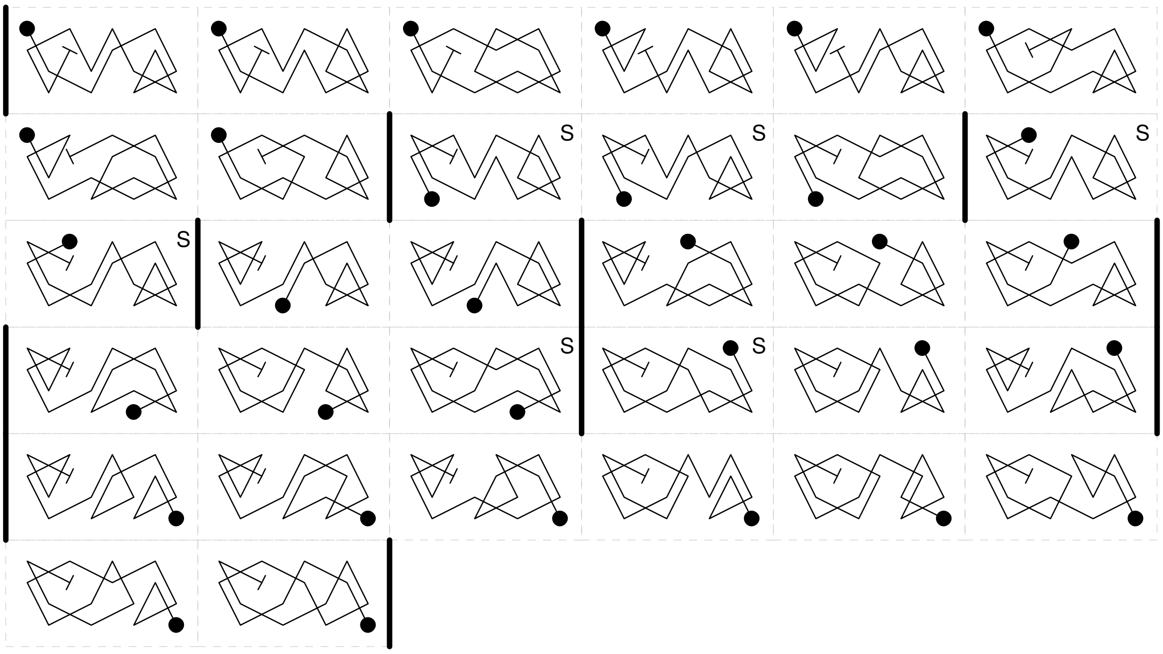
H(1) continued



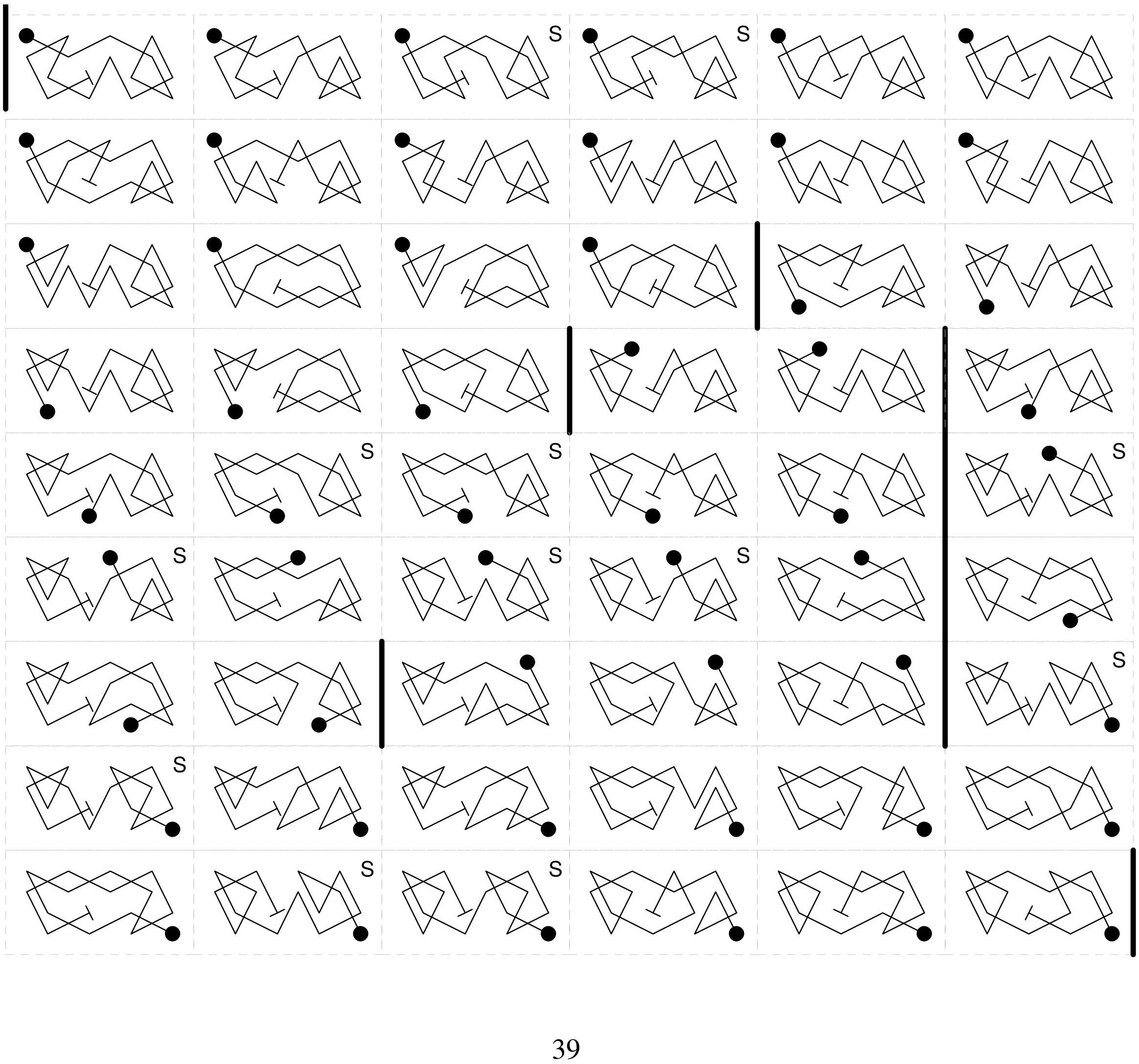
b-file: b1-b42



c-file: c1-c32



d-file: d1-d54.



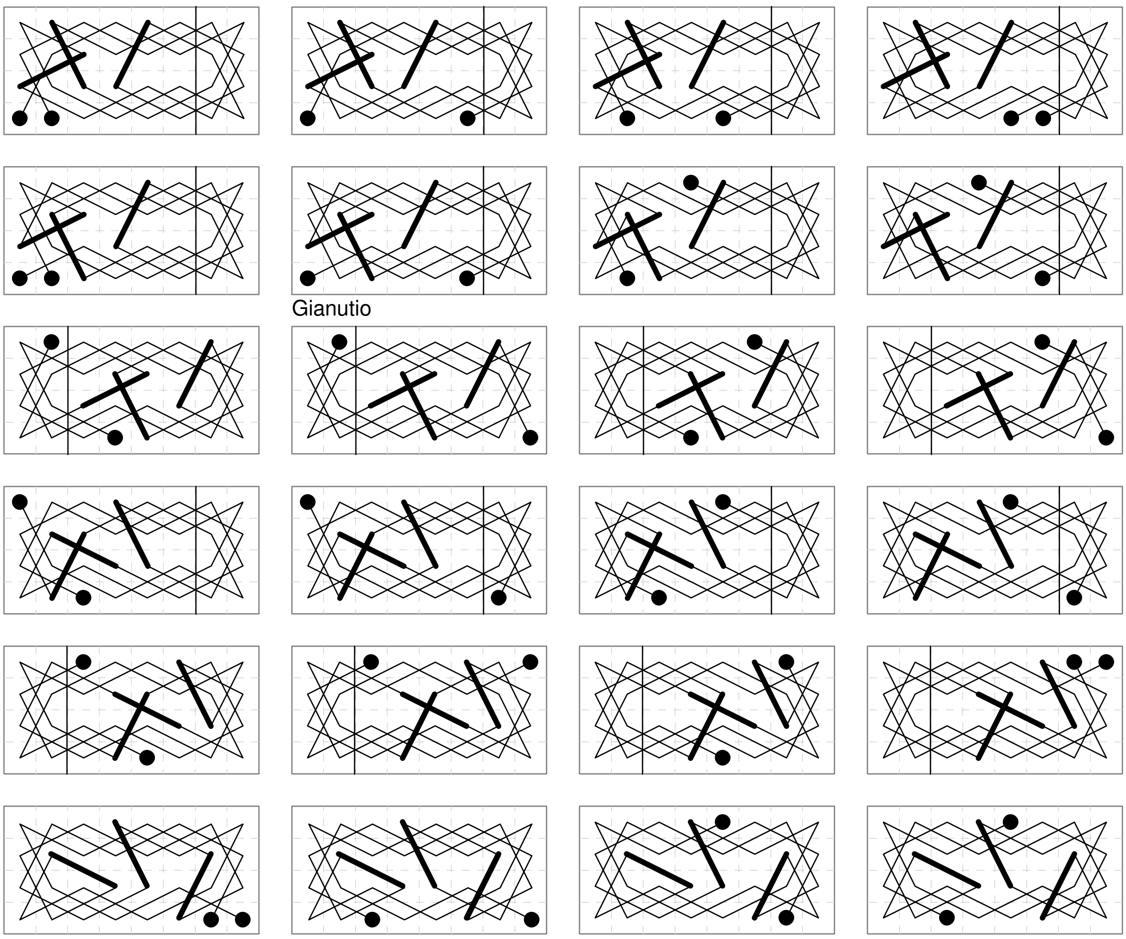
Counting the numbers of half-tours between given files we find:
H(1,1) = 22, H(1,2) = 8, H(1,3) = 8, H(1,4) = 16, H(1,5) = 13, H(1,6) = 8, H(1,7) = 10, H(1,8) = 33
H(2,1) = 8, H(2,2) = 4, H(2-3) = 3, H(2,4) = 5, H(2,5) = 3, H(2,6) = 3, H(2,7) = 6, H(2,8) = 10
H(3,1) = 8, H(3,2) = 3, H(3,3) = 2, H(3,4) = 2, H(3,5) = 3, H(3,6) = 3, H(3,7) = 3, H(3,8) = 8
H(4,1) = 16, H(4,2) = 5, H(4,3) = 2, H(4,4) = 6, H(4,5) = 6, H(4,6) = 3, H(4,7) = 3. H(4,8) = 13

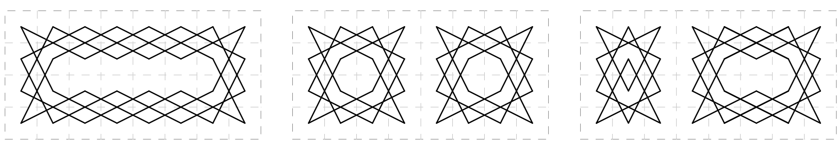
Adding these we find the numbers of half-tours beginning in files 1 to 4 are: H(1) = 118, H(2) = 42, H(3) = 32, H(4) = 54, (total 246). Thus the number of geometrically distinct complete 4×8 tours with 1-3 central move is H(1)·H(3) = 118·32 = 3776 and with 2-4 centre is H(2)·H(4) = 42·54 = 2268 and with 3-5 centre is H(3)·H(5) = 32·54 = 1728. These four figures add to **7772**. This total was first given by Sainte-Marie (1877). The numbers of 4×8 tours with end-points in the same side is the same as the number with end-points in opposite sides, i.e. half of 7772 = 3886. All are asymmetric.

G = 3776 + 2268 + 1728 = 7772 = A, S = 0, T = 4·G = 31088.

We use these figures in ⌘ 6 when we try to enumerate double half-board tours.

4×8 Tours of Edge-Hugging Type. There are 24 tours of this type on the 4×8. Six can be joined to a copy themselves to give a full-board closed tour. Twenty can be regarded as extensions of the 12 tours on the 4×6 by a two-file braid at one end, but in one case this is not possible, instead there is a distinctive 4×8 case with four tours. Here are the 24 with extensions indicated: The 4×6 design in the first row cannot be extended to the left. The four in the last row are the distinctive 4×8 design.



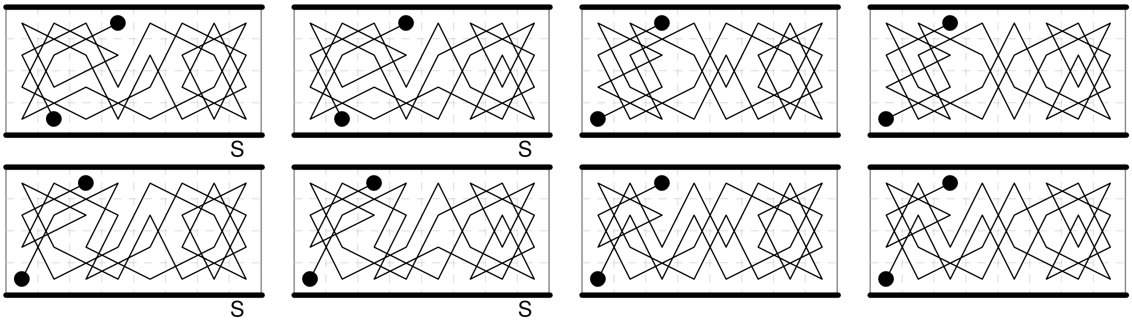


The border braid is one of the three crosspatch pseudotours on this board.

4×8 Tours of Squares and Diamonds. Examination of the half-tours shows that those formed of squares and diamonds number 12 from the a and h files, 6 from the b and g files, 6 from the c and f files and 12 from the d and e files. Thus the number of squares and diamonds tours with middle moves a-c, b-d, c-e are 6·12 = 72 in each case, total **216**. The structure of these tours can be expressed alternatively in terms of the succession in which the squares (S) and diamonds (D) are visited.

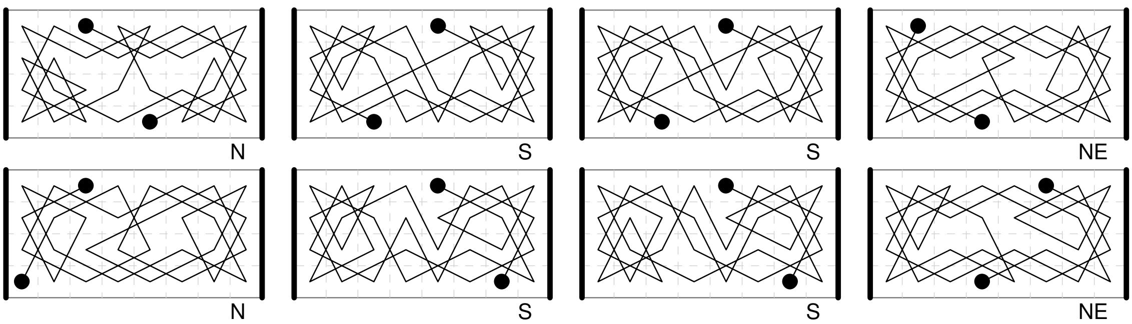
The half-tours are either of DSDS or DDSS type, with the middle move joining an S to a D. This gives three types: DSDSDSDS, DSDSDDSS, DDSSDDSS. Numbers of half-tours of DSDS type are 8 from a and d, 4 from b and c, all others being DDSS. Thus the totals in each of these classes can be calculated. DSDSDSDS: 3·(4·8) = 96. DDSSDDSS: 3·(2·4) = 24. DSDSDDSS: 3·(8·2 + 4·4) = 96. It may be noted that one end rhomb is always a diamond and the other a square. There are 108 tours with the end rhombs in the same half of the board and 108 with them in opposite halves.

Semi-Magic 4×8 Tours. The earliest example of a 4×8 semi-magic tour was given by F. J. Brede (1844) and the subject was studied in more detail by Carl Wenzelides (1849) with several more examples. The question was also considered by H. J. R. Murray (1917). No tour magic in both ranks and files is possible, but there are 8 with the lines of four all adding to 66.



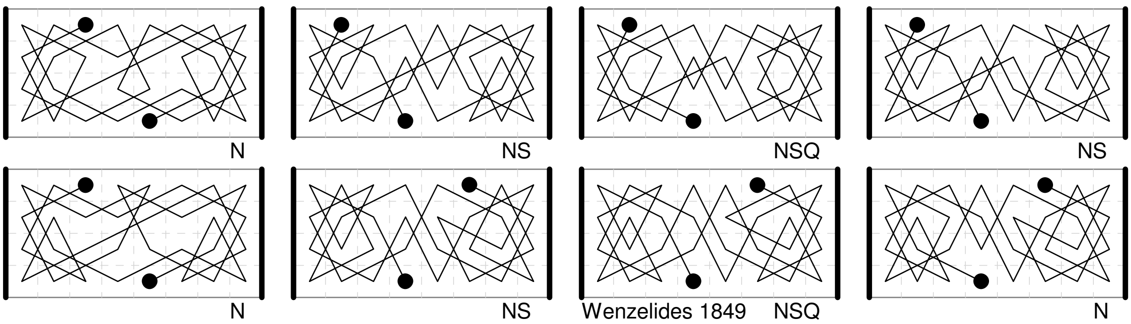
The first two are quasimagic, two ranks sum 130 and two to 134. The second two are near-magic, and use Beverley quartes in the first half. The ranks sum to 128, 132, 132, 136. The four on the left are of squares and diamonds type. The eighth tour is in Brede (1844) in the form of a semi-magic 8×8 tour adding to 260 in all files, formed by joining this 4×8 semi-magic tour to a copy of itself. The second and the fourth in the first line are in Wenzelides (1849 Fig, 61 and 1858 Fig..D).

There are 68 tours in which the lines of eight all give the total 132. (The following diagrams are the results of my own work, but I understand these figures were confirmed by computer methods by G. Stertenbrink in Aug 2003). Among these 68 tours 30 have end-points separated by a {2,3} move.

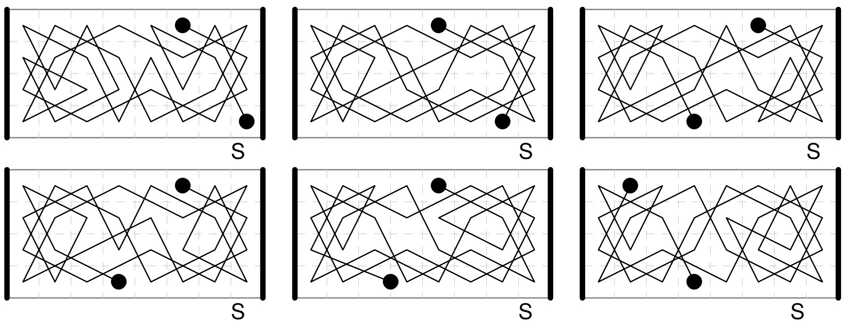


These include 8 near-magic tours (in which the files add to the magic value and two others). Four (marked S) are of squares and diamonds type, two have file sums 2×64, 4×66, 2×68 and two have 1×62, 6×66, 1×70. These latter are those diagrammed by Murray (1917) as being the "nearest approaches" to magic on the 4×8. The other four (marked N) are of the near symmetric type with diametral difference 4 (two marked E also being of edge-hugging type). In these the end-points will join to the ends of the middle move to form a symmetric pseudotour of two 16-move circuits. The file sums of the NE cases are 1×58, 6×66, 1×74. The other N cases have file sums 2×58, 5×66, 1× 82.

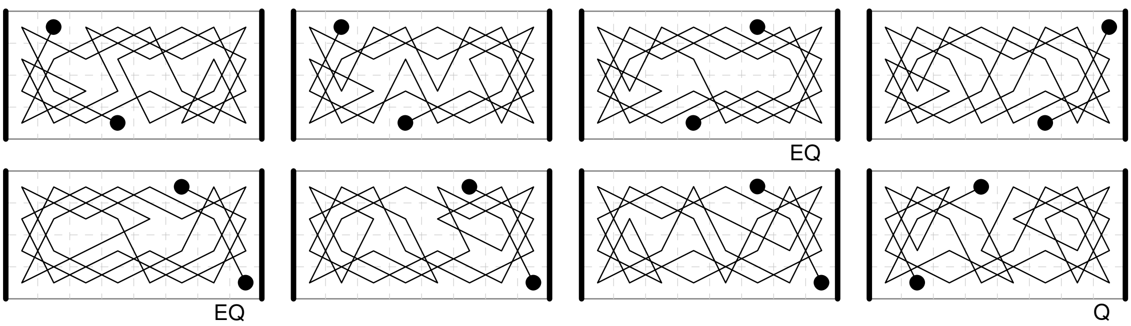
The remaining {2,3} cases include 8 more of the near symmetric type (N) , five of which are of squares and diamonds, and two of these are quasi-magic (files add to two values). The two marked NSQ have diametral difference 8, and file sums 4×58, 4×74. Their symmetry is biaxial. One of these appears in Wenzelides (1849, Figs.65 and 66).



There are 6 more of squares and diamonds type:

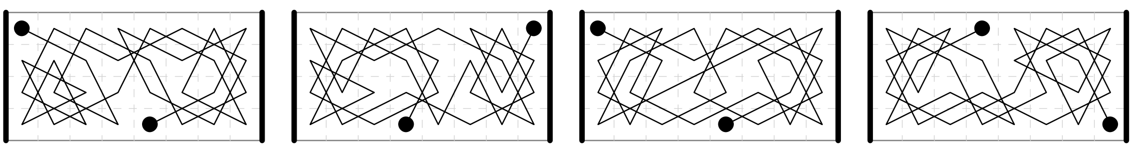


The remaining eight {2,3} tours include 2 of edge-hugging type (E) which are also quasi-magic, and one other quasimagic. The EQ tours have diametral difference 4, and file sums 6×64, 2×72 (or 6×68, 2×60 in the reverse numbering). The other Q has file sums 6×70, 2×54 (reverse 6×62, 2×78).

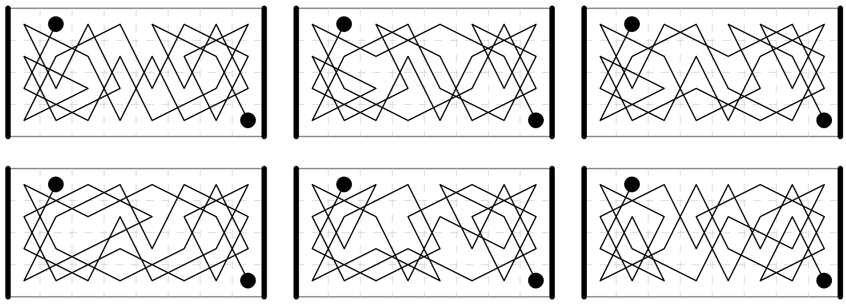


Diagrams of the other semi-magic tours follow classified by end-point separation.

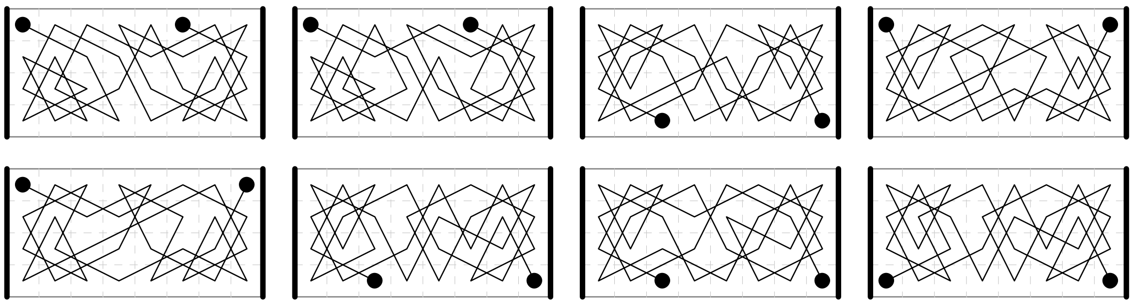
The {4,3} cases:



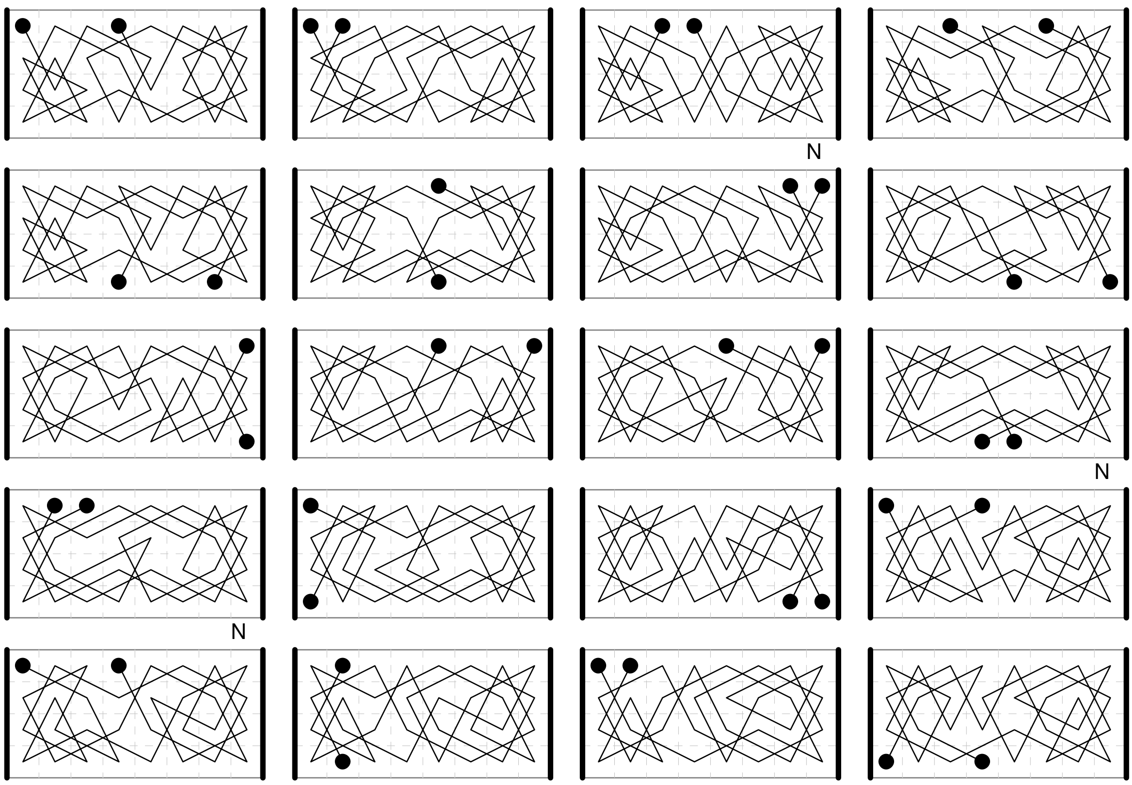
The {6,3} cases:



The {0,5} and {0,7} cases:

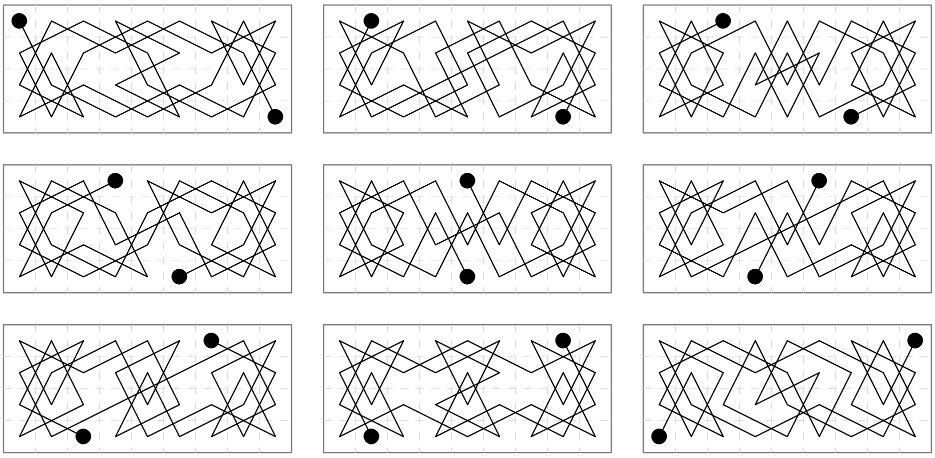


The {0,1} and {0,3} cases. Three of the {0,1} type are near-symmetric (N).

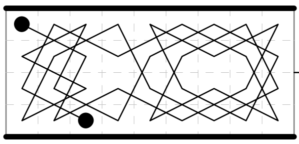


4×9

On the 4×9 board: There are 112 symmetric knight tours, all open of course. One example of each end position is shown with the central move oriented d2-f3.



As I was preparing this book for publication important new work by Awani Kumar (2018) on boards 4×9 up to 4×28, including fully magic tours, was published. Results from his work enhance the next sections considerably. On the 4×9 board Kumar shows a semi-magic tour. The files add to 74 and the ranks to four values (161, 166, 167, 172).



1	34	17	22	9	32	13	26	7
18	21	2	33	14	27	8	31	12
35	16	19	4	23	10	29	6	25
20	3	36	15	28	5	24	11	30

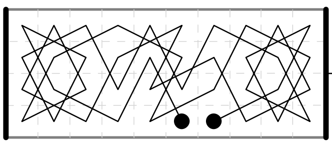
There are 15 edge-hugging tours two of which are symmetric (see introduction to the 4-rank section). G = 47478, S = 112, A = 47366, T = 189688.

4×10

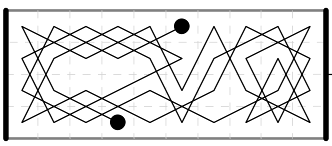
G = 303278 = A, S = 0, T = 1213112. These tours were enumerated by Kraitchik (1927).

Awani Kumar (2018) has examined magic properties on this board and finds 3102 semi-magic, adding to 205 in the ranks, of which 30 are quasimagic (two totals in the files) and 22 near-magic (magic total and two others in the files).

We show one of each from that source. The files in the first add to 66 (6 times) and 106 (4 times). In the second, near-magic, they add to:82 (the magic total, 8 times) and to 78 and 86 (once each).

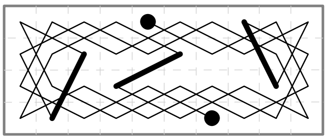
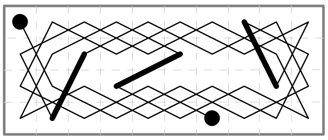
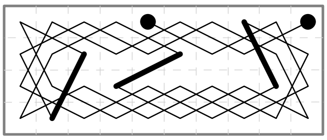
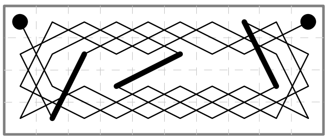


5	26	9	30	11	22	13	34	17	38
8	29	6	23	2	31	20	37	14	35
25	4	27	10	21	12	33	16	39	18
28	7	24	3	32	1	40	19	36	15



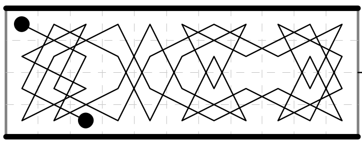
3	38	17	22	5	40	7	28	13	32
18	23	4	39	16	21	14	31	10	29
37	2	25	20	35	6	27	8	33	12
24	19	36	1	26	15	34	11	30	9

Edge-hugging tours total 40, 8 use a-c, 8 use b-d, 8 use c-e, since they link directly to a corner cell, whereas 16 use d-f. Here are four tours of d-f type, all with the same linkage pattern. The other tours can be regarded as extensions of 4×8 edge-hugging tours.

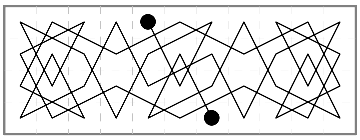
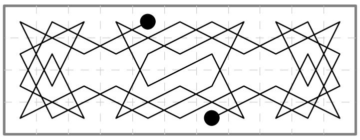
4×11

Awani Kumar (2018) has enumerated 3341926 arithmetically distinct tours on this board, of which 267 are semi-magic, adding to 90 in the files, as in this example he gives. The longer lines add to four different values (248, 249, 246, 247) four successive numbers.



1	42	21	26	5	36	17	38	13	34	15
22	25	2	41	18	39	8	35	16	31	12
43	20	23	4	27	6	37	10	29	14	33
24	3	44	19	40	9	28	7	32	11	30

The following two symmetric examples (Jelliss undated) were formed by linking together circuits forming pseudotours on partial boards.

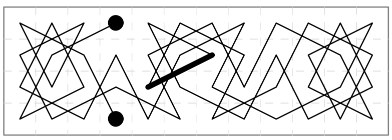
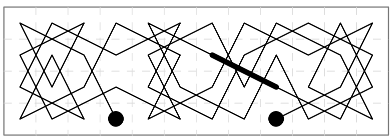



There are 19 edge-hugging tours, of which two are symmetric.

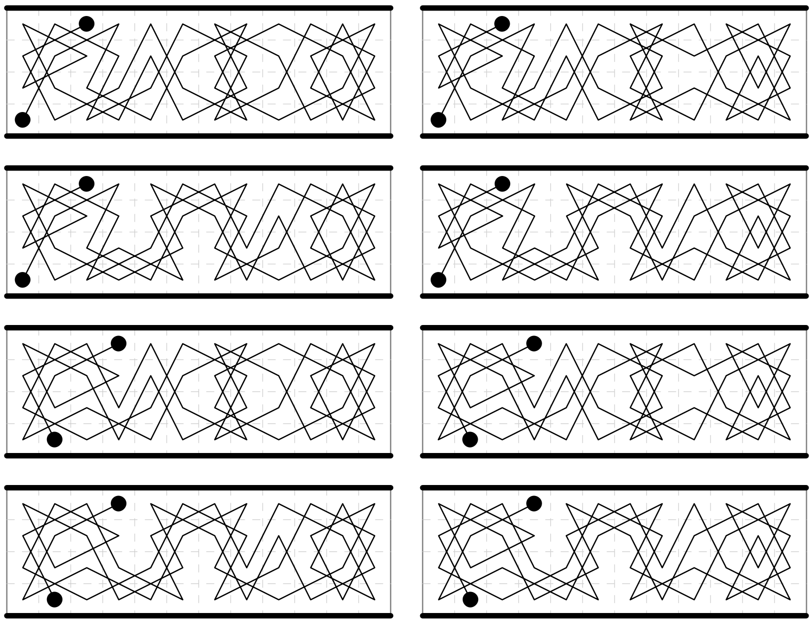
4×12

Edge-hugging: These tours total 56, from 10 cases as for 4×10 plus four extra.

Squares and Diamonds. Examination of the half-tours of squares and diamonds type shows that there are 40 from a and d, 12 from c and b, 16 from e and f. Thus the number of S&D tours with middle moves a-c, b-d, c-e, d-f, e-g are respectively $40\cdot 12 = 480$, $40\cdot 12 = 480$, $12\cdot 16 = 192$, $16\cdot 40 = 640$ and $16\cdot 16 = 256$, totalling 2048. It is necessary for a half-tour to complete two circuits in an end quad before making a second circuit in the middle quad, otherwise the middle quad acts as a barrier to completion of the path. Examples:

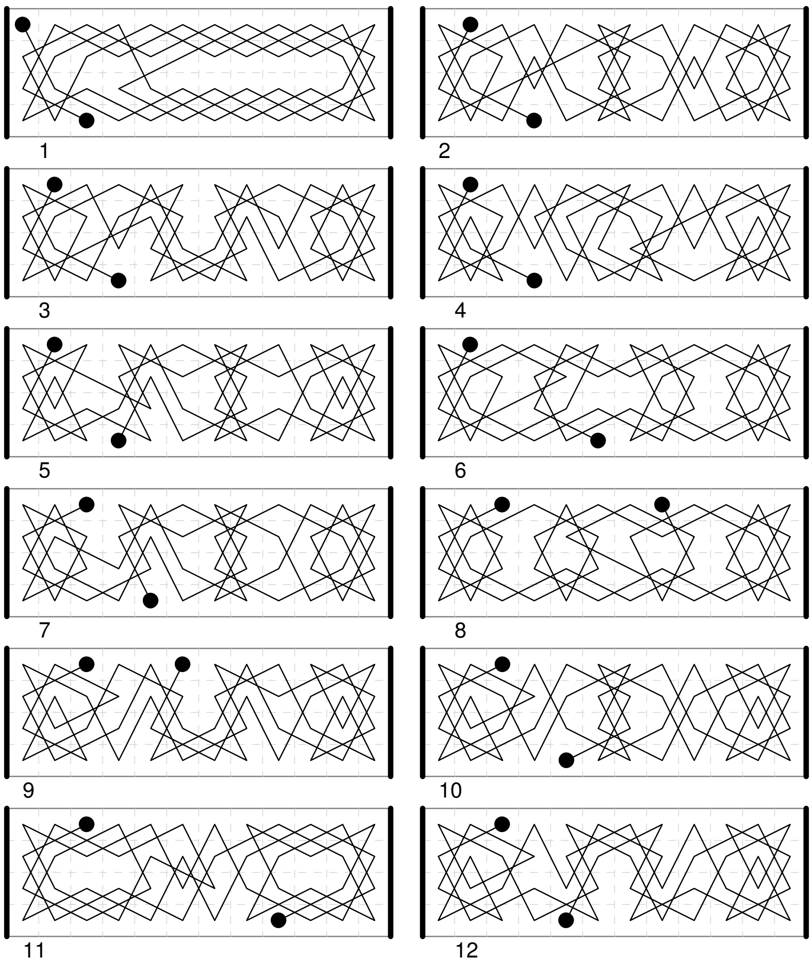



Semi-magic S&D tours with short lines magic. On this board I have only examined tours of squares and diamonds type, seeking for a possible magic tour, and finding 8 semi-magic tours. The short lines (files) are all magic adding to 98.

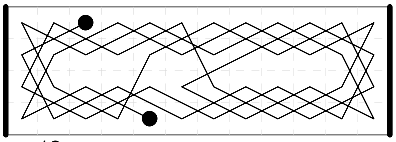


In 1 and 2 the long lines add to 284, 288, 300, 304. In 3, 4, 5, 6 they add to 288, 292, 296, 300. Tours 7 and 8 are quasimagic, long lines sum to 292 twice and 296 twice.

Quasi-magic tours 4×12 with long lines magic. Jean-Charles Meyrignac reported (26 August 2003) that his program computed all the 4×12 quasi-magic tours of this type, finding 48. We show them in two batches. Batch 1 (1-12).



Batch two (13-24):



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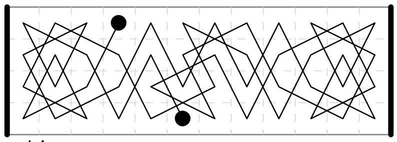

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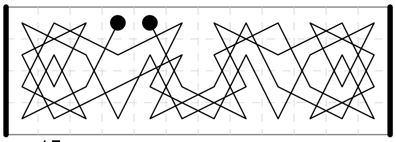

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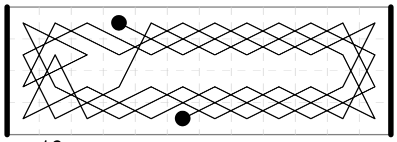

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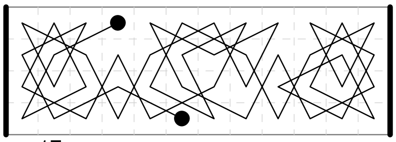

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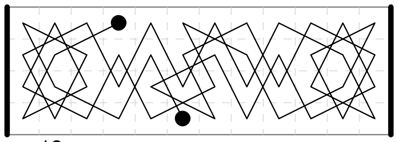

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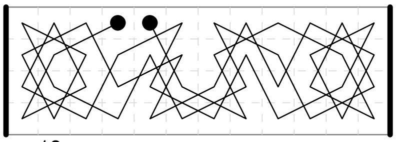

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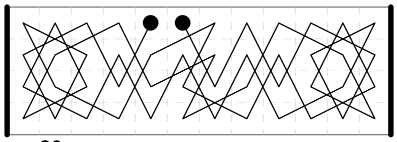

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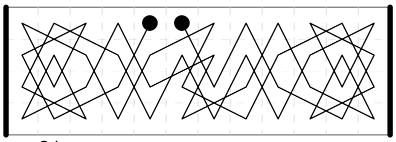

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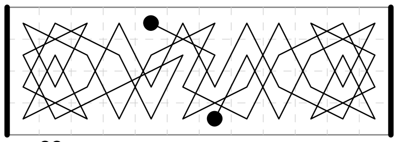

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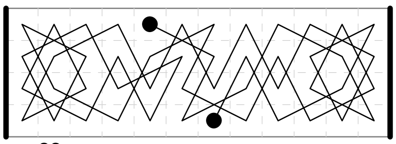

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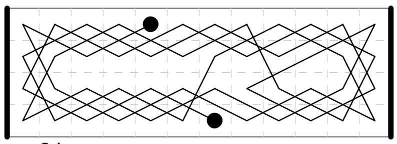


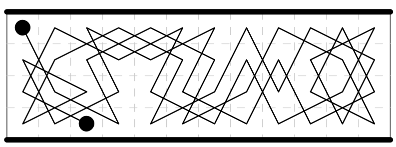
24

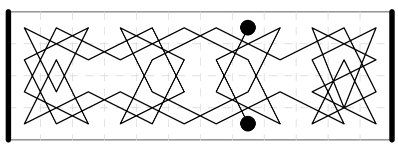
Horizontal lines have their sum equal to 294, and vertical lines have sums taking two values.
Tours 1, 4, 13, 16, 17, 24 have file sums 10 of 96 and 2 of 108 (reverse 10 of 100 and 2 of 88).
Tours 2, 3, 6, 8, 9, 10 and 12 have file sums 8 of 90 and 4 of 114 (reverse 8 of 106 and 4 of 82).
Tours 5 and 7 have file sums 9 of 110 and 3 of 62 (reverse 9 of 86 and 3 of 134).
Tour 11 has file sums 6 of 110 and 6 of 86.
Tours 14, 18, 22, 23 have file sums 10 of 94 and 2 of 118 (reverse 10 of 102 and 2 of 78).
Tours 15 and 19 have file sums 8 of 114 and 4 of 66 (reverse 8 of 82 and 4 of 130).
Tours 20 and 21 have file sums 5 of 70 and 7 of 118 (reverse 5 of 126 and 7 of 78).

No 2 has a three-move line.

Tours 1, 13, 16, 24 of these 4×12 tours can be seen to be a braid extensions of 4×8 tours.

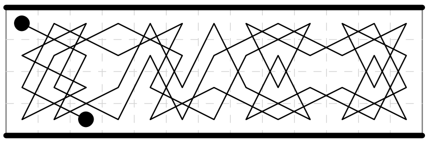
Awani Kumar (2018) reports 18243164 arithmetically distinct tours, of which 608 are semi-magic in the short lines (of which 28 are quasi-magic and 72 near-magic) and 58356 semi-magic in the long lines (48 quasi-magic and 112 near-magic). Here are two near-magic tours he gives. The first adds to 98 in the files (ranks 290 294 294 298). This uses Beverley quartes in the middle section. The second adds to 294 in the ranks (files 98 ten times, and 96 and 100 once each).





4×13

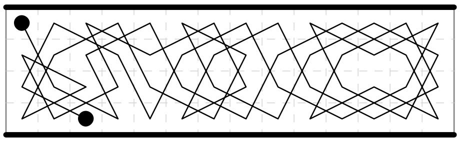
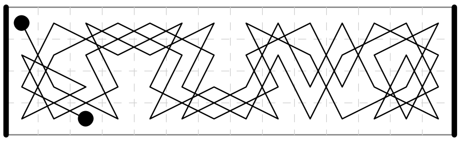
Kumar (2018) reports 100641235 arithmetically distinct tours of which 1444 are semi-magic with short lines adding to 106. In this example the ranks add to the successive totals 343 344 345 346.



PUZZLE: To arrange the 52 cards of a standard pack in a knight`s tour in the sequence Ace to King in each of the four suits so that each file contains four cards of the same rank (sol. at end).

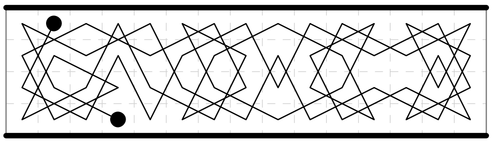
4×14

Kumar (2018) reports 526152992 arithmetically distinct tours. Of these 3480 are semi-magic in the short lines, constant 114, with 136 quasimagic and 244 near-magic (in the example the ranks add to 399 403 395 399). Also 1092618 semi-magic in the long lines, constant 399, of which 170 are quasimagic and 330 near-magic (in the example 12 files add to 114 the others to 110 and 118).

4×15

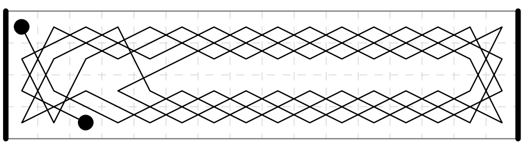
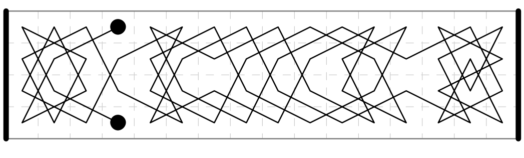
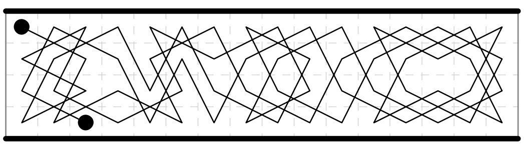
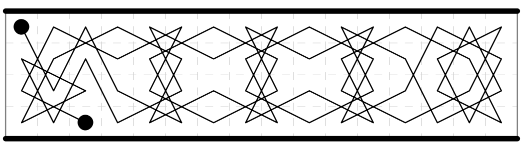
Kumar (2018) reports 8221 semi-magic tours, with file sum 122, but none quasimagic. He gives an example where the four rank sums are consecutive numbers (456, 457, 458, 459).



4×16

The braid extension method for 4-rank boards can extend shorter quasi-magic tours to the 4×16 board, and to any larger board 4×4k. The result is magic in the long lines..

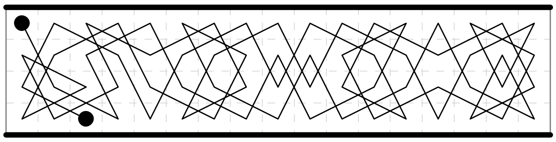
Kumar (2018) has not enumerated the semi-magic tours with ranks adding to 520, but finds 710 quasimagic and 1304 near-magic (in his example 14 files add to 130 and the others to 128 and 132).

Kumar finds 19212 semi-magic tours with files adding to 130, of which 488 are quasimagic (his example above adds to 522 522 518 518 in the ranks) and 1012 are near-magic (his example above adds to 520 516 524 520). This has a repeatable mid-section.

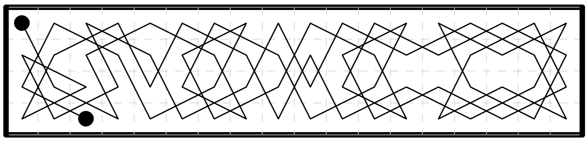
4×17

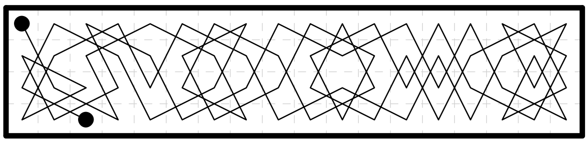
Kumar (2018) finds 45262 semi-magic tours with files adding to 138. His example has ranks adding to consecutive values (587 588 585 586), one of 30 such arithmic forms. None quasimagic.

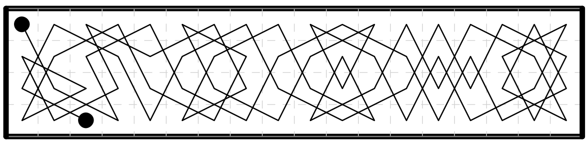


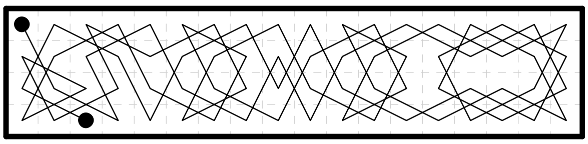
4×18

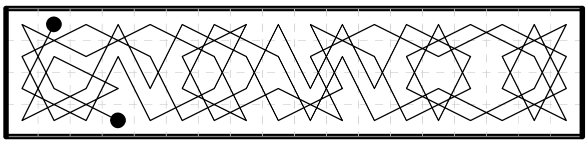
Awani Kumar (2018) finds 16 arithmetically distinct magic tours, numbered 1 to 16, but these are only 8 counted geometrically since they occur in pairs that are the reversals of each other (1=15, 2=16, 3=9, 4=10, 5=11, 6=12, 7=13, 8=14).

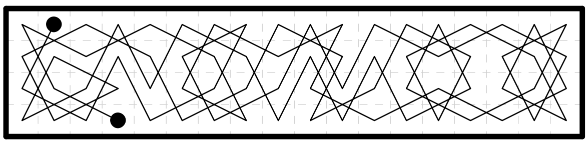


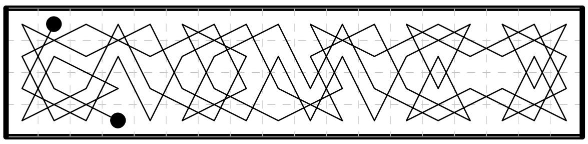


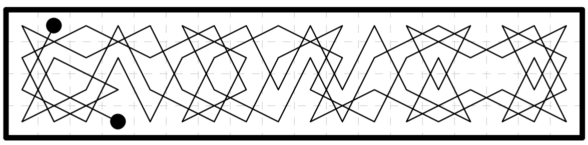










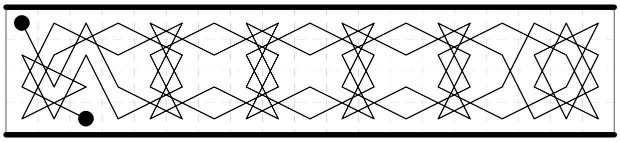


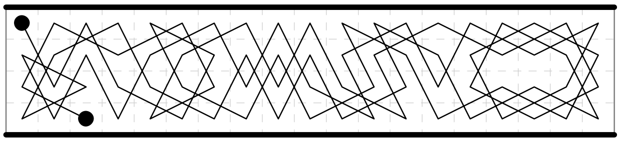
Kumar also reports finding 213280 semi-magic tours with constant file sum 146 of which 2624 are quasi-magic and 3976 near-magic, and an unknown number of semi-magic tours with constant rank sum 657 of which 2492 are quasi-magic and 6322 near-magic.

The magic tours are further to these.

4×19

Tour formed by putting in another occurrence of the three-file repeating pattern from the Kumar 4×16 near-magic example. It remains semi-magic with file sum 154. Ranks add to 722 699 764 741. The other example is from Kumar (2018) and has arithmic rank sums 732 733 730 731.

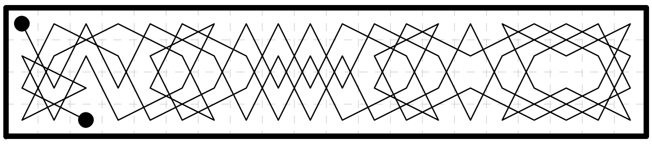


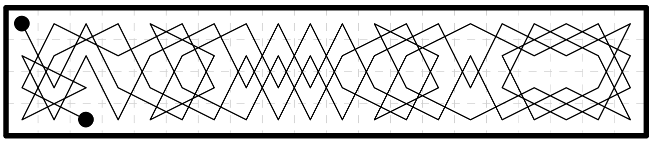


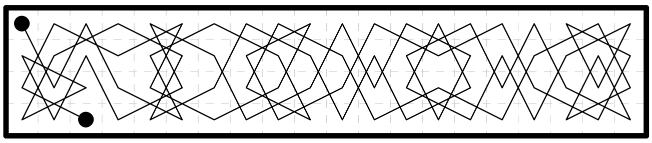
Kumar finds 250247 semi-magic tours, but none quasi-magic.

4×20

Awani Kumar (2018) finds 88 arithmetically distinct magic tours, which means 44 geometrically distinct since symmetry is impossible. He gives three examples:







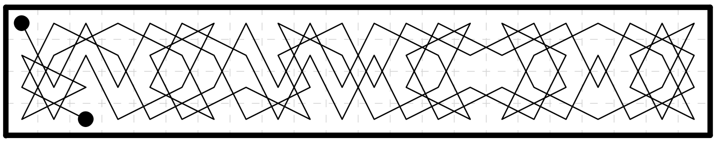
Diagrams of all 44 are now shown on the website.

4×2k

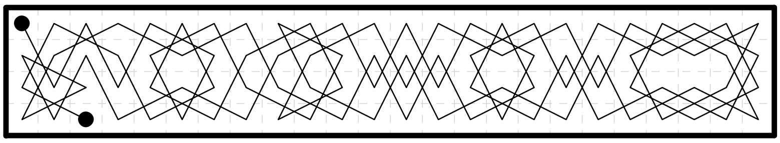
It is clear that magic tours can be constructed on all boards 4×2k with k = 9 or greater, for example by simple braid extension from 4×18 to 4×22 and from 4×20 to 4×24 and so on. In a separate communication Awani Kumar reports finding 464 arithmetically distinct magic tours (232 geometrically distinct) on the 4×22 board.

Here are single examples of magic tours of sizes 4×22 up to 4×28:

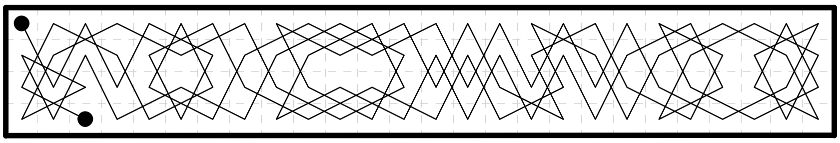
4×22



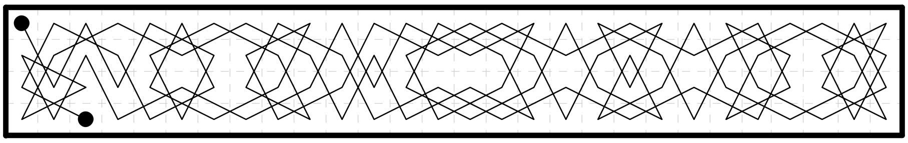
4×24



4×26



4×28

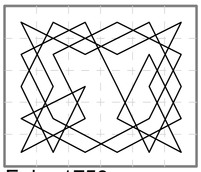


For further dagrams, and XL files listing the tours in arithmetical form, see the website.

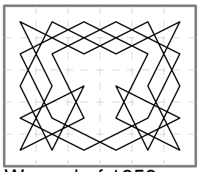
Five-Rank Boards

5×6

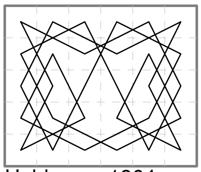
The 5×6 rectangle shares, with the 3×10, the title of smallest rectangle (30 cells) that will admit of a closed knight's tour. On the 5×6 there are, surprisingly, only three closed tours. The asymmetric one was found by Euler (1759) and those with axial symmetry by Warnsdorf (1858) and Haldeman (1864). The asymmetric tour combines the two halves of the symmetric tours. By rotation and reflection they thus can be shown in 2×2 + 4×1 = 8 different 5×6 diagrams.



Euler 1759



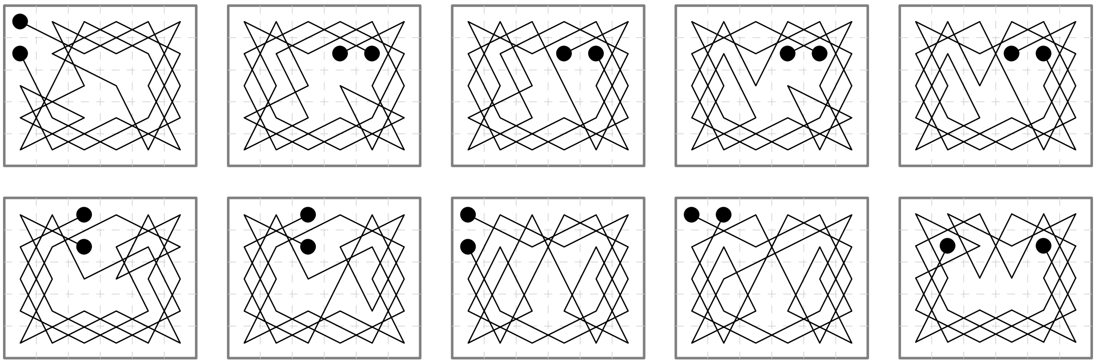
Warnsdorf 1858



Haldeman 1864

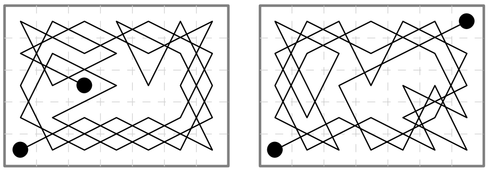
If the central cells c3, d3 are omitted then there is a border-pattern pseudotour of four circuits (two of 8 cells, two of 6 cells) on the remaining cells. The closed tours delete one move in each of these circuits and connect them with four new connections — two single links and two double links that pass through the central cells.

There are no symmetric open tours on the 5×6 board. This is perhaps surprising, since a symmetric open tour is possible on the 3×4 board, which is the middle area of the 5×6 board. Here are some open tours. The first from Willis (1821), then six from Kraitchik (1926), and two from Murray (1942) that show three two-move lines. The last (Jelliss 2015) approximates axial symmetry.



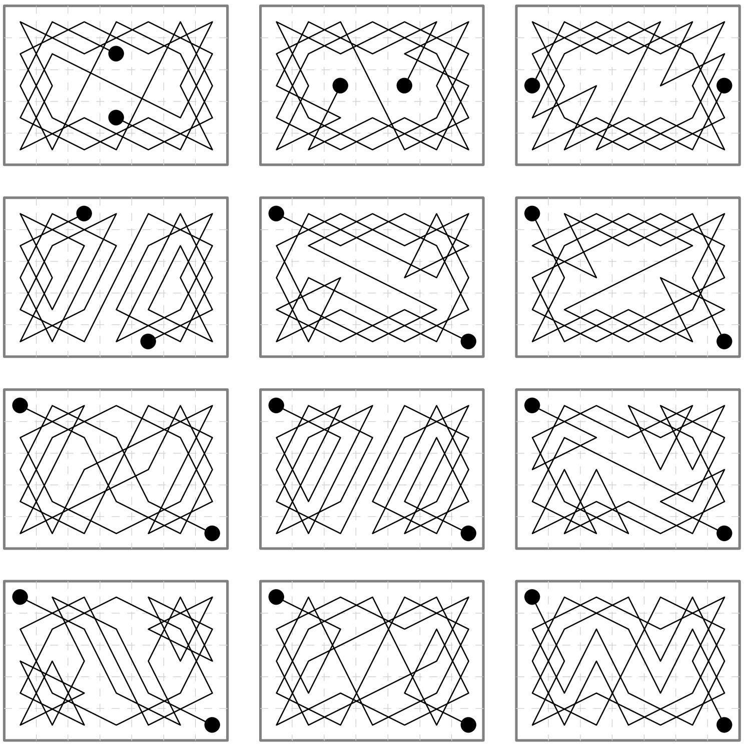
5×7

Willis (1821) and Jaenisch (1862) gave asymmetric examples.



I find 288 symmetric tours on this board. U. Papa (1920) claimed 260. Classified by separation of end-points they are: {0,2} 18 (4 horizontal c3-e3, and 14 vertical d2-d4); {0,6} 14 (all a3-g3); {2,4} 56 (12 horizontal, b2-g4 or b4-g2, and 44 vertical, c1-e5 or c5-e1); {4,6} 200 (a1-g5 or a5-g1). The corner-to-corner tours comprise much the larger class. They can be subclassified according to the direction of the straight line though the centre cell. If they are all drawn a5-g1 then the central two moves are b2-f4 (32 cases), b4-f2 (28 cases), c1-e5 (64 cases) or c5-e1 (76 cases).

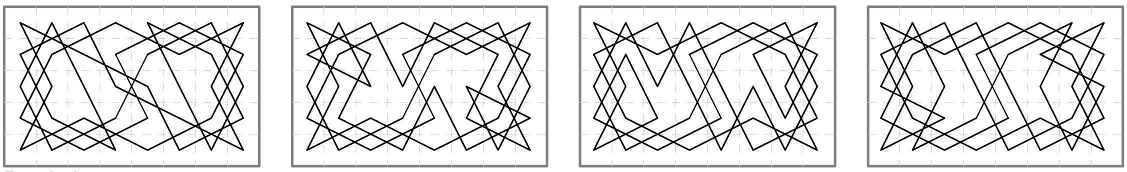
Examples of each case are shown. The first diagram is from Papa (1920), the second from Murray (1942), the third Jelliss (1985), the others Jelliss (1997).



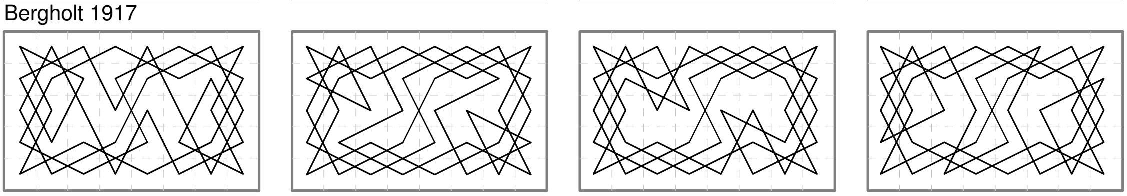
5×8

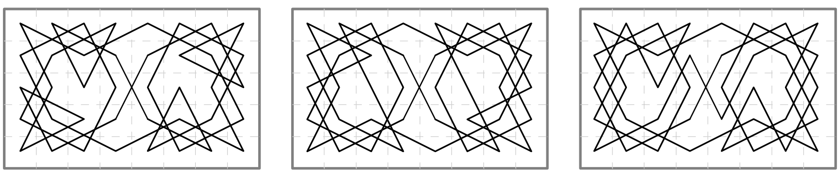
Murray (1942) wrote: “The main interest in this board is that it can be used to build up tours on the chessboard by compartments. We [i.e. Murray, Bergholt and Moore] have found 11 tours in central diametral symmetry on this board” (using Papa’s term for Bergholtian symmetry). The 11 diagrams are then given. This agrees with my own enumeration, done independently in 1988.

The first tour shown here is due to Bergholt (1917), it is the only one in which the centre moves are cut four times. There are four in which they are cut twice, and in the remaining six they are not cut. The last of these was given by Papa (1922).



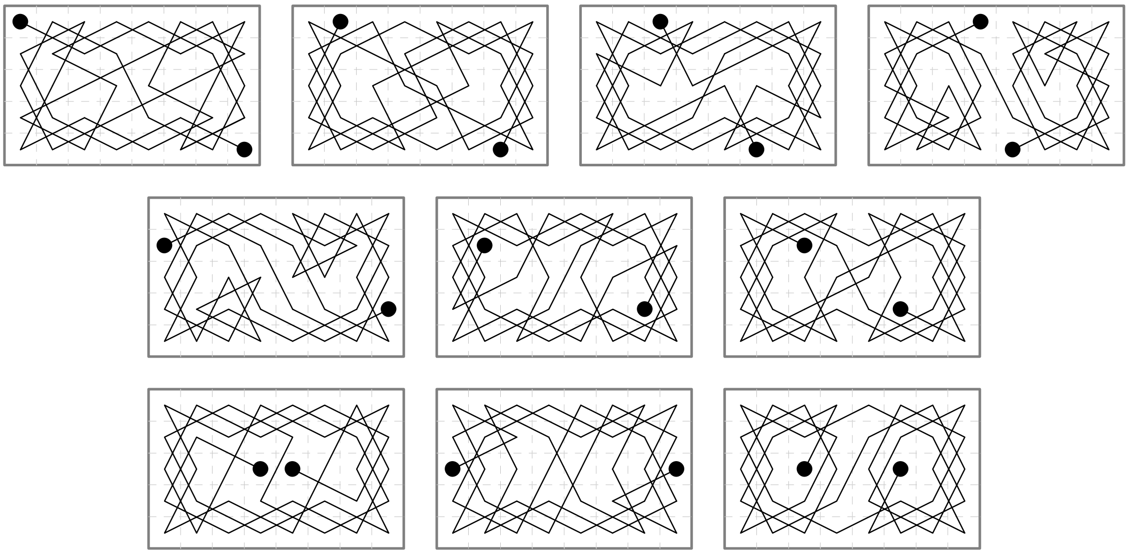
Bergholt 1917



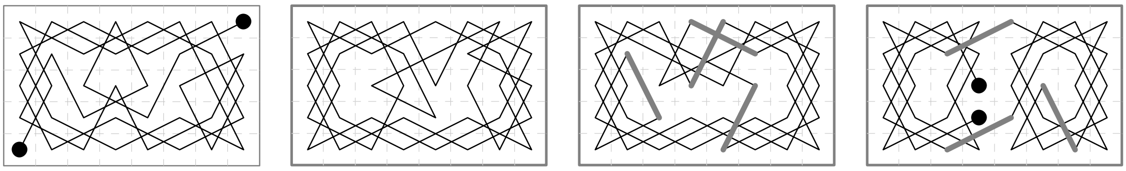


Papa 1922

There are 22 reentrant symmetric open tour solutions (two from each of the 11 closed solutions). A few non-reentrant examples are shown, one for each pair of end positions.



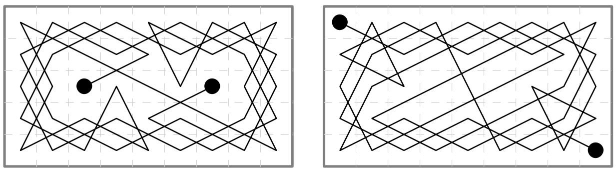
There are also 11045 asymmetric closed tours making 11056 geometrically distinct tours in all, and giving 2×11 + 4×11045 = 44202 diagrams (totals from D. E. Knuth). Willis (1821) gave an asymmetric open tour,and Jaenisch (1862) an asymmetric closed tour. My examples are formed by simple linking from two crosspatch pseudotours on this board.



Willis 1821 Jaenisch 1862 Jelliss 2018 Jelliss 2018

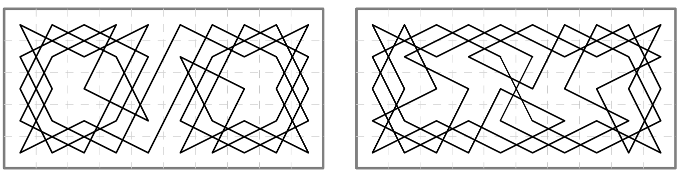
5×9

Two example symmetric tours, necessarily open, by Murray (1942) and Jelliss (1985).



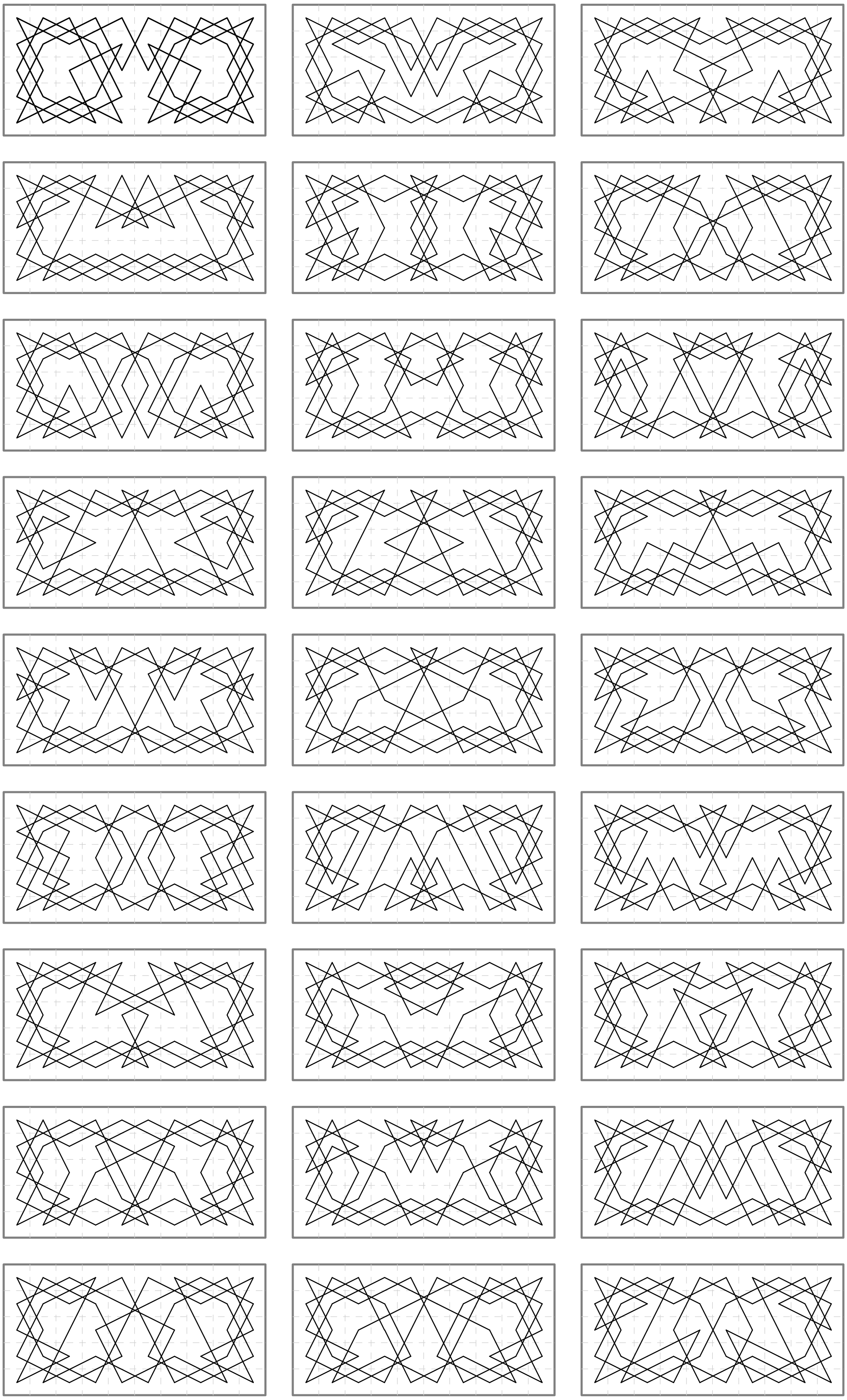
5×10

Eulerian and Bergholtian examples by Murray (1942).



The 5×10 board of 50 cells admits 1986 geometrically distinct symmetric closed knight tours, of which 1133 are Sulian, 606 Eulerian and 247 Bergholtian. This information from D. E. Knuth who has applied computers to the enumeration of tours on small boards.

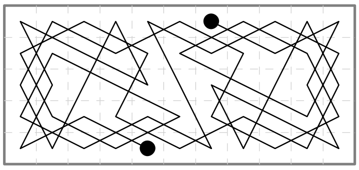
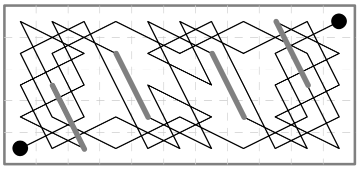
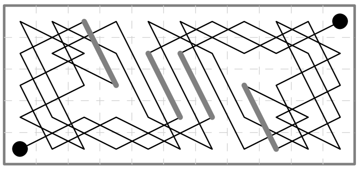
The following diagrams show 27 striking examples of tours with Sulian symmetry.



The first is by Murray (1942). The others are my own from over 250 that I constructed.

5×11

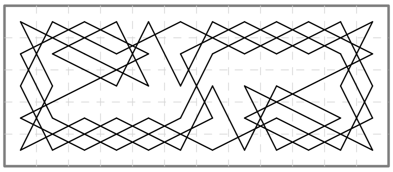
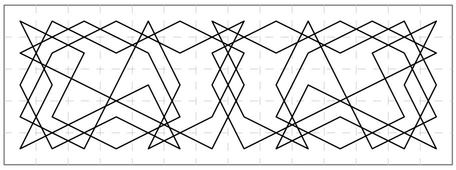
Symmetric open tour of 55 cells (Murray 1942) and 2 of 12 symmetric tridirectional tours (Jelliss 1994). Two others are found by taking the alternative sides of the marked rhombs.

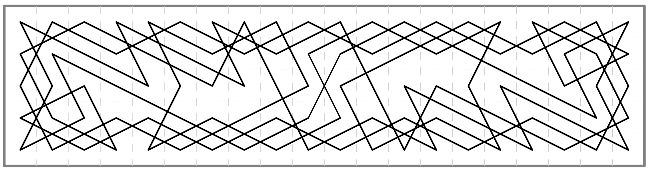




For other examples of tri-directional tours see boards 3×4k, 6×9 and 7×7.

5×12, 5×14, 5×20

Only Bergholtian tours are possible 5×12 and there are 4429 in all (Knuth). One example shown, a closed tour (Jelliss undated). The 5×14 is another board like the 5×6 and 5×10 and 7×10 that admit tours with Sulian axial symmetry. One example (Jelliss 2015). Finally a 5×20 Bergholtian closed tour of 100 cells (Jelliss 1988). Collings (1987) indicated how tours of any length 5×n can be constructed given components 5×5, 5×6, 5×7, 5×8.

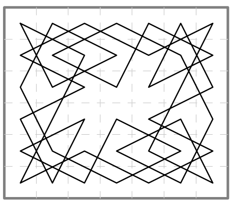
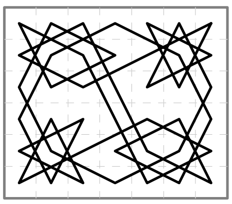
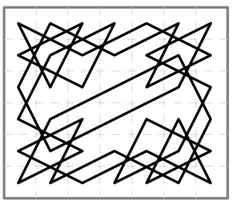
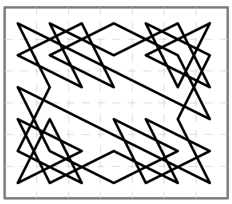



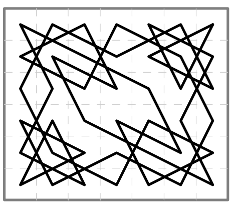
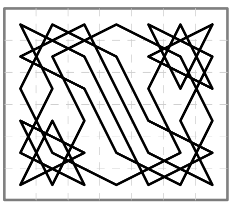
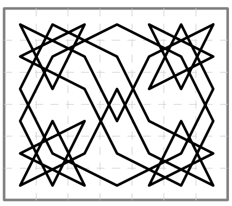


Six-Rank Boards

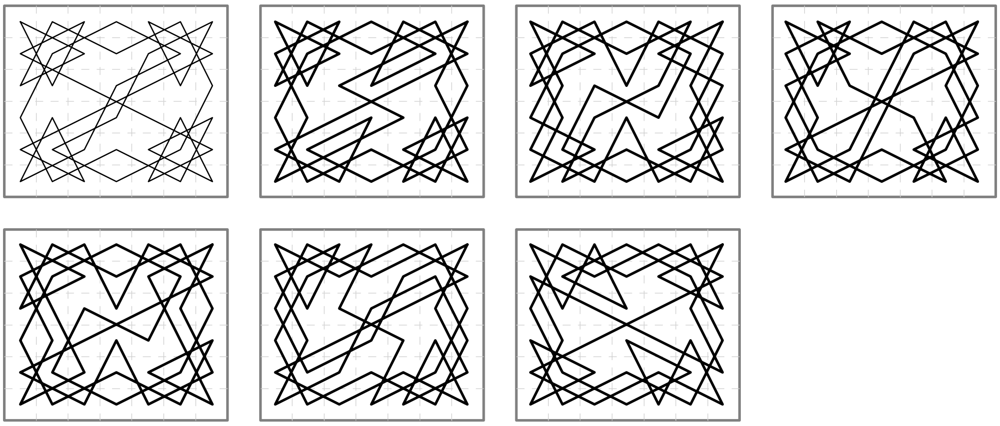
6×7

There are closed tours of all three symmetric types: Bergholtian (19), Eulerian (263), and Sulian (265) making 547 in all. Diagrams of all the Bergholtian tours are shown on the KTN website. We show seven of each type here. First, Eulerian (the first diagram is by Bergholt):

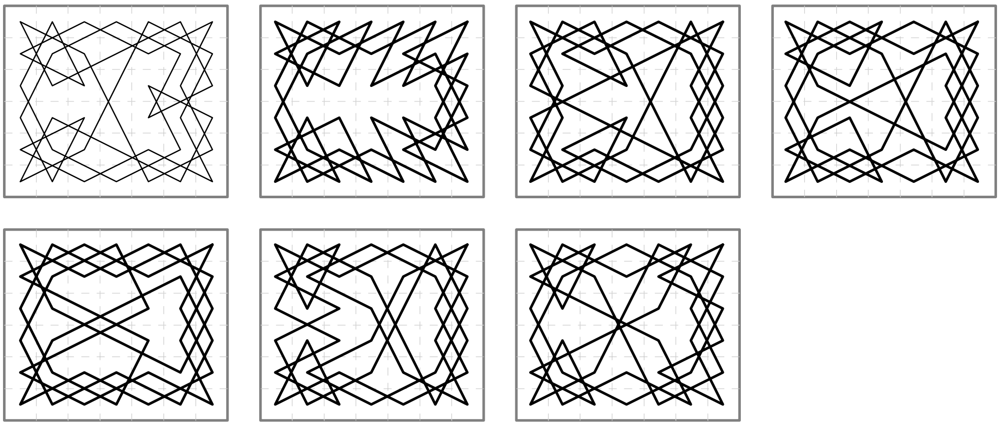





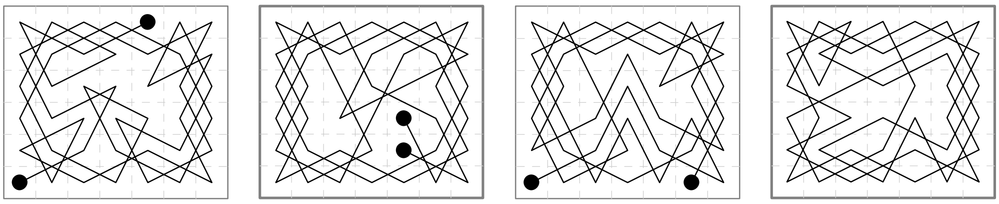
Bergholtian (the first is by Bergholt himself):



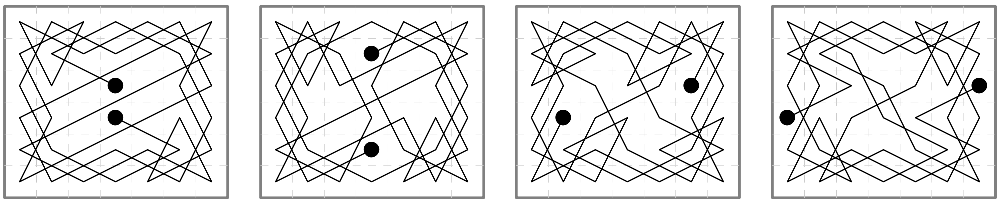
Sulian (the first is by Bergholt). The boards are shown here in the 6×7 orientation for reasons of space, but 7×6 orientation would show the Sulian axis vertical which makes the symmetry clearer.



Asymmetry is also possible of course. The first asymmetric open tour here is from Willis (1821) and the others from Warnsdorf (1823) and the closed tour from Warnsdorf (1858).



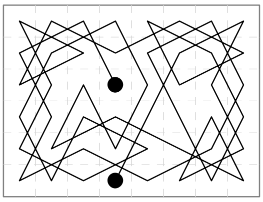
We also show four symmetric open tours. The total symmetric open tours is unknown, but there are 38 reentrant solutions (two from each of the 19 closed solutions of Bergholtian type).

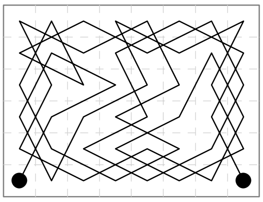


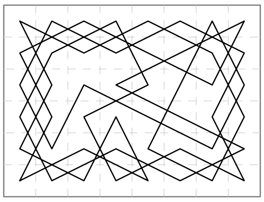
I find the patterns on this board particularly attractive and varied.

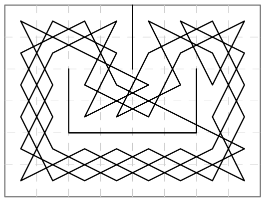
6×8

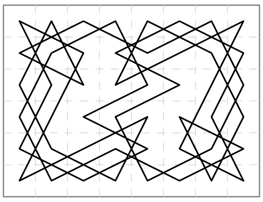
Open tour by Warnsdorf (1823) illustrating his rule, starting from d4 (reoriented). Open and closed tours by Willis (1821). The theory of straits and slants applies on this board since it divides into twelve 2×2 blocks. My asymmetric closed tour, by simple linking from 6×8 crosspatch (Jelliss 2018) has the minimum of four slants. Symmetric closed tours by Haldeman, Bergholt and Jelliss (2) require at least 6 slants. Only Eulerian symmetry is possible in closed knight tours on the 6×8 = 48-cell board, and there are 2817 of this type (according to D. E. Knuth). The asymmetric tours amount to over 10 million, so are beyond our recreational interest here.

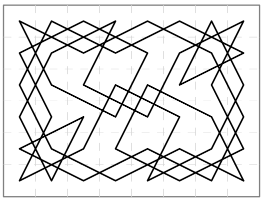

Warnsdorf 1823

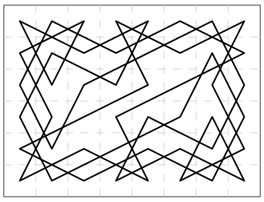

Willis 182

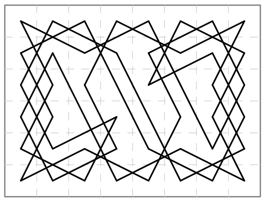

Willis 1821


Jelliss 2018

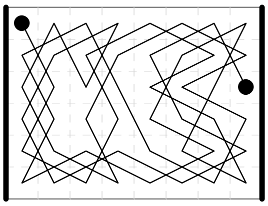

Haldeman 1864

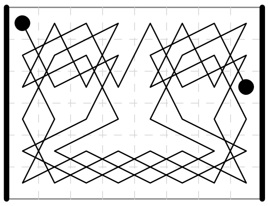

Bergholt 1918

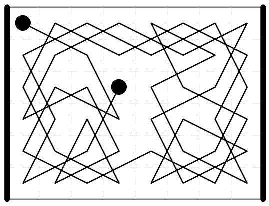

Jelliss

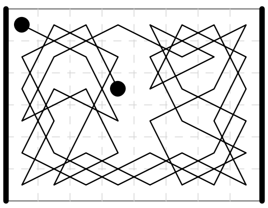

Jelliss

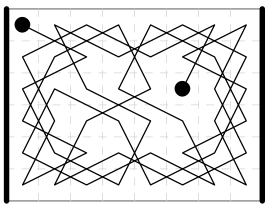
Semi-magic knight tours on 6×8 board. Jean-Charles Meyrignac (26 Aug 2003) reported the following results for quasi-magic tours on the 6×8 board, his program (total running time: 6 minutes) found the following five tours (it only searched for tours beginning in a corner). The ranks all add to 196. First tour file sums are alternately 121 and 173. Second tour files sums alternately 123 and 171. Third and fourth tours file sums are in alternate pairs of 145 and 149. Fifth tour file sums 6 of 153 and 2 of 129. The second tour is evidently of Beverley type.

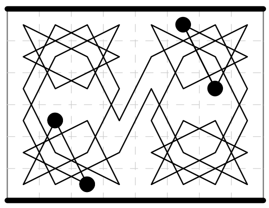










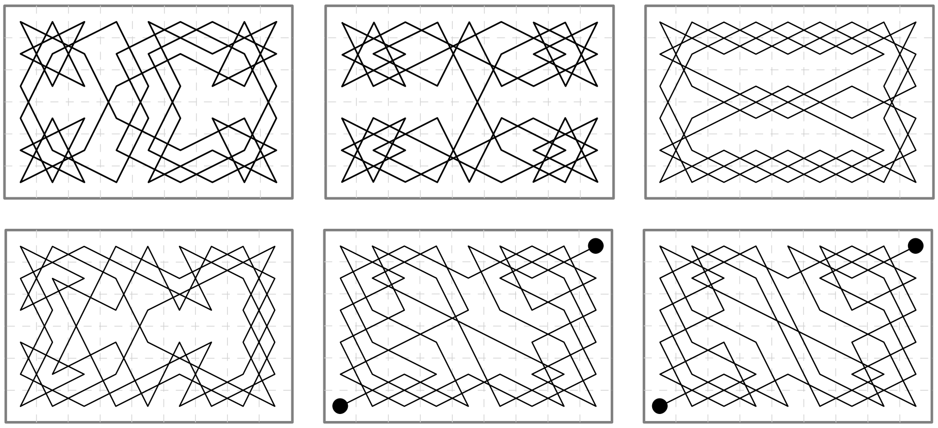


The sixth diagram shows a quasi-magic symmetric tour in which the files add to 147 and the ranks to 4 of 200 and 2 of 188 (with 1 at g4, or b3). I seem to have mislaid the source of this.

6×9

On the 6×9 = 54-cell board there are 10264 Sulian, 11105 Eulerian and 1862 Bergholtian symmetric tours (data from D. E. Knuth). A Sulian example by Murray (1942) and three by Jelliss (1994 and two 2015) are shown. My first example joins an asymmetric 3×9 open tour to a copy of itself. I'm not sure if a semi-magic tour of this type is possible. The ranks would add to 165 and the three smallest numbers in each rank must add to 42. One such set of triplets is:

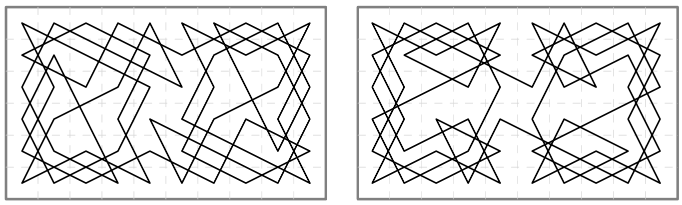
1 17 24, 2 13 27, 3 18 21, 4 12 26, 5 14 23, 6 16 20, 7 10 25, 8 15 19, 9 11 22.



The open tours (Jelliss 1994) are tridirectional and symmetric.

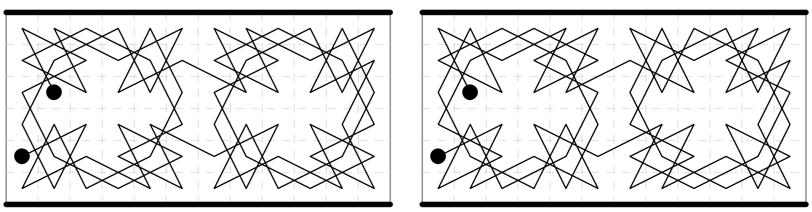
6×10

T. W. Marlow (17 Jan 1998) reported that by a computer search he found the number of geometrically different closed tours with 2-fold rotary symmetry on this board was **197064**. Example tours of 60 cells by Murray (1942) and Jelliss (1998).

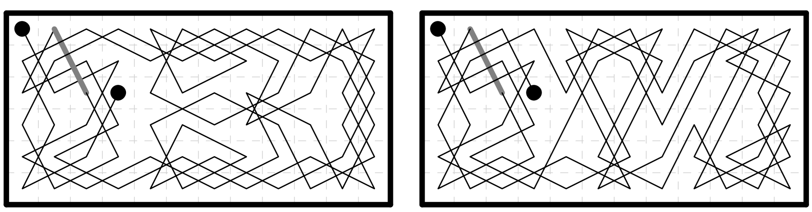


6×12

Semi-magic and magic! knight tours on 6×12 board. Here are two 72-cell, 6×12, quasi-magic tours constructed by joining together two of Awani Kumar's 6×6 semi-magic tours, suitably chosen. The 6-cell lines add to 219 and the 12-cell lines to 510 and 366. Similar tours can be constructed, joining 6×6 semi-magic tours end-to-end to any length 6k.



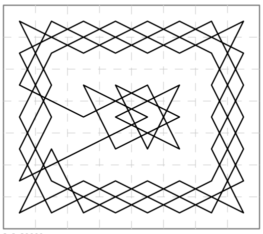
Stop Press! Awani Kumar has now (email 26 Jun 2018) constructed two fully magic tours on this board. The 12-cell lines add to 438.

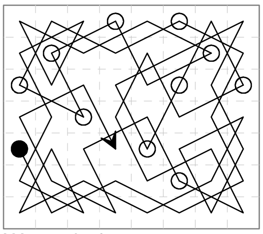


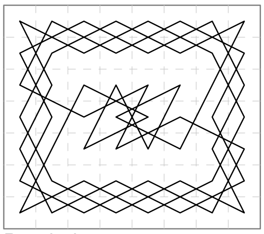
Seven-Rank Boards

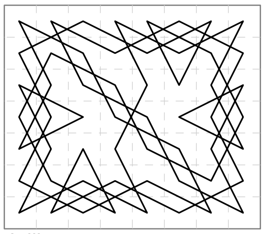
7×8

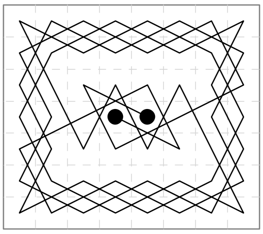
A closed tour from Willis (1821), he also gave a similar open tour. Then one from Warnsdorf (1823), where white circles mark ambiguous choices. Only Bergholtian symmetry in a closed tour is possible on this board and there are 10984 such tours in all [Knuth]. The earliest by Ernest Bergholt from *British Chess Magazine* 1918, and one of my own. Then a symmetric open tour with ends on the central cells. Then a piece-wise symmetric closed tour with constant difference 7 between numbers in diametrally opposite cells when numbered from any of the eight dotted cells. The other two diagrams are asymmetric closed tours by simple linking from the two crosspatch patterns.

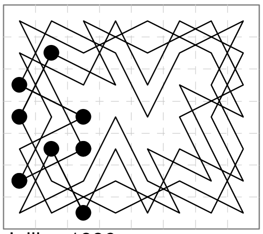

Willis 1821

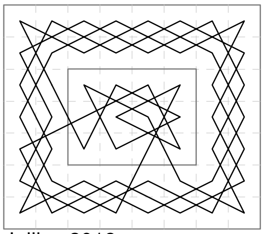

Warnsdorf 1823

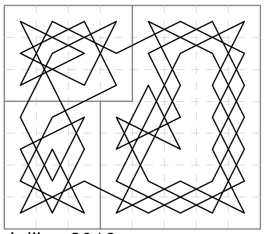

Bergholt 1918


Jelliss 1998


Jelliss 1998

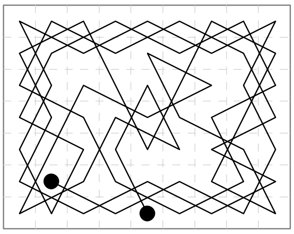

Jelliss 1996

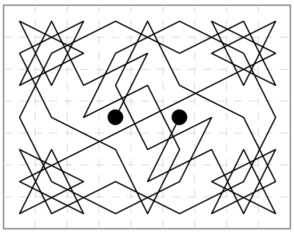

Jelliss 2018

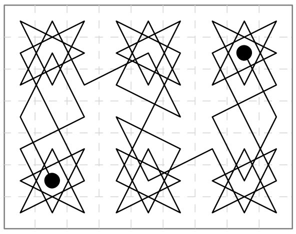

Jelliss 2018

7×9

Board 7×9. Asymmetric by Jaenisch (1862), symmetric by Murray (1942) and Jelliss (1998).

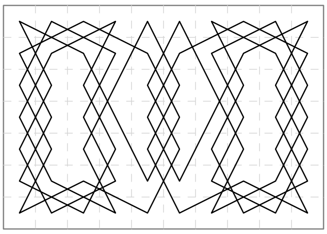


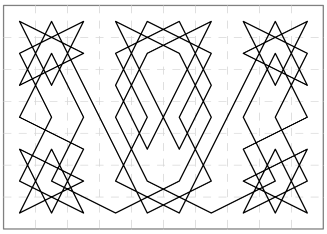


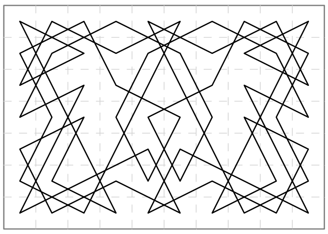


7×10

This is another board like the 7×6 and 5×14 above that admit tours with Sulian axial symmetry Three examples of 70 cells (Jelliss 2015).

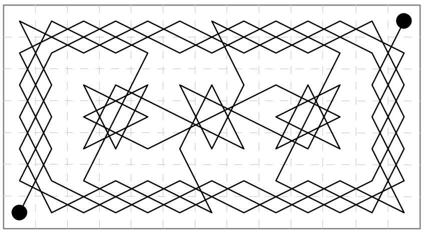






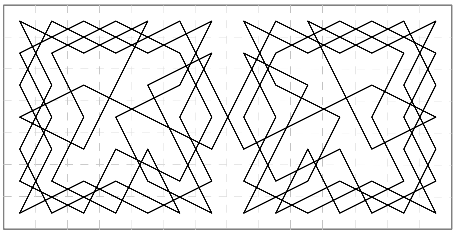
7×13

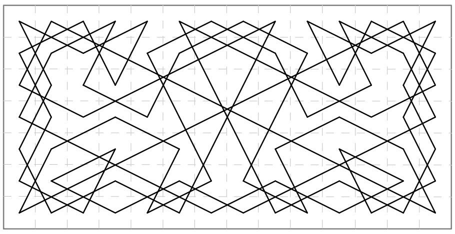
Symmetric open tour of 91 cells (Jelliss undated).



7×14

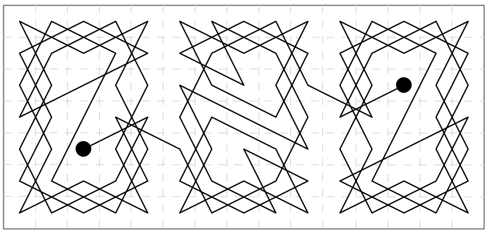
Sulian examples. A simple design is to join two tours 7×7.





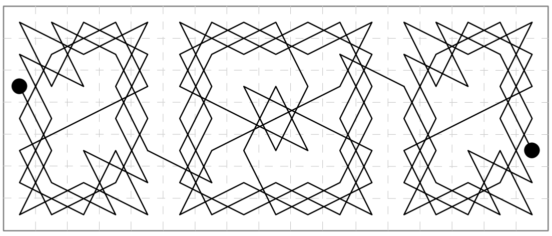
7×15

Board 7×15. A symmetric open tour of 105 cells (Jelliss 2015).



7×17

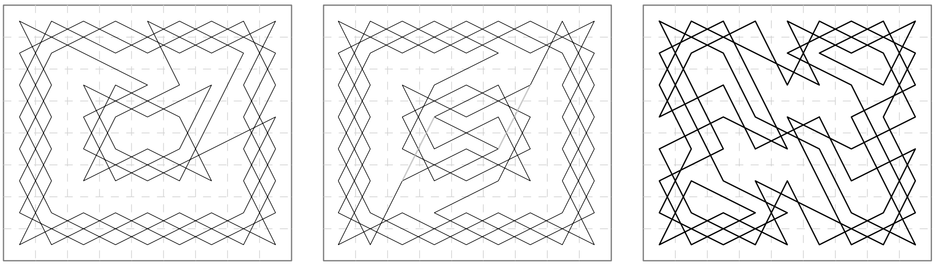
A symmetric open tour of 119 cells (Jelliss undated).



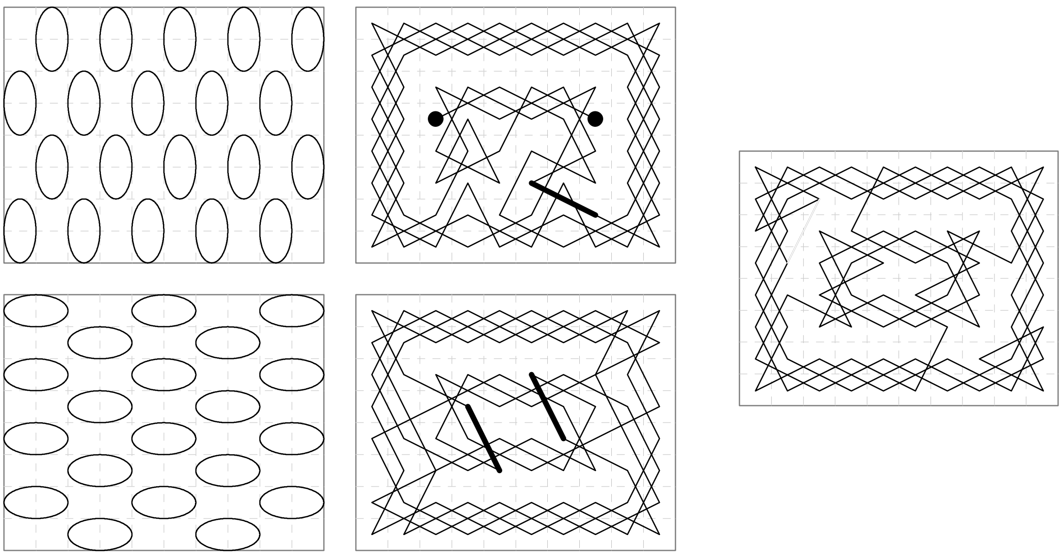
Eight-Rank Boards

8×9: On the 8×9 board the two concentric braids each consist of two equal strands (of 26 cells on the outer and 10 cells on the inner). So to join them by simple linking requires only four deletions and insertions, but the result is asymmetric as in the first diagram. The symmetric, Bergholtian, tour in the second diagram uses six deletions.

The third tour (Jelliss 2001) is a symmetric Celtic tour (i.e. with no size 1 triangles) and including a central move that is intersected five times (the maximum possible in a Celtic tour).

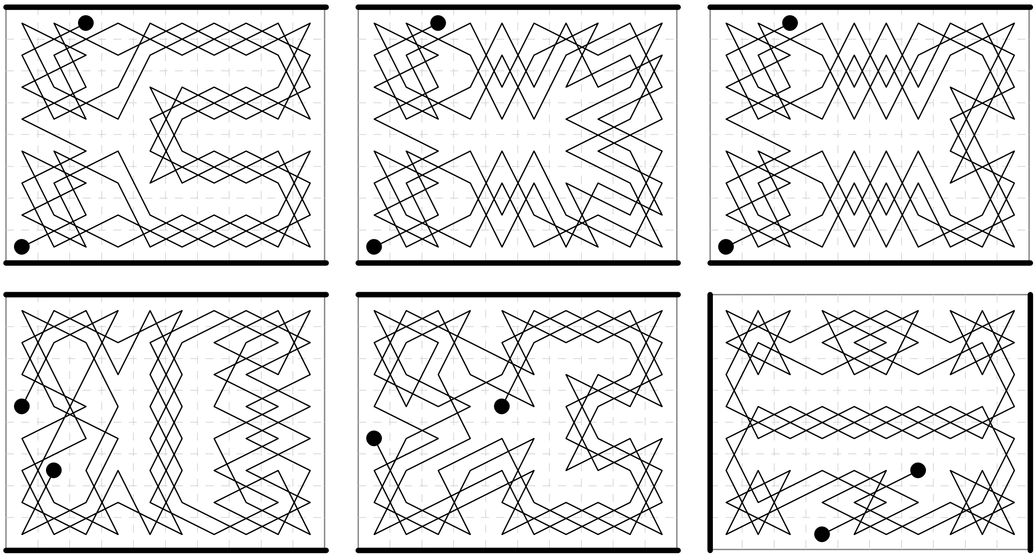


8×10: Two tours on this board appear in Ernest Bergholt's *Fifth Memorandum* (1 April 1916). “Explanation of a simple and general method applicable to all squares and rectangles having an even number of cells in both their sides.” This ‘domino method’ is in effect an extension of the method described by Sainte-Marie (1877) for the 4×8 board. The two groups of dominoes are toured separately, and a single link connects them (or two in the case of a closed tour). Bergholt notes that one path can be a half-turn of the other, and thus form a tour of Euler type, only when the dominoes are taken along a 4·n +2 side (as in the lower diagram) but not along a 4·n side.



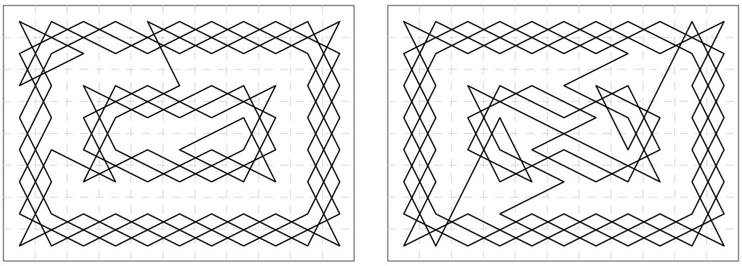
On this board the two concentric braids each consist of four equal strands (of 14 cells on the outer and 6 cells on the inner). So to join them by simple linking requires eight deletions and insertions. The symmetric (Eulerian) tour shown on the right (Jelliss) achieves this.

Semimagic tours 8×10: I have constructed nine tours (three typical examples shown here) based on extending the braid in the Beverley type tours on the 8×8 to cover the extra two files. They all add to 324 in the files, as required in a magic tour, but the ranks sum to two alternating values, making them quasi-magic. The first has the sums 365 and 445, the second 395 and 415, the third 393 and 417. Then two others with file sum 324. The rank sums are 445, 365 and 361,449.



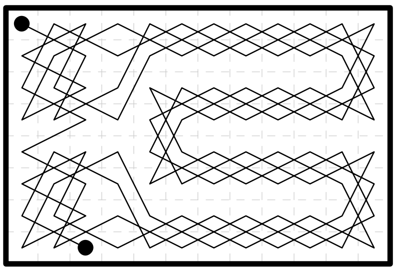
The sixth example is constructed by the ‘lozenge’ method that I found for 12×12 magic tours, but due to the limitations of this board the result is only quasi-magic. The 10-cell lines add to 405 (consisting mainly of 5 pairs adding to 81). The 8-cell lines add to 364 and 284 (the magic constant would be 324). [Some results appeared in *The Games and Puzzles Journal*, issue 26, April 2003.]

8×11: On this board of 88 cells the number of strands in each braid reverts to two, as for the 8×9 board (30 cells in the outer and 14 cells in the inner) so simple linking requires four deletions as in the asymmetric tour on the left. The symmetric Bergholtian tour uses eight deletions.

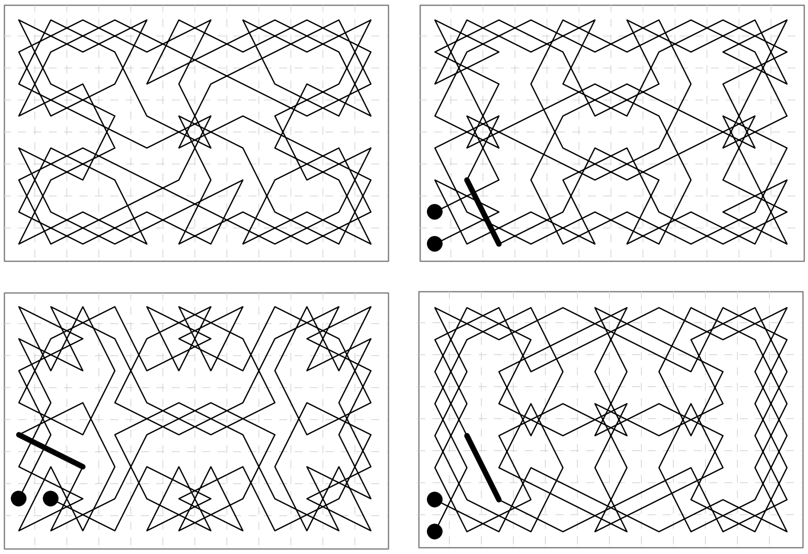


8×12: Although it is known that magic knight tours are possible on all 4·h × 4·k boards 8×8 and larger (e.g. by braid extension of 8×8 magic tours), the only non-square example that has actually been published as far as I know is this one, a simple braid extension of Beverley's tour (p.347), that I gave in *Variant Chess* 1992 (vol.1 #8 p.105) just to show it is possible. File sum 388, Rank sum 582.

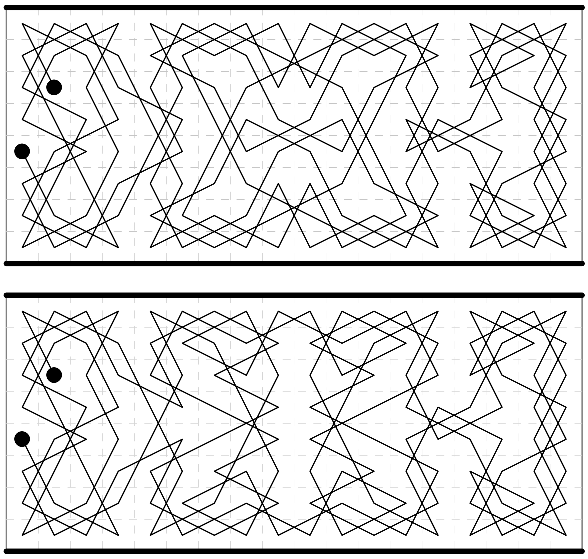
1	46	71	76	5	44	67	78	7	42	65	80
72	75	2	45	68	77	6	43	66	79	8	41
47	70	73	4	37	12	83	62	39	10	81	64
74	3	48	69	84	61	38	11	82	63	40	9
49	94	23	28	13	36	59	86	15	34	57	88
24	27	50	93	60	85	14	35	58	87	16	33
95	22	25	52	29	20	91	54	31	18	89	56
26	51	96	21	92	53	30	19	90	55	32	17



The symmetric closed tour below of 96 cells is by Murray 1942. The three open tour examples of near biaxial symmetry are by Victor Gorgias from the *Dubuque Chess Journal* 1871.

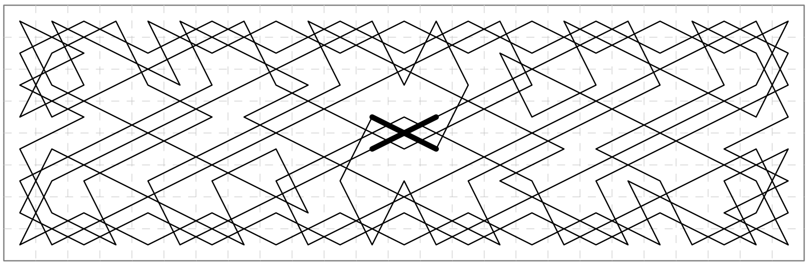


8×18: These two quasi-magic tours of 144 cells (Jelliss 24 April 2003) were constructed by splitting the 8×8 magic tour 00b in two and joining up the loose ends by four paths in direct quaternary symmetry. In memory of H. S. M. Coxeter (9 Feb 1907 - 31 Mar 2003).



The files all add to 580. The ranks add to 1269 or 1341, each occurring 4 times.

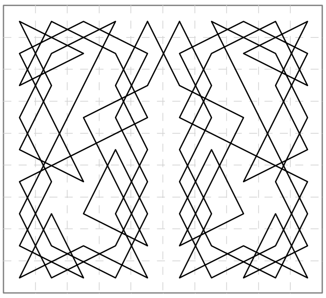
8×25: Bergholtian 200-cell tour (Jelliss, Jan 1988) for Australian bicentenary year.

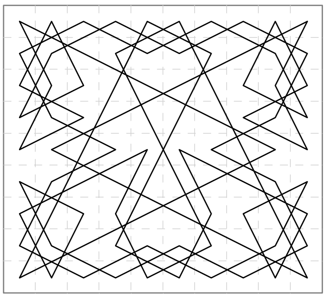


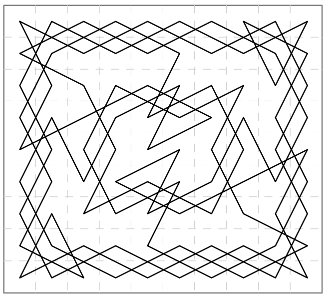
A closed centrosymmetric tour on any board 4·h × (2·k + 1) <u>must</u> cross in the centre.

Larger Oblongs

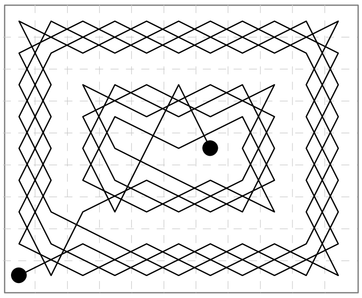
9×10: This board (being odd by singly-even) allows Sulian axial symmetry. The first example is from Kraitchik 1927. The second (Jelliss 2015) shows three consecutive four-move lines, twice. The border braids are formed of four equal strands (16 or 14 cells outer, 8 or 6 inner), so eight deletions are necessary for simple linking. The shorter circuits are symmetric, so for a symmetric tour two deletions must be made in each of these, making a minimum of 12 deletions; this is achieved in the example shown (Jelliss 1999). There are two cells in the centre that also have to be joined into the tour, the two moves through these can be treated like a single insertion.

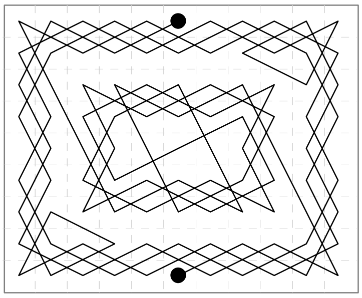




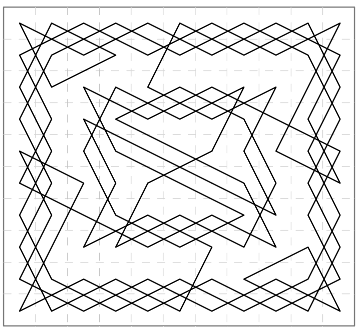


9×11: There are two strands in the border braids (48 and 24 outer, 16 and 8 inner) and three central cells. Each strand is in biaxial symmetry. Asymmetric and symmetric examples (Jelliss 1999).

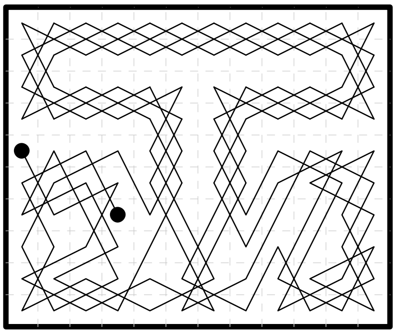




10×11: There are four strands in each braid (18 or 16 outer, 10 or 8 inner), two being centro-symmetric, and there are 2×3 central cells to be joined in. Thus twelve deletions at least for a symmetric tour. The example, Bergholtian, uses 14 deletions (Jelliss 1999)

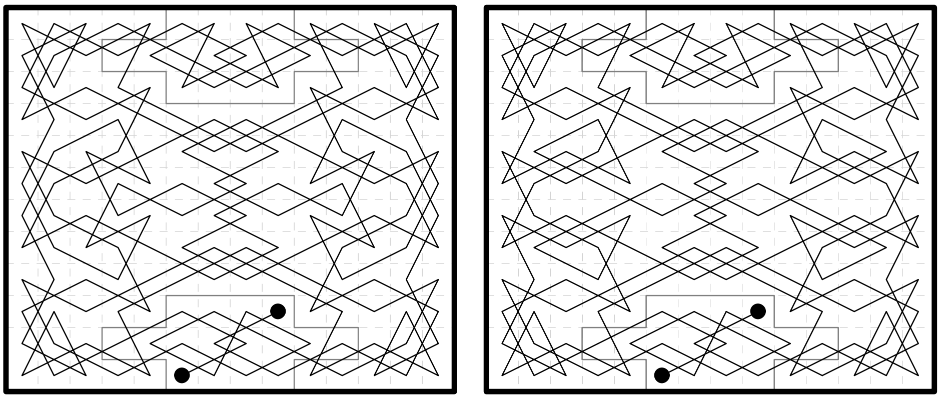


10×12: Magic tour formed by extending one of Awani Kumar's 6×12 magic tours with a braid. It probably works for the other 6×12 tour as well. The file and rank sums are 605 and 726.



11×13: Tours on boards of this size will be found under Space Chess in ⌘ 11 where they form the floors of a 7-storey 1001-cell Sheherazade tour.

12×14: The construction of two magic tours on this rectangle (Jelliss 2011) uses a method developed for the 12×12 board. They differ from biaxial symmetry only in the two marked regions.

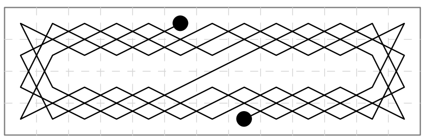


‘A Magic Tour on a 12×14 Board’ on the KTN website and ‘Proving The Possibility’ Jeepyjay Diary pages. This is a board (4·m)×(4·n + 2).

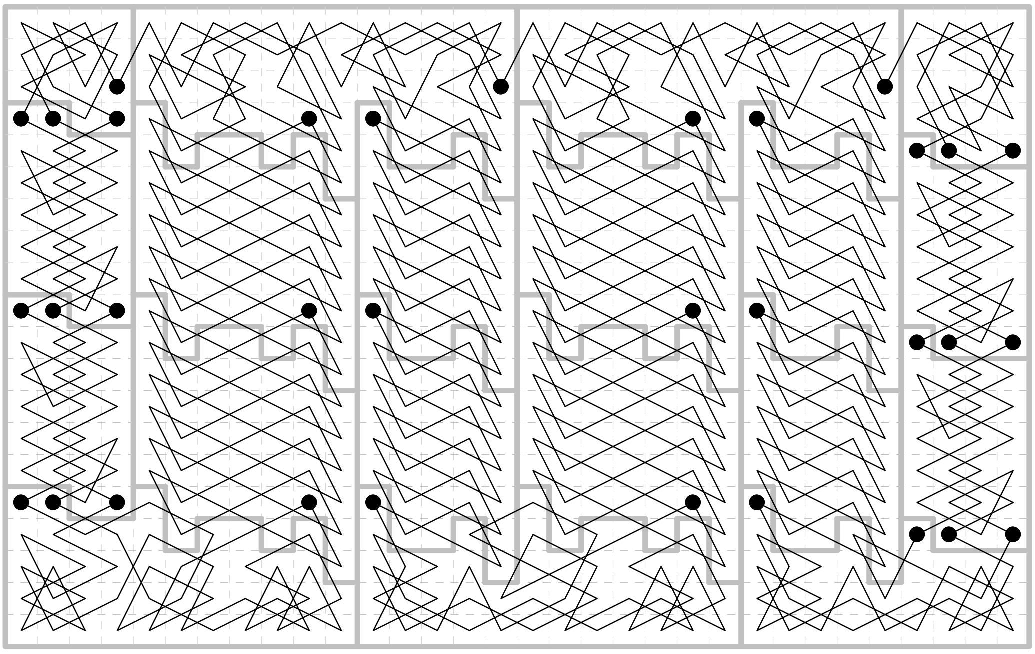
17×19: Tours on boards of this size will be found under Space Chess (p.729) where they form the floors of a 3-storey 969 cell ‘Methuselah Tour’.

Puzzle Solution

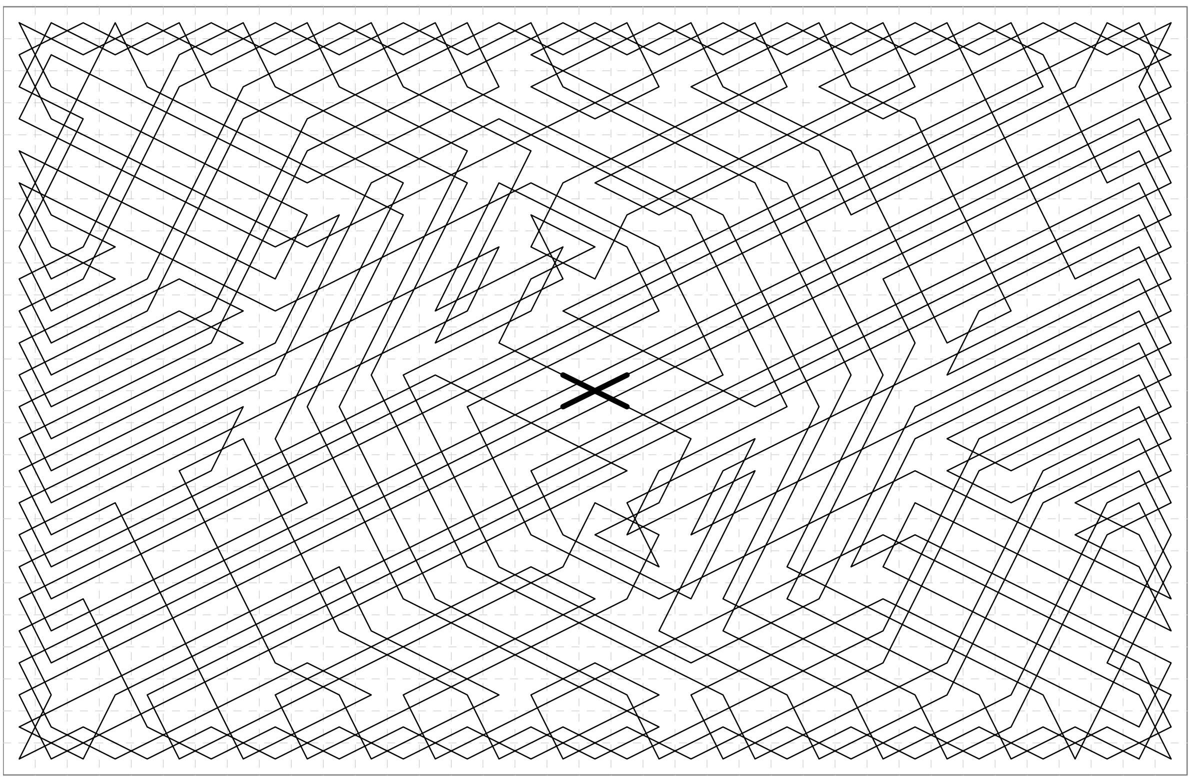
PUZZLE (p.48): **The 4×13 Card Pack Tour.** The sequence of the cards along the ranks runs: 4 3 5 2 6 A 7 K 8 Q 9 J X (where X is 10). A simple edge-hugging tour.



20×32: This 640-cell expandable tour by Pierre Dehornoy (2003) shows how to construct a tour with most moves in two directions. The board can be expanded lengthwise in units of 12 and vertically in units of 6 by duplicating edge and central components. See also his 16×16 tour p.628.



24×37: An 888-cell tour (Jelliss 1985) with oblique binary symmetry of Bergholtian type.



This is mainly in the form of a patchwork of areas each of which exhibits one of the eight possible patterns in which an area of board can be covered by straight lines of knight moves, one passing through each cell (see ⌘ 1) This was my 1985/6 Christmas/New Year card.