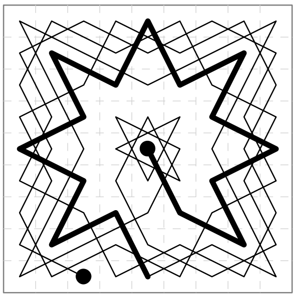
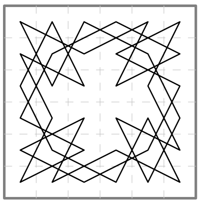
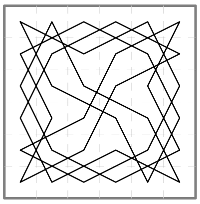
⌘ 5

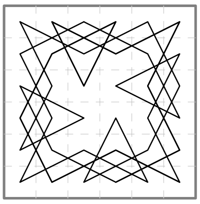
Odd & Oddly Even Boards

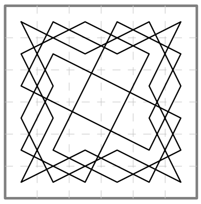


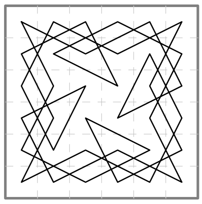
by G. P. Jelliss



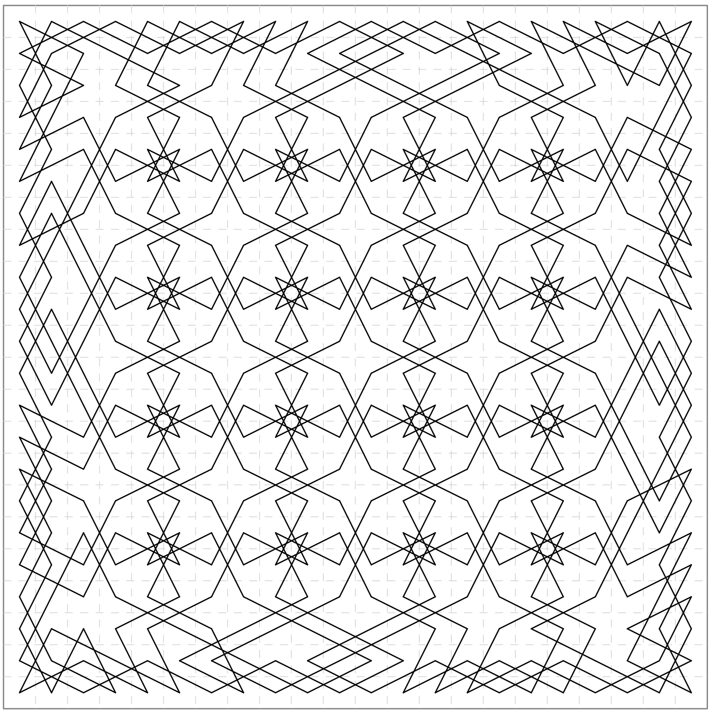








2019



==

Title Page Illustrations:

A **9×9** pictorial and figured tour (Jelliss 1997).

The five tours with birotary symmetry on the **6×6** board (Paul de Hijo 1882).

22×22. Tour (Jelliss Oct 2017) with oblique quaternary symmetry using a repeating pattern in the central area shown by Sharp (1925).

Contents

© George Peter.Jelliss 2019
http://www.mayhematics.com/

Knight's Tour Notes, Volume 5, Odd and Oddly Even Boards
If cited in other works please give due acknowledgment of the source as for a normal book.

==

Odd Square Boards

==

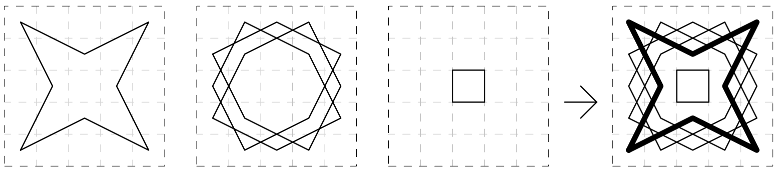
The 5×5 Board

==

This is the smallest knight-tourable square board. Classified by separation of end-points there are 8 types of 5×5 tour. The numbers of each type are: {4,4} 18, {0,4} 30, {3,3} 6, {2,4} 14, {2,2} 8, {1,3} 14, {1,1} 8, {0,2} 14. We diagram all 112 tours in this section.

If you prefer larger numbers and wish to count the different diagrams possible, then each symmetric tour can be viewed in 4 distinguishable orientations, and each asymmetric tour in 8, thus the total becomes: 8×4 + (112 – 8)×8 = 864. If further you wish to present the tours in numerical or directed form this total has to be doubled since each tour can be numbered from either end, giving the total 1728. This was the total given by Charles Planck in his ‘Chessboard Puzzles’ series in *Chess Amateur* (¶25 and ¶26, Dec 1908 p.83 and Feb 1909 p.147). This is the earliest reference I have found to this enumeration.

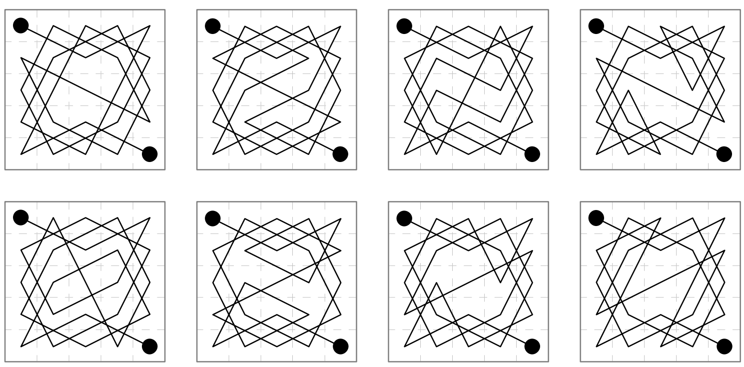
The moves through the corner cells form a circuit of 8. It follows that every 5×5 tour must have one end on a corner cell, since at least one of these corner paths must be disrupted. The other end can be any other cell of the same colour. Euler (1759) was the first to publish examples and he noted that every tour must start or end on a corner cell. The moves through the other cells, excluding the centre, form a circuit of 16. Together these circuits make a pseudotour of the centreless 5×5 board.



In tours the moves through the corners can form three different patterns: (A) two pieces, each of 4 cells, (B) two pieces, of 6 and 2 cells, (C) one path of 8 cells. The way the 16-move circuit is broken corresponds to whether the path through the centre is (1) straight, two pieces 8 + 8, (2) acute, 6 + 10, (3) right, 4 + 12, (4) obtuse, 2 + 14, or (5) terminates at the centre, one piece. Combining these classifications we get 13 classes: A1, A2, A3, A4; B1, B2, B3, B4; C1, C2, C3, C4, C5.

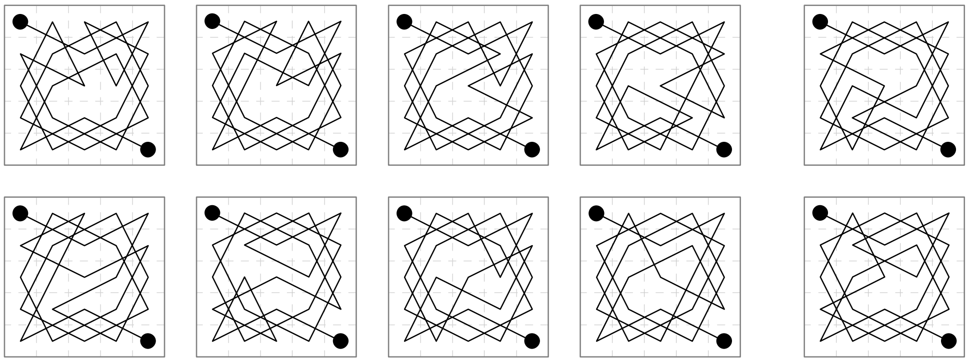
The A classes comprise the 18 corner-to-opposite-corner tours {4,4}.

Class A1 consists of the eight symmetric tours, with straight through centre:



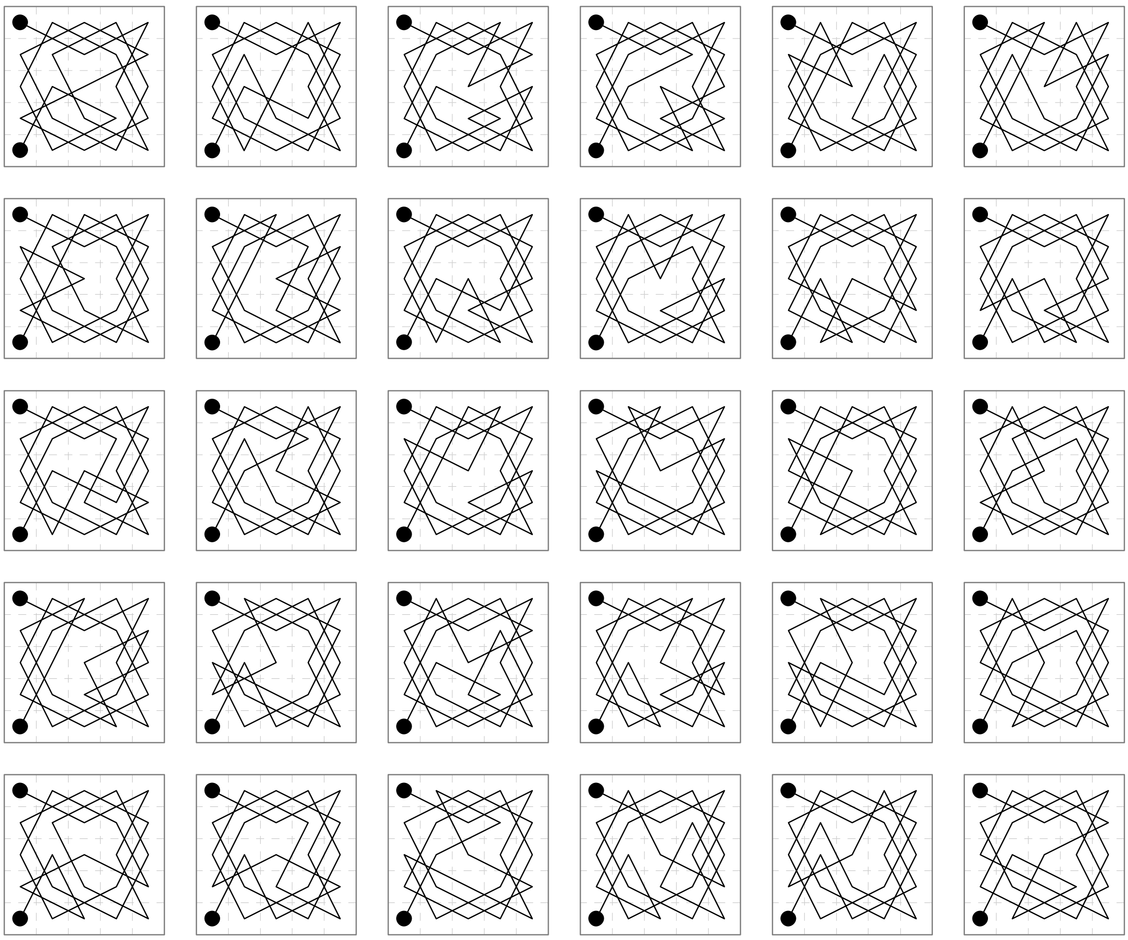
These were all diagrammed, in numerical form, in Euler's paper of 1759, and he gave formulae for 34 others. We show them here in the same sequence as Euler but inverted in orientation. Murray (1942) noted that the maximum number of two-unit lines in a 5×5 tour is three, and there are two such tours, first and last in this set.

Classes A2, A3, A4 are also corner-to-opposite-corner tours: A2 four with acute centre. A3 two with right-angled centre. A4 four with obtuse centre.



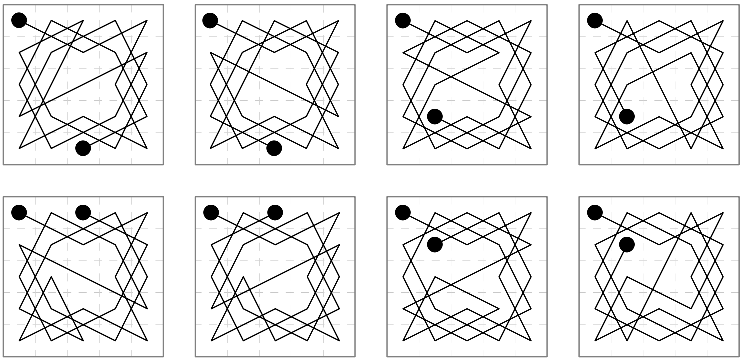
The B classes comprise 30 tours from corner to adjacent corner {0,4}.

B1 straight centre 2 tours. B2 acute centre 8 tours. B3 right-angled centre 12 tours. B4 obtuse centre 8 tours.

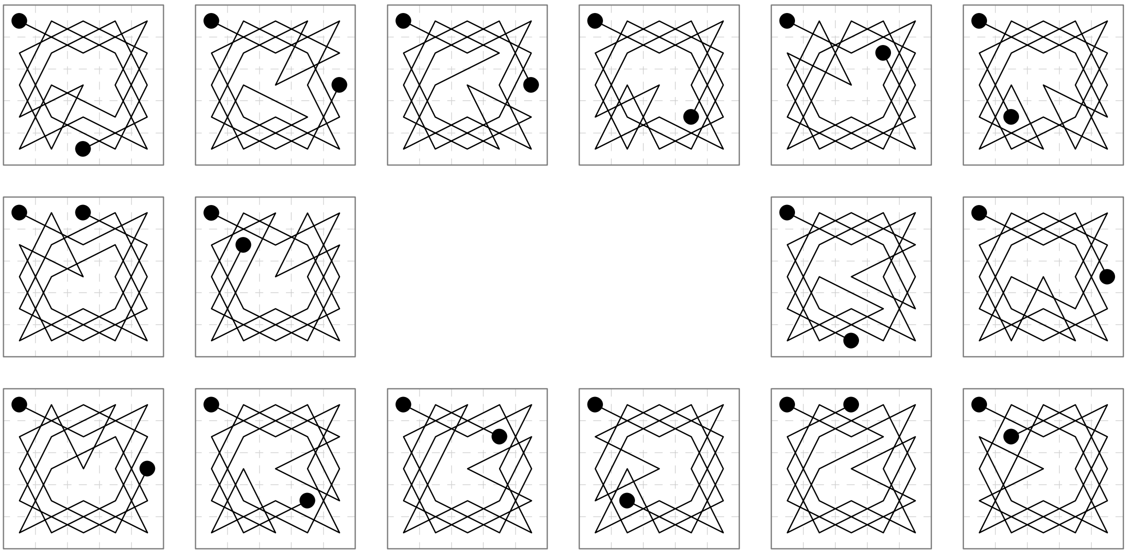


The C Class comprises those tours with one end point not in a corner.

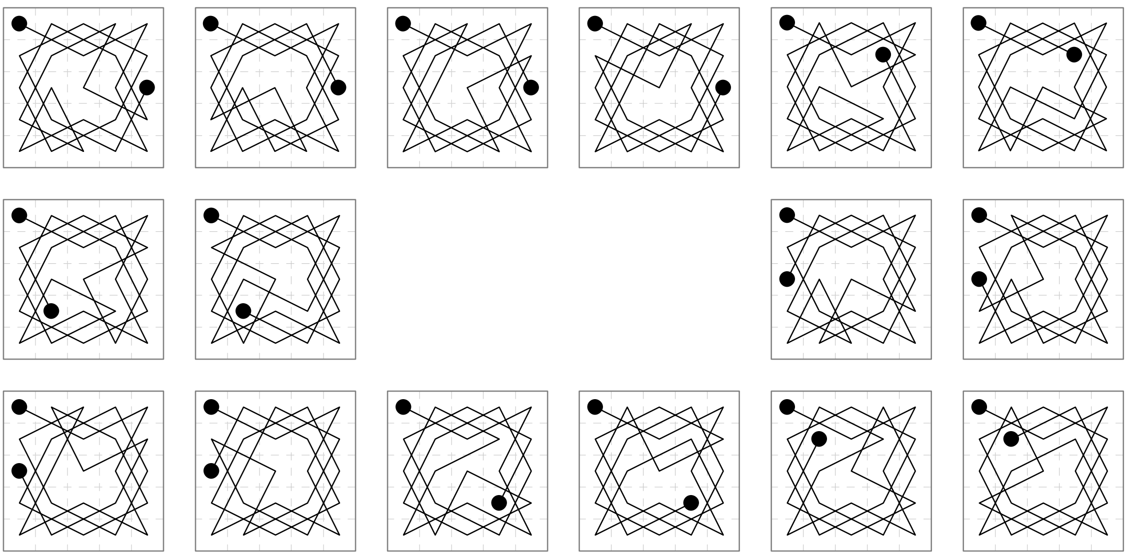
C1 has 8 tours with straight move through centre.



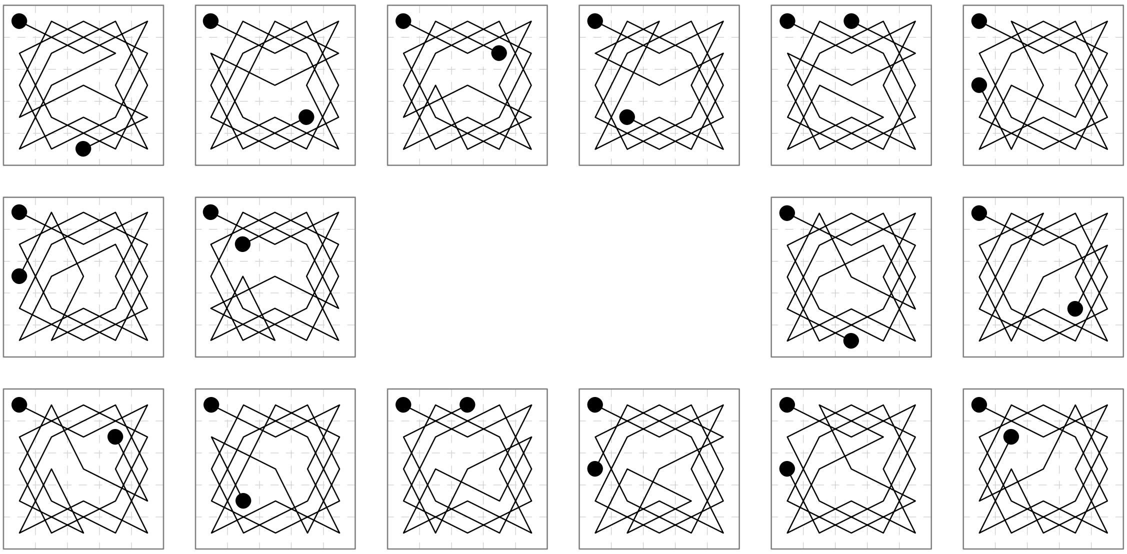
C2 has 16 tours with acute centre, 8 diagonal and 8 lateral:



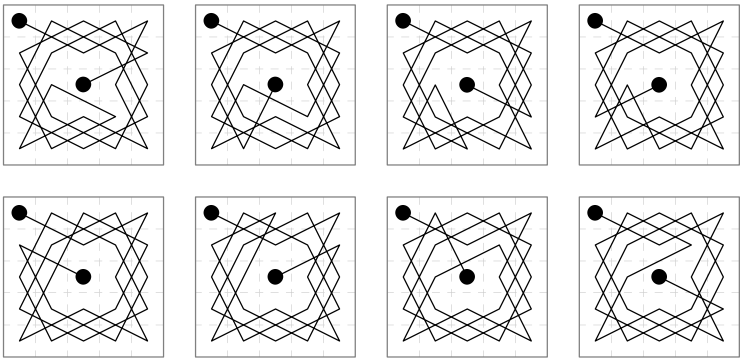
C3 has 16 tours with right-angled centre.



C4 has 16 tours with obtuse angle at centre, 8 lateral and 8 diagonal:



C5 has 8 tours terminating at the centre:



EXERCISE: Complete the following completely defined 5×5 tours:

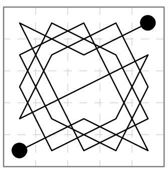
(1) a1...a4-c3. (2) a1-c2...b2 with central moves forming angles (a) diagonal-acute (b) orthogonal-acute (c) right angle containing b2 (d) right angle excluding b2 (e) orthogonal-obtuse (f) diagonal obtuse (g) straight line. (3) a1-c2... d4 with (a)–(f) as in (2).

(4) a1 ... e3 with centre angle (a) orthogonal-obtuse (b) diagonal-obtuse.

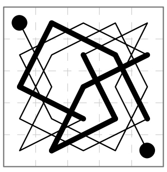
(5) a1 ... c1 with centre angle (a) diagonal-acute (b) orthogonal-acute.

(6) Identify which are ‘tours of inspection’, i.e. they ‘pass through’ every unit square — not only the squares of the board but also the squares whose corners are the centres of the cells.

Euler's first symmetric 5×5 tour has arithmetic progression 1, 7, 13, 19, 25 along the diagonal. This makes it a Figured Tour. We show similar results on the larger odd boards.

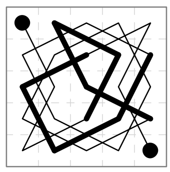


23	18	5	10	**25**
6	11	24	**19**	14
17	22	**13**	4	9
12	**7**	2	15	20
1	16	21	8	3



1	6	17	12	23
16	11	22	**7**	18
5	2	**19**	24	13
10	15	4	**21**	8
3	20	9	14	25

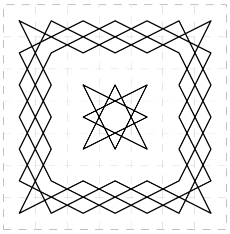
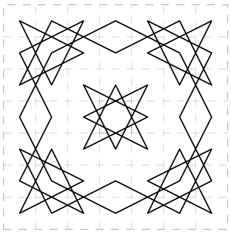
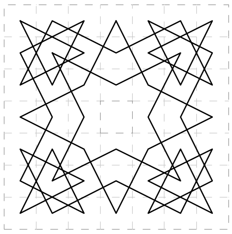
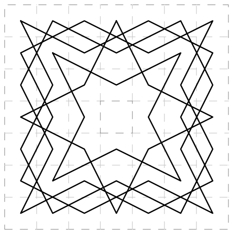
1	6	17	12	23
16	11	22	**5**	18
21	2	**7**	24	13
10	15	4	**19**	8
3	20	9	14	25



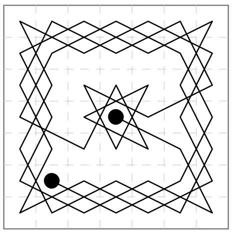
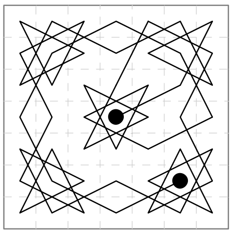
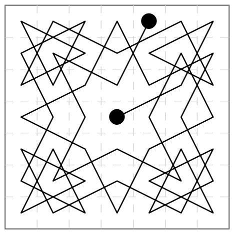
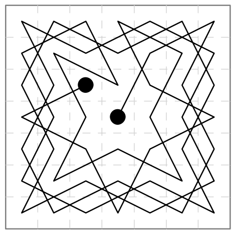
The other 5×5 pair of tours (Jelliss 2000) are a new type of figured tour in which four numbers 5-7-19-21 can be permuted but still form a knight tour!

The 7×7 Board

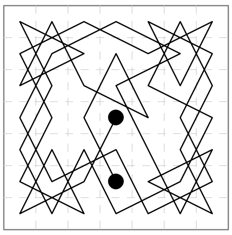
There are four octonary pseudotours, omitting the centre cell, formed of three circuits. The pairs differ according to the direction of the 7-4 move. 1-2 + 5-6 + 3-8-7-4-9 and 1-5 + 2-6 + 3-8-7-4-9.

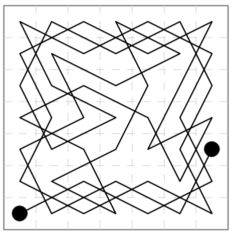





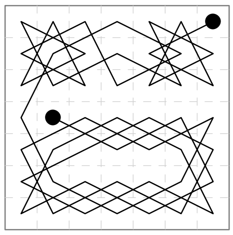
The four asymmetric tours below are derived from the above 7×7 octonary pseudotours with minimum deletions. The first is by Dudeney (cited by Murray). The other three are my own.

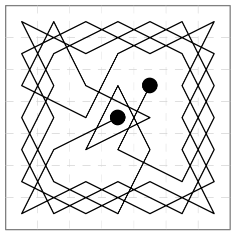





The next four asymmetric tours are from Willis (1821), Jaenisch (1862), Kraitchik (1927) and Dickins (1981). Other compartmental tours consisting of linked 3×7 and 4×7 tours are easily formed.


Willis 1821

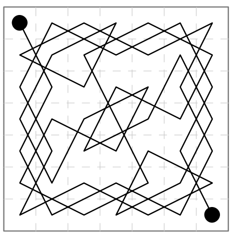

Jaenisch 1862

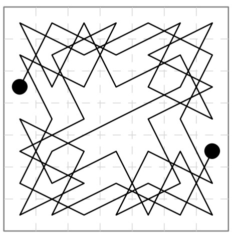

Kraitchik 1927

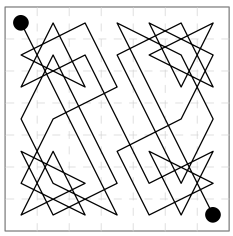

Dickins 1981

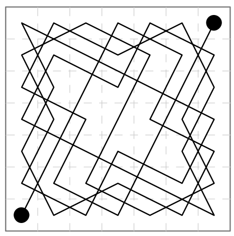
A tour between any two cells of the majority colour is possible. To show all geometrically distinct cases of end-point placement would require 50 diagrams.

We now show some assorted centro-symmetric tours. One each from Jaenisch (1862), Bergholt (1916), Dudeney (cited by Murray but without date), and Murray (1942). The Bergholt shows maximum birotary symmetry in a symmetric open tour (deleting b3-d4-f5 and inserting a5-b3, f5-g3 converts it into a centreless tour with 90° rotational symmetry.)

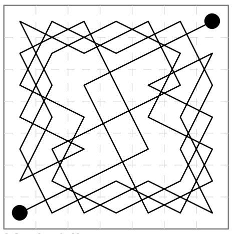

Jaenisch 1862

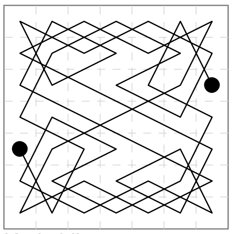

Bergholt 1916

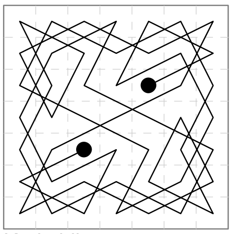

Dudeney

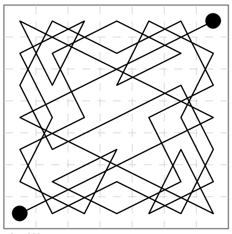

Murray 1942

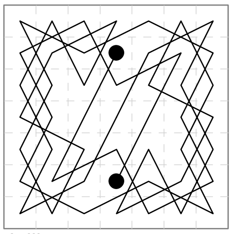
Now three from Kraitchik (1927), and three of my own, one early and two later composed so that, examples of all five symmetric end-point positions could be shown.

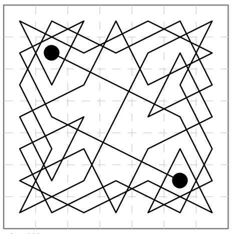

Kraitchik 1927


Kraitchik 1927

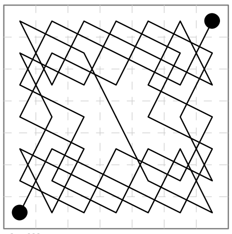

Kraitchik 1927

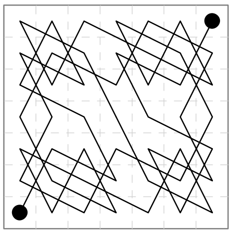

Jelliss 1981

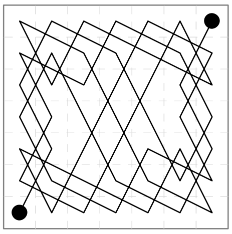

Jelliss 2004

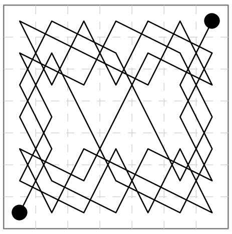

Jelliss 2004

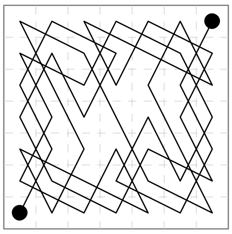
These eight are symmetric tridirectional tours, out of 28 I found on this board. This problem was suggested to me by Donald Knuth in a letter dated 8 March 1994.

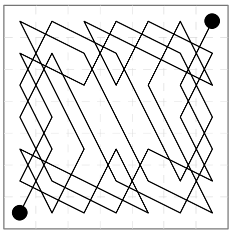

Jelliss 1994

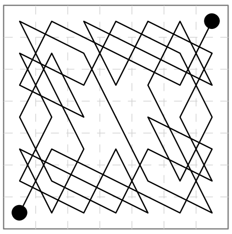


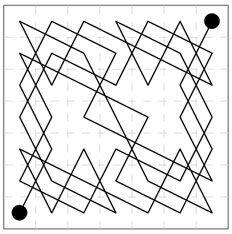






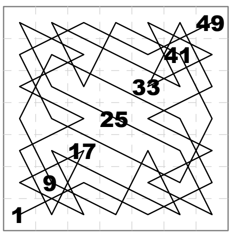






For other tridirectional tours see the 3×4k, 5×11, 6×9 boards in ⌘ 4.

My article on ‘Figured Tours’ in *Mathematical Spectrum* (1992/3) included a symmetric open tour 7×7 with arithmetical progression with CD 8 along a diagonal, analogous to the 5×5 with CD 6 shown by Euler (see p.6).

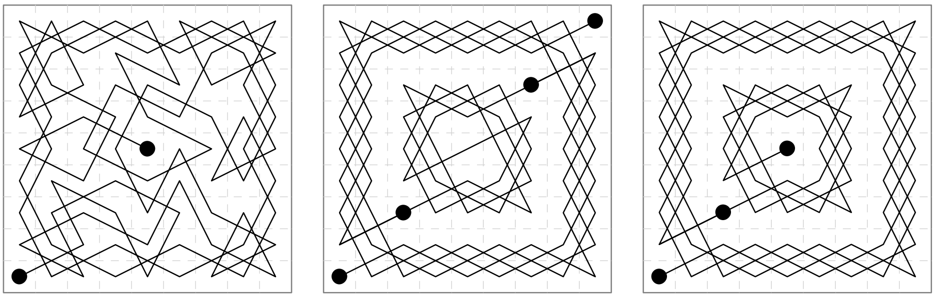


===

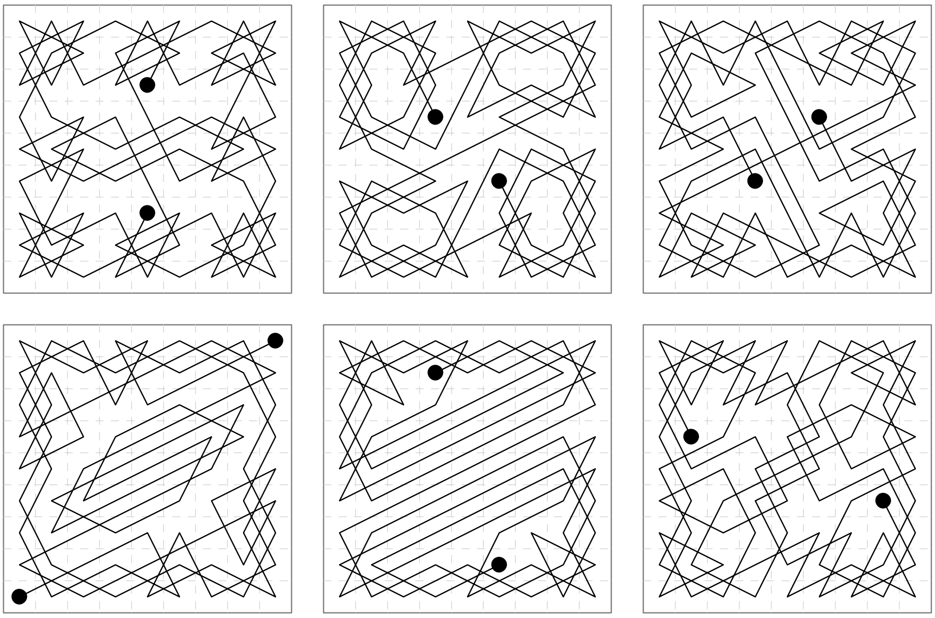
The 9×9 Board

===

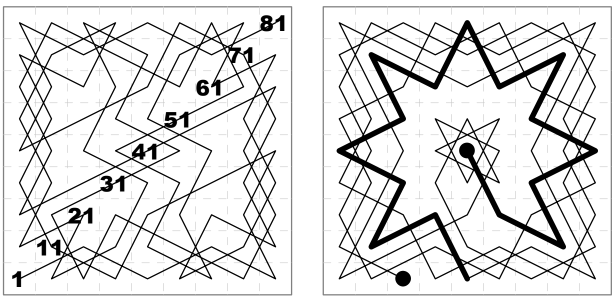
Much the earliest tour known on this board is this corner to centre example by Chapais (1780), his Figure 29 [diagram supplied by Herbert Bastian]. The next work I know of was over 100 years later, by Lucas (1894), shown by the other two examples. These incorporate a central 5×5 tour. On the 9×9 the border braid consists of two strands of unequal length, one of 16 and the other of 40 cells. The symmetric example is easily extended to 13×13, 17×17 and so on.



Here are six varied symmetric tours of my own construction.



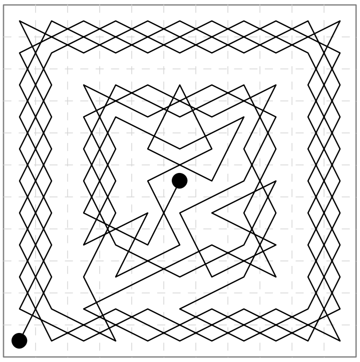
A symmetric open figured tour 9×9 showing AP with CD 10 from *Math Spectrum* (1992/3).



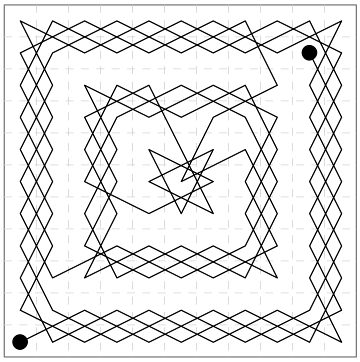
The open tour 9×9 with a starburst was set as a puzzle in my booklet on *Figured Tours* (1997). When numbered from the centre it has even and odd numbers circling the centre cell.

The 11×11 Board

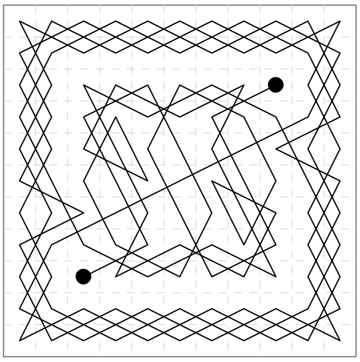
Here are examples from Chapais (1780 Fig.31), Lucas (1894), Kraitchik (1927) and Murray (1942). These are all based on the border method. The braid consists of two unequal strands.



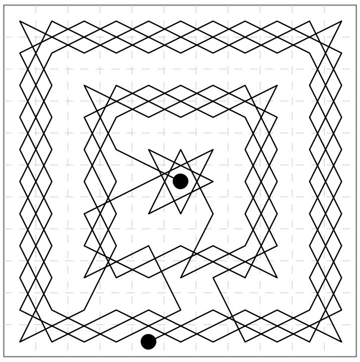
Chapais 1780



Lucas 1894



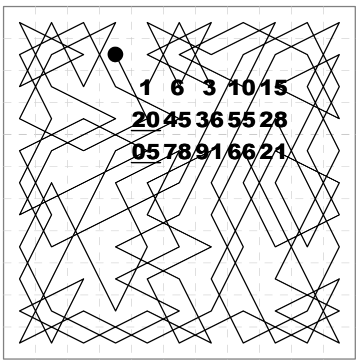
Kraitchik 1927

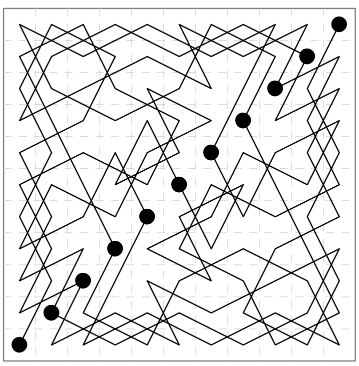


Murray 1942

The Murray example shows an "expanding tour on the 4n + 3 boards" (i.e. sides 7, 11, 15, 19 and so on, starting from the Dudeney 7×7 example).

This open tour 11×11 was set as a puzzle in my booklet on *Figured Tours* (1997). It shows the triangular numbers in a rectangle. Taken in sequence they follow a rook tour of alternating 2 and 1 steps. Add 100 to the underlined numbers.





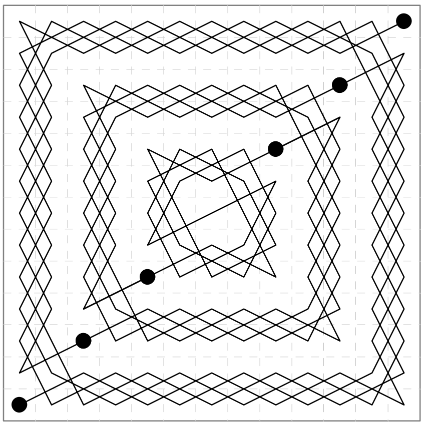
The dots in the symmetric tour (Jelliss 20 Apr 2019) represent the numbers 1, 13, 25, 37, 49, 61, 73, 85. 97, 109, 121 forming the arithmetic progression with common difference 12. This is similar to the 5×5 and 7×7 and 9×9 figured tours shown in the previous sections, but has the extra constraint that none of the linkage paths cross the diagonal.

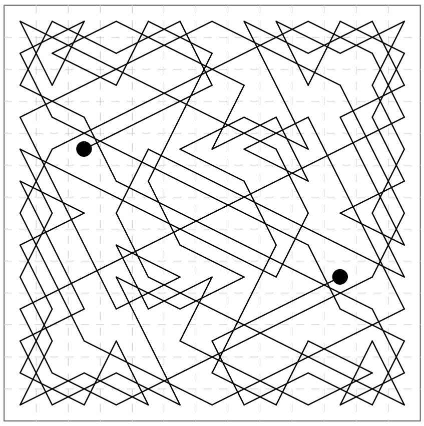
==

The 13×13 Board

==

The following tour by Lucas (1894) solves all cases 5×5, 9×9, 13×13 and easily extends to 17×17, 21×21, and so on by adding similar borders. My example was constructed beginning with the long line through the centre, other cells being gradually and somewhat randomly joined in.



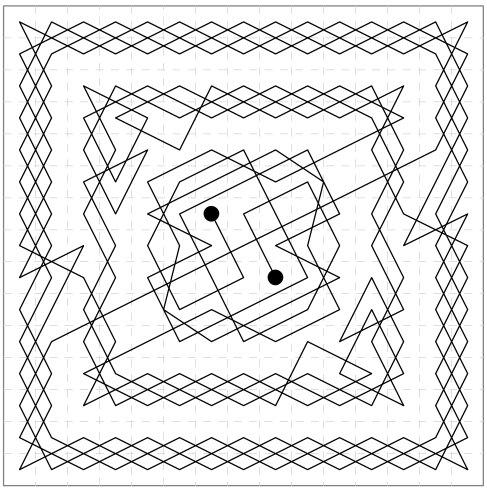


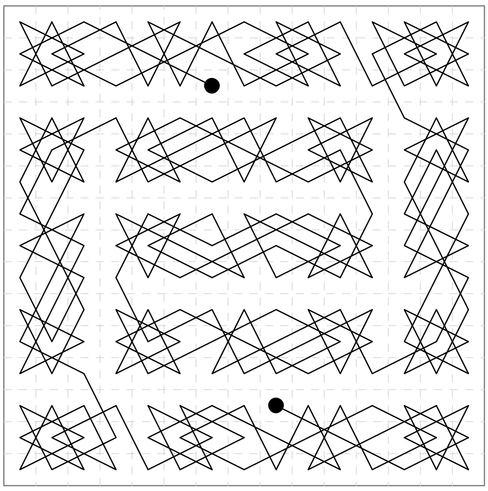
==

The 15×15 Board

==

Here is a symmetric tour 15×15 by Kraitchik (1927) based on borders. It is also easy to link together nine 5×5 tours, as was also done by Kraitchik.





He also gave a compartmental open but asymmetric tour formed of tours of 3-rank boards linked together. Shown is a similar symmetric tour of my own.

Oddly Even Squares

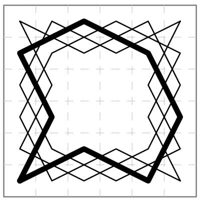
The 6×6 Board

The 6×6 board is the smallest square on which a closed tour is possible. The centre-outwards cell coding numbers the cells 0 to 5. There are ten generic (or geometrically distinct) knight moves on the board 0-1, 1-1, 0-2, 0-3, 1-3, 2-3, 0-4, 1-4, 3-4, 1-5.

In the pre-computer era attempts to enumerate all the closed knight tours on the 6×6 board were made by F. Fitting (1924, 1953) who cited a total of 10298 diagrams, and by H. J. R. Murray (1942) who arrived at a total of 1240 geometrically distinct tours. However the correct figure is now known to be **1245**. These consist of the **5** with quaternary symmetry, the **17** with binary symmetry and **1223** asymmetric. Thus since an asymmetric tour can be presented in 8 different orientations, a binary symmetric tour in 4 different orientations and a quaternary symmetric tour in 2 different orientations there are 5×2 + 17×4 + 1223×8 = 10 + 68 + 9784 = **9862** possible distinct tour diagrams. This total was first correctly reported in a computer study by J. J. Duby (1964).

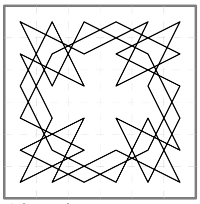
Within the 6×6 frame, omitting the centre cells, the border can be filled in one way by four 8-move equal circuits. Moves in a tour are either round the four circuits (coded 514323415), or from one circuit to another (codes 11, 13, 31), or into and out of the centre cells (01, 02, 03, 04, 10, 20, 30, 40). This appears to have been the basis of Fitting's method of enumeration.

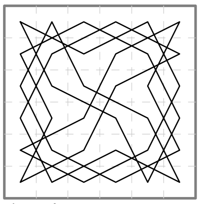
5	4	3	3	4	5
4	2	1	1	2	4
3	1	0	0	1	3
3	1	0	0	1	3
4	2	1	1	2	4
5	4	3	3	4	5

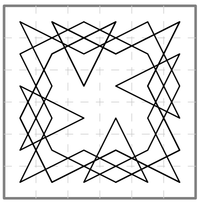


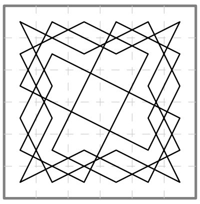
Quaternary Symmetry on the 6×6 Board

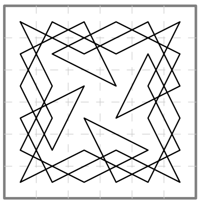
The five closed knight tours with 90-degree rotational symmetry that exist on the 6×6 board were found by the Abbé Phillippe Jolivald (writing under the pen name of Paul de Hijo) as an offshoot to his enumeration of symmetric pseudotours on the 8×8 board, published in 1882. Four of these tours had been found by Carle Adam le jeune, in his ms *Kaleidiscope Echiquienne* written in 1867 (the missing one being that with 12 slants). The tour with the centre lines forming a Greek cross was diagrammed by Laisement (1782). These five most attractive tours have been independently rediscovered many times, e.g. by Bergholt (1918), Papa (1920), Kraitchik (1927), Cozens (1940).


12 aa/aa


4 ee/ee


4 gg/gg


4 ii/ii

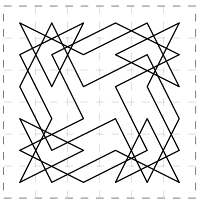

4 ll/ll

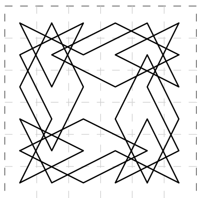
The central angles of the 5 quaternary tours are in our lettering (see ⌘ 1) a, e, g, i, l, all four angles in a quaternary tour being of the same type of course. The ‘a’ tour uses 12 slants (8 provided by the ‘a’s, and 4 others) but the ‘e’, ‘g’, ‘i’ and ‘l’ tours use only 4 slants (in these, each central angle provides one slant). These tours all have 24 moves octonary and 12 birotary. Tour ‘i’ has the maxima of 16 right angles and 4 straight angles, and the mimima of 4 diagonal-acute, 0 lateral-acute, 0 diagonal obtuse. Tour ‘e’ has the maxima of 12 diagonal-obtuse angles and the minima of 4 diagonal-acute and 0 straight (as do‘a’,‘g’,‘l’). Tour ‘a’ also shows the minimum 8 lateral obtuse.

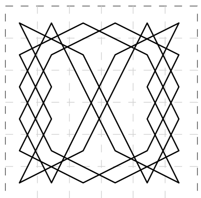
Probably the simplest method of finding all the tours with quaternary symmetry is the graphical one of using blank diagrams and drawing in all possible patterns of angles on the four central cells, then step by step filling in all quaternary linkages between these given moves and the vacant cells.

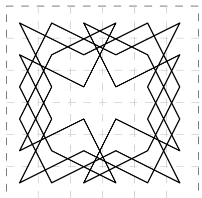
For example in the impossible case of four f-angles first put in the f-angles and the moves through the corners, which are common to all closed tours. Then the moves through the next-to-corner cells a5, b1, e6, f2 are fixed (since moves like a5-c4 are blocked). Then the moves through the edge cells a4, c1, d6, f3 are fixed (since moves like a4-c3 and a4-c5 are blocked). We now have two eight-move short circuits (a2-c3-b5-d6-f5-d4-e2-c1 is one), showing that a tour with these centre angles is impossible. We can however join up the loose ends (a3-c2 etc) to form a quaternary pseudotour of three component paths. See the first diagram below. The c-angles similarly produce a quaternary pseudotour of five circuits, see the second diagram below.

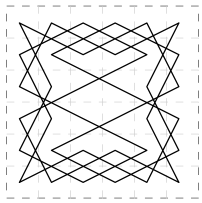
Besides the five quaternary tours, there are 15 distinct quaternary pseudotours of closed paths. Two oblique and thirteen direct, comprising seven with lateral axes and six with diagonal axes.

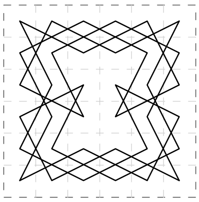

1x20 + 2x8

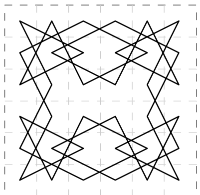

1x20 + 4x4

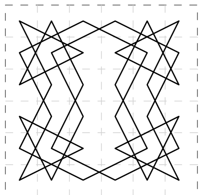

2x18

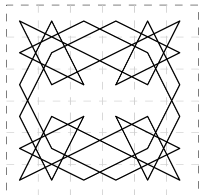

2x18

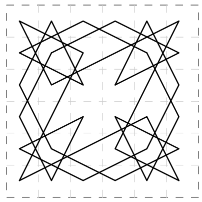

2x18

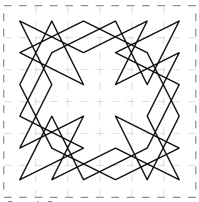

2x18

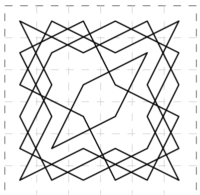

2x18

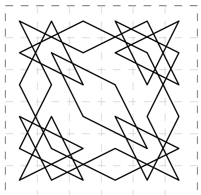

2x18

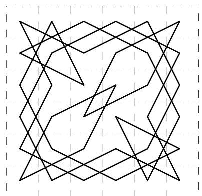

2x12 + 2x6

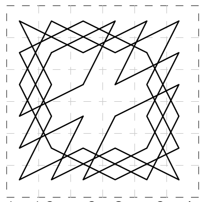

3x12


3x12


4x8 + 1x4


1x20 + 1x12 + 1x4


1x12 + 2x8 + 2x4


1x12 + 2x8 + 2x4

The legends indicate the numbers of circuits and their lengths. Six of the lateral patterns are of two circuits of 18 moves ('demitours'), while two diagonal patterns have three circuits of 12 moves. The fifth diagram is shown by D. B. Price (*Maths Gazette* 1961) in a note on symmetry.

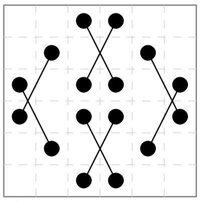
A method that may be considered more 'systematic' or mechanical is to use the octonary numbering of the board, listing all possible sequences of numbers, from any starting number, seeking a sequence of nine numbers in which each diagonal number (0, 2, 5) occurs once and each off-diagonal number (1, 3, 4) twice, using the permissible knight-move transitions. If one arrives at a sequence of less than nine numbers which cannot be extended because the numbers have already been used one deletes that sequence. To shorten the procedure one can also take account in advance other features; such as that the transition 1-1 cannot occur in oblique quaternary symmetry (since by itself it forms a four-move circuit) and that if an off-diagonal number is repeated one of the numbers between the two occurrences must be a diagonal number.

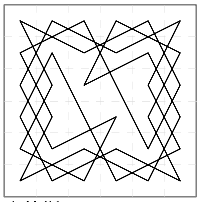
Beginning at the centre (0) one ends up with the ten sequences: 0151432340, 0234151340, 0234151430, 0323415140, 0341514320, 0415132340, 0415143230, 0431514320, 0432315140, 0432341510 (where the 0 has been repeated at the end to indicate closure). These occur in pairs that are reversals of each other. This duplication serves as a check on the correctness of the count. This type of method, with different numberings, was used by de Hijo (1882) and Bergholt (1918) in solving this problem.

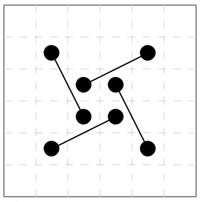
Binary Symmetry on the 6×6 Board

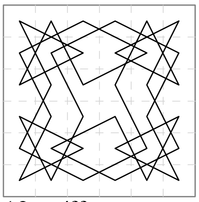
Besides the quaternary tours there are a further 17 closed knight tours with binary symmetry, invariant to 180-degree rotation, on the 6×6 board. Fourteen were diagrammed in Carle Adam's work (1867), but the complete set was first given by Kraitchik (1927).

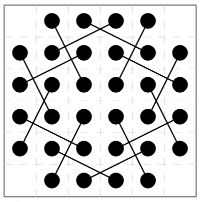
Two of these tours show mixed quaternary symmetry (see ⌘ 7). They have just 4 moves in pure oblique quaternary, shown in the middle diagram. The first has 24 octonary and 8 purely direct (this tour is in the note by Price 1961) while the other has 16 octonary and 16 purely direct (this tour is in Jaenisch 1862 where it is described as ‘three-times reentrant’) The h:j:k values of these tours are (2:6:1) and (4:4:1) respectively. The octonary components consist of the corner moves like a1-b3, a1-c2, and the edge-to-edge moves a2-c1, b1-a3, plus in the first tour the moves like a2-b4, b1-d2.


(h=2)


4 ii/ll

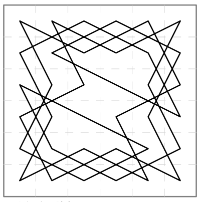

(k=1)

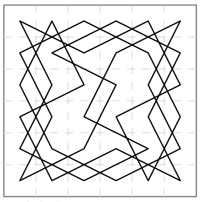

12 cc/ff

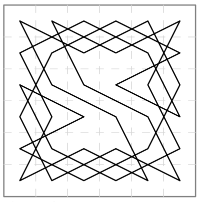

(h=4)

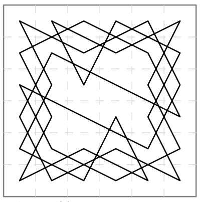
Besides the two in mixed quaternary there are a further 15 closed knight tours with binary symmetry (rotational symmetry of Eulerian type). Euler (1759) gave a single example. The first five below have 4 slants and the others 8 slants (the mixed symmetry cases above are of 4 and 12 slants).

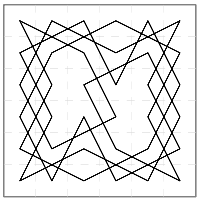
The central angles of the 17 binary tours are a/b, a/l, b/c, b/i, b/n, c/d, c/f, c/g (3 times), c/l, e/g, f/g, f/i, g/i, g/n, i/l, two angles being of one type and two of another. No case is possible in which all four angles are alike; tours with angles all alike turn out to be either quaternary or asymmetric. Using the graphical approach it is quite simple to check this enumeration of the tours that have binary symmetry. Tour 8aa/ll is the only 6×6 closed tour with the maximum 14 diagonal-acute angles.

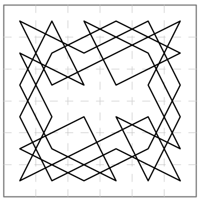

4bb/ii

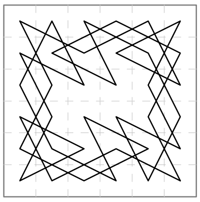

4 bb/nn

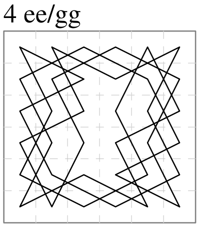

4 ee/gg

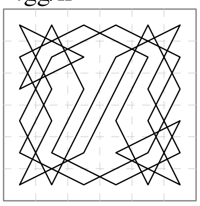

4gg/ii

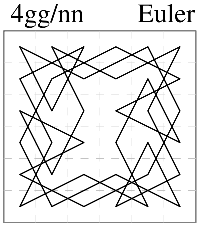

4gg/nn Euler

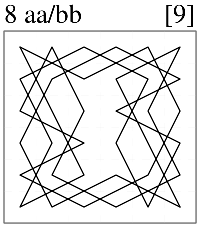

8 aa/bb [9]

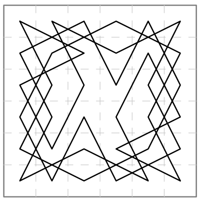

8 aa/ll

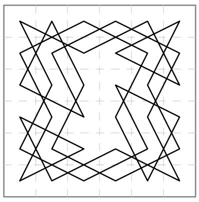

8 bb/cc

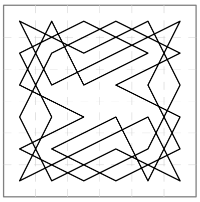

8 cc/dd

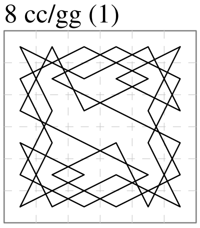

8 cc/gg (1)


8 cc/gg (2) [7,10]


8 cc/gg (3)


8 cc/ll


8 ff/gg


8 ff/ii

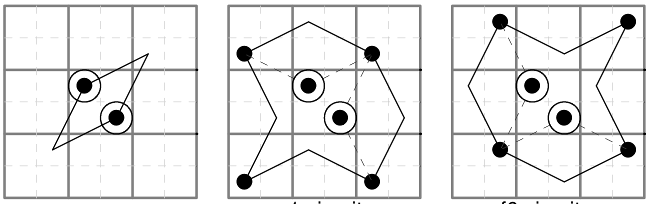
As noted earlier there are 105 symmetric combinations of central angles, however the above tours show that in fact on the 6×6 board only 20 are possible in actual tours. This is very unlike the 8×8 case where all 105 combinations are possible (183 if we distinguish direct and oblique cases). The difference is due to border effects where the central moves meet the surrounding braid.

Central Angle Method on the 6×6 Board

In *Chessics* (#22 1985 p.70) I outlined a way of classifying the 6×6 tours by means of the angles on the central cells (see ⌘ 1). This divides the patterns into sufficiently small classes to make an enumeration of the asymmetric closed tours by this method feasible. Progress was further reported in *Chessics* (#30 1987 p.160) and brought to completion during October 1992.

The first stage in the enumeration is to consider all ways the knight can pass through a pair of diametrically opposite centre cells. The 15 angles, lettered a to o, form 15 pairs of type xx and 105 (that is 15·14/2) of type xy, thus giving 120 pairs in all. The angles a, h, m are symmetric by reflection in the a6-f1 diagonal. If angles x and y are asymmetric then the pairs xx and xy each occur in two geometrically distinct forms. The number of pairs of asymmetric angles is 12 of type xx and 66 (that is 12·11/2) of type xy, thus giving 78 pairs of this type. Thus the total of pairs of diametrally opposite moves to be considered is 120 + 78 = 198.

However, many of these 198 pairs can easily be shown to be impossible in a closed 6×6 tour, according to the way they impinge on the surrounding border circuits. The case mm eliminates itself by forming a four-move circuit in the middle of the board. If no moves from the centre cells c4, d3 go to the three cells a5, e5, e1 then an eight-move circuit (one of the strands of the border braid) is forced through these cells and the corner a1. Similarly for the three cells b2, b6, f2 and the corner f6.



mm-case a1 circuit f6 circuit

The angles a, f and m disrupt both these circuits and so can potentially form tours in combination with any other angle. The angles h, j and k disrupt neither circuit and so cannot form tours unless in combination with a, f or m. The other angles disrupt only one of the circuits (c disrupts the same circuit in two places). Pairs of cases occur with angle f since it is asymmetric; the other two angles that disrupt both circuits, a and m, are symmetric.

By these arguments we eliminate 100 cases and are left with 98 to consider: aa, ab, ac, ad, ae, af, ag, ah, ai, aj, ak, al, am, an, ao, bb, bc, bd, be, 2bf, bg, bi, bl, bm, bn, bo, cc, cd, ce, 2cf, cg, ci, cl, cm, cn, co, dd, de, 2df, dg, di, dl, dm, dn, do, ee, 2ef, eg, ei, el, em, en, eo, 2ff, 2fg, fh, 2fi, 2fj, 2fk, 2fl, fm, 2fn, 2fo, gg, gi, gl, gm, gn, go, hm, ii, il, im, in, io, jm, km, ll, lm, ln, lo, mn, mo, nn, no, oo.

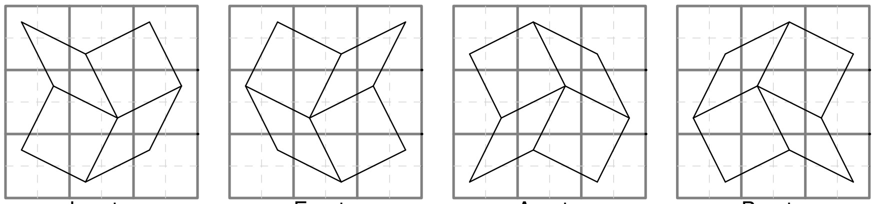
Each closed tour contains four central angles and shows two of the 98 cases listed. We can indicate the type of tour by a code of the form uv/wx where uv is one pair of diametrally opposite centre angles and the other pair is wx. The 5 doubly symmetric tours are of type uu/uu (reduced to ‘u’) and the 17 singly symmetric tours are of the form uu/vv (reduced to ‘u/v’).

In enumerating the tours, we work through systematically from aa/-- to oo/--. The procedure was to first enumerate all tours with ‘a’ on c4 and successively ‘a’, ‘b’, ..., ‘o’ on d3, then all tours with ‘b’ on c4, but no ‘a’s anywhere, then all tours with ‘c’ on c4, but no ‘a’s or ‘b’s anywhere, and so on in alphabetical order. To reduce the labour I followed the procedure of first eliminating any symmetry so that duplicates due to rotation or reflection are not generated. This enumeration was done by hand, connecting dots on sheets with twelve diagrams to a page.

My attempted enumeration of the tours by the angle method, as noted above, was carried out while not being aware of the nature of the work by Duby (1964). It proved necessary to cross-check my results by the method of slants to ensure accuracy. I reported this work to Prof. D. E. Knuth and in December 1992 he sent me a complete computer print-out enumerating the tours, in a numerically coded form, produced by a method based on slants. This revealed discrepancies in two sections which I then had to recheck. The catalogue of tours printed here is a record of my work on this subject. Diagrams of all 1223 geometrically distinct asymmetric closed tours are given.

Straits and Slants on the 6×6 Board

The method of straits and slants (see ⌘ 1) applied to the 6×6 board is simpler than on the 8×8 since the four nets of straits are all of the same shape. Each passes through one corner of the board.

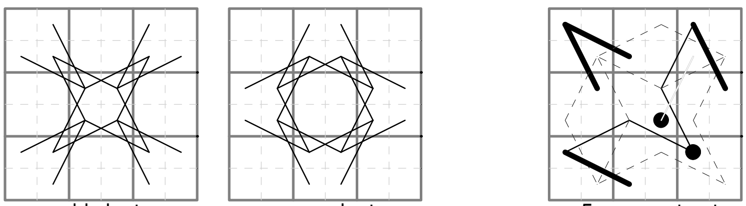


L-net E-net A-net P-net

Reflection of the board in the principal diagonal interchanges the A and E nets. Reflection in the secondary diagonal interchanges the L and P nets. Rotation 180° combines both these effects. Reflection in the vertical median interchanges L with E and P with A. Reflection in the horizontal median interchanges L with A and P with E. Rotation 90° clockwise makes cyclic changes L-E-P-A-L. Rotation 90° anticlockwise causes the reverse cycle L-A-P-E-L. By these rotations and reflections we can always orient an open directed tour so that the first slant is of LE type.

The total number of slants on the 6×6 board is 32, since there are 2 vertical grid lines on each of which there are 4 strands of 2 slants, and 2 horizontal grid lines on each of which there are also 4 strands of 2 slants. The maximum number of slants that can occur in a closed tour is 4 less than half the number of slants, that is 12 on the 6×6 board.

As noted in the introductory section on straits and slants, if we number the block diagram by a wazir tour, then we can classify the slants as **odd** or **even** according as they connect odd or even numbered blocks (a diagram of the even slants appeared in Fitting 1953). On the 6×6 board the odd slants all have one end in the central block.



odd slants even slants 5-segment net

In any net on the 6×6 board there are 5 **odd** cells (i.e. in the odd-numbered blocks) and 4 **even** cells. A net traversed in one path begins and ends on odd cells, from which it is linked to the other nets by odd slants. Other breaks in the net introduce one odd and one even cell and corresponding slant. Thus in a closed tour the number of odd slants is always 4 more than the number of even slants. In tours with 4, 6, 8, 10, 12 slants the even:odd distribution is 0:4, 1:5, 2:6, 3:7, 4:8 respectively. Since the odd slants are all incident with the centre cells, the numbers of pairs of slants meeting at centre cells is respectively 0, 1, 2, 3, 4.

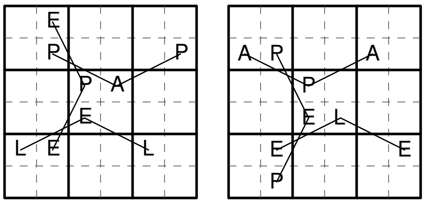
On the 6×6 board a division of a net into 5 segments is possible, but does not lead to any tour. For example the L-net can be split into the segments b4-a6-c5, e6-f4, a2-c1, d3, e2 including two isolated cells, but the slants incident at e2, a2, e6 meet at c3, d4, thus eliminating any slants incident with b1, b5, f5 in the P-net, forcing the f1-circuit. Thus the partitions possible are: 4 (1:1), 6 (1:2), 8 (1:3) (2:2), 10 (1:4) (2:3), 12 (2:4) (3:3).

In cataloguing the asymmetric tours we use the central angles and straits and slants methods together. We list the tours according to their number of slants (4, 6, 8, 10, 12) and secondly according to their angle codes, in the partial alphabetical order a, c, f, m // b, d, e, g, i, l, n, o // h, j, k, p. Pairs of slants meeting at the centre cells form the angles a, c, f, m, while pairs of straits form the angles h, j, k, p. Other angles use one slant and one strait.

Open Tours 6×6 with Three Slants

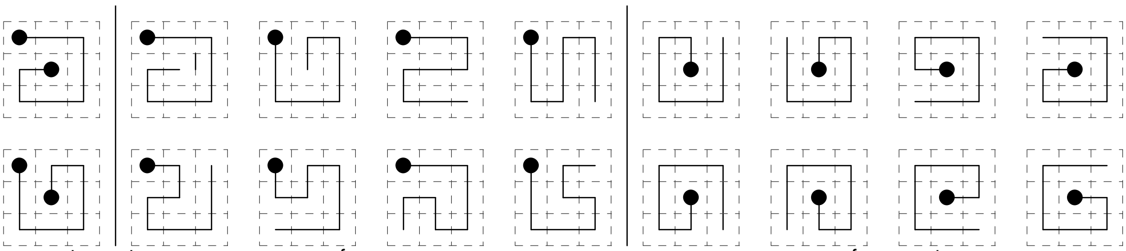
The open tours with three slants were succesfully enumerated by H. J. R. Murray in 1942 using the method of straits and slants. The total geometrically distinct is 2^11 = **2048**.

We can always rotate the board so that the first slant in the tour is an LE move. The three slants must be LE, EP, PA. The positioning must be either as in the first or second of these diagrams, where there are two choices for each slant. The second diagram however is equivalent to the first by reflection top to bottom. It generates the reverse tours of those given by the first diagram.



The number of EE and PP routes is 2 each, corner to centre routes. The number of LL routes, from a corner, is 8. The number of AA routes, from the centre, is 8.

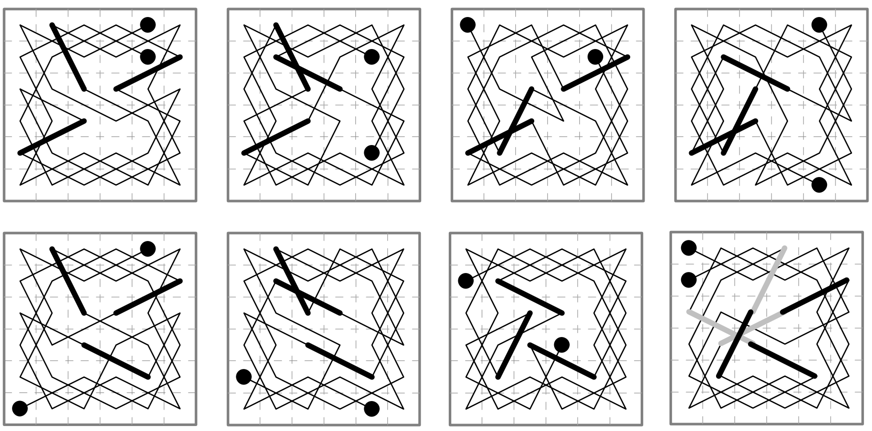
From this data we can conclude, by multiplying together the choices at each stage, that the number of tours is: 2 (choice of LE) · 2 (choice of EP) · 2 (choice of PA) · 2 (routes EE) · 2 (routes PP) · 8 (routes LL) · 8 (routes AA) = 2^11 = 2048.



corner to centre from corner from centre

From the 60 asymmetric closed tours with 4 slants (see the following catalogue) by deleting one slant we can derive 240 tours with 3 slants. There are 6 tours with binary symmetry with 4 slants. These generate 2 reentrant tours each, that is 12. Similarly the four quatersymmetric tours with four slants generate 4 reentrant tours. Thus of the 2048 tours with 3 slants **256** = 2^8 are reentrant.

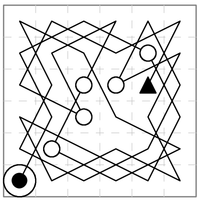
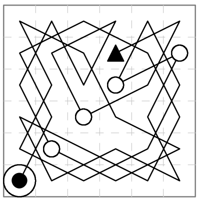
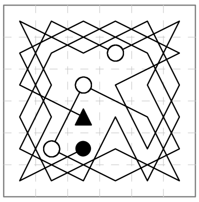
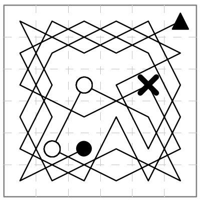
Here are some diagrams of non-reentrant tours with three slants, showing all 8 slant positions and a number of different end-point positions.



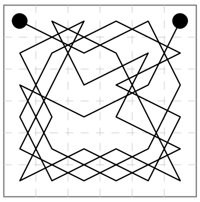
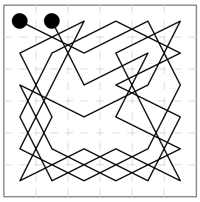
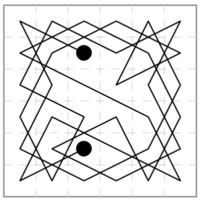
In the final example the straits extend to form three two-move lines delineating K and enclosing a size 1 triangle, so it seems appropriate to dedicate it to Prof D. E. Knuth (see also a similar tour on the 8×8 board, in the section on Rogetian tours in ⌘ 6).

Perhaps surprisingly Warnsdorf (1823) first tried out his rule for generating tours (move to the cell with fewest exits) on the 6×6 board. We show his 6×6 tours here in geometrical form. (For more historical details see ⌘ 6.)

A black dot and triangle indicate the start and end cells. A circle round the initial dot indicates that there are choices of first move, though this may only be because of symmetry. The white dots mark points where the rule provides a choice of two or more moves, so alternative routes could have been taken. In the fourth 6×6 example the move at e4 (marked by a cross) to c3 does not conform to the rule, which requires e4-f6 as in the third 6×6 example.

Here are three other open tours he gave. (See also the account in the historical chapter).

He also gave three asymmetric closed tours. These are noted in the catalogue that follows They are, in terms of number of slants and central angles used. 4 gi/ii, 6 gi/gm, 8 ab/ef. I have also labelled two that appear in Takefuji (1992). These are 8 bi/cf, 12 ac/ff.

==

Catalogue of Asymmetric 6×6 Closed Tours

Diagrams of all 1223 geometrically distinct asymmetric closed tours are given on the following pages. We list them according to number of slants and subclassify them according to the angles used. Each individual tour can be designated by its number of slants, its centre angles and a further digit (n) where there are two or more with the same angles.

Asymmetric tours with 4 slants: (p.269) Total **60** (+ 4 quaternary and 6 binary = **70**). The central angles occurring are b, e, g, i, l, n, using one straight and one slant.

Asymmetric tours with 6 slants: (p.271) Total 304 (No symmetric tours have 6 slants.) Two nets are toured in an unbroken eight-move path, while the other two nets are toured in two parts. Since there is one even slant and five odd slants a pair of slants must meet on one of the centre cells (forming an angle of type a, c, f or m). This means that one of the broken nets must be toured as a seven-move path of straights and an isolated cell. The numbers of these four types are: a-type 80, c-type 32, f-type 136, m-type 56.

Asymmetric tours with 8 slants: (p.281) Total 527. (+ 10 with binary symmetry = 537). The 527 consist of 22 of triple type (having three of acfm with one of hjk: 10 h, 8 j, 4 k) and 505 of double type: (aa) 65, (ac) 53, (af) 149, (am) 39, (cc) 15, (cf) 61, (cm) 8, (ff) 79, (fm) 33, (mm) 3.

Asymmetric tours with 10 slants: (p.297) Total 288 (No symmetric tours have 10 slants.)

Asymmetric tours with 12 slants: (p.307) Total 44 (+ 1 quaternary and 1 binary = 46). These tours have four central angles of the types a, c, f, m, each formed of 2 slants.

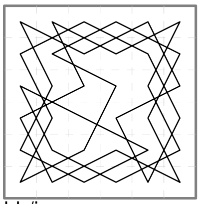
In the 1992 list sent to me by Donald Knuth the tours are in 16 ‘classes’ according to how the consonant and vowel nets are divided.

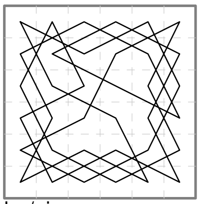
Notes on crossings, lines and triangles in tours are interspersed between the catalogue pages.

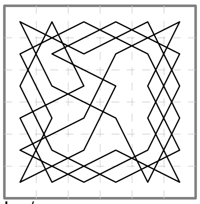
Asymmetric Closed Tours with 4 Slants

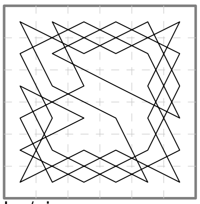
Total **60**.

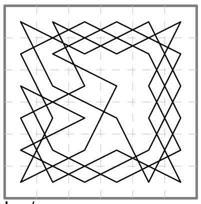
4 slants 1-30 of 60

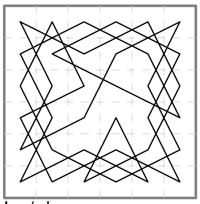

bb/in

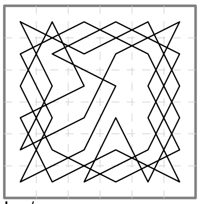

be/ei

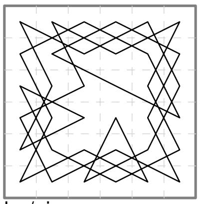

be/en

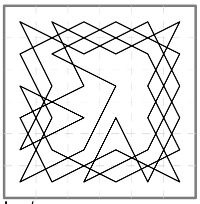

be/gi

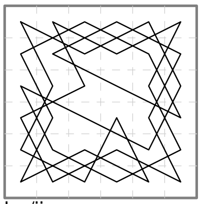

be/gn

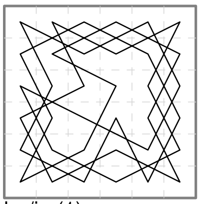

bg/ei

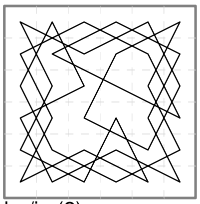

bg/en

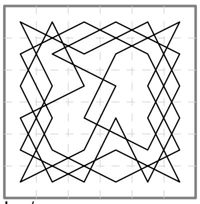

bg/gi

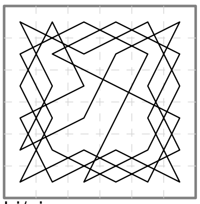

bg/gn

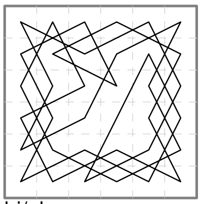

bg/ii

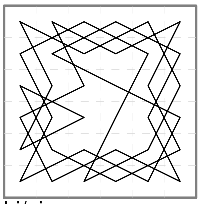

bg/in (1)

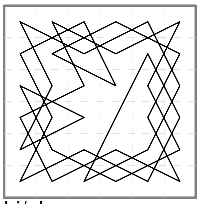

bg/in (2)

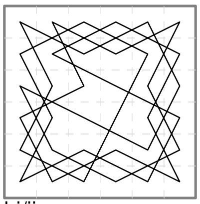

bg/nn

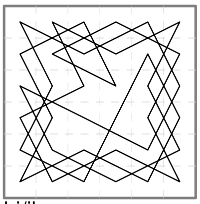

bi/ei

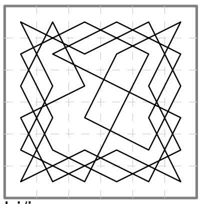

bi/el

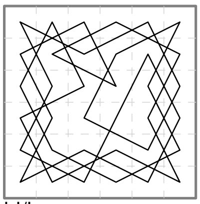

bi/gi

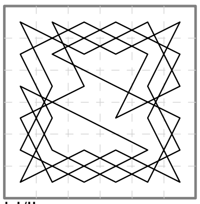

bi/gl

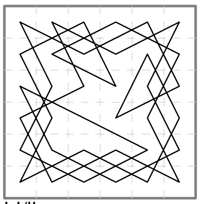

bi/ii

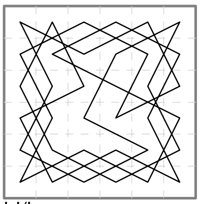

bi/il

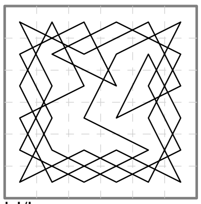

bi/in

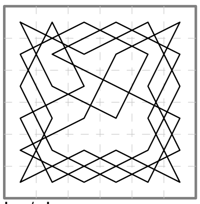

bi/ln

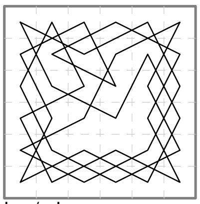

bl/ii

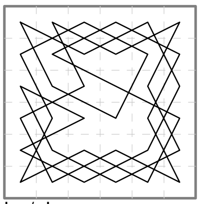

bl/il

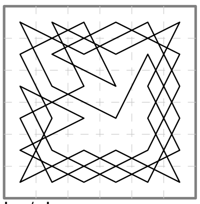

bl/in

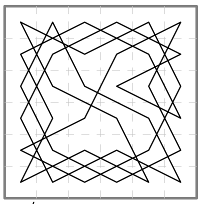

bl/ln


bn/ei


b n / e l

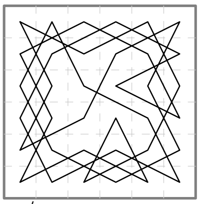

bn/gi

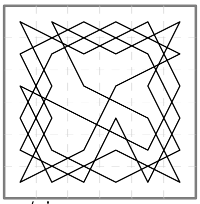

bn/gl

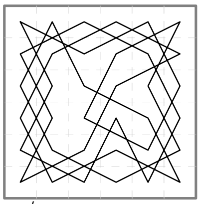

ee/eg

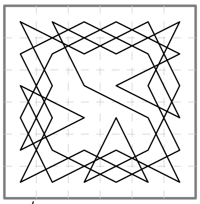
Tours bg/in (1) and (2) could be distingushed as bg/in and bg/ni if the second part is read from the bottom left corner, but this method of naming unfortunately breaks down from 6bc/ei onwards.

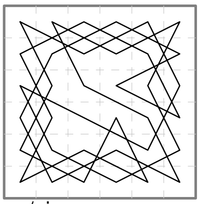
4-slants 31-60 of 60:

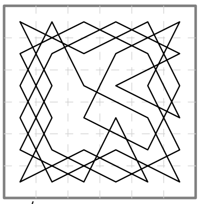

eg/eg

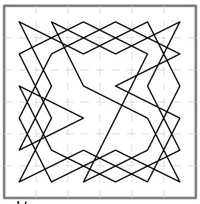

eg/ei

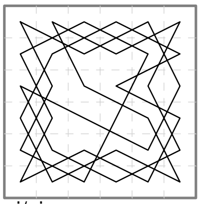

eg/en

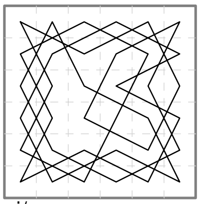

eg/gg

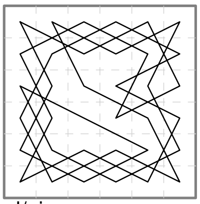

eg/gi

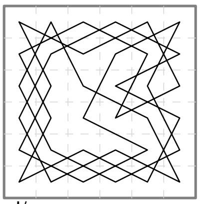

eg/gn

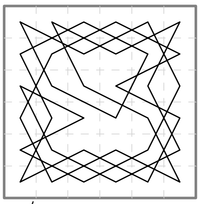

ei/gg

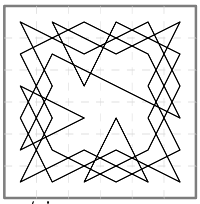

ei/gi

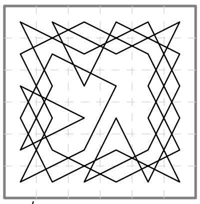

ei/gn

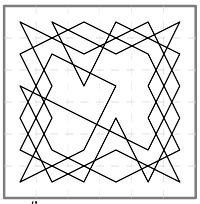

el/gi

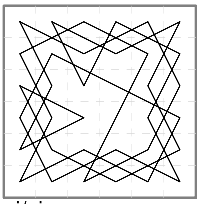

el/gn

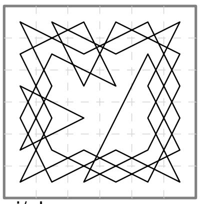

en/gg

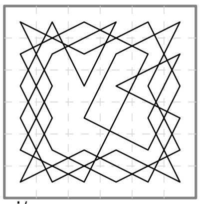

gg/gi

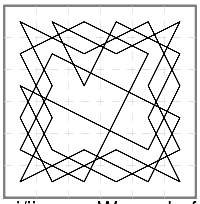

gg/gn

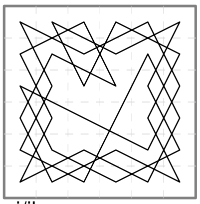

gg/in

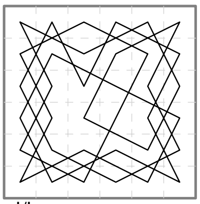

gi/gi

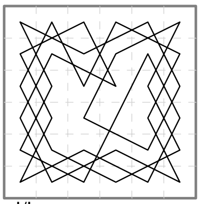

gi/gl

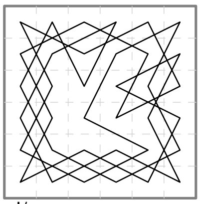

gi/gn

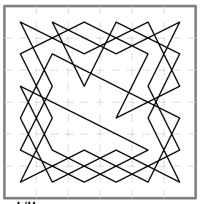

gi/ii Warnsdorf

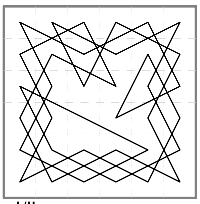

gi/il

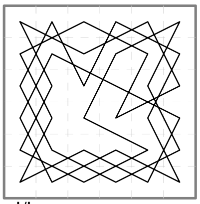

gi/in

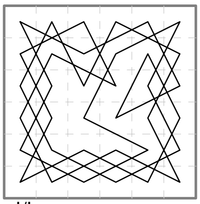

gi/ln

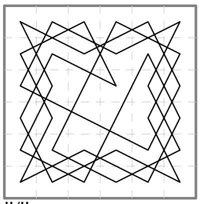

gl/gn

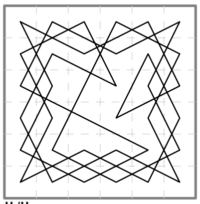

gl/ii

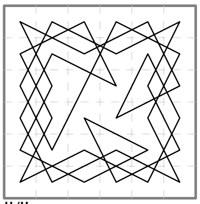

gl/il


gl/in


gl/ln


ii/il


il/il


il/ll

In the Knuth list Class 1 consists of the 70 4-slant tours (including symmetric).

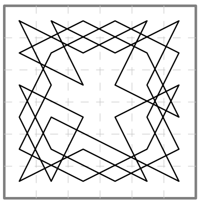
Crossings. Four uncrossed moves are found in the symmetric tours 4ll/ll where they are birotary and 4cc/dd where they are parallel, and three are found in 4il/ll (above) and 8ae/gm (p.289).

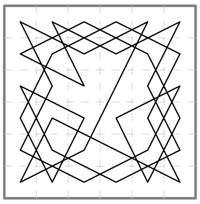
A Seven-fold Crossed Move is found in 6 tours: 10aa/bf(1) 10ab/cf(2) 10ac/bf(3) 10af/fg(6) 10ag/cf(1) and 12ac/af (2). These diagrams are labelled 7x.

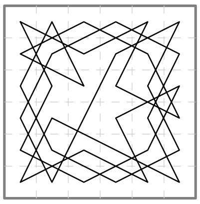
Asymmetric Closed Tours with 6 Slants

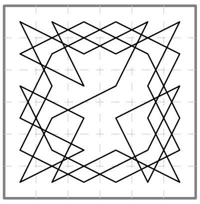
Total **304**.

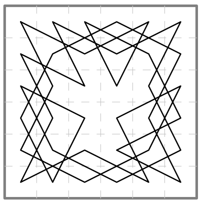
6 slants a-type 1-30 of 80:

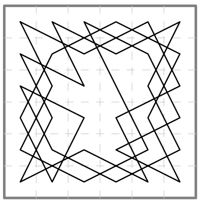

ab/bb

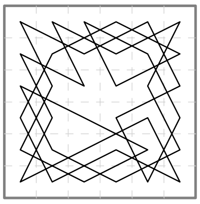

ab/bd (1)

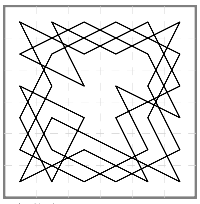

ab/bd (2)

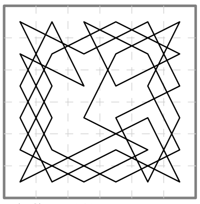

ab/be

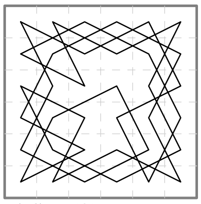

ab/bg

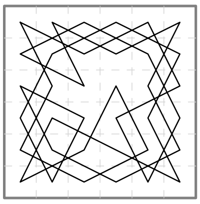

ab/bi (1)

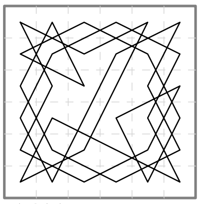

ab/bi (2)

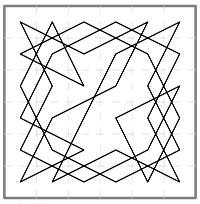

ab/bl

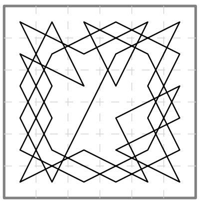

ab/bn (1)

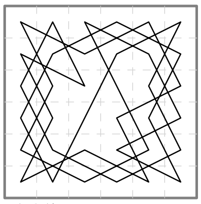

ab/bn (2)

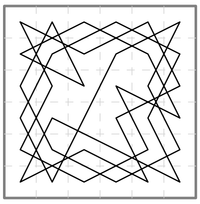

ab/bo

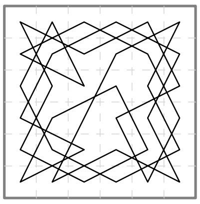

ab/dd

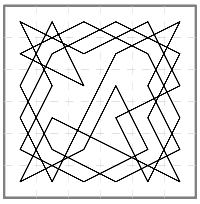

ab/de

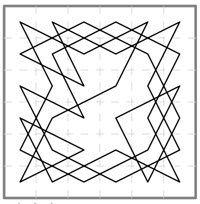

ab/dg

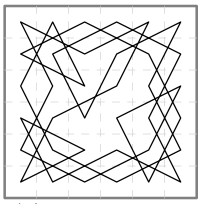

ab/di

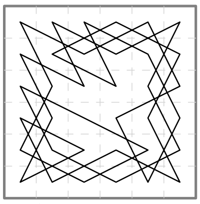

ab/dl

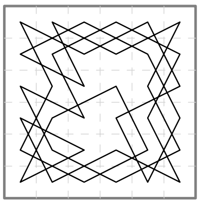

ab/dn

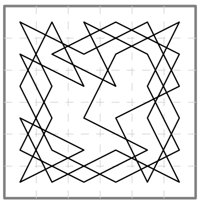

ab/do

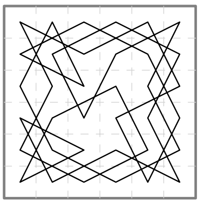

ab/el

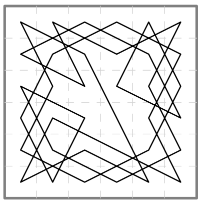

ab/eo

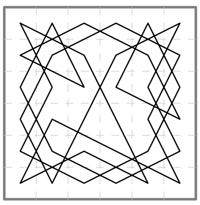

ab/il

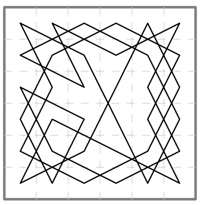

ab/ln (1)

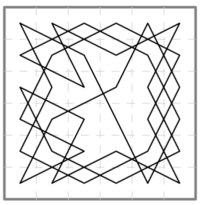

ab/ln (2)

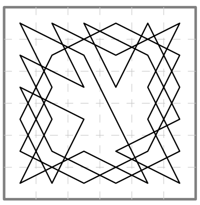

ab/no

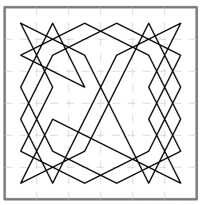

ad/bb


ad/bd (1) [14]

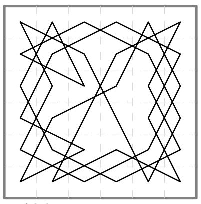

ad/bd (2) [11]

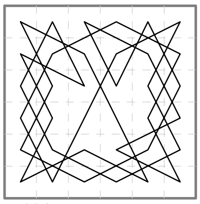

ad/be

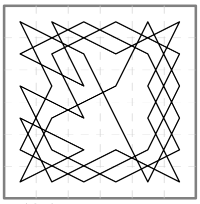

ad/bg

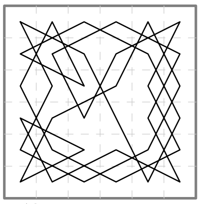

ad/dd [11,14]

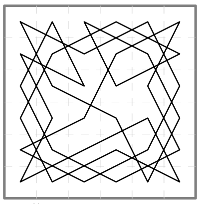
6 slants a-type 31-60 of 80:

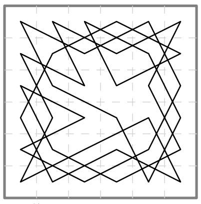

ad/de

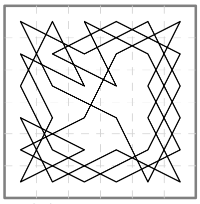

ad/dg

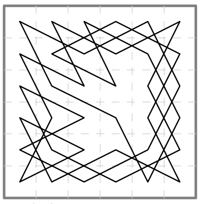

ad/el

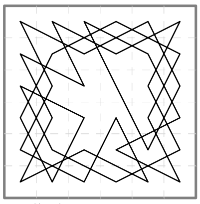

ad/eo

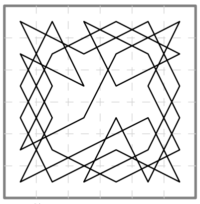

ae/be

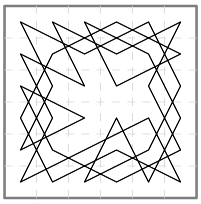

ae/bg

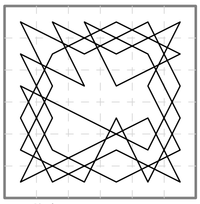

ae/el

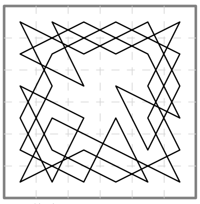

ae/gl

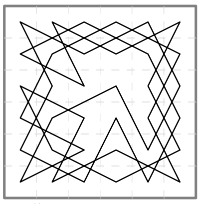

ag/bd

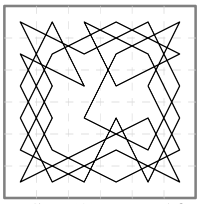

ag/be [7]

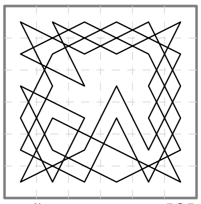

ag/bg [7]

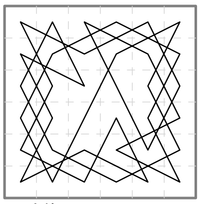

ag/bi [7,10]

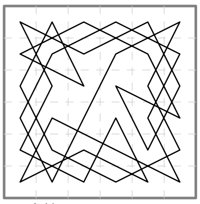

ag/bl [8]

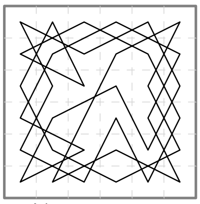

ag/bn (1)

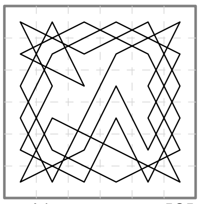

ag/bn (2) [7,10]

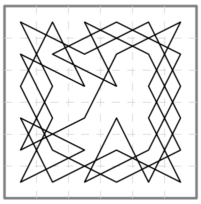

ag/bo [8]

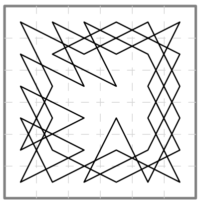

ag/di

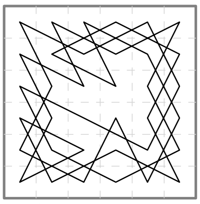

ag/dl

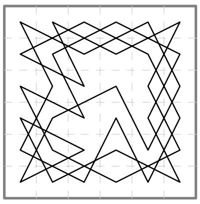

ag/dn

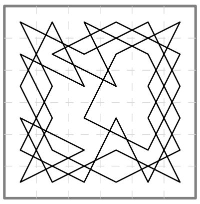

ag/do [8]

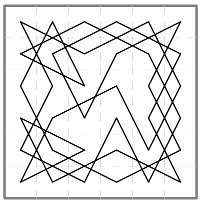

ag/el

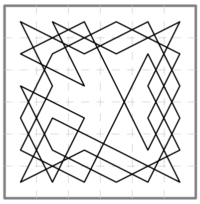

ag/gl

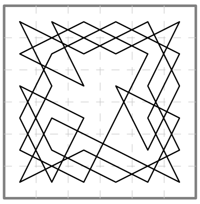

ag/il

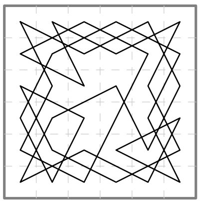

ag/ln (1)

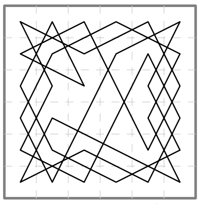

ag/ln (2)


ag/no


ai/bi


ai/bl


ai/bn

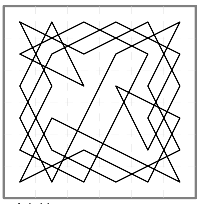

ai/di 4 lnes

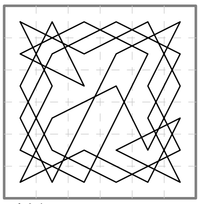
Triangles in closed 6×6 tours.

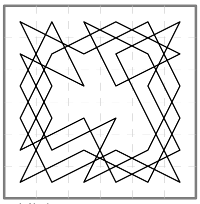
Knuth has noted that all triangles of sizes [1] to [12] can occur and also sizes [14], [15], [18], though no tour includes all sizes [1] to [12]. The smaller triangles [1] to [6] and [12] use only single moves, the [12] being three successive knight moves. A 7-move 'star' counts as 5 [12]s, while 6 moves give 4, 5 give 3 and 4 forming a 'shield' count as 2. The maximum is 12 [12]s, e.g. in 12cc/ff.

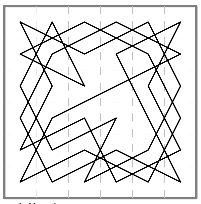
Continued on p.280.

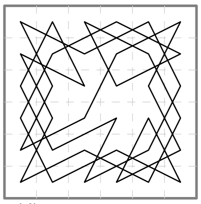
6 slants a-type 61-80 of 80.

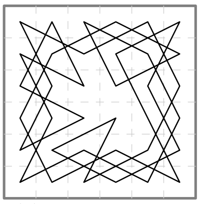

ai/dl

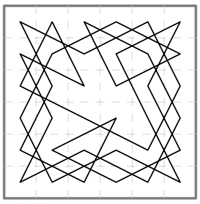

ai/dn

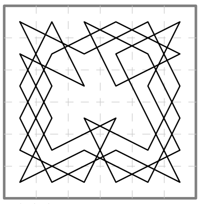

al/bb

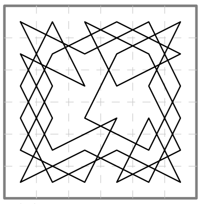

al/bd

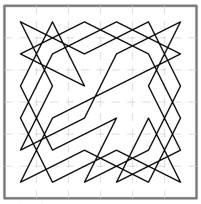

al/be

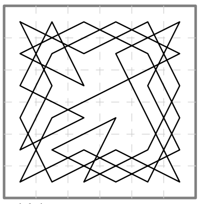

al/bg

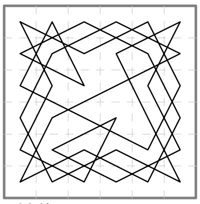

al/bi

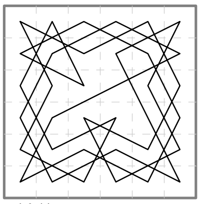

al/bl

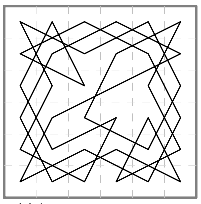

al/bn

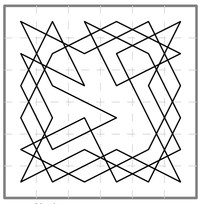

al/de

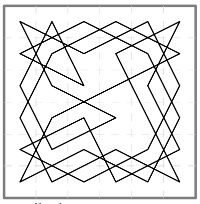

al/dg

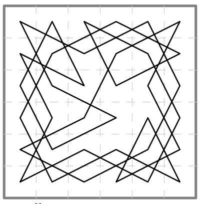

al/di

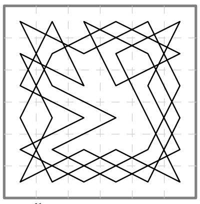

al/dl

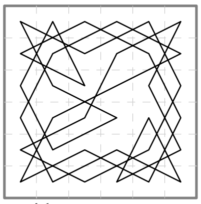

al/dn

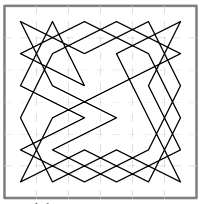

ao/bb


ao/bd

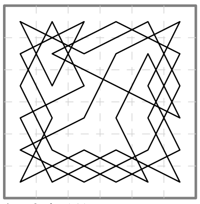

ao/be

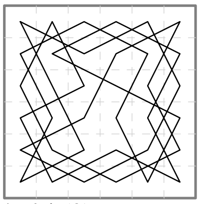

ao/bg

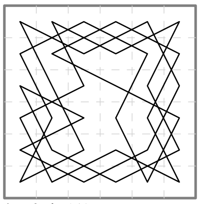

ao/de

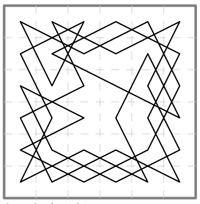

ao/dg

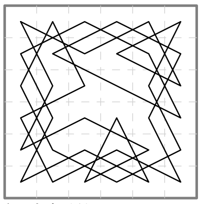
6 slants c-type 1-10 of 32:

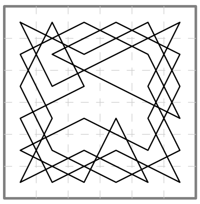

bc/ei (1)

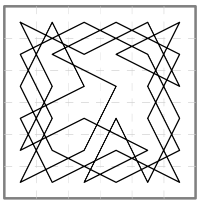

bc/ei (2)

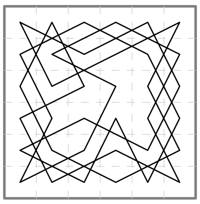

bc/gi (1)

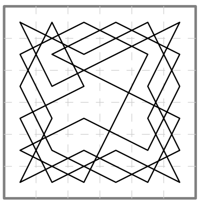

bc/gi (2)

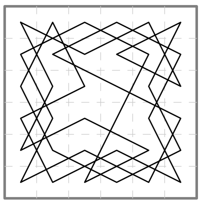

bg/ci (1)


bg/ci (2)

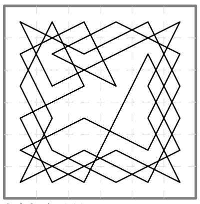

bg/cn (1)

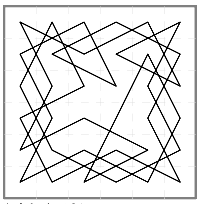

bg/cn (2)

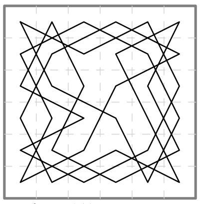

bi/ci (1)

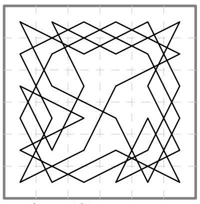

bi/ci (2)

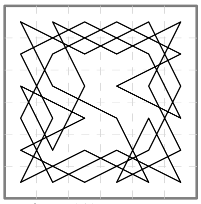
6 slants c-type 11-32 of 32:

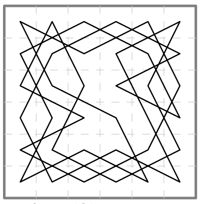

bi/cl (1)

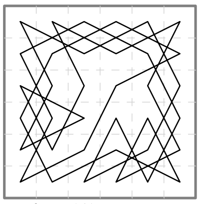

bi/cl (2)

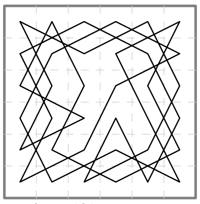

ce/eg (1)

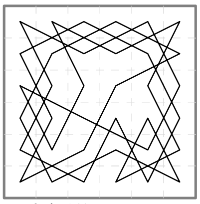

ce/eg (2)

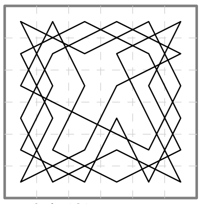

ce/gg (1)

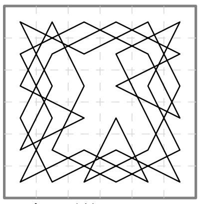

ce/gg (2)

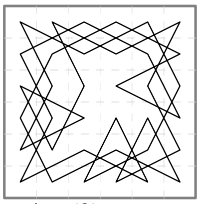

cg/eg (1)

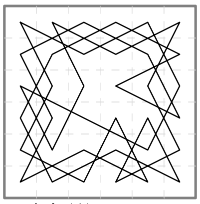

cg/eg (2)

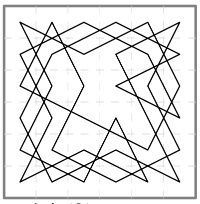

cg/ei (1)

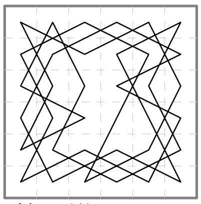

cg/ei (2)

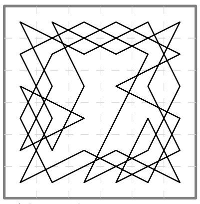

cg/gg (1)

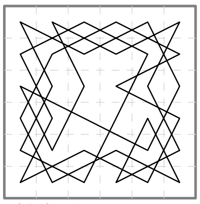

cg/gg (2)

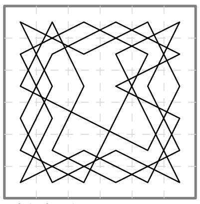

cg/gi (1)

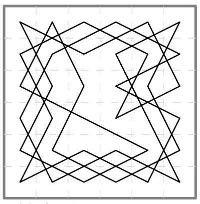

cg/gi (2) [7]

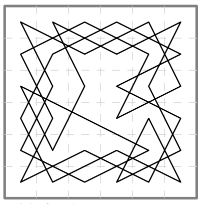

ci/gg (1) [7,10]

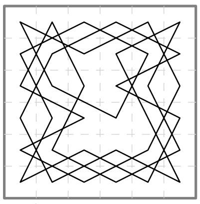

ci/gg (2)

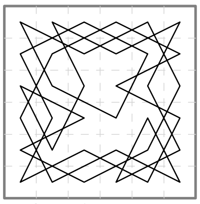

ci/gi (1)


ci/gi (2) [7,10]

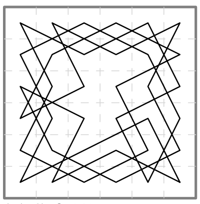

cl/gi (1) [10]

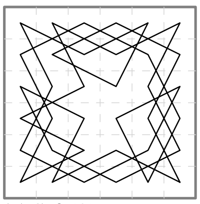

cl/gi (2)

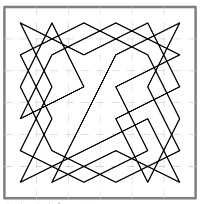

cn/gg (1)

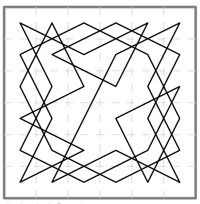

cn/gg (2)

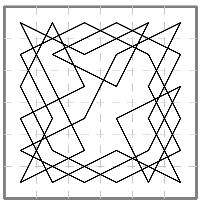
6 slants f-type 1-5 of 136:


bb/bf (1)

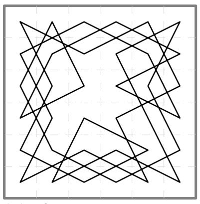

bb/bf (2)

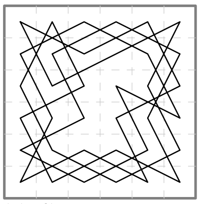

bb/df (1)

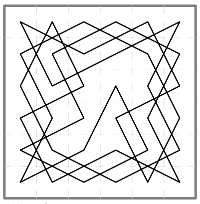

bb/df (2)

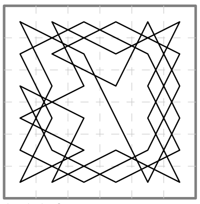

bb/ef

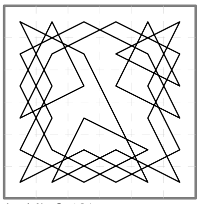
6 slants f-type 6-40 of 136:

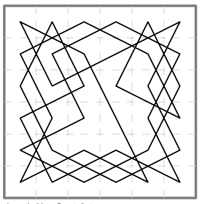

bb/fg

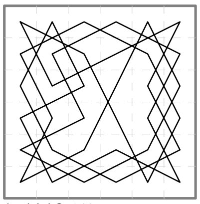

bb/fl

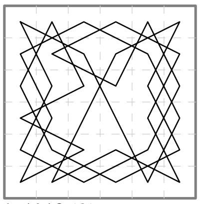

bb/fo

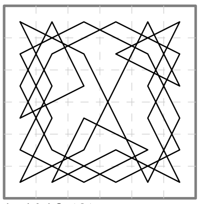

bd/bf (1)

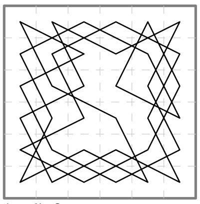

bd/bf (2)

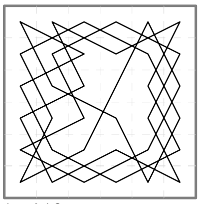

bd/bf (3)

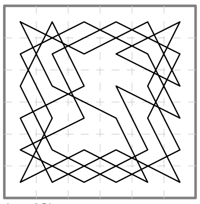

bd/df (1) [11]

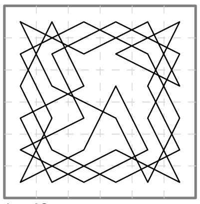

bd/df (2)

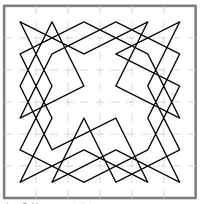

bd/df (3) [7]

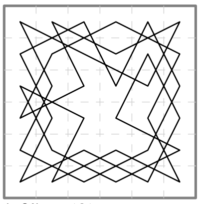

be/bf

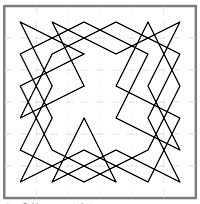

be/df

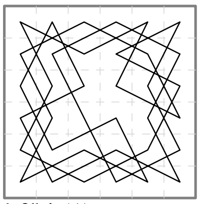

be/fl

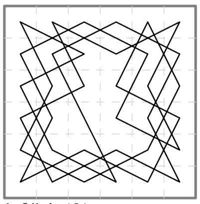

be/fo

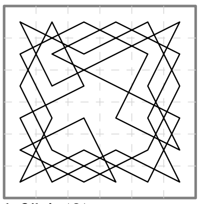

bf/bg (1)

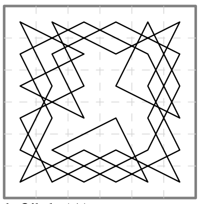

bf/bg (2)

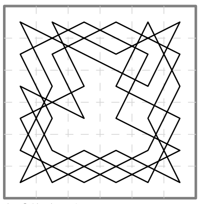

bf/bg (3)

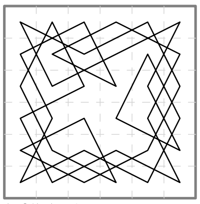

bf/bi (1)

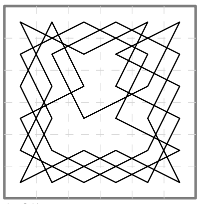

bf/bi (2)

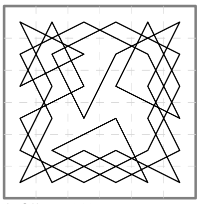

bf/bi (3)

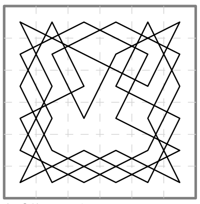

bf/bl (1)

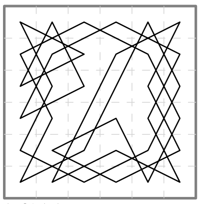

bf/bl (2)

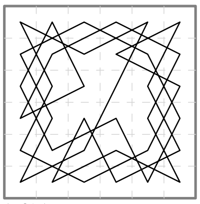

bf/bl (3)

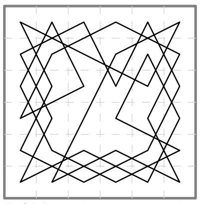

bf/bn

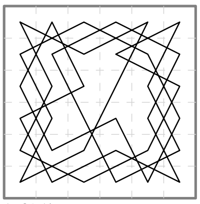

bf/bo (1)

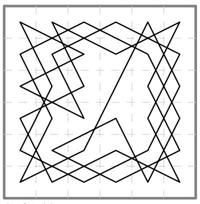

bf/bo (2)

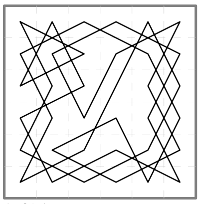

bf/dd

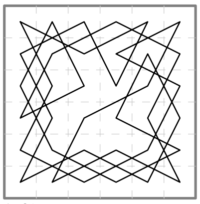

bf/dg (1)

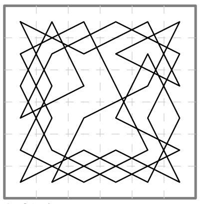

bf/dg (2) [7,10]

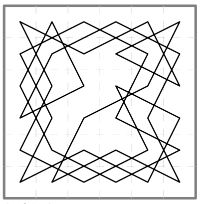

bf/di

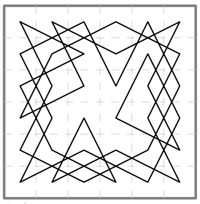

bf/dl


bf/do

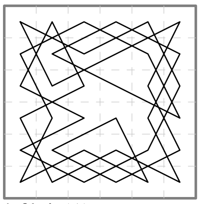

bf/eg

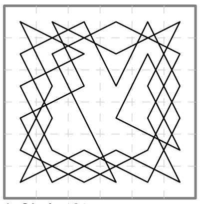

bf/ei

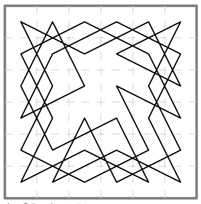

bf/el

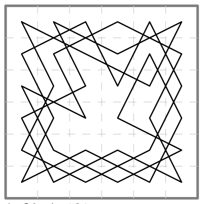

bf/gg

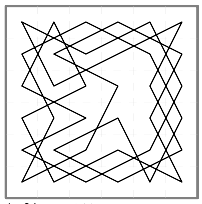
6 slants f-type 41-75 of 136:

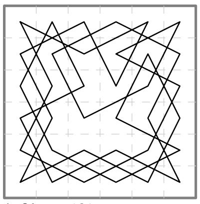

bf/gi (1)

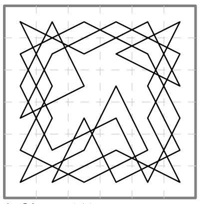

bf/gi (2)

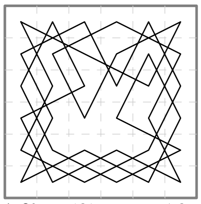

bf/gl (1)

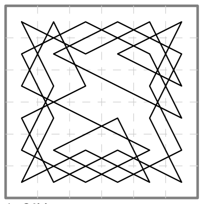

bf/gl (2) [7,10]

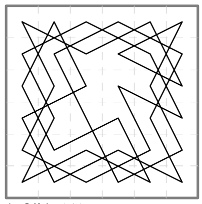

bf/gn (1)

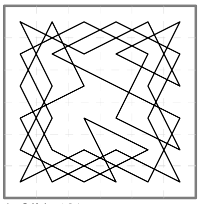

bf/gn (2)

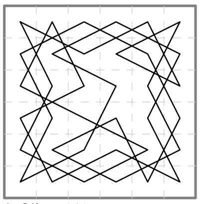

bf/go (1)

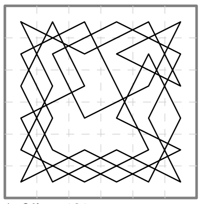

bf/go (2) [7,10]

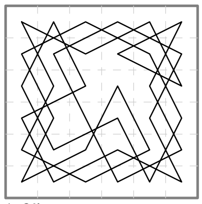

bf/ii

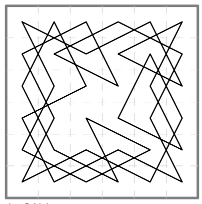

bf/il (1)

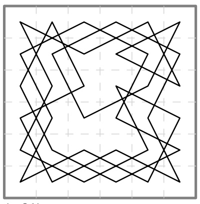

bf/il (2)

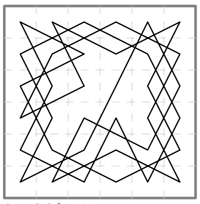

bf/in (1)

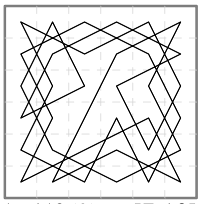

bf/in (2)

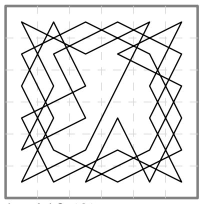

bf/io

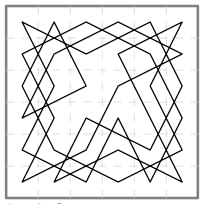

bf/ll

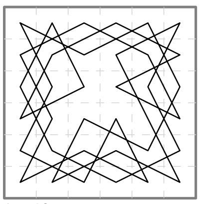

bf/ln

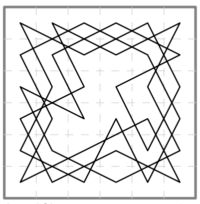

bg/df (1)

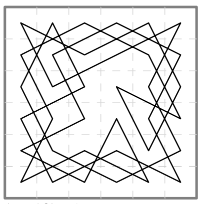

bg/df (2) [7,10]

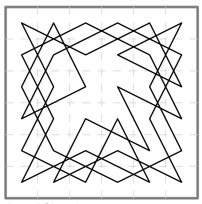

bg/df (3)

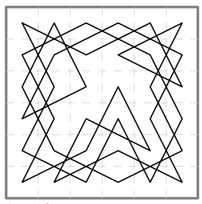

bg/ef

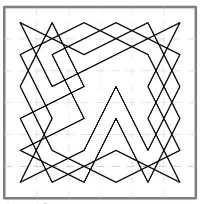

bg/fg [7]

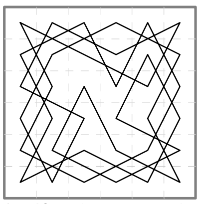

bg/fl (1) [7,10]

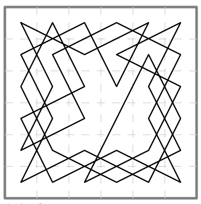

bg/fl (2)

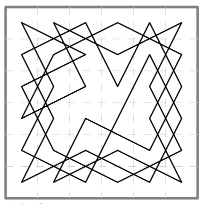

bg/fl (3)

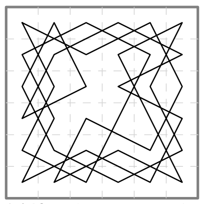

bg/fo (1)

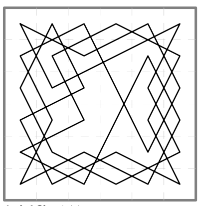

bg/fo (2)

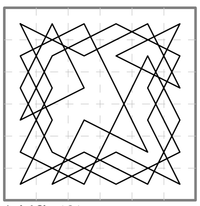

bg/fo (3) [7,10]

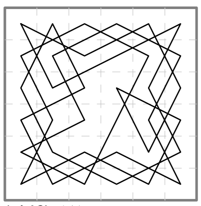

bi/fg (1)

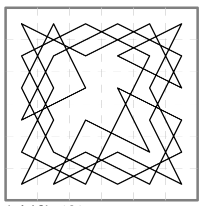

bi/fg (2)

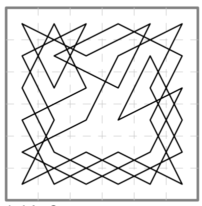

bi/fg (3) [7,10]


bi/fi (1)

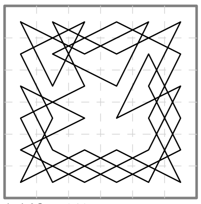

bi/fi (2) [9]

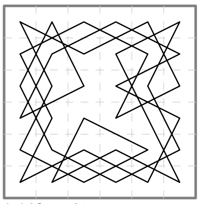

bi/fl (1)

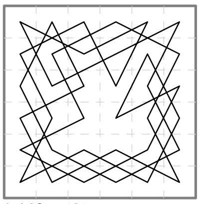

bi/fl (2) [9]

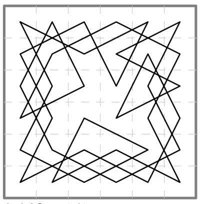

bl/ef

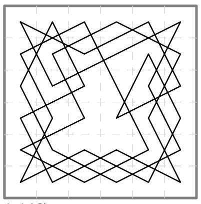
6 slants f-type 76-110 of 136:

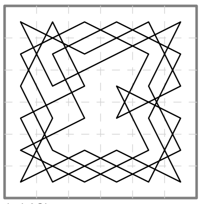

bl/fg (1)

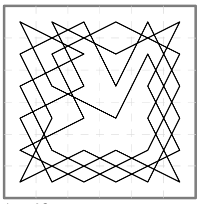

bl/fg (2) [10]

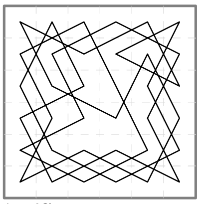

bl/fg (3) [8]

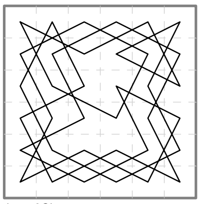

bl/fg (4)

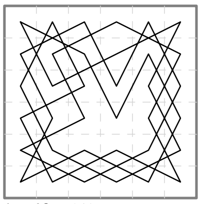

bl/fi

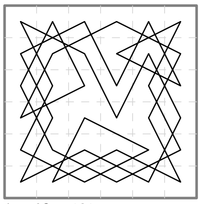

bl/fl

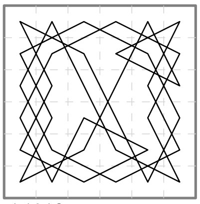

bn/fg

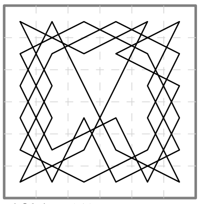

bn/fi

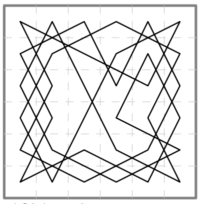

bn/fl

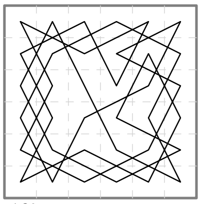

bo/fg (1) [8]

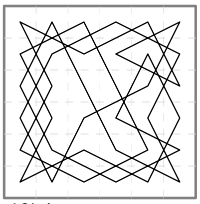

bo/fg (2)

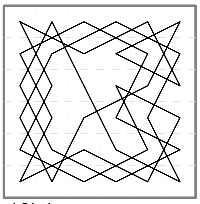

dd/df

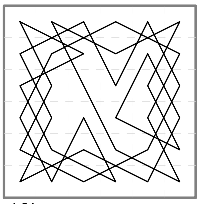

df/dg (1)

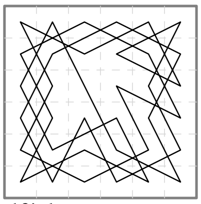

df/dg (2) [7,10,11]

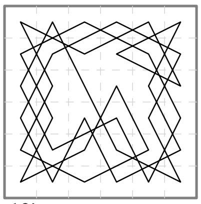

df/eg

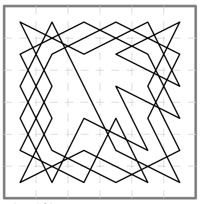

df/ei

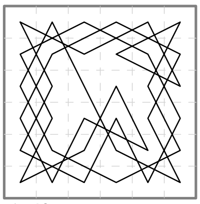

df/el

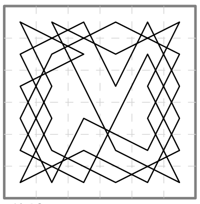

df/gg

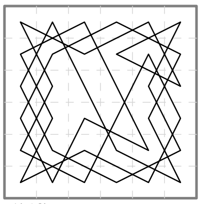

df/gl

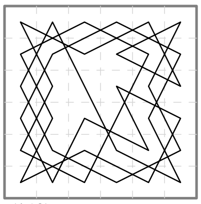

df/go

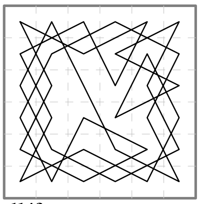

dg/fl

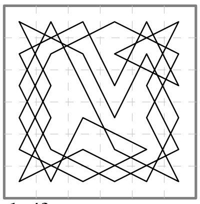

dg/fo

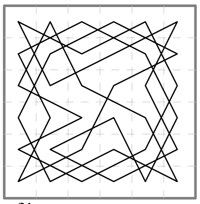

di/fg

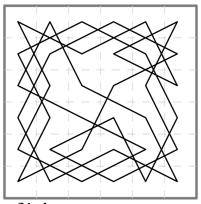

di/fi [9]

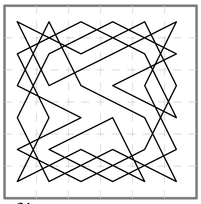

di/fl [9]

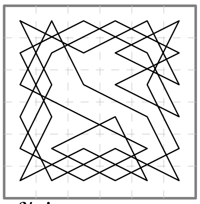

dl/fg

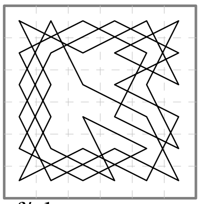

do/fg

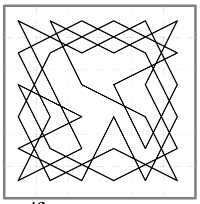

ef/eg

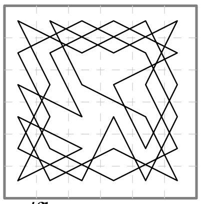

ef/ei

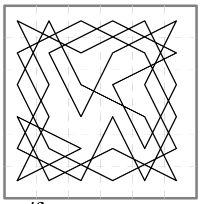

ef/gg


ef/gi

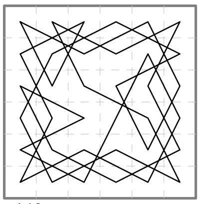

ef/gl

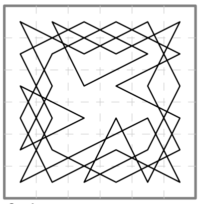

eg/fg

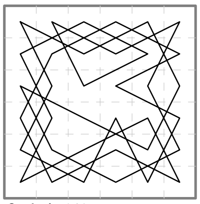

eg/fl

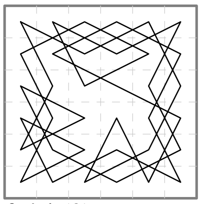

eg/fo

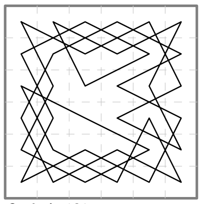
6 slants f-type 111-136 of 136:

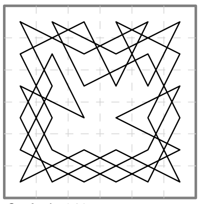

ei/fg

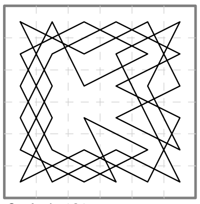

fg/gg [7]

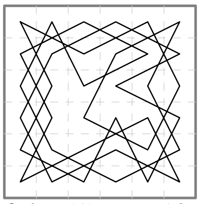

fg/gi (1) [7,10]

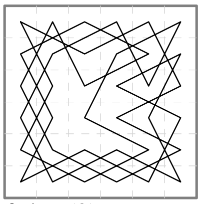

fg/gi (2)

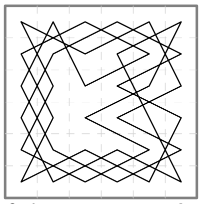

fg/gi (3)

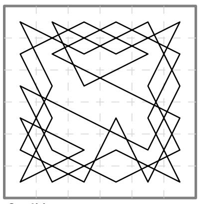

fg/gl (1)

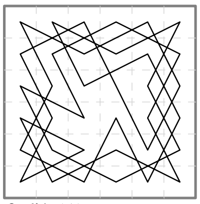

fg/gl (2) [8]

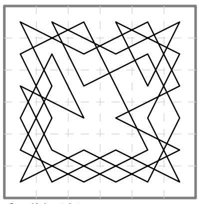

fg/gn (1) [7,10]

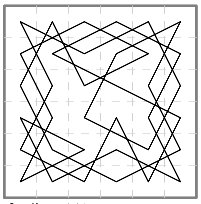

fg/gn (2)

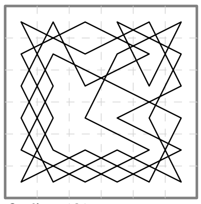

fg/go [8]

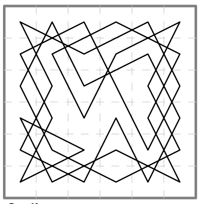

fg/ii

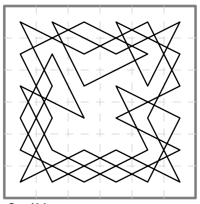

fg/il (1)

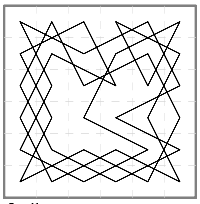

fg/il (2)

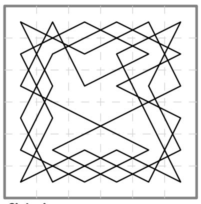

fg/in (1)

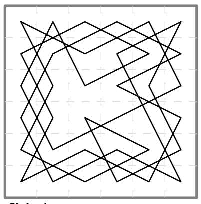

fg/in (2)

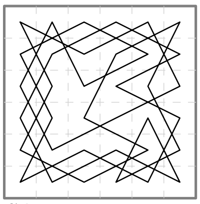

fg/io

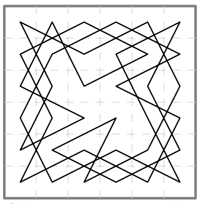

fg/ll

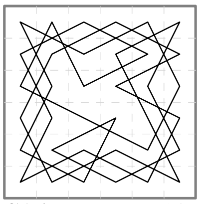

fg/ln

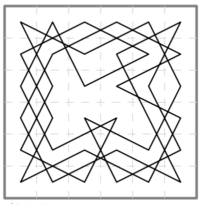

fi/gi

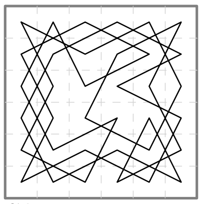

fi/gl [10]

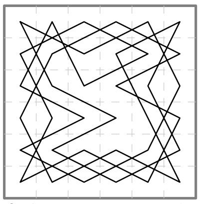

fi/gn


fl/gg [7,10]

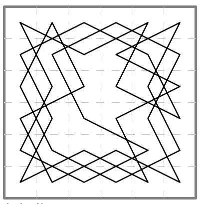

fl/gi [7,10]

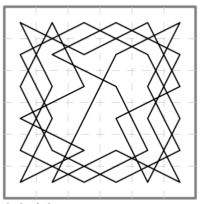

fl/gl [7,10]

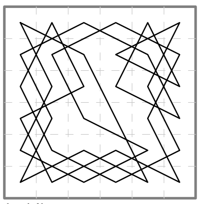

fl/gn

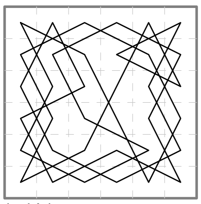

fo/gg [7,10]

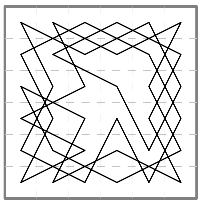
6 slants m-type 1-5 of 56:


bb/bm

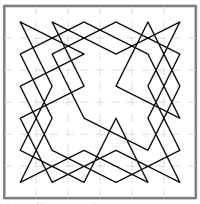

bb/dm

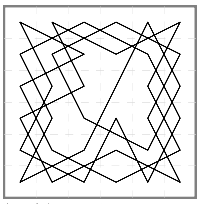

bd/bm

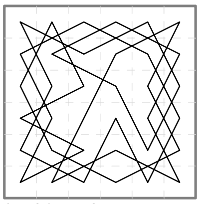

bd/dm [7]

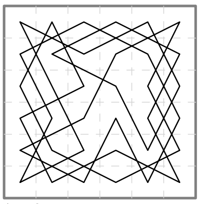

bg/bm (1)

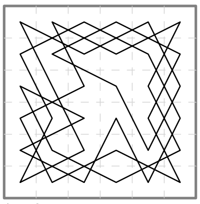
6 slants m-type 6-40 of 56:

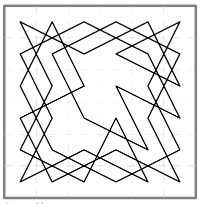

bg/bm (2)

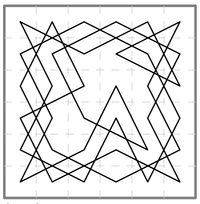

bg/dm (1)

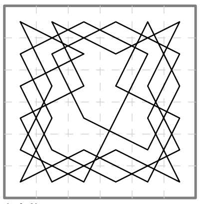

bg/dm (2)

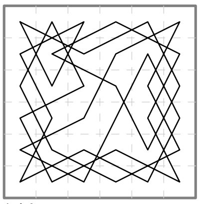

bg/em

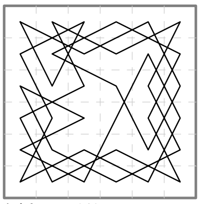

bg/gm

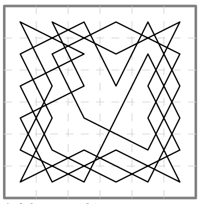

bg/lm

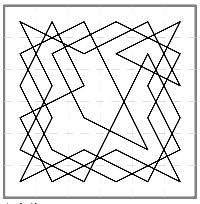

bg/mo

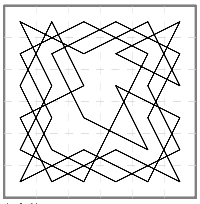

bi/bm

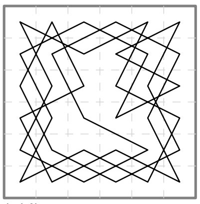

bi/em

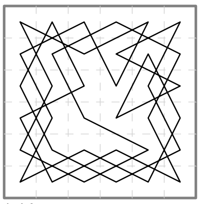

bi/gm (1)

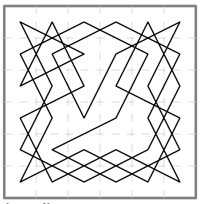

bi/gm (2)

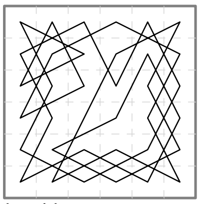

bi/im [9]

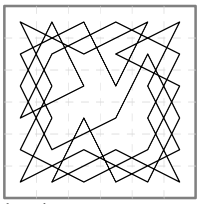

bi/lm [9]

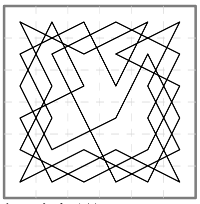

bl/bm

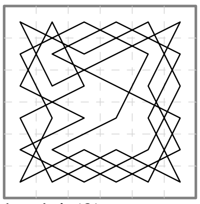

bl/gm

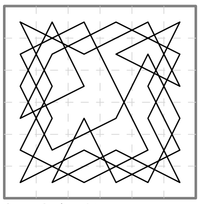

bm/bo

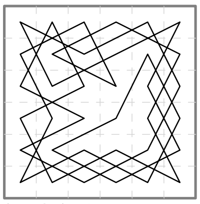

bm/dg

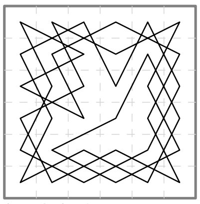

bm/gg

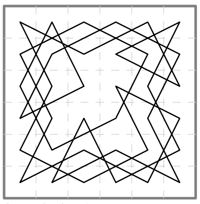

bm/gi (1)

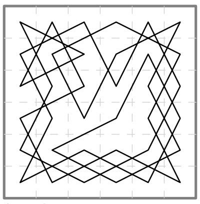

bm/gi (2)

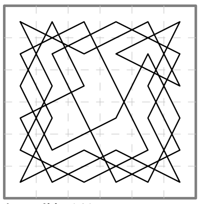

bm/gi (3)

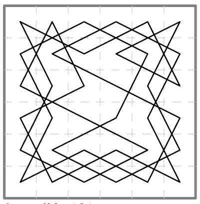

bm/gl (1)

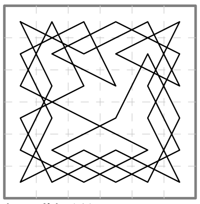

bm/gl (2)

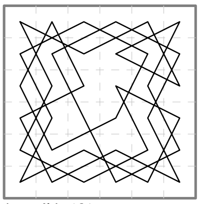

bm/gl (3)

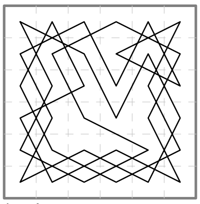

bm/go

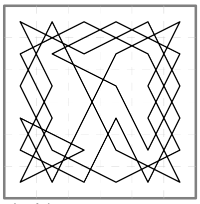

bm/ii (1)

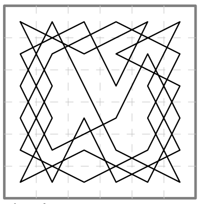

bm/ii (2)

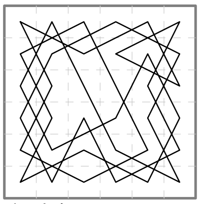

bm/il (1)

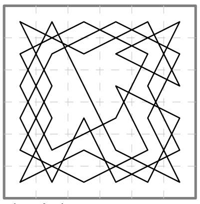

bm/il (2)

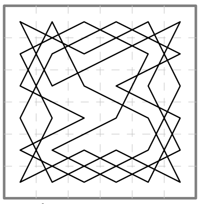

bo/gm


dg/dm [7]


dm/gg

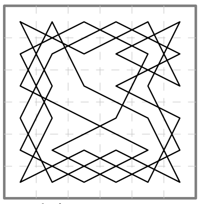

dm/gi

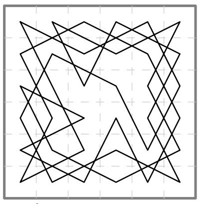

dm/gl

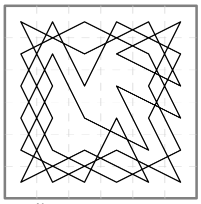

em/gg

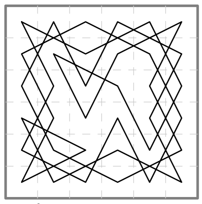
In the Knuth list the 6-slant tours are divided into 96 in class 2 and 208 in class 3 according to whether the two unbroken nets are toured successively or are separated.

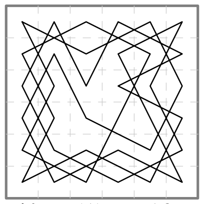
6 slants m-type 41-56 of 56: W = Warnsdorf

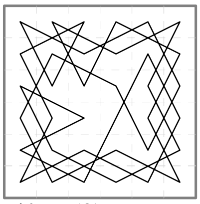

em/gi

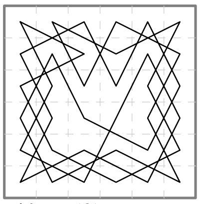

gg/gm

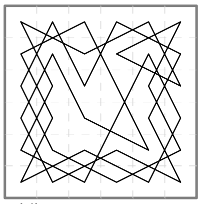

gg/lm

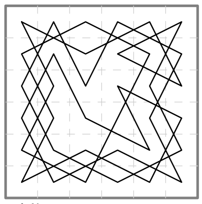

gg/mo

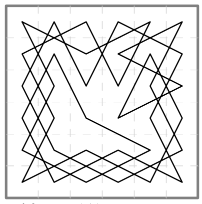

gi/gm (1) [7,10]W

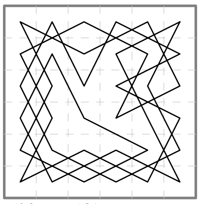

gi/gm (2)

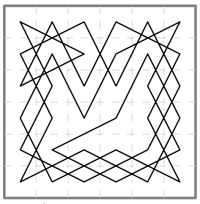

gi/gm (3)

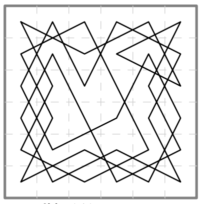

gi/im

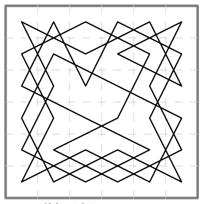

gi/lm [9]

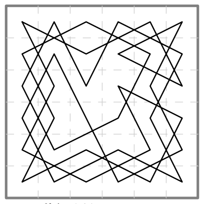

gl/gm (1)

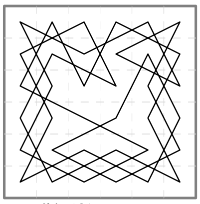

gl/gm (2) [10]


gm/go


gm/ii (1)


gm/ii (2)


gm/il (1)


gm/il (2)

Two-move lines in 6×6 closed tours.

The maximum of 4 two-move lines occurs in three tours: The symmetric 4ii/ii and the asymmetric 6ai/di (outlining a clear quadrilateral) and 6ad/dd (forming size [14] and [11] triangles).

There are 43 tours with 3 two-move lines: 4bi/ii 4gi/ii 4ii/il 6ab/dd 6ad/bd(1) 6ai/bi 6ad/bd(2) 6ai/dl 6al/di 6bd/df(1) 6bi/ci(1) 6bi/fi(1) 6ci/gi(2) 6dd/df 6df/dg(2) 6di/fi 6fi/gi 8aa/bd 8ab/ad 8ab/ai(2) 8ad/ai 8ad/df(1) 8ad/df(2) 8af/dd 8af/dg(4) 8ag/ai(1) 8ag/cd 8ai/ai 8ai/al(2) 8ai/bf(1) 8ai/cd 8ai/df 8ai/fg(4) 8ai/fi(1) 10aa/ab(1) 10aa/ai 10ac/ad(3) 10ac/ad(5) 10ac/ad(6) 10ac/ag(7) 10ac/ai(3) 10af/ag(7) 10af/ai. The 10ac/ad(6) is the only one to have the three lines parallel.

Small Triangles in 6×6 closed tours.

There are 54 tours in which all the triangles are size [6] or smaller. I call these ‘compact’. These are easily recognised by the absence of type [12] triangles. Many of these include all six sizes, but some exclude [3] and two (4ii/ll symmetric and 4ii/il asymmetric) exclude both [3] and [4]. There are just 2 with 8-slants (8bcfg and 8ffgg). Among these is Euler's symmetric tour 4gg/nn which has no size [1] triangles, making it a ‘Celtic’ tour.

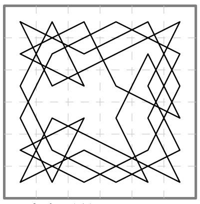
There are 28 other tours that have no [12]s. These consist of 20 showing [7]+[10], including the symmetric 8cc/gg, and four with [10] alone, namely 6bl/fg(2), 6cl/gi(1), 6fi/gi and 6gl/gm(2) which is the only other Celtic tour. The other four show: [8] in 6bo/fg(1), [8]+[10] in 6fg/go, [11] in 6bd/df(1), and [7]+[10]+[11] in 6df/dg(2). These last also include three two-move lines.

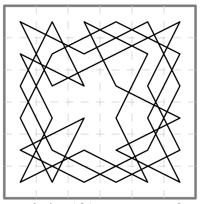
Continued on p.296.

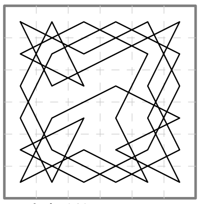
Asymmetric Closed Tours with 8 Slants

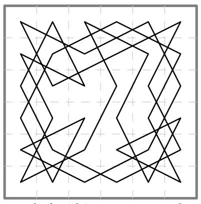
Total 527.

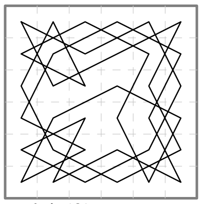
8 slants, triple type, 22 (aac 7, acf 10, acm 1, cff 3, cfm 1):

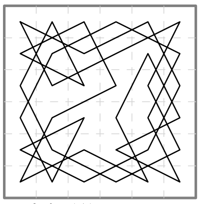

ac/ah (1)

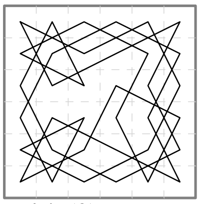

ac/ah (2) [9]

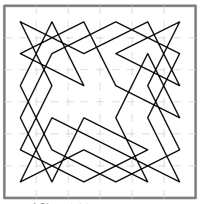

ac/aj (1)

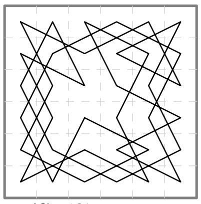

ac/aj (2) [9]

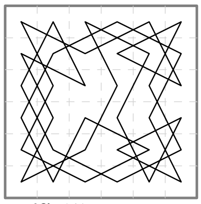

ac/aj (3)

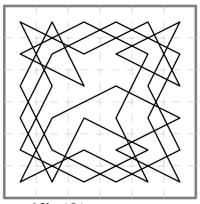

ac/ak (1)

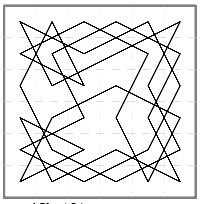

ac/ak (2)

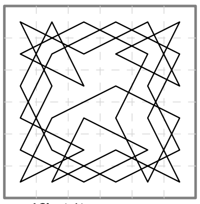

ac/fh (1)

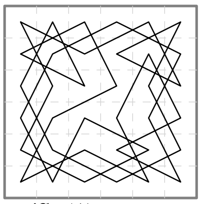

ac/fh (2)

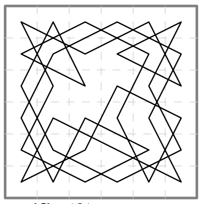

ac/fj (1)

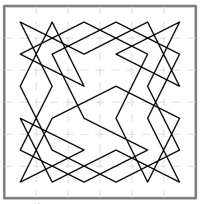

ac/fj (2)

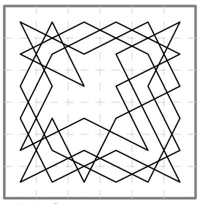

ac/fj (3)

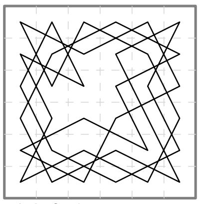

ac/fj (4)

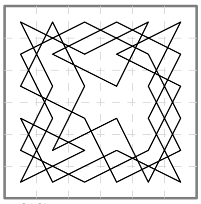

ac/fk (1)

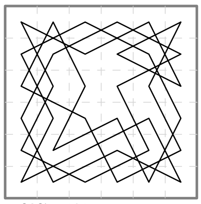

ac/fk (2)

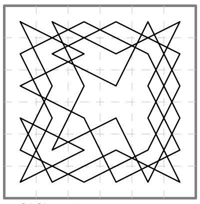

ac/jm

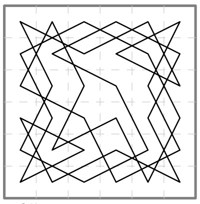

ah/cf (1)


ah/cf (2)


cf/fh (1)

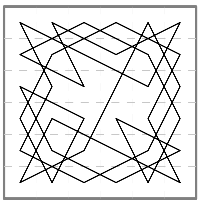

cf/fh (2)

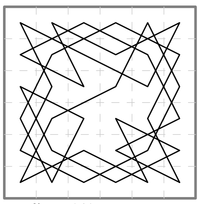

cf/fh (3)

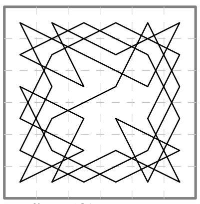

cf/hm

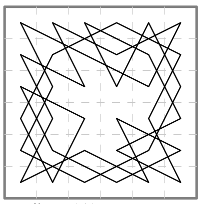
The tours 8ac/aj(1) and 8ac/fj(2) show the maximum 13 lateral obtuse angles.

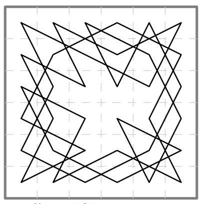
8 slants, double type: aa, 1-5 of 65:


aa/bd

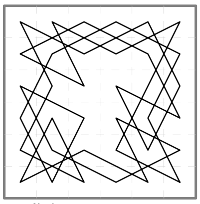

aa/be (1)

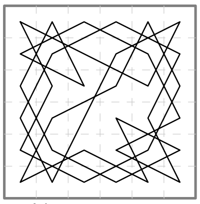

aa/be (2)

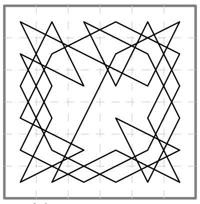

aa/bg (1) [7]

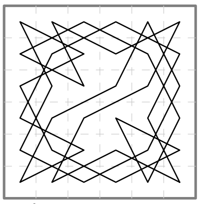

aa/bg (2) [7]

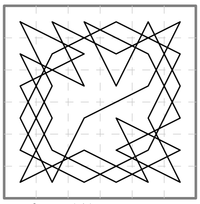
8 slants, aa-type, 6-20 of 65:

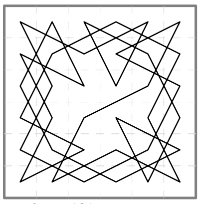

aa/bl [9]

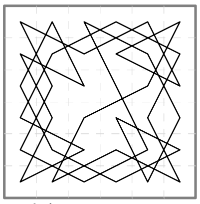

aa/de

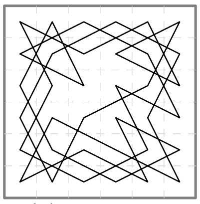

aa/dg [7]

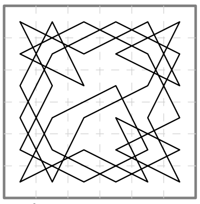

aa/ee

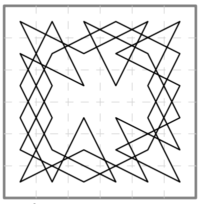

aa/eg (1)

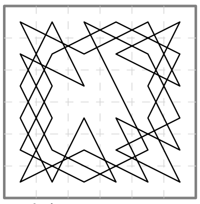

aa/eg (2)

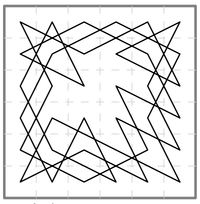

aa/ei

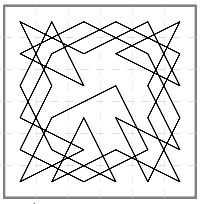

aa/el

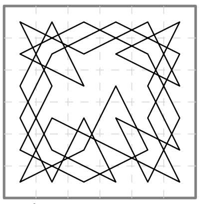

aa/en

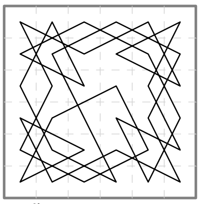

aa/gg


aa/gi

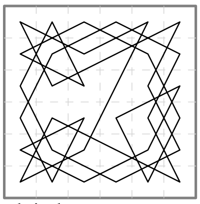

aa/gl

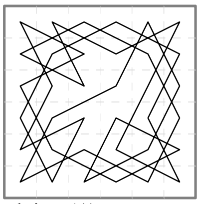

aa/gn

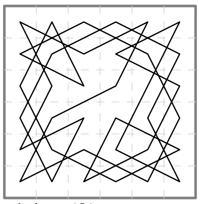

aa/go [7]

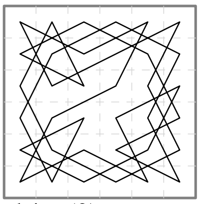

aa/in

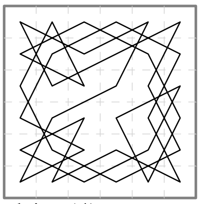
8 slants, aa-type, 21-40 of 65:

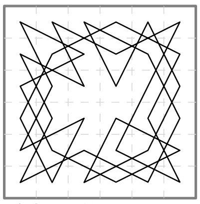

ab/ad

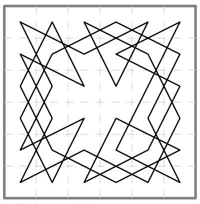

ab/ae (1)

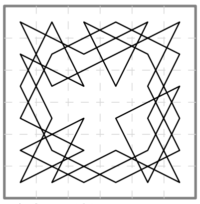

ab/ae (2)

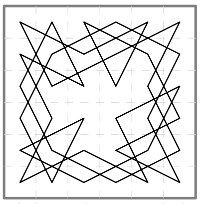

ab/ae (3)

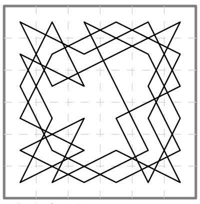

ab/ae (4)

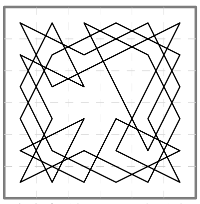

ab/ag (1)

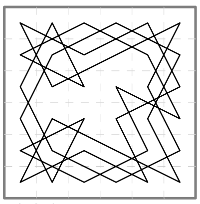

ab/ag (2)

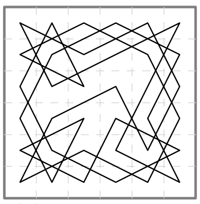

ab/ag (3) [8,9]

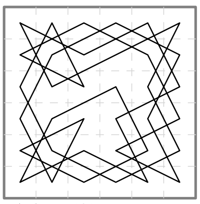

ab/ag (4) [8,9]

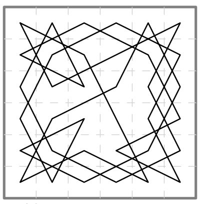

ab/ai (1) [9]

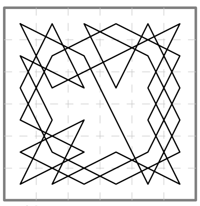

ab/ai (2) [9,18]

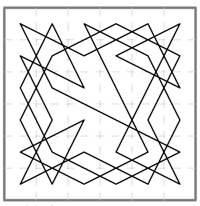

ab/al

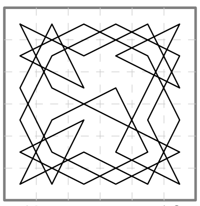

ab/an (1)

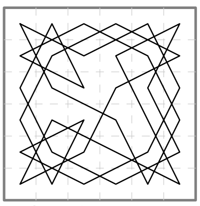

ab/an (2)

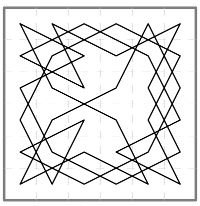

ad/ae


ad/ag [8,9]

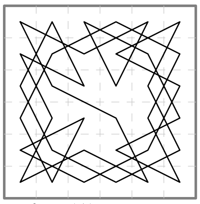

ad/ai

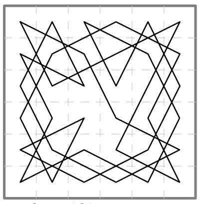

ad/an [10]

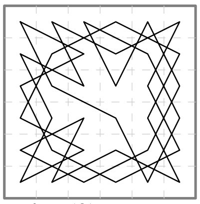

ae/ae (1)

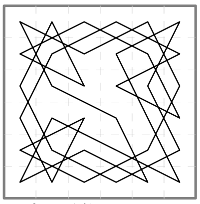

ae/ae (2)

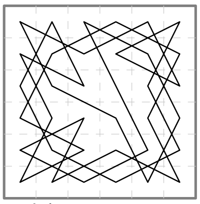
8 slants, aa-type, 41-65 of 65:

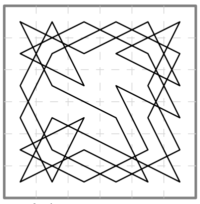

ae/ag (1)

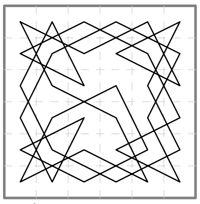

ae/ag (2) [8,9]

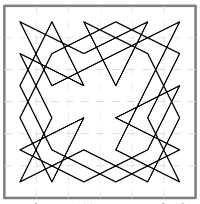

ae/ag (3)

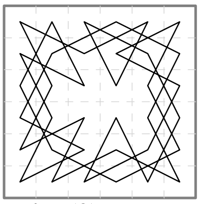

ae/ag (4) [7]

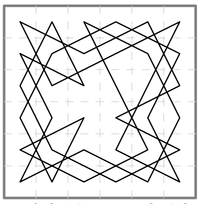

ae/ai

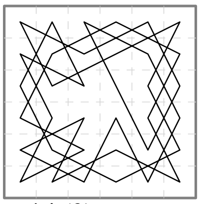

ae/al

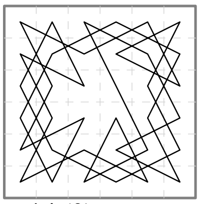

ae/an

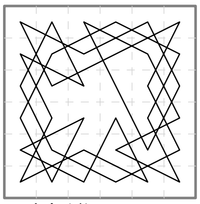

ag/ag (1) [8,9]

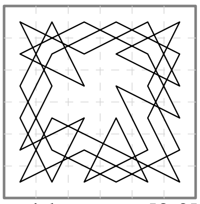

ag/ag (2)

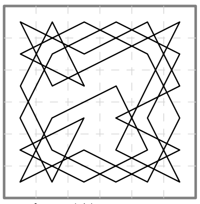

ag/ai (1) [7,9,18]

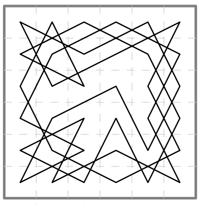

ag/ai (2) [9]

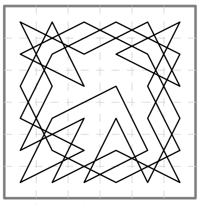

ag/ai (3)

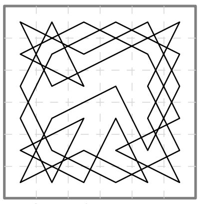

ag/ai (4) [9]

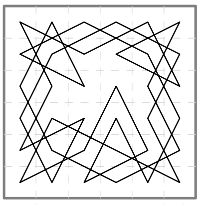

ag/al [8,9]

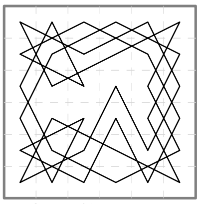

ag/an (1)

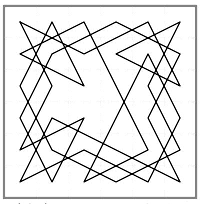

ag/an (2)

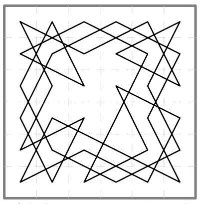

ag/an (3)

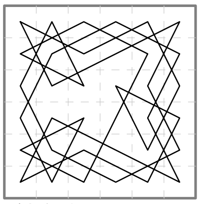

ag/an (4)

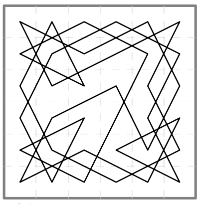

ag/ao (1) [8,9]

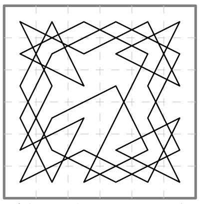

ag/ao (2) [8]


ai/ai [9,15]

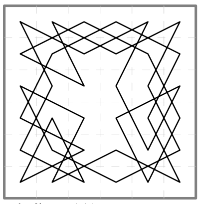

ai/al (1) [9,15]

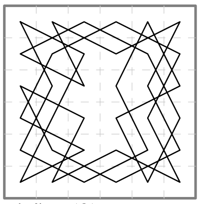

ai/al (2)

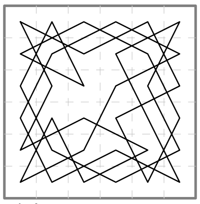

ai/an (1)

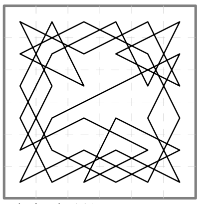

ai/an (2) [15]

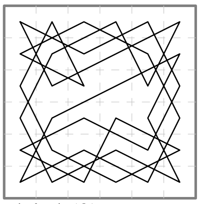
8 slants, ac-type, 1-10 of 53:

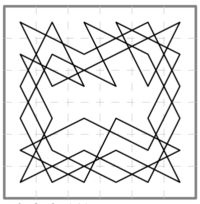

ab/bc (1)

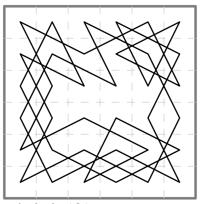

ab/bc (2)

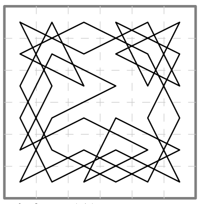

ab/ce

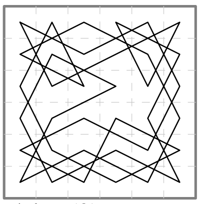

ab/cd (1)

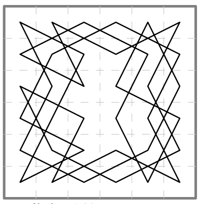

ab/cd (2)


ab/cl (1) [9]

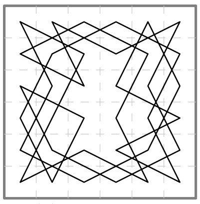

ab/cl (2)

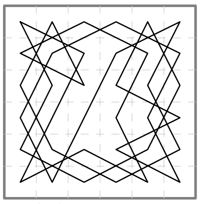

ab/co (1)

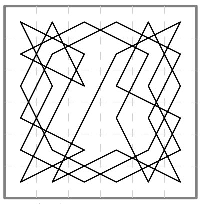

ab/co (2)

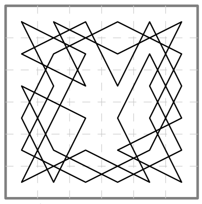

ac/bb (1) [9]

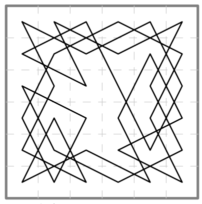
8 slants, ac-type, 11-30 of 53:

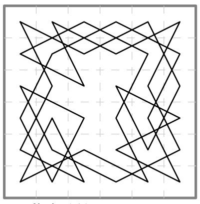

ac/bb (2) [9]

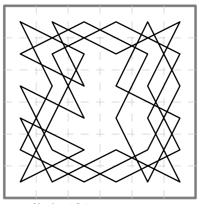

ac/bd (1)

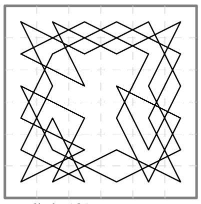

ac/bd (2)

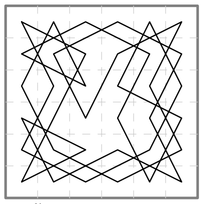

ac/bg [9]

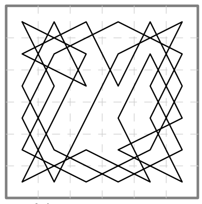

ac/bi

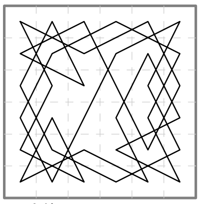

ac/bl (1)

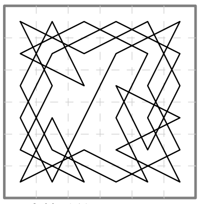

ac/bl (2) [9]

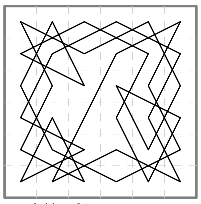

ac/bl (3)

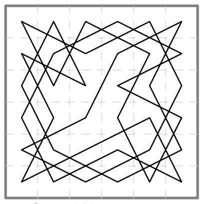

ac/bo

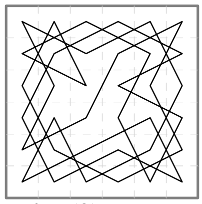

ac/dg

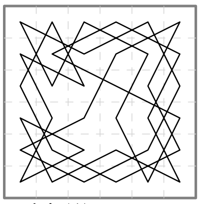

ac/di

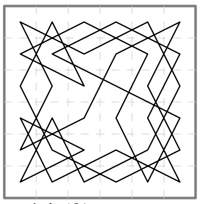

ac/dl (1)

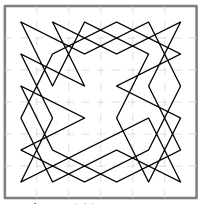

ac/dl (2)

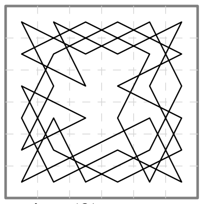

ac/eg (1)

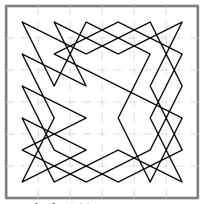

ac/eg (2)


ac/ei (1)

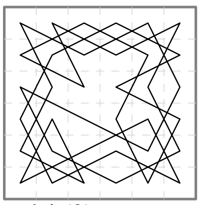

ac/ei (2)

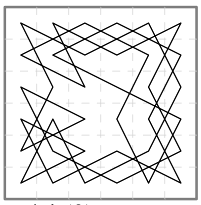

ac/gg (1)

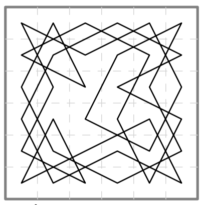

ac/gg (2)

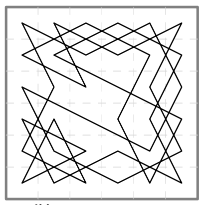

ac/gi (1)

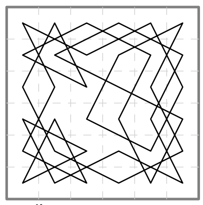
8 slants, ac-type, 31-45 of 53:

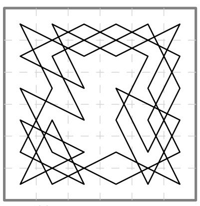

ac/gi (2)

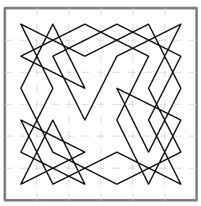

ac/gi (3)

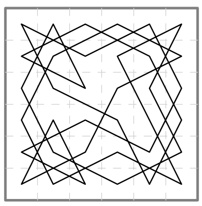

ac/gn

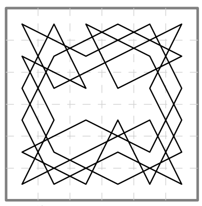

ac/ii

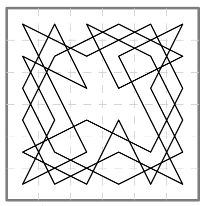

ac/in

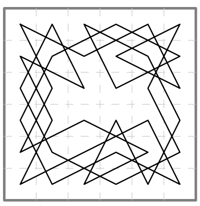

ac/ll

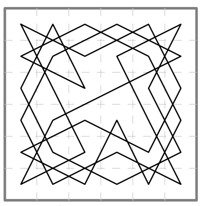

ac/lo

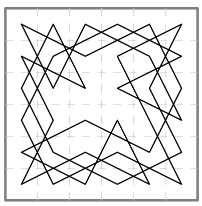

ae/ce

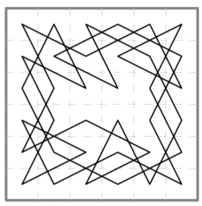

ag/bc (1) [7,9,10]

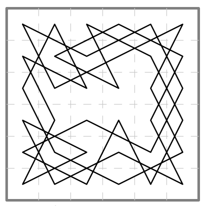

ag/bc (2)


ag/bc (3) [7]

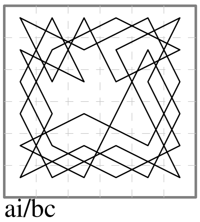

ag/cd

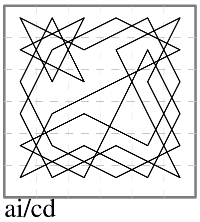

ag/cg

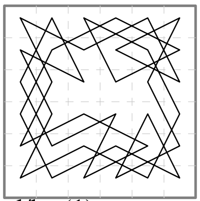

ag/cl (1)

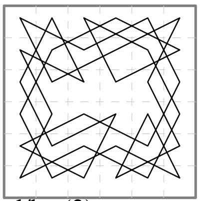

ag/cl (2) [9]

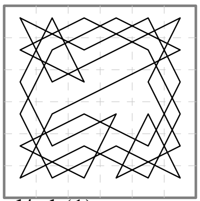
8 slants, ac-type, 46-53 of 53:

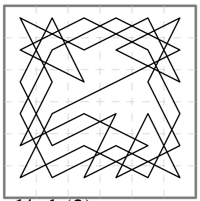

ai/bc

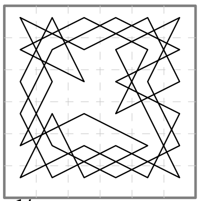

ai/cd

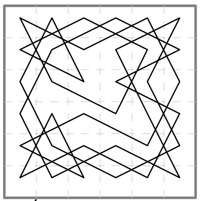

al/bc (1)


al/bc (2) [9]

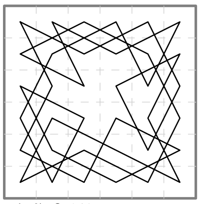

al/cd (1)

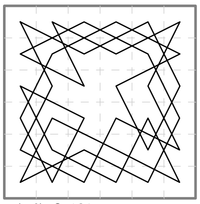

al/cd (2)

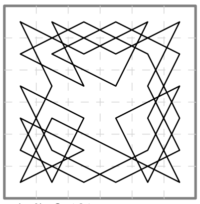

al/cg [10]

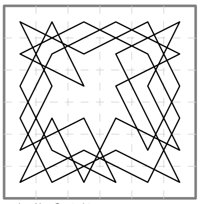

an/cg [7,10]

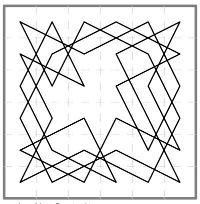
8 slants, af-type, 1-25 of 149: (W = Warnsdorf)

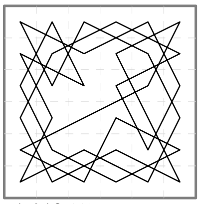

ab/bf (1)

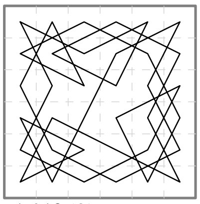

ab/bf (2)

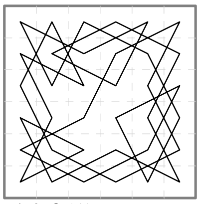

ab/bf (3)

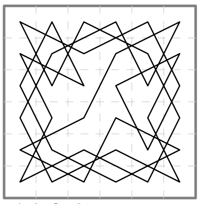

ab/bf (4)

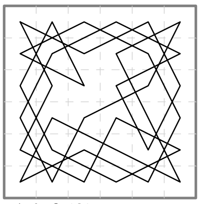

ab/bf (5)

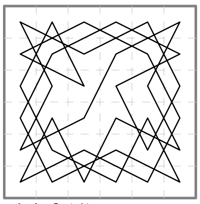

ab/df (1)

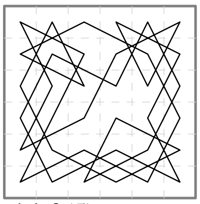

ab/df (2)

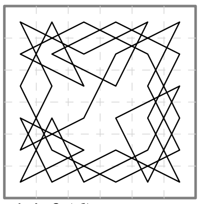

ab/ef (1)

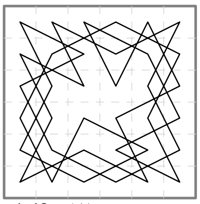

ab/ef (2)

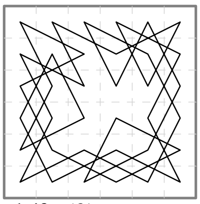

ab/ef (3) W

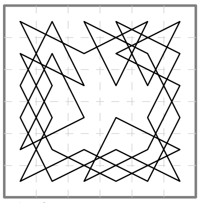

ab/ef (4)

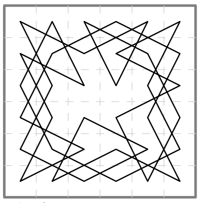

ab/ef (5)

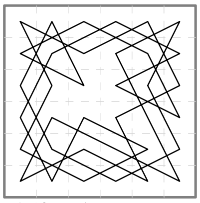

ab/ef (6)

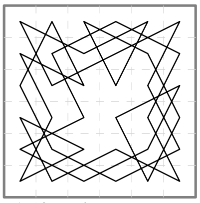

ab/fg (1)

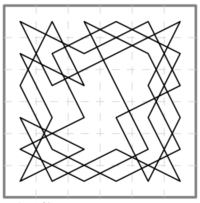

ab/fg (2)

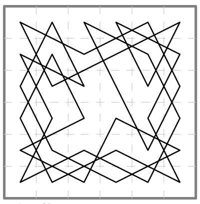

ab/fg (3)

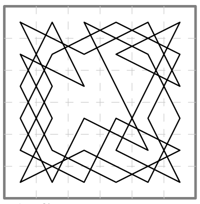

ab/fg (4)

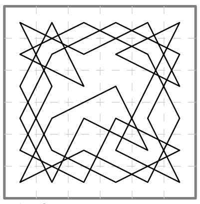

ab/fg (5)

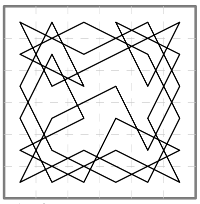

ab/fg (6) [8,9]

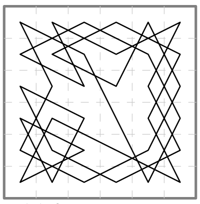

ab/fi (1) [9]


ab/fi (2) [9]

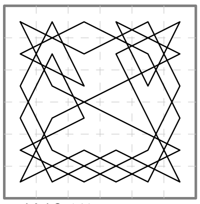

ab/fi (3)

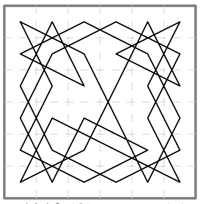

ab/fn (1)

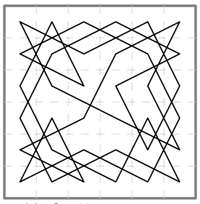

ab/fn (2)

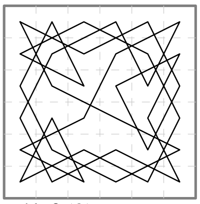

ad/bf

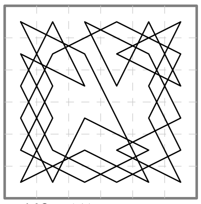
8 slants, af-type, 26-50 of 149:

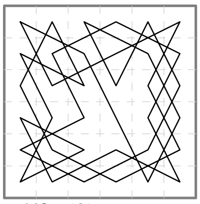

ad/df (1) [14]

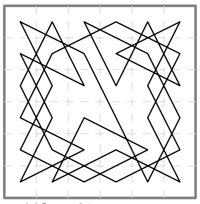

ad/df (2) [7,11]

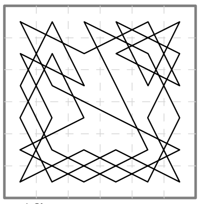

ad/ef (1)

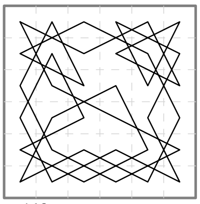

ad/ef (2)

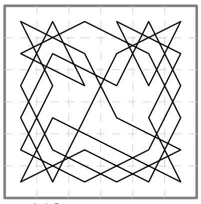

ad/fg (1)

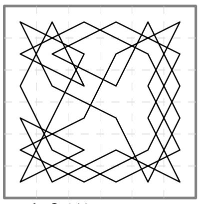

ad/fg (2) [8,9]

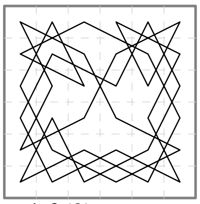

ad/fg (3)

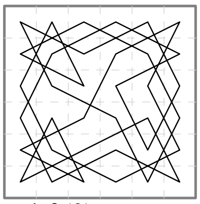

adfi

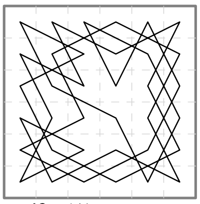

ad/fn [10]

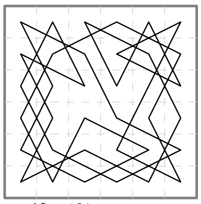

ae/df

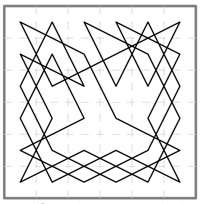

ae/ef (1)

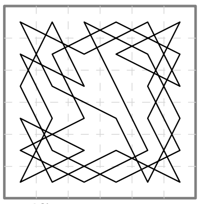

ae/ef (2)

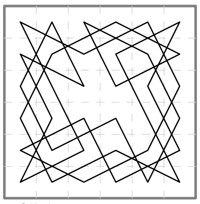

ae/ef (3)

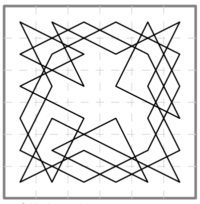

ae/fg (1)

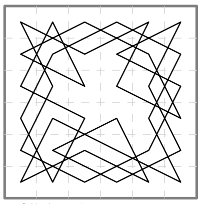

ae/fg (2)

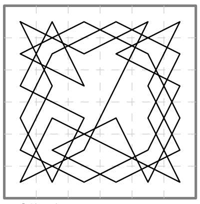

ae/fg (3) [8,9]

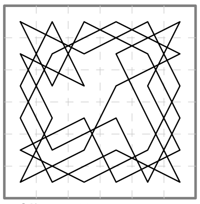

ae/fi

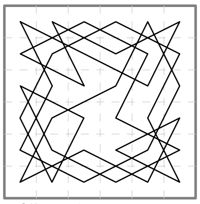

af/bb (1)

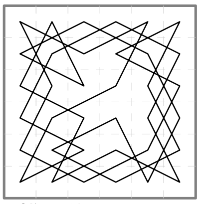

af/bb (2) [10]

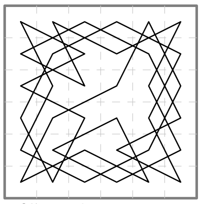

af/bb (3) [10]


af/bd [10]

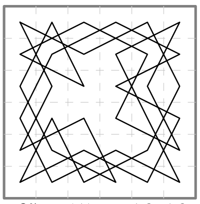

af/be (1)

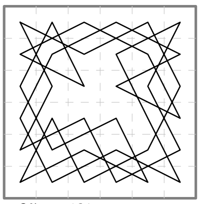

af/be (2)

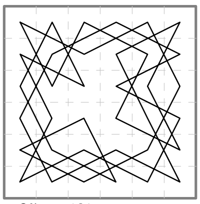

af/be (3)

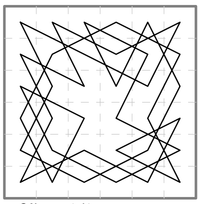

af/be (4)

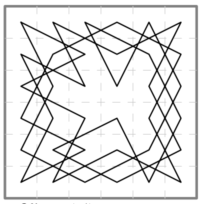
8 slants, af-type, 51-60 of 149:

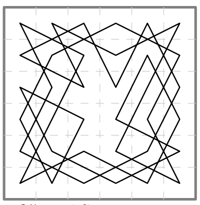

af/bg (1) [7,10,10]

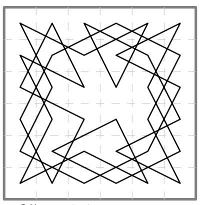

af/bg (2) [7]

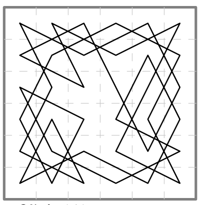

af/bg (3) [7,10,10]

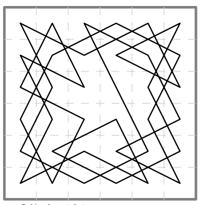

af/bg (4)

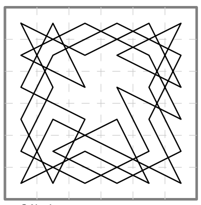

af/bg (5)


af/bg (6) [9]

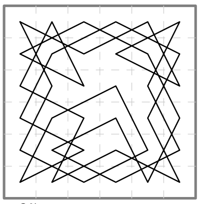

af/bg (7)

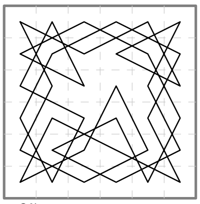

af/bi (1)

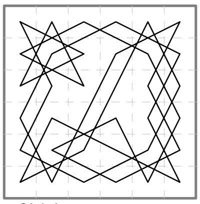

af/bi (2)

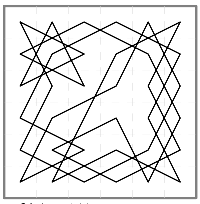

af/bl [10]

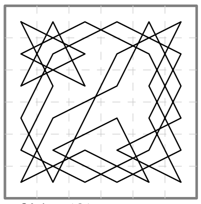
8 slants, af-type, 61-70 of 149:

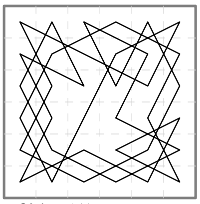

af/bn

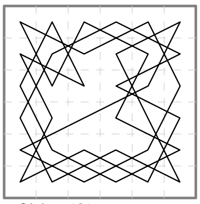

af/bo [10]

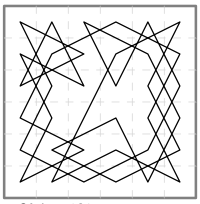

af/dd [10]

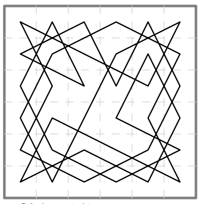

af/de (1)

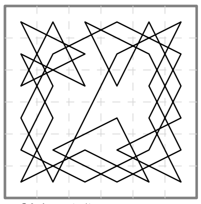

af/de (2)


af/dg (1) [7]

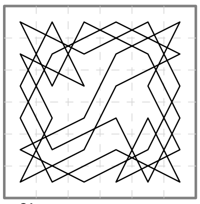

af/dg (2) [7,10,10]

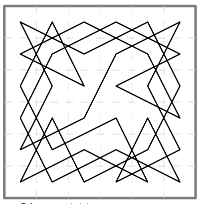

af/dg (3)

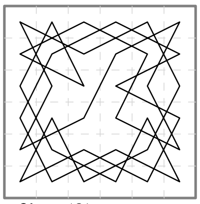

af/dg (4) [7,10]

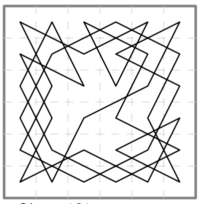

af/dg (5)

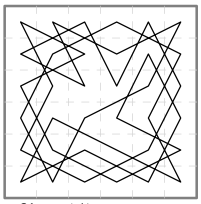
8 slants, af-type, 71-95 of 149:

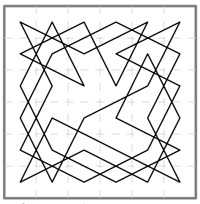

af/ee

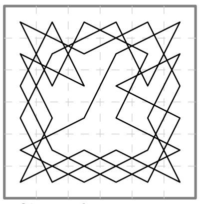

af/eg (1)

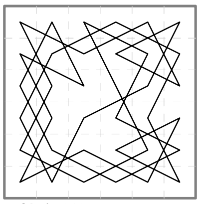

af/eg (2)

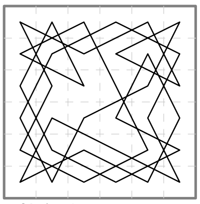

af/eg (3)

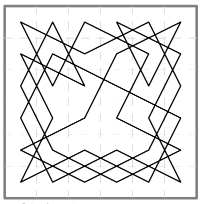

af/eg (4)

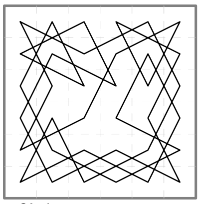

af/eg (5)

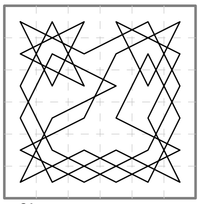

af/eg (6)

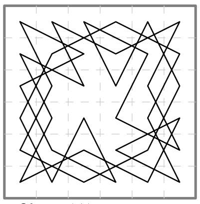

af/ei (1) [15]

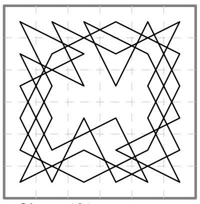

af/ei (2)

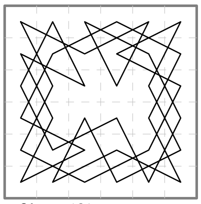

af/ei (3)

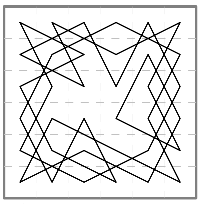

af/el

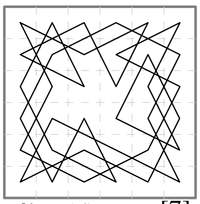

af/eo

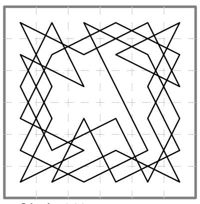

af/gg (1)

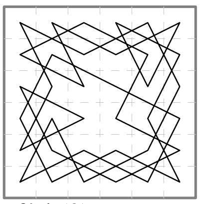

af/gg (2)

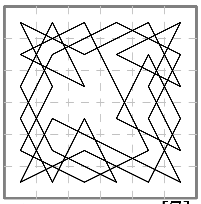

af/gg (3)

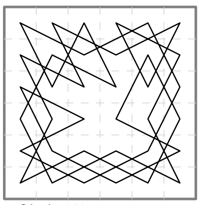

af/gg (4) [7]

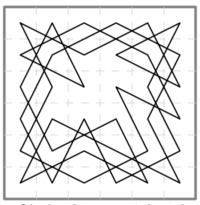

af/gg (5) [7]

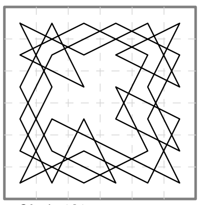

af/gi (1)

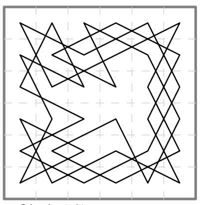

af/gi (2)

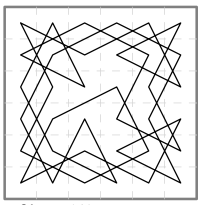

af/gi (3) [7]


af/gl (1)

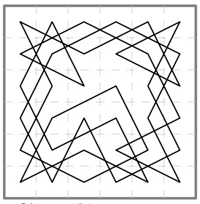

af/gl (2) [7,10,10]

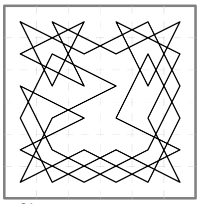

af/gl (3) [7]

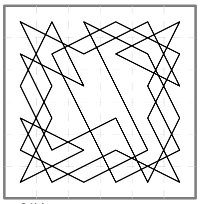

af/gl (4) [9]

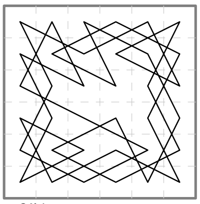

af/gn (1)

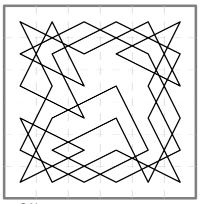
8 slants, af-type, 96-120 of 149:

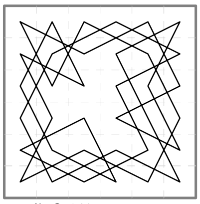

af/gn (2)

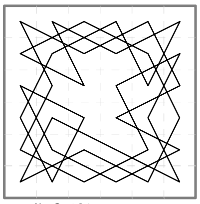

af/go

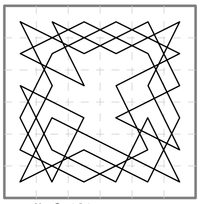

af/ii

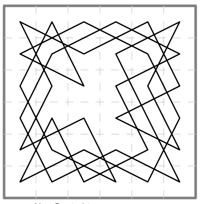

af/il

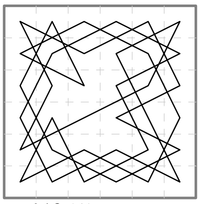

af/ln

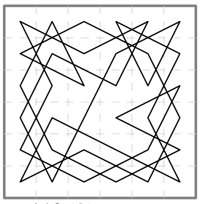

ag/bf (1) [8]

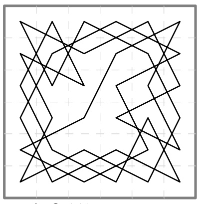

ag/bf (2)

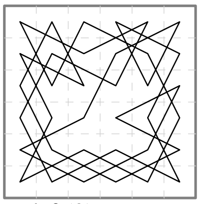

ag/bf (3)

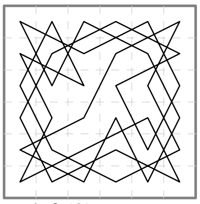

ag/bf (4) [8]

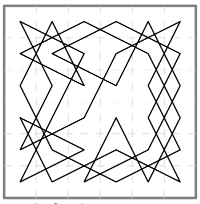

ag/df (1) [8]

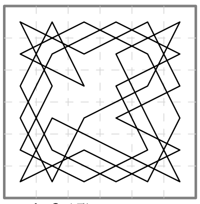

ag/df (2)

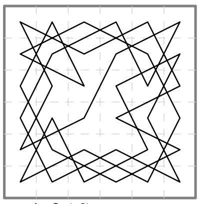

ag/ef (1)

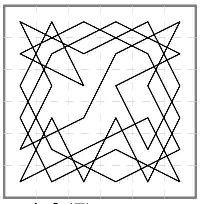

ag/ef (2)

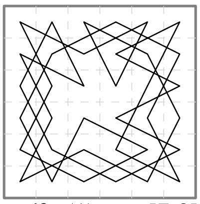

ag/ef (3)

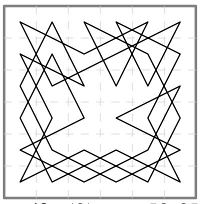

ag/ef (4)

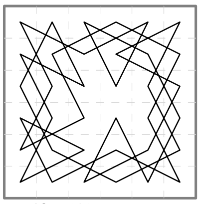

ag/ef (5) [8]

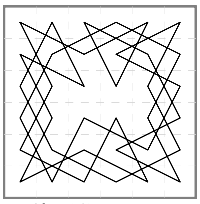

ag/ef (6)

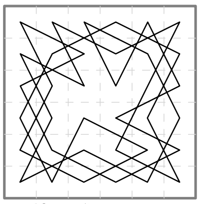

ag/ef (7)

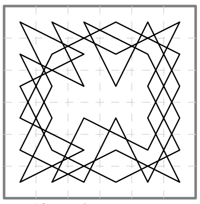

ag/fg (1) [7,8]

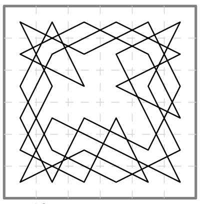

ag/fg (2) [8,9]


ag/fg (3)

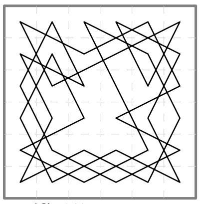

ag/fg (4)

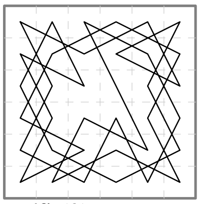

ag/fg (5) [7,10]

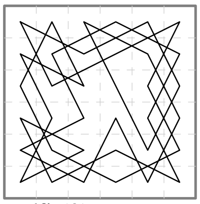

ag/fg (6)

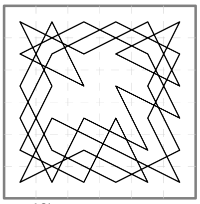

ag/fg (7)

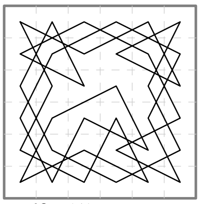
8 slants, af type, 121-130 of 149:

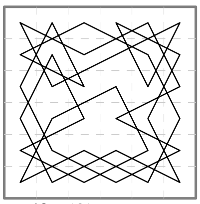

ag/fi (1) [9]

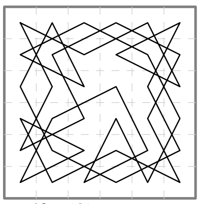

ag/fi (2)

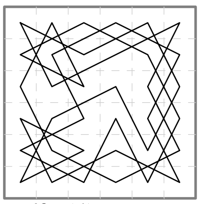

ag/fi (3) [9]

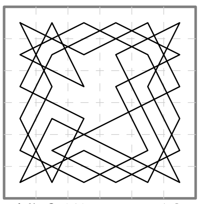

ag/fl [8]

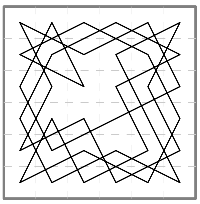

ag/fn (1)


ag/fn (2)

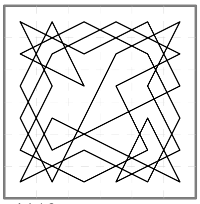

ag/fn (3)

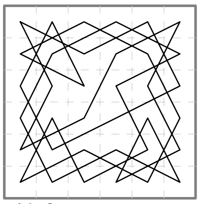

ag/fn (4)

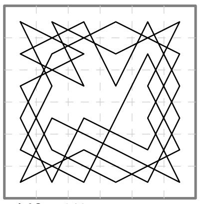

ai/bf (1) [18]

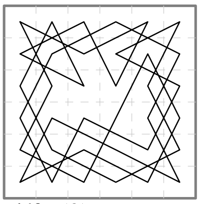

ai/bf (2)

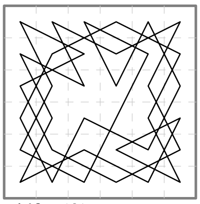
8 slants, af-type, 131-149 of 149:

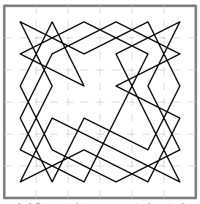

ai/df

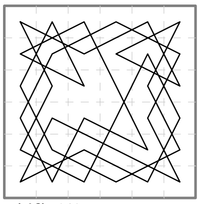

ai/ef

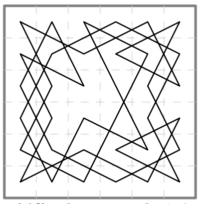

ai/fg (1)

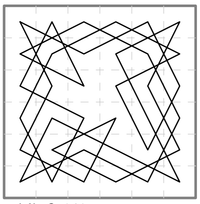

ai/fg (2)

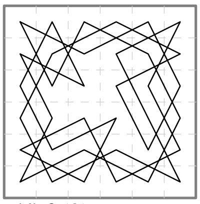

ai/fg (3)

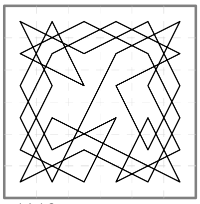

ai/fg (4) [7,10,18]

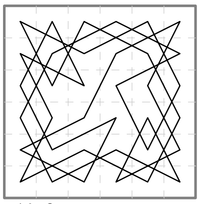

ai/fi (1) [9]

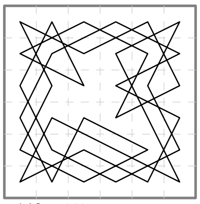

ai/fi (2) [9,15]

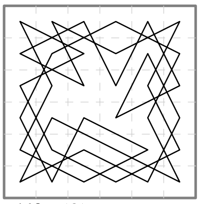

al/bf (1)

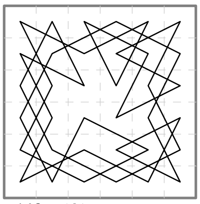

al/bf (2)

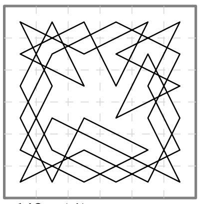

al/df

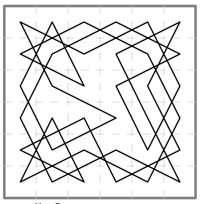

al/ef

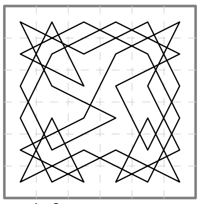

al/fg (1)

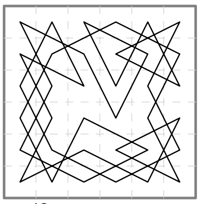

al/fg (2)


al/fg (3)

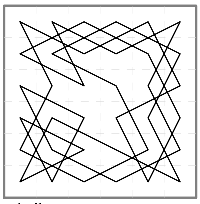

al/fg (4)

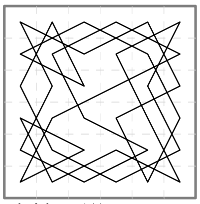

ao/bf

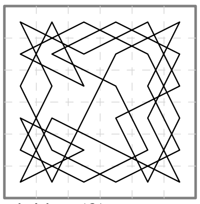

ao/ef

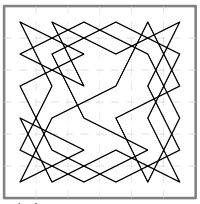

ao/fg

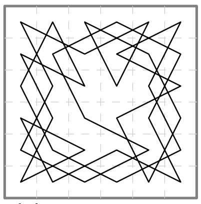
8 slants, am type, 1-15 of 39:

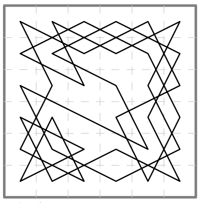

ab/bm

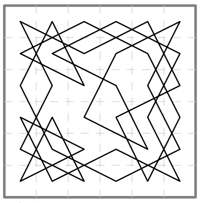

ab/dm (1)

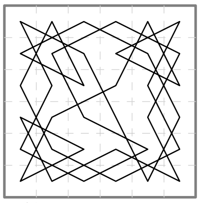

ab/dm (2)

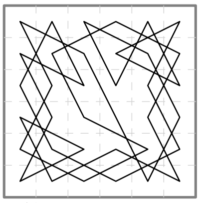

ab/em

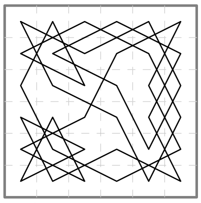

ab/gm

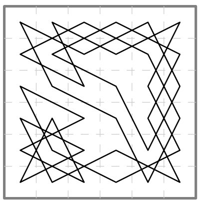

ab/im

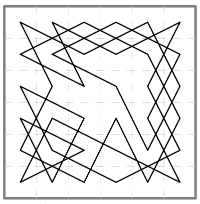

ab/mn

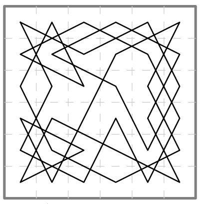

ad/em

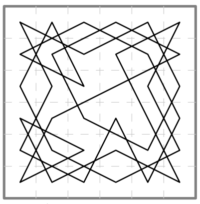

ad/gm

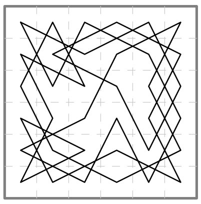

ae/em


ae/gm

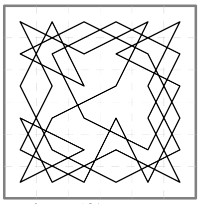

ag/bm [8]

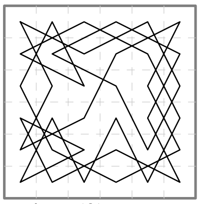

ag/dm (1) [8]

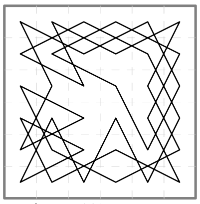

ag/dm (2)

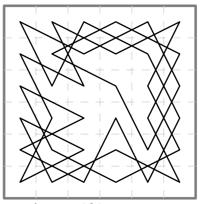

ag/em (1)

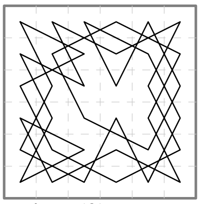
8 slants, am type, 16-39 of 39:

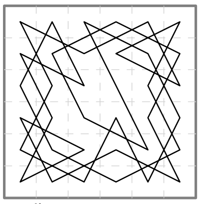

ag/em (2)

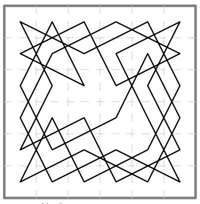

ag/em (3)

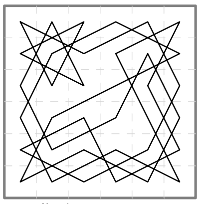

ag/gm (1)

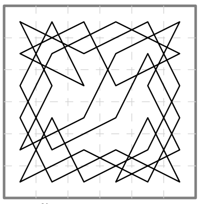

ag/gm (2)

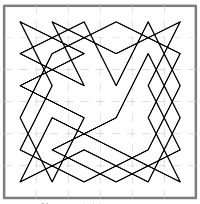

ag/gm (3)

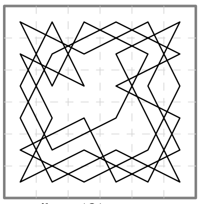

ag/im

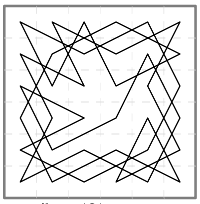

am/bb

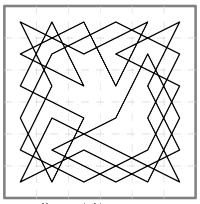

am/bd

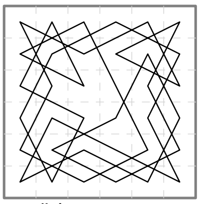

am/be

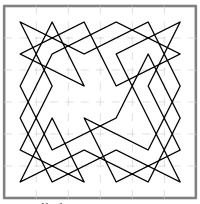

am/bg (1)

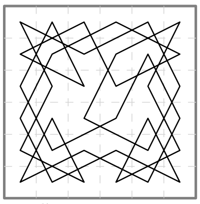

am/bg (2) [7,10]

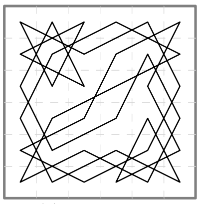

am/bg (3)

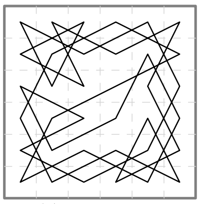

am/bg (4)

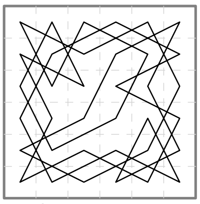

am/bi

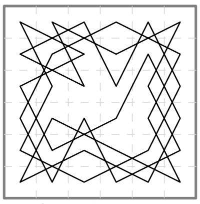

am/bl

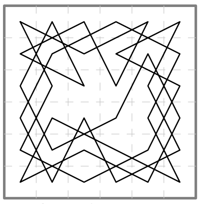

am/bn

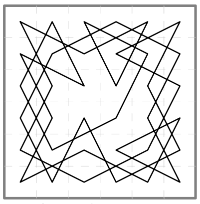

am/de

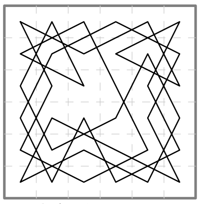

am/dg

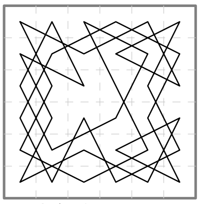

am/eg


am/gg (1)

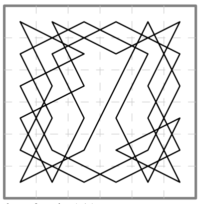

am/gg (2)

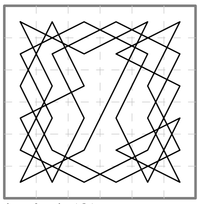

am/gg (3)

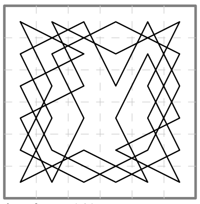

am/gi (1)

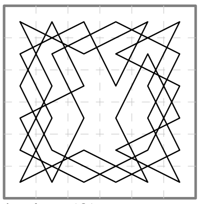

am/gi (2) [15]

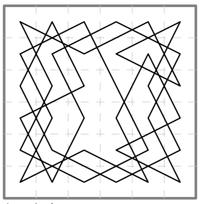
8 slants, cc type, 1-10 of 15:

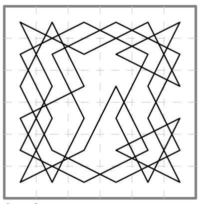

bc/cd (1)

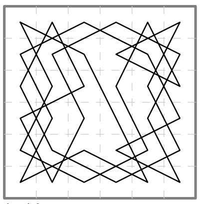

bc/cd (2)

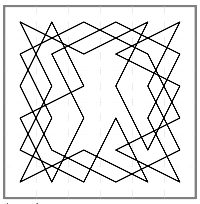

bc/cg (1)

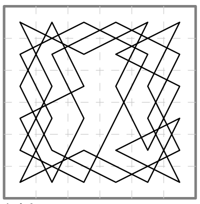

bc/cg (2)

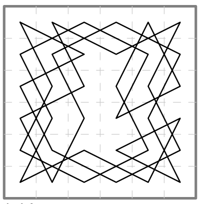

bc/ci


bc/co

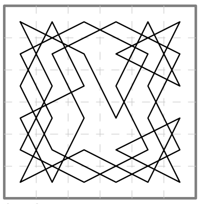

bd/cc

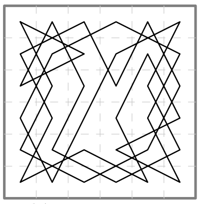

bg/cc

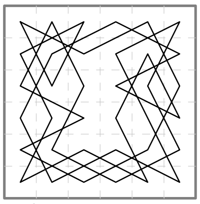

bi/cc

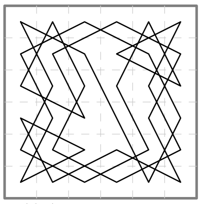

bl/cc

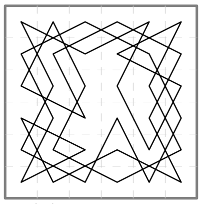
8 slants, cc type, 11-15 of 15:


bo/cc

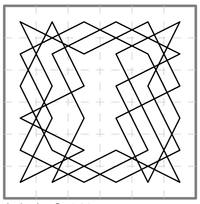

cc/dg

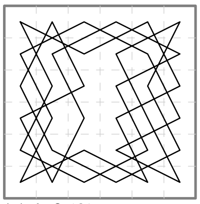

cc/gg [7]

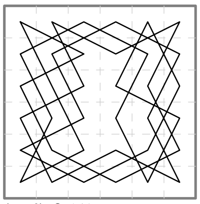

cd/cl

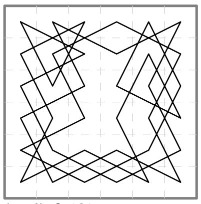

cg/cl

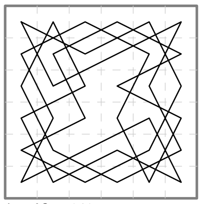
8 slants, cf type, 1-30 of 61: T = Takefuji

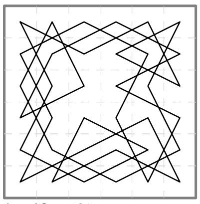

bb/cf (1)

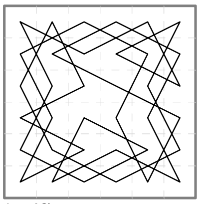

bb/cf (2)

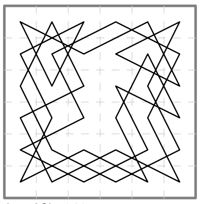

bc/bf (1)

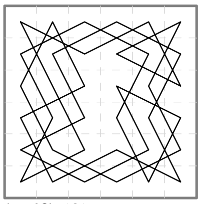

bc/bf (2)

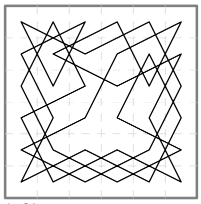

bc/fg (1)

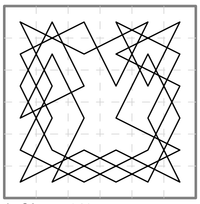

bc/fg (2) [10]

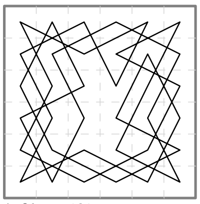

bc/fi

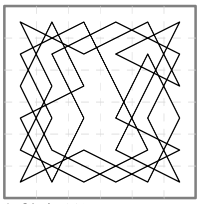

bc/fl (1)

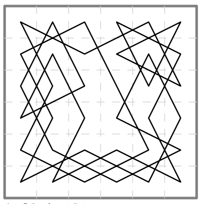

bc/fl (2)

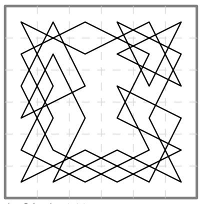

bf/ce

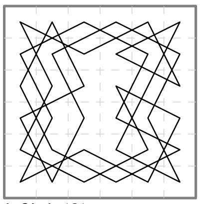

bf/cg (1)

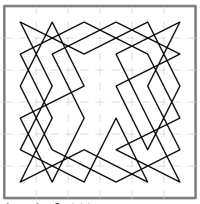

bf/cg (2)

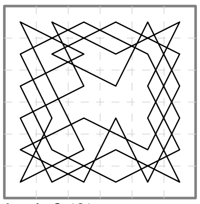

bf/ci (1)

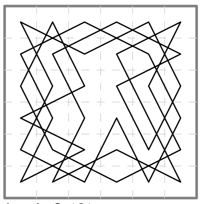

bf/ci (2)

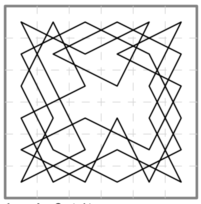

bf/cl (1)

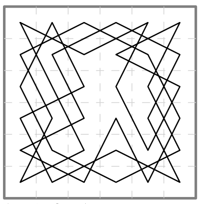

bf/cl (2)

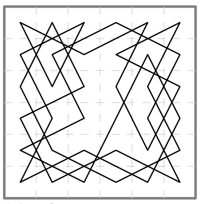

bg/cf (1)

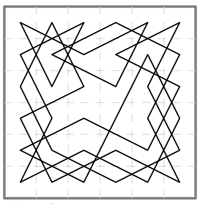

bg/cf (2)

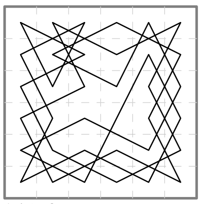

bg/cf (3)

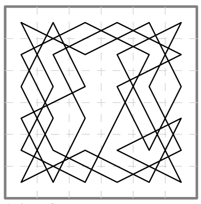

bg/cf (4)

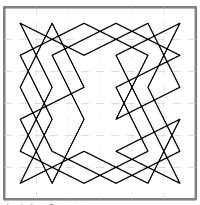

bg/cf (5)

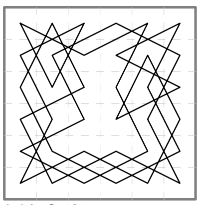

bi/cf (1)

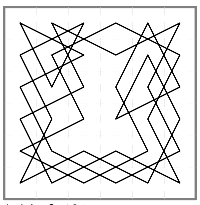

bi/cf (2) T

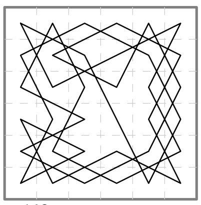

bi/cf (3)

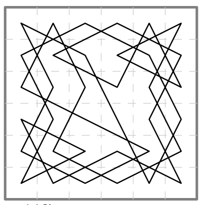

bi/cf (4)


bl/cf (1)

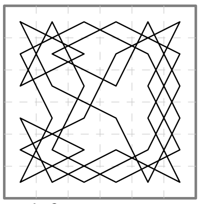

bl/cf (2)

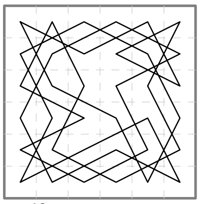

bl/cf (3)

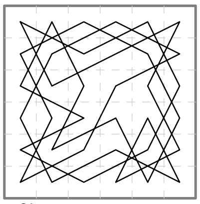

cd/fg [10]

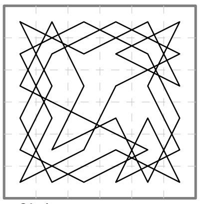

cd/fi

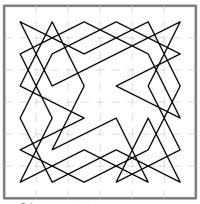
8 slants, cf type, 31-40 of 61:

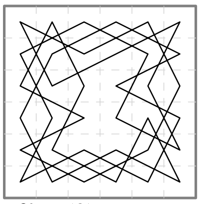

ce/ef

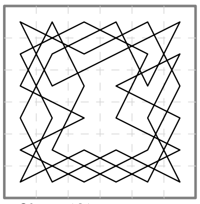

ce/fg

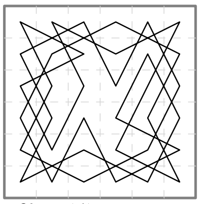

cf/eg

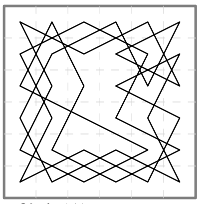

cf/ei [9]

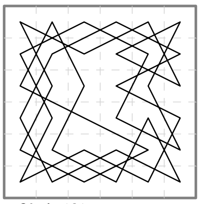

cf/gg (1)


cf/gg (2)

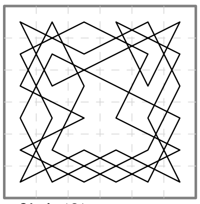

cf/gg (3)

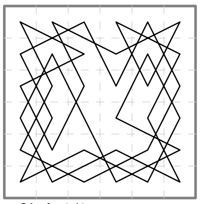

cf/gg (4)

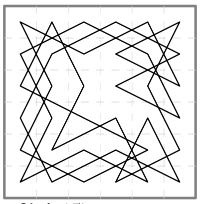

cf/gi (1)

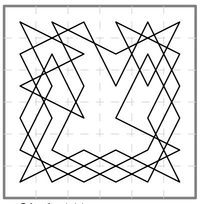

cf/gi (2)

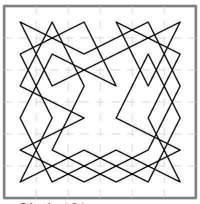
8 slants, cf type, 41-61 of 61:

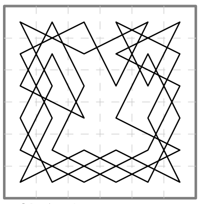

cf/gi (3)

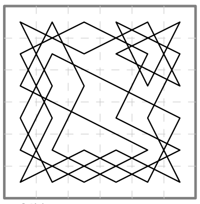

cf/gi (4)

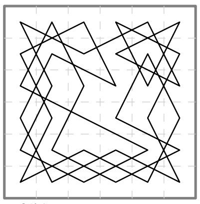

cf/gi (5) [9]

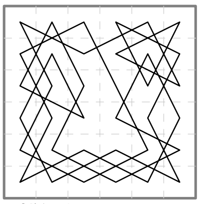

cf/gl (1)

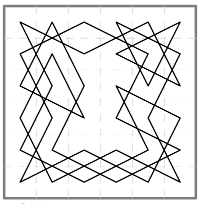

cf/gl (2)

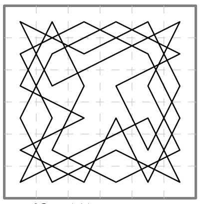

cf/gl (3)

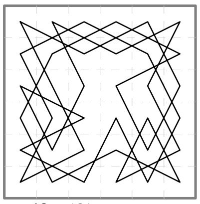

cf/ii

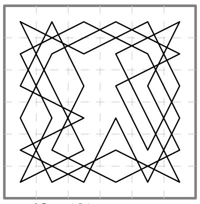

cf/il (1)

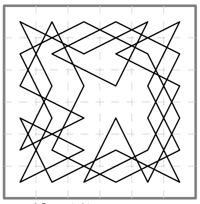

cf/il (2)

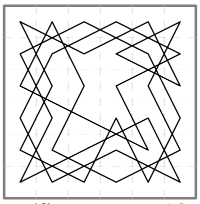

cf/ll

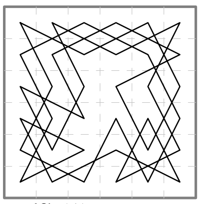

cg/fg (1) [7,10]

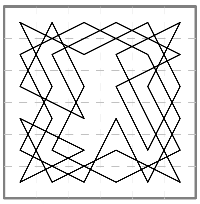

cg/fg (2)

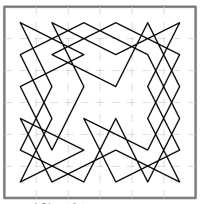

cg/fg (3)

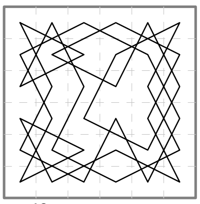

cg/fg (4)

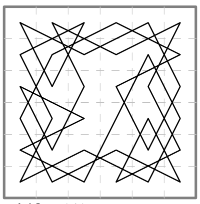

cg/fi [7,10]

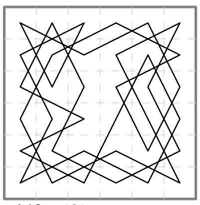

cg/fl (1)


cg/fl (2)

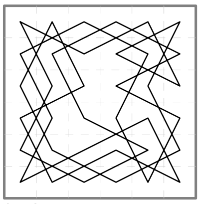

cg/fl (3)

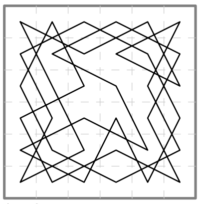

cg/fn

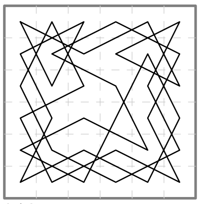

ci/fg (1)

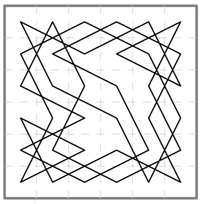

ci/fg (2)

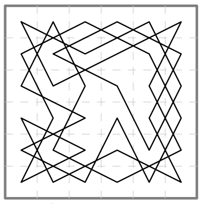
8 slants, cm type, 8:

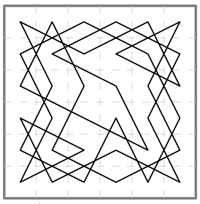

bc/gm

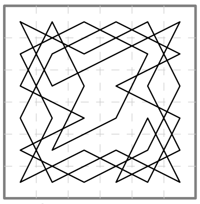

bg/cm

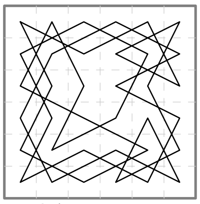

bi/cm [9]


ce/gm

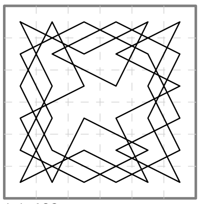

cg/gm

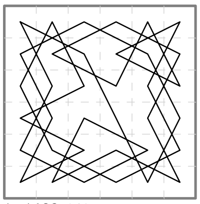

cg/im

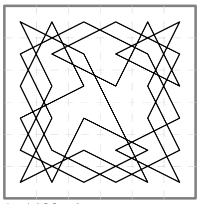

cm/gg

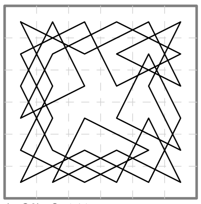

cm/gi [9]

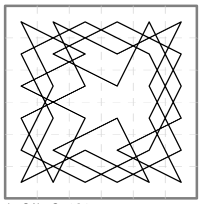
8 slants, ff type, 1-25 of 79:

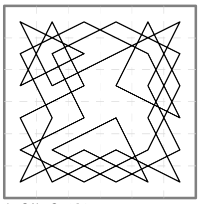

bb/ff

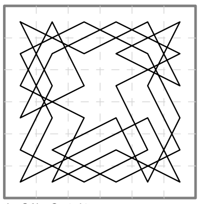

bd/ff (1)

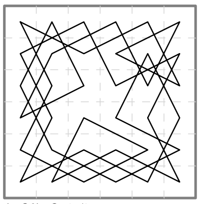

bd/ff (2)

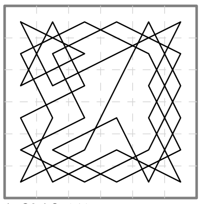

bf/bf (1)

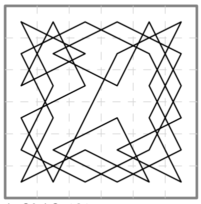

bf/bf (2)

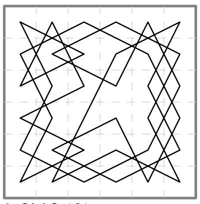

bf/bf (3)

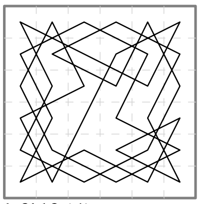

bf/bf (4)

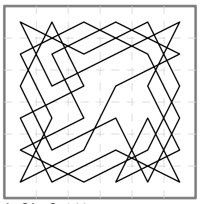

bf/bf (5)

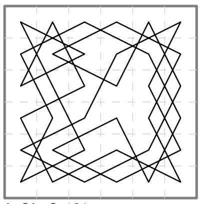

bf/df (1)

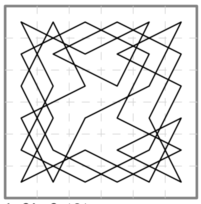

bf/df (2)

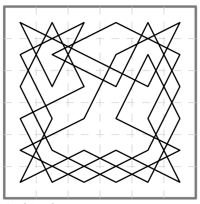

bf/df (3)

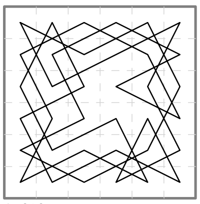

bf/df (4)

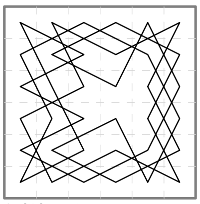

bf/ef (1)

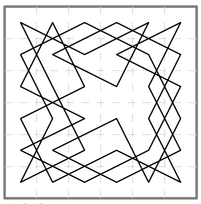

bf/ef (2)

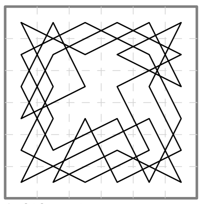

bf/ef (3)

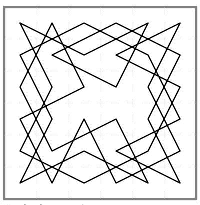

bf/ef (4)

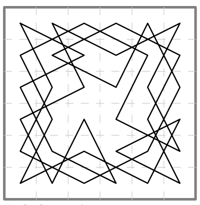

bf/fg (1)

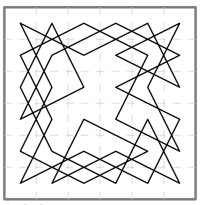

bf/fg (2)

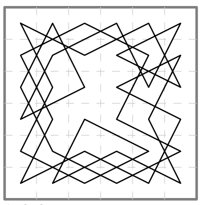

bf/fg (3)

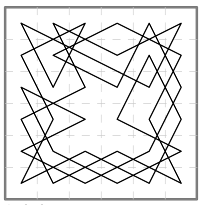

bf/fg (4)


bf/fg (5)

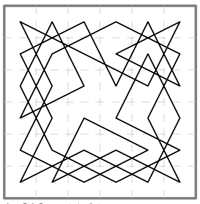

bf/fg (6)

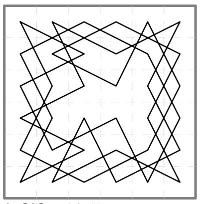

bf/fg (7)

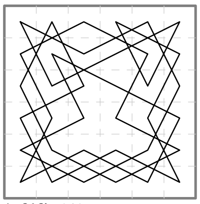

bf/fg (8)

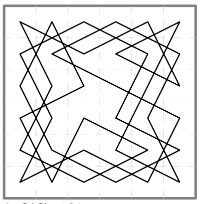

bf/fg (9)

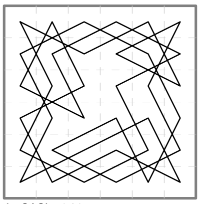
8 slants, ff type, 26-35 of 79:

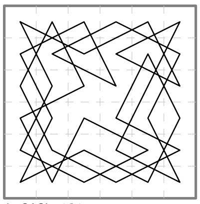

bf/fg (10) [7,10]

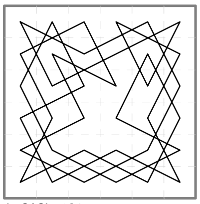

bf/fg (11)

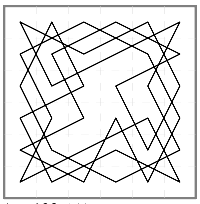

bf/fi (1)

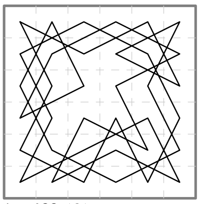

bf/fi (2)

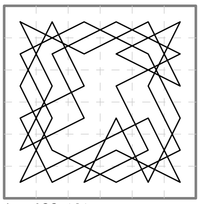

bf/fl (1)


bf/fl (2)

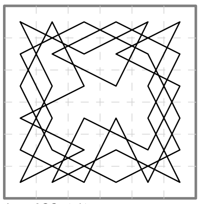

bf/fl (3)

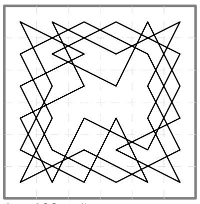

bg/ff (1) [7,10]

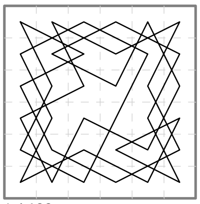

bg/ff (2) [7,10,10]

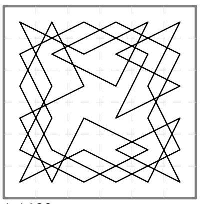

bg/ff (3) [7]

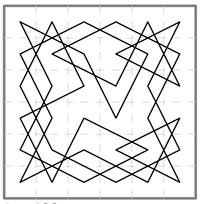
8 slants, ff type, 36-60 of 79

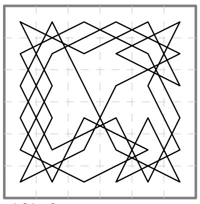

bg/ff (4)

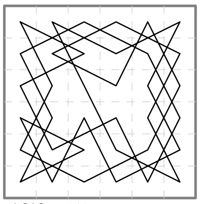

bg/ff (5)

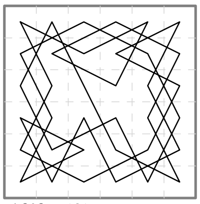

bi/ff

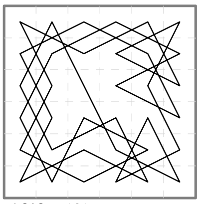

bl/ff

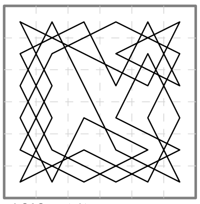

bo/ff

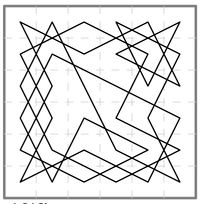

df/ef

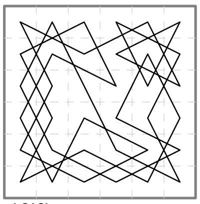

df/fg (1)

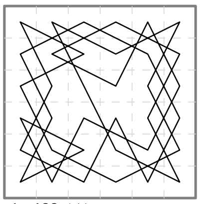

df/fg (2)

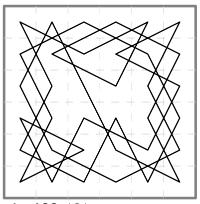

df/fg (3)

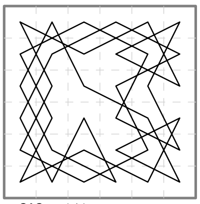

df/fg (4) [7,10]

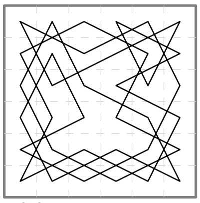

df/fi

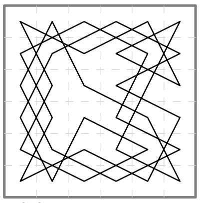

df/fl

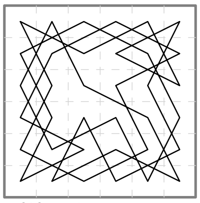

dg/ff (1)

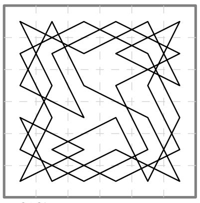

dg/ff (2)

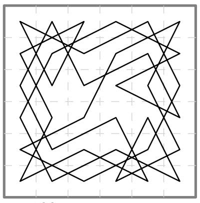

ef/fg (1)

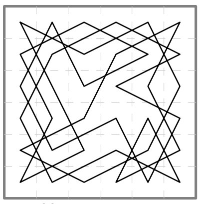

ef/fg (2)

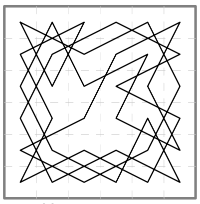

ef/fg (3)

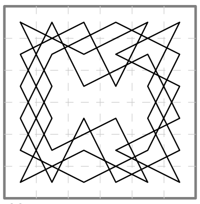

ef/fg (4)

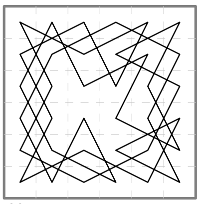

ef/fl

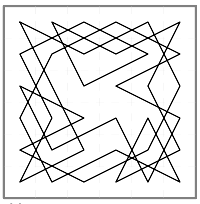

eg/ff (1)


eg/ff (2)

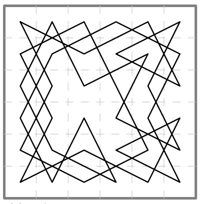

eg/ff (3)

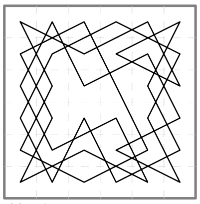

ff/gg (1)

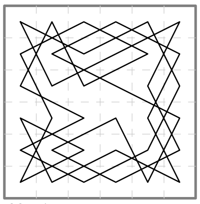

ff/gg (2)

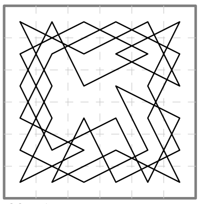

ff/gg (3)

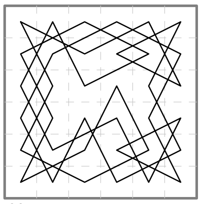
8 slants, ff type, 61-79 of 79:

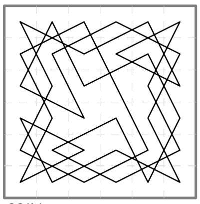

ff/gi (1) [9]

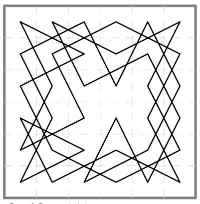

ff/gi (2)

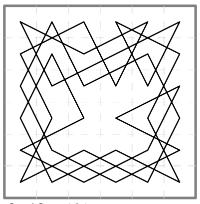

ff/gi (3)

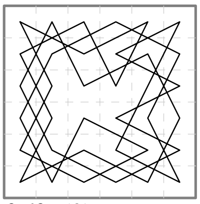

ff/gl

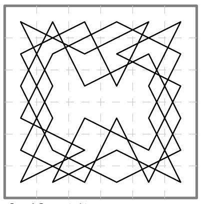

ff/go

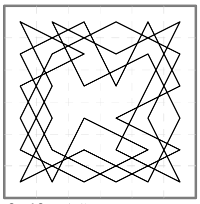

ff/il

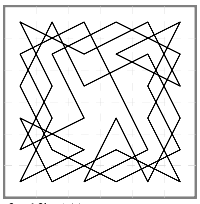

fg/fg (1)

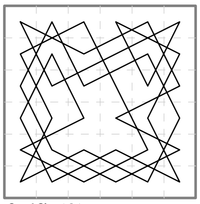

fg/fg (2)

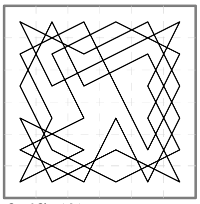

fg/fg (3)

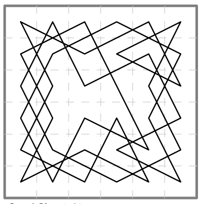

fg/fg (4)

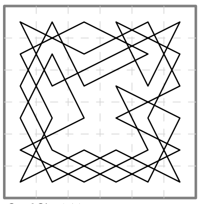

fg/fg (5)

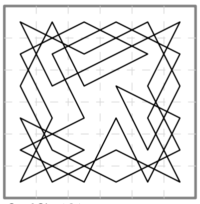

fg/fi (1)

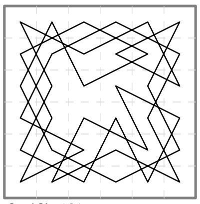

fg/fi (2)

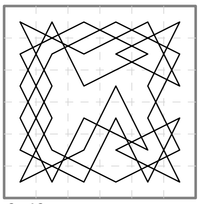

fg/fi (3)


fg/fi (4)

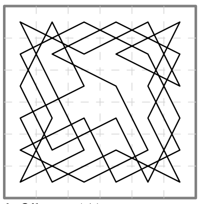

fg/fl (1)

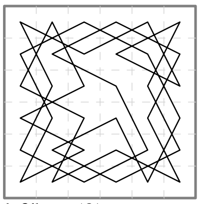

fg/fl (2)

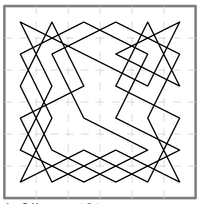

fg/fl (3)

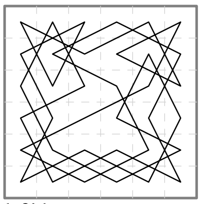

fg/fo

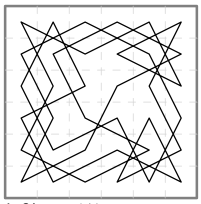
8 slants, fm type, 1-15 of 33:

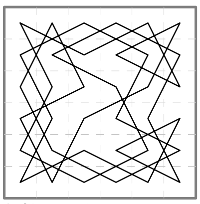

bf/bm (1)

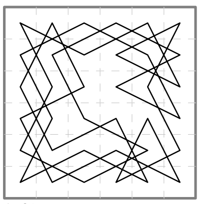

bf/bm (2)

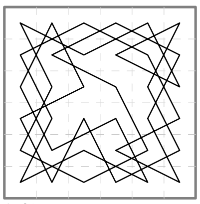

bf/bm (3)

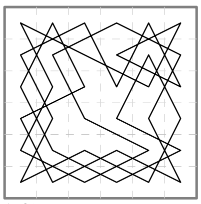

bf/dm

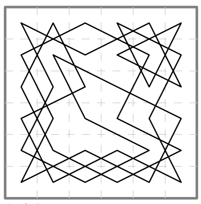

bf/em (1)

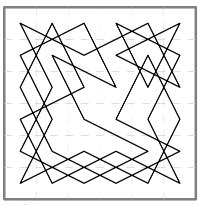

bf/em (2)

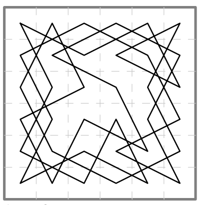

bf/gm (1)

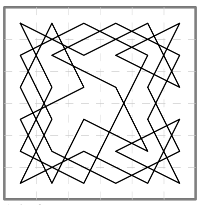

bf/gm (2)

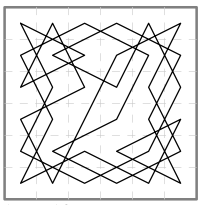

bf/gm (3) [7,10]

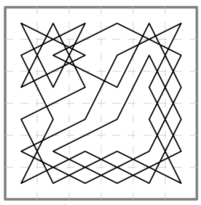

bf/im


bf/lm

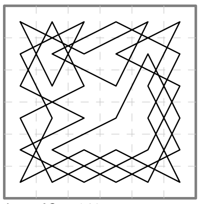

bg/fm

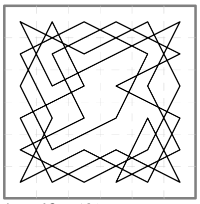

bi/fm [9]

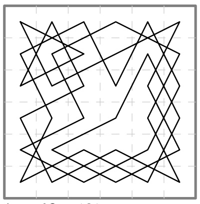

bm/df

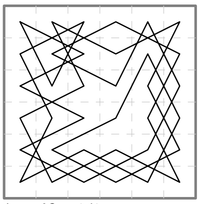

bm/ef

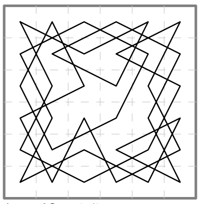
8 slants, fm type, 16-33 of 33:

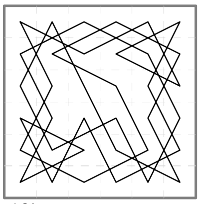

bm/fg (1)

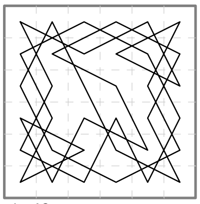

bm/fg (2)

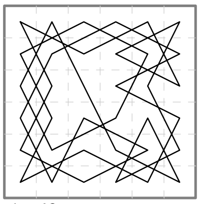

bm/fg (3) [8]

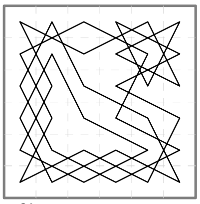

bm/fg (4)

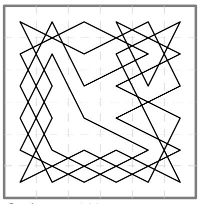

bm/fg (5)

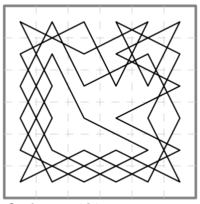

df/gm

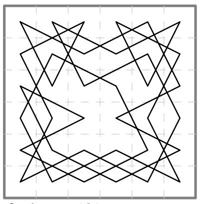

dg/fm

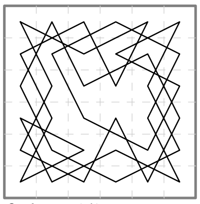

dm/fg

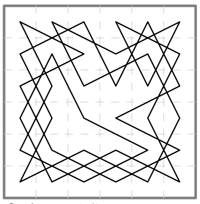

ef/gm

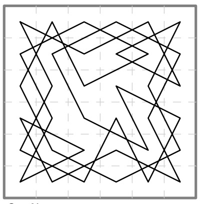

fg/gm (1) [8,10]

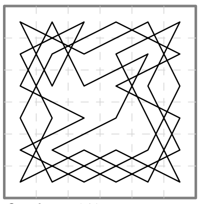

fg/gm (2)

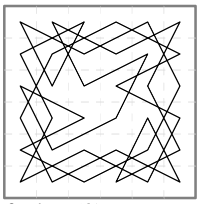

fg/gm (3)

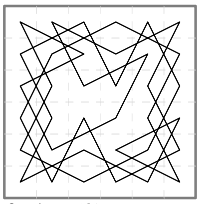

fg/gm (4)


fg/gm (5)


fg/lm

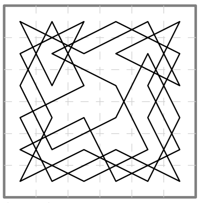

fm/gg (1)

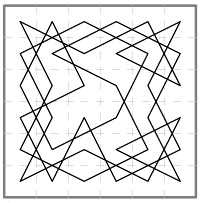

fm/gg (2)

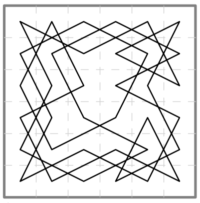

fm/gg (3)

Tour 8dg/fm has a centred 3×5 diamond (parallelogram).

8 slants, mm type, 3:


bm/bm


bm/gm (1)


bm/gm (2)

In the Knuth list the 8-slant tours (including symmetric) are in classes 4 to 8 with 290, 39, 136, 56 and 16 tours respectively.

Triangles in closed 6×6 tours (continued from p.280)**.**

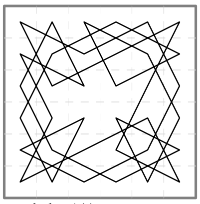
The tours with three two-move lines that form a triangle are: size [2] in 8ad/ai 8ai/cd 8ai/df size [11] in 6ad/bd(2) 6bd/df(1) 6df/dg(2) 8ad/df(2) size [14] in 6ad/bd(1) and 8ad/df(1). See also the four-line case. Size [15] is in 8ai/ai and size [18] in 8ab/ai(2) 8ag/ai(1) 8ai/bf(1) 8ai/fg(4) 10af/ai. The size [11] occurs only in the four tours listed above, and in the 4-line tour 6ad/dd, though it only needs two two-move lines. Sizes [7] to [10] and [15] need only one two-move line. Size [15] also occurs in 8af/ei(1) 8ai/al(1) 8ai/an(2) 8ai/fi(2) 8am/gi(2) 10ac/ci(3).

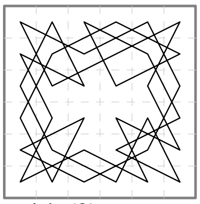
Tours where the larger triangles [7] to [11] and [14] [15] [18] occur are noted below diagrams.

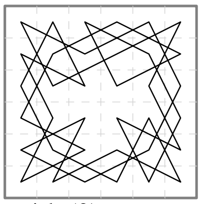
Asymmetric Closed Tours with 10 Slants

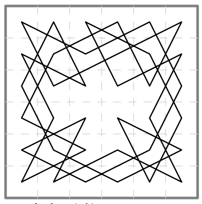
Total 288.

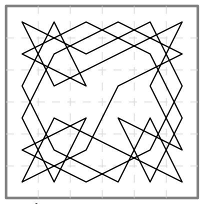
10 slants, aaa type, 9:

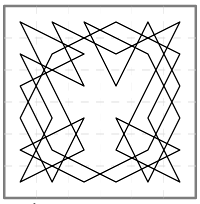

aa/ab (1) [9]

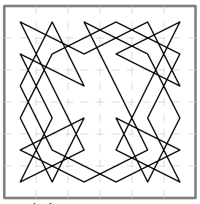

aa/ab (2) [9]

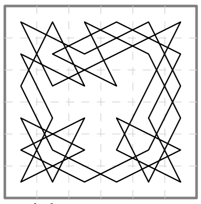

aa/ab (3) [9]

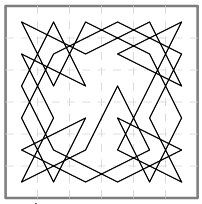

aa/ab (4) [9]


aa/ae

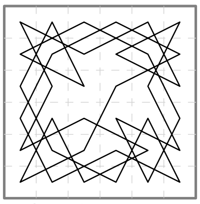

aa/ag

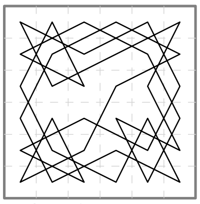

aa/ai

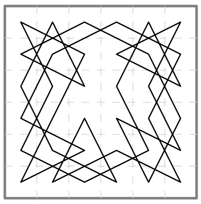

aa/al [9]

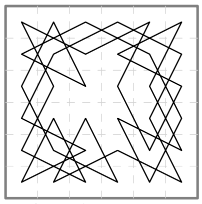

aa/ao

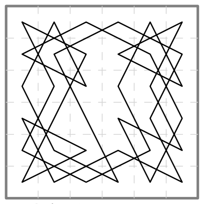
10 slants, aac type, 1-20 of 52:

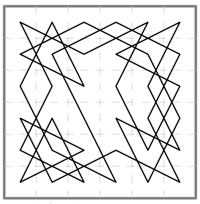

aa/ce (1)

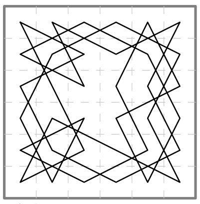

aa/ce (2)

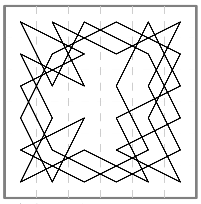

aa/cg (1)

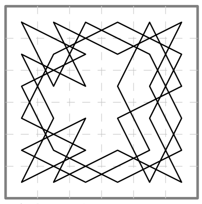

aa/cg (2)

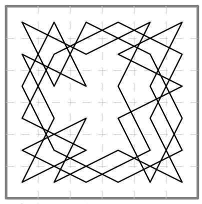

aa/ci (1)

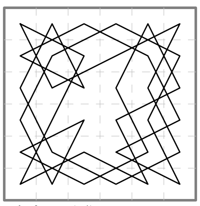

aa/ci (2)

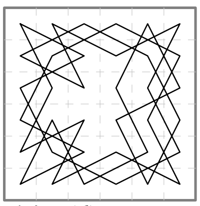

ab/ac (1)

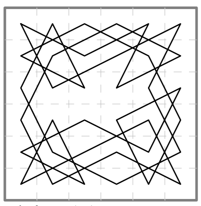

ab/ac (2)

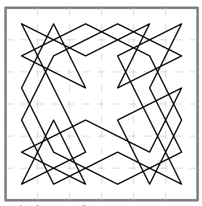

ab/ac (3)

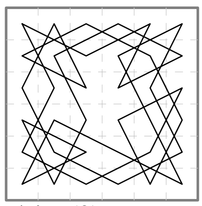

ab/ac (4)

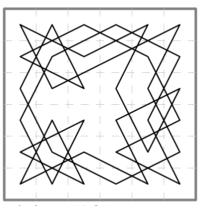

ab/ac (5)

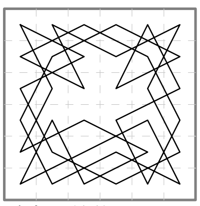

ab/ac (6)

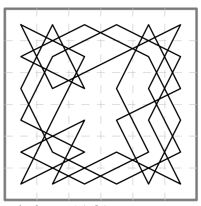

ab/ac (7)

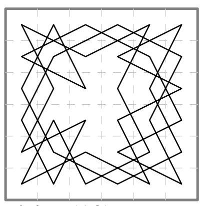

ab/ac (8) [9]

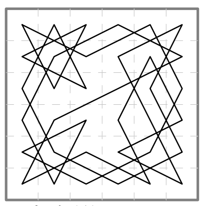

ab/ac (9) [9]


ab/ac (10)

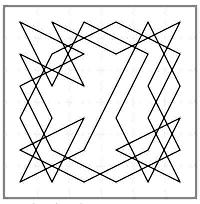

ab/ac (11)

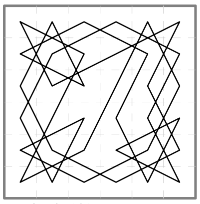

ab/ac (12)

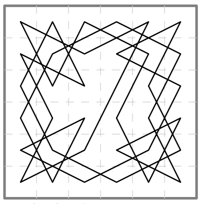

ab/ac (13)

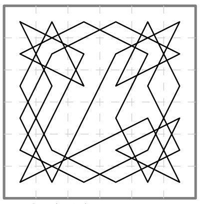

ac/ad (1)

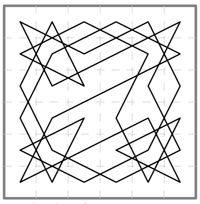
10 slants, aac type, 21-35 of 52:

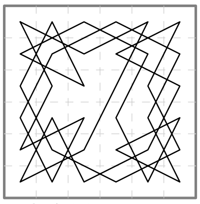

ac/ad (2)

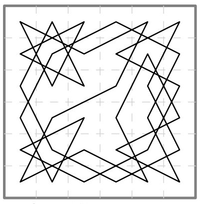

ac/ad (3)

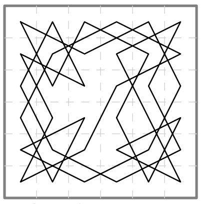

ac/ad (4)

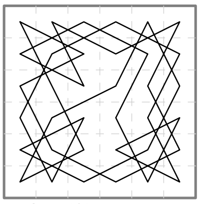

ac/ad (5)

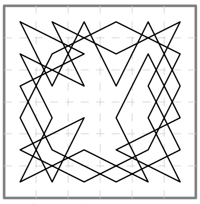

ac/ad (6)

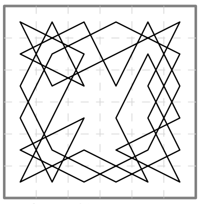

ac/ad (7)

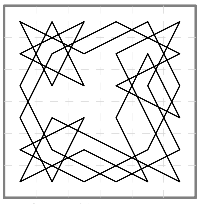

ac/ae (1)

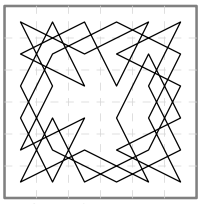

ac/ae (2)

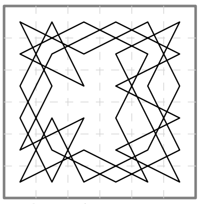

ac/ae (3)

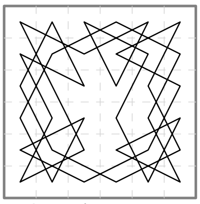

ac/ag (1)


ac/ag (2) [8]

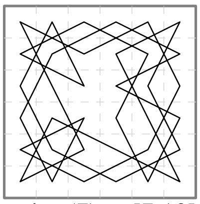

ac/ag (3) [7]

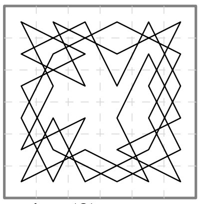

ac/ag (4)

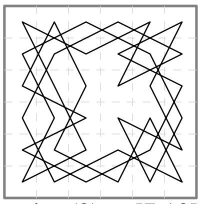

ac/ag (5) [8]

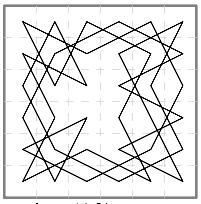

ac/ag (6)

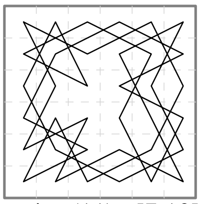
10 slants, aac type, 35-52 of 52:

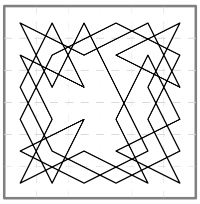

ac/ag (7) [7,10]

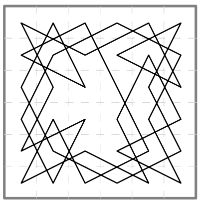

ac/ag (8)

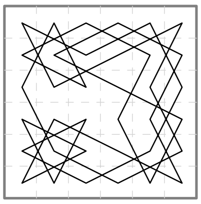

ac/ag (9) [7,10]

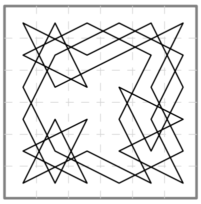

ac/ag (10) [8]

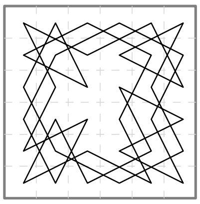

ac/ag (11) [7,10]

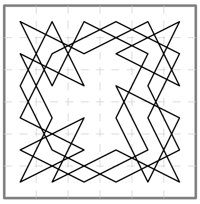

ac/ai (1)

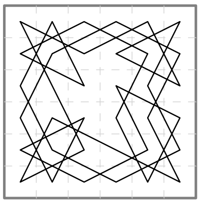

ac/ai (2)

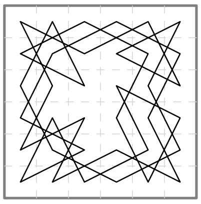

ac/ai (3)

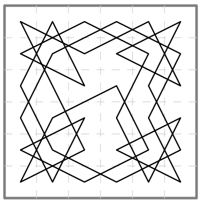

ac/al (1)

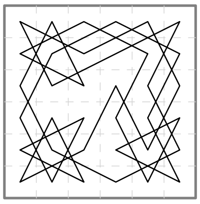

ac/al (2)

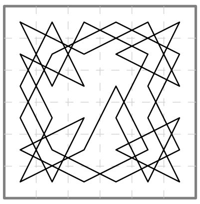

ac/al (3)

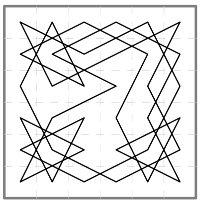

ac/al (4)


ac/al (5)

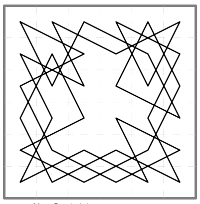

ac/an

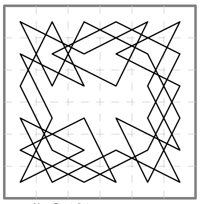

ac/ao (1)

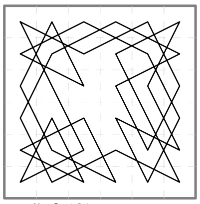

ac/ao (2)

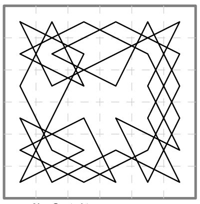

ac/ao (3)

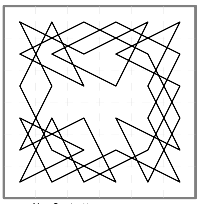
10 slants, aaf type, 1-15 of 33:

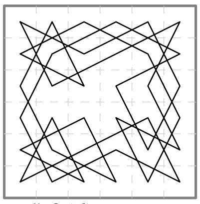

aa/bf (1) 7x

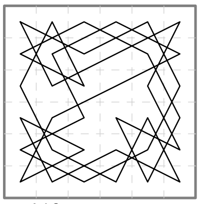

aa/bf (2)

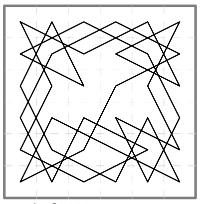

aa/bf (3)

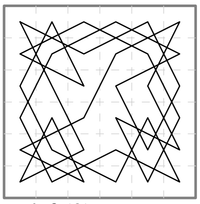

aa/bf (4)

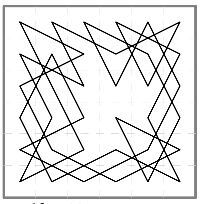

aa/bf (5)

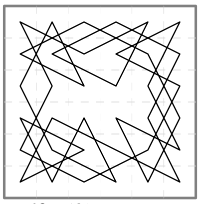

aa/bf (6) [9]

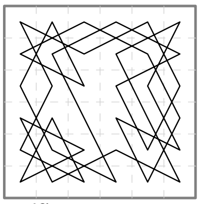

aa/df

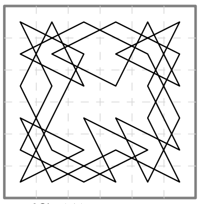

aa/ef (1)

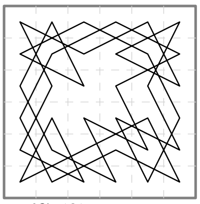

aa/ef (2)

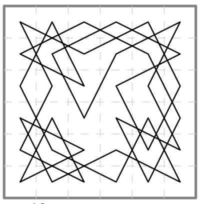

aa/fg (1)


aa/fg (2)

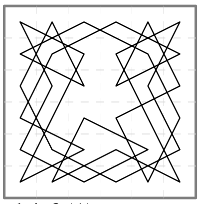

aa/fi

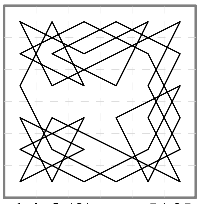

aa/fl (1)

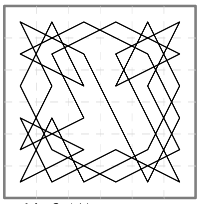

aa/fl (2) [9]

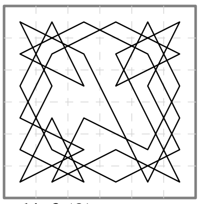

aa/fo

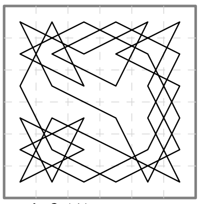
10 slants, aaf type, 16-33 of 33:

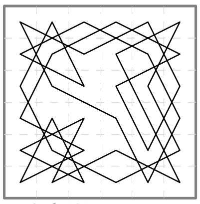

ab/af (1)

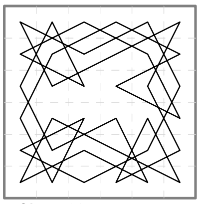

ab/af (2) [10]

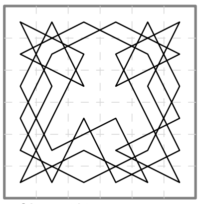

ad/af (1)

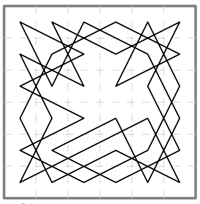

ad/af (2)

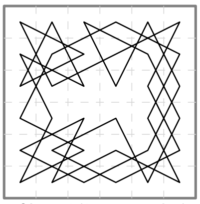

ae/af (1)

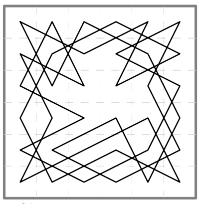

ae/af (2)

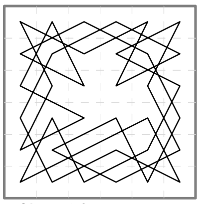

af/ag (1) [10]

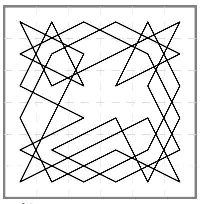

af/ag (2)

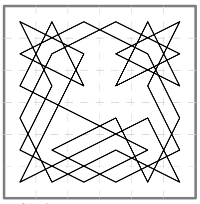

af/ag (3)

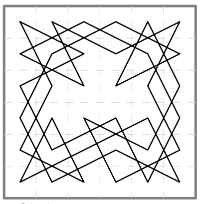

af/ag (4) [8,9]

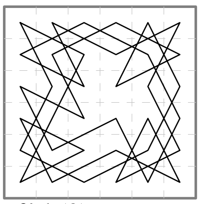

af/ag (5)

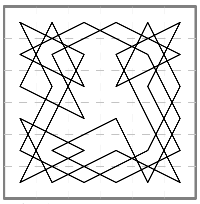

af/ag (6)

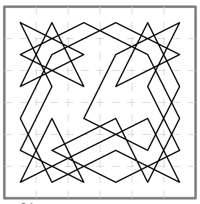

af/ag (7) [8]


af/ai

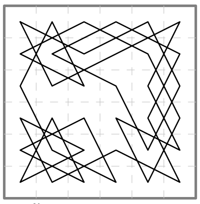

af/al (1)

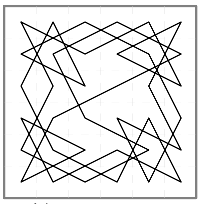

af/al (2) [9]

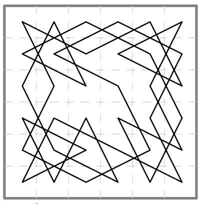

af/al (3)

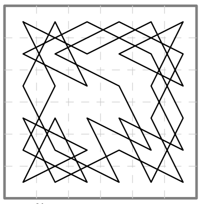

af/an

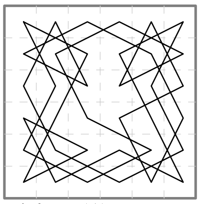
10 slants, aam type, 8:

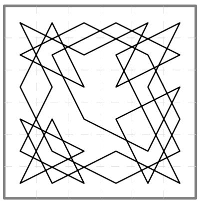

aa/bm

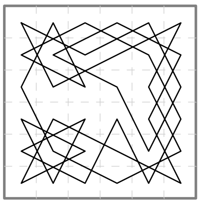

aa/dm

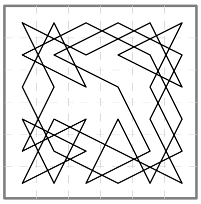

aa/gm [7]


aa/lm

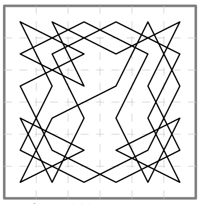

ab/am (1)

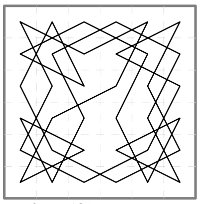

ab/am (2) [9]

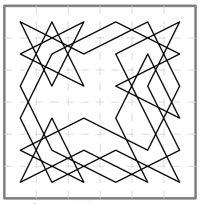

ag/am (1) [8]

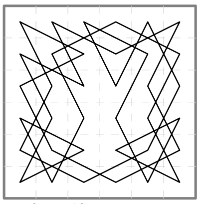

ag/am (2) [8,9]

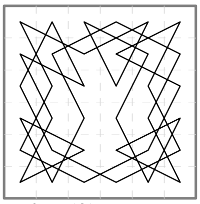
10 slants, acc type, 11:

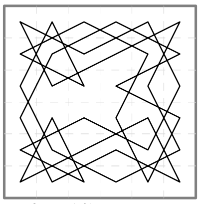

ac/ce (1)

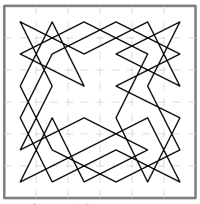

ac/ce (2)

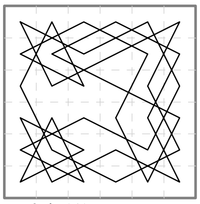

ac/cg (1) [7]

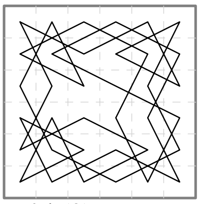

ac/cg (2)

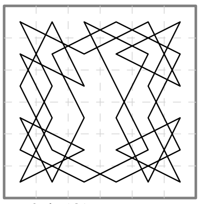

ac/cg (3)

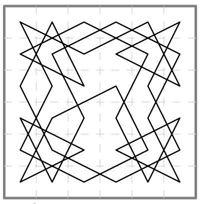

ac/cg (4)


ac/cg (5)

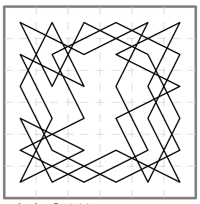

ac/ci (1)

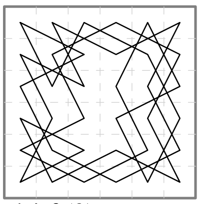

ac/ci (2)

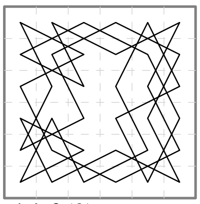

ac/ci (3) [15]

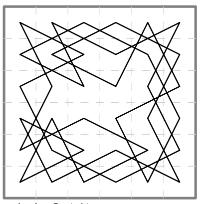

ac/cn

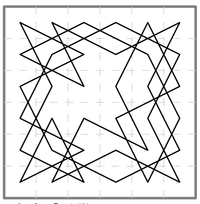
10 slants, acf type, 1-5 of 69:


ab/cf (1)

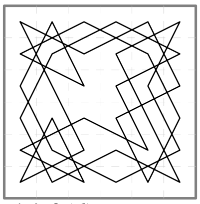

ab/cf (2) 7x

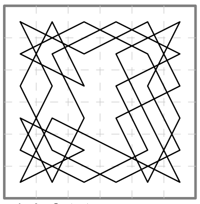

ab/cf (3)

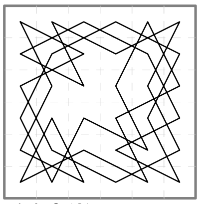

ab/cf (4)

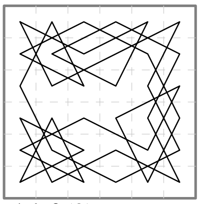

ab/cf (5)

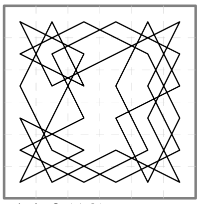
10 slants, acf type, 6-20 of 69:

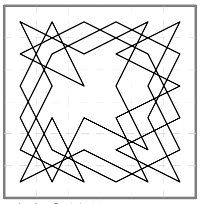

ab/cf (6)

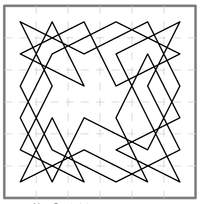

ab/cf (7)

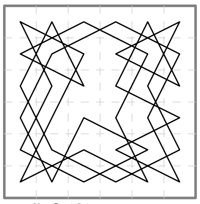

ab/cf (8)

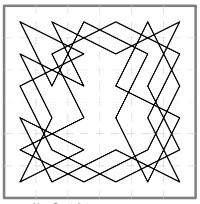

ab/cf (9) [10]

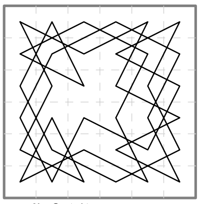

ab/cf (10) [10]

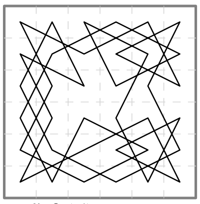

ab/cf (11)

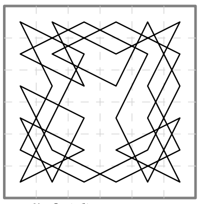

ac/bf (1)

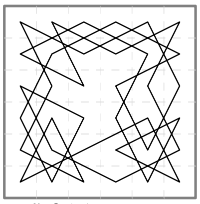

ac/bf (2)

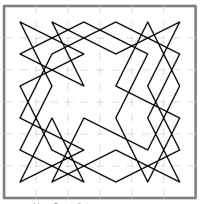

ac/bf (3) 7x

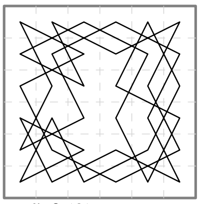

ac/bf (4)


ac/bf (5)

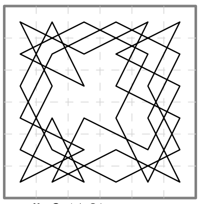

ac/bf (6)

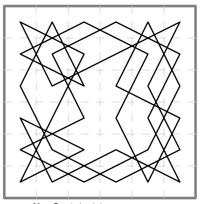

ac/bf (7)

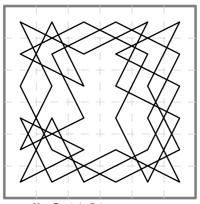

ac/bf (8)

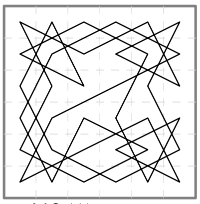

ac/bf (9)

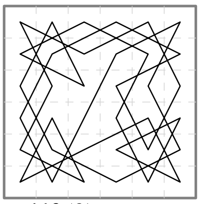
10 slants, acf type, 21-40 of 69:

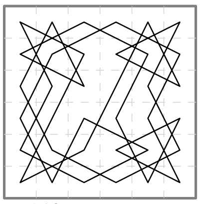

ac/bf (10)

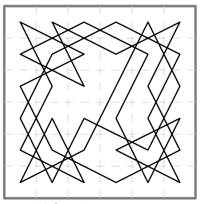

ac/bf (11) [10]

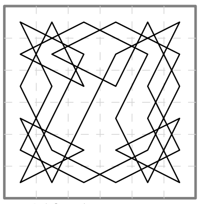

ac/bf (12)

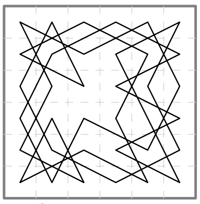

ac/df (1)

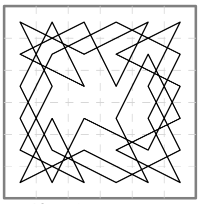

ac/df (2)

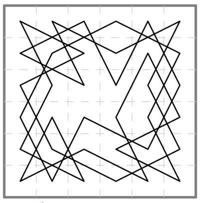

ac/df (3)

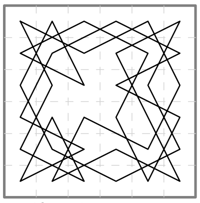

ac/df (4)

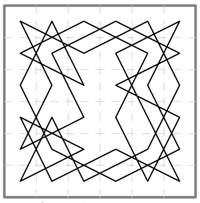

ac/df (5)

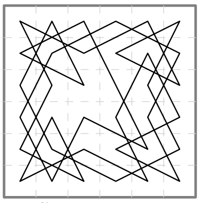

ac/fg (1)

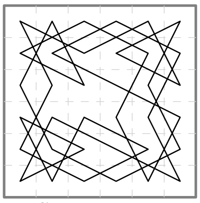

ac/fg (2)

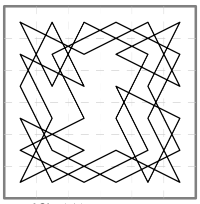

ac/fg (3)

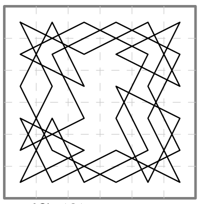

ac/fg (4) [7,10]

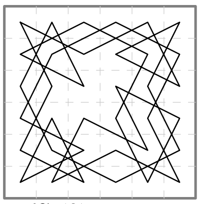

ac/fg (5) [7,10]

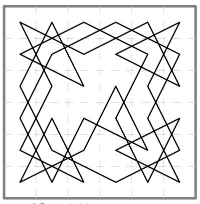

ac/fi (1)

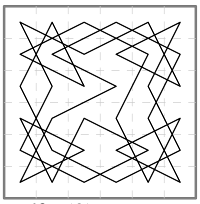

ac/fi (2)


ac/fl (1)

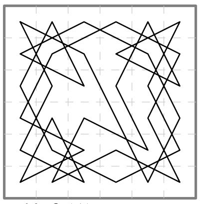

ac/fl (2)

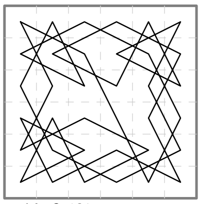

ac/fl (3)

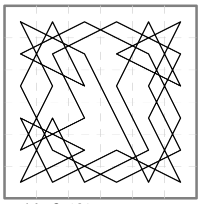

ac/fo (1)

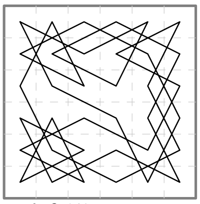

ac/fo (2)

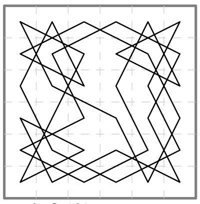
10 slants, acf type, 41-55 of 69:

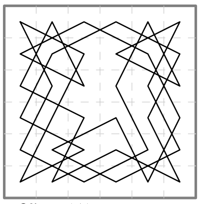

ad/cf (1)

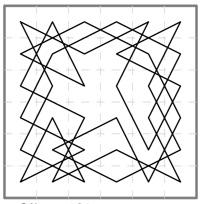

ad/cf (2)

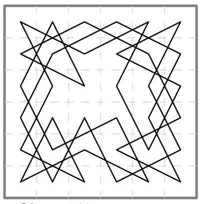

ad/cf (3)

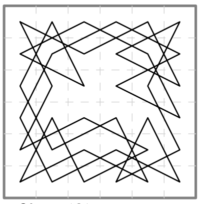

ae/cf (1)

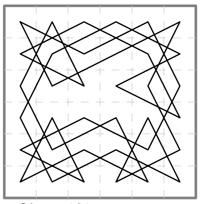

ae/cf (2) [10]

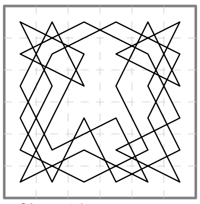

af/bc (1)

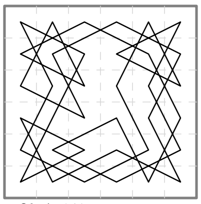

af/bc (2)

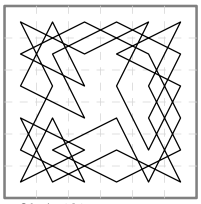

af/cg (1)

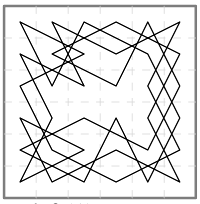

af/cg (2)

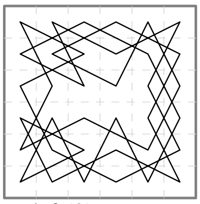

af/cg (3)


af/cg (4)

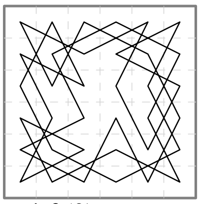

af/cl (1)

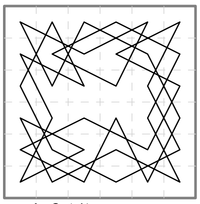

af/cl (2)

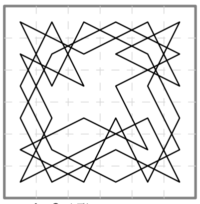

ag/cf (1) 7x

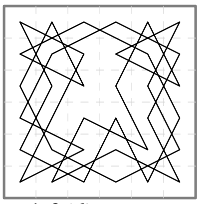

ag/cf (2)

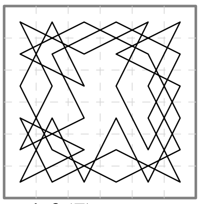
10 slants, acf type, 56-69 of 69:

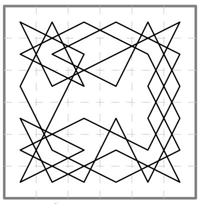

ag/cf (3)

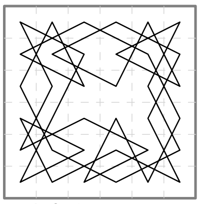

ag/cf (4)

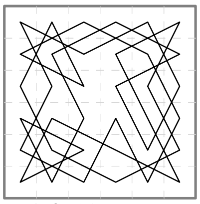

ag/cf (5)

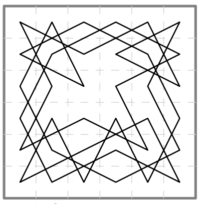

ag/cf (6)

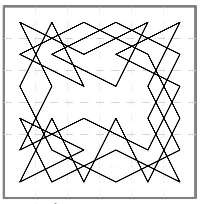

ag/cf (7)

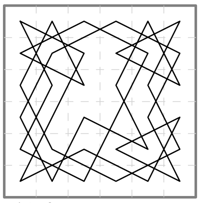

ag/cf (8) [10]

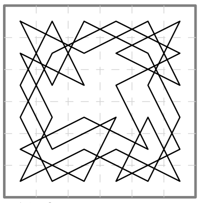

ag/cf (9)

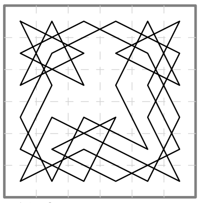

ag/cf (10) [8]

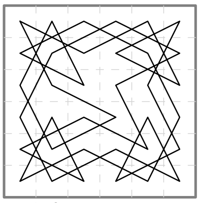

ag/cf (11)


ag/cf (12)

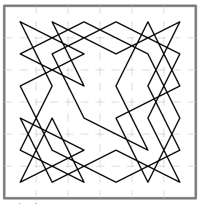

ai/cf [9]

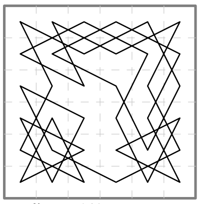

al/cf (1)

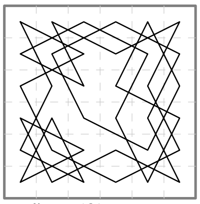

al/cf (2)

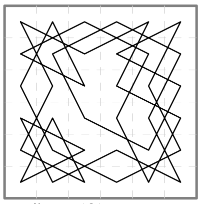

ao/cf

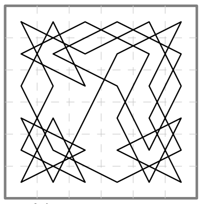
10 slants, acm type, 13:

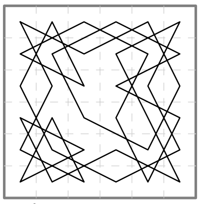

ab/cm

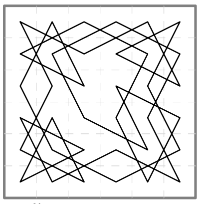

ac/bm (1)

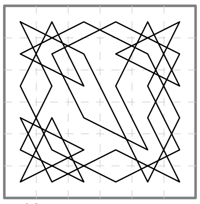

ac/bm (2)

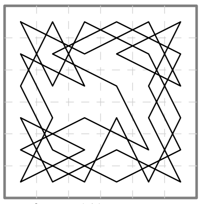

ac/bm (3)

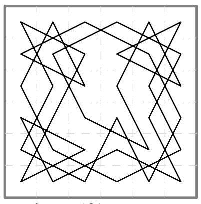

ac/dm

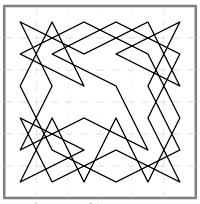

ac/gm [7,10]

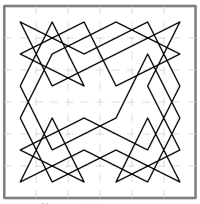

ac/lm

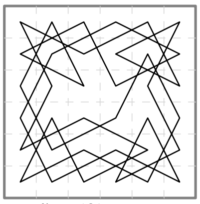

ad/cm


ag/cm (1)

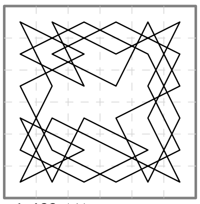

ag/cm (2)

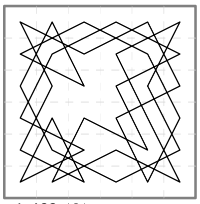

ag/cm (3)

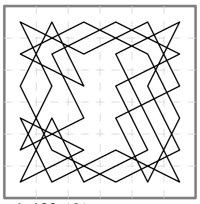

am/bc (1)

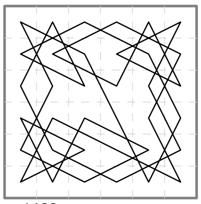

am/bc (2)

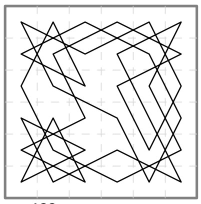
10 slants, aff type, 1-20 of 35:

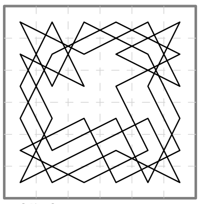

ab/ff (1)

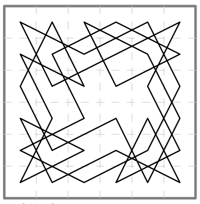

ab/ff (2)

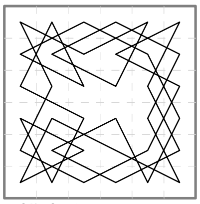

ab/ff (3)

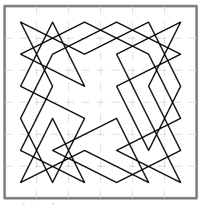

ad/ff

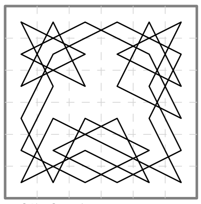

ae/ff

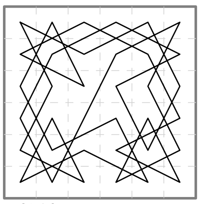

af/bf (1)

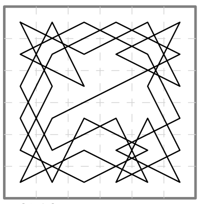

af/bf (2) [9]

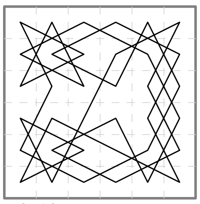

af/bf (3) [10]

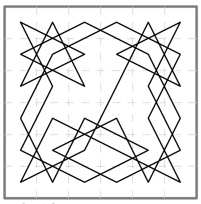

af/bf (4)

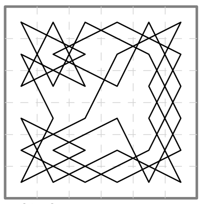

af/bf (5) [10]

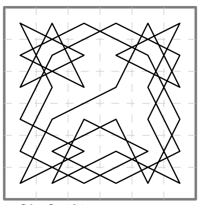

af/df (1)

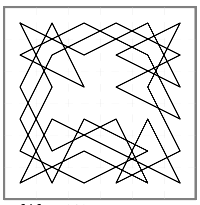

af/df (2)

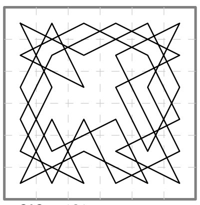

af/df (3) [10]

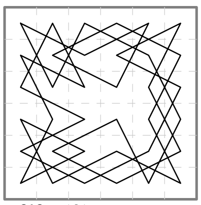

af/df (4) [10]

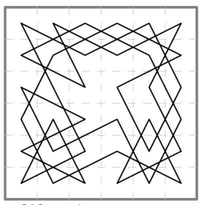

af/ef (1)


af/ef (2)

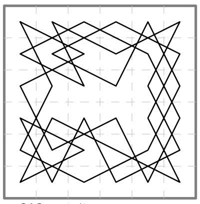

af/fg (1) [10]

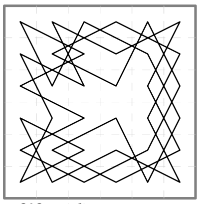

af/fg (2)

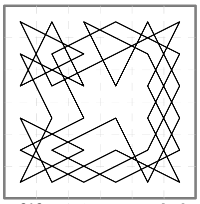

af/fg (3)

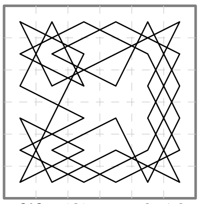

af/fg (4)

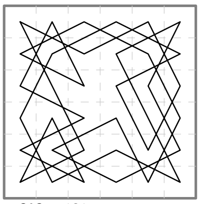
10 slants, aff type, 21-35 of 35:

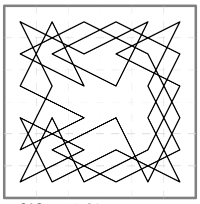

af/fg (5) [7,10,10]

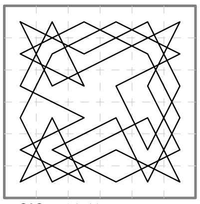

af/fg (6) 7x

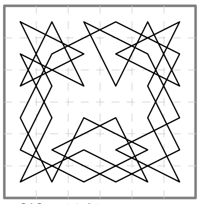

af/fg (7) [8,9]

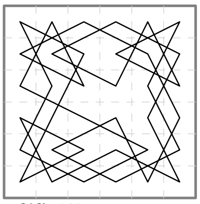

af/fg (8) [8,10]

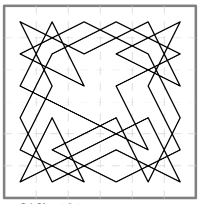

af/fg (9)

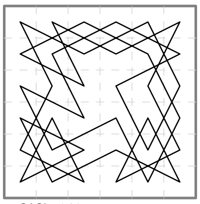

af/fg (10)

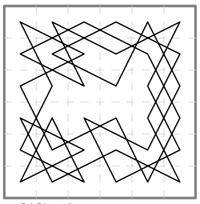

af/fg (11)

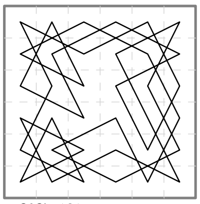

af/fg (12)

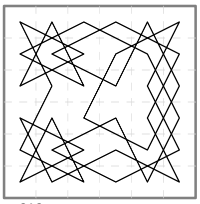

af/fi (1)

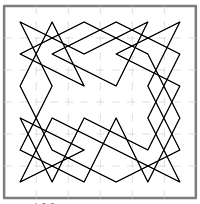

af/fi (2)


af/fl (1)

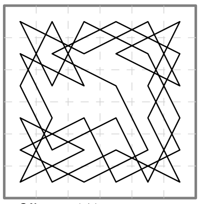

af/fl (2)

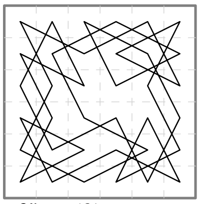

af/fl (3)

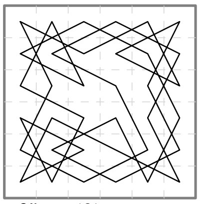

af/fn

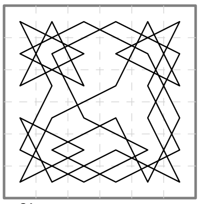

ag/ff

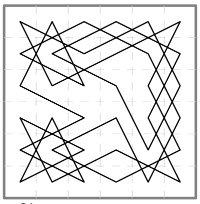
10 slants, afm type, 11:

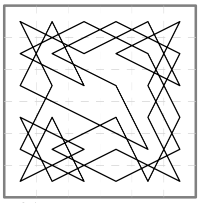

af/bm (1)

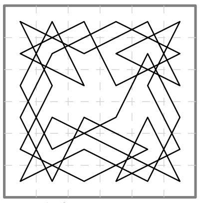

af/bm (2)

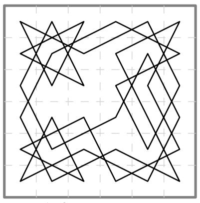

af/bm (3) [10]

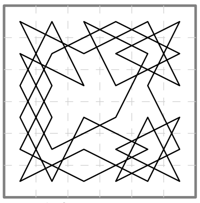

af/em

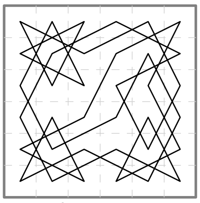

af/gm

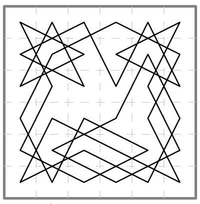

af/im


am/bf (1)

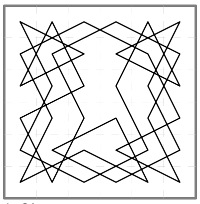

am/bf (2)

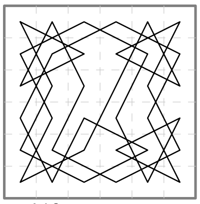

am/bf (3)

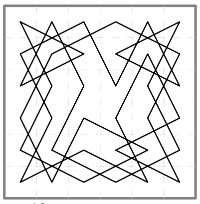

am/ef

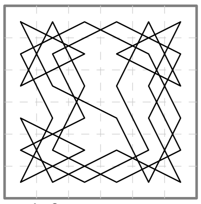

am/fg

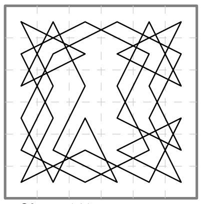
10 slants, ccf type, 9:

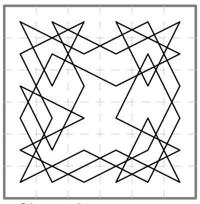

bf/cc

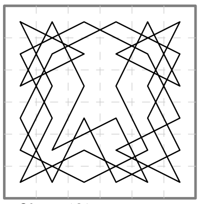

cc/df

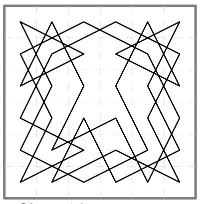

cc/fg

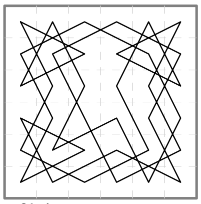

ce/cf


cf/cg (1)

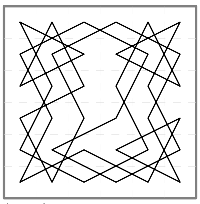

cf/cg (2)

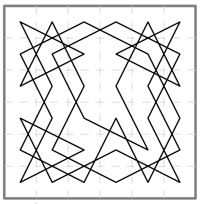

cf/cg (3)


cf/cg (4)

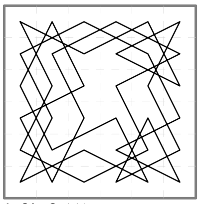

cf/ci [9]

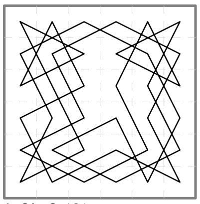
10 slants, ccm type, 2:

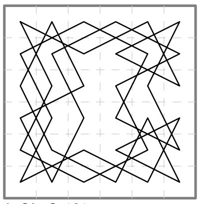

bm/cc

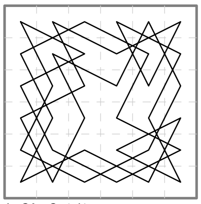

cg/cm

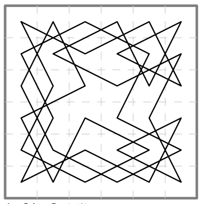
10 slants, cff type, 1 - 15 of 19:

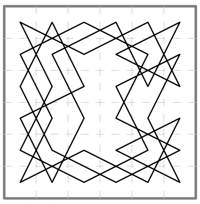

bf/cf (1)

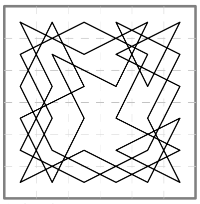

bf/cf (2)

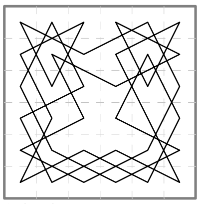

bf/cf (3)

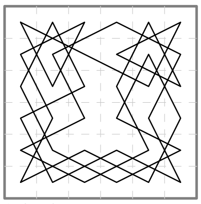

bf/cf (4)

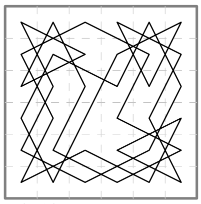

bf/cf (5)

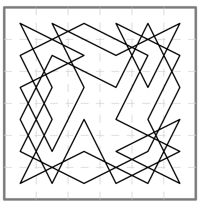

bf/cf (6)

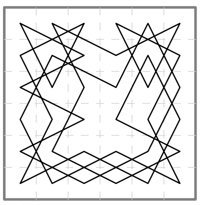

bf/cf (7)

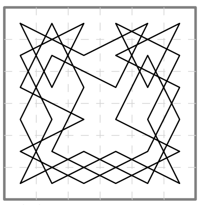

bf/cf (8) [10]

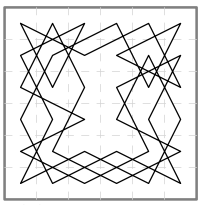

bf/cf (9)

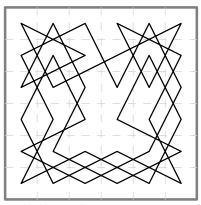

cf/df


cf/fg (1)

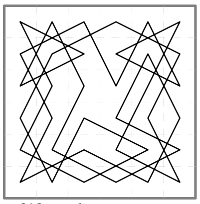

cf/fg (2)

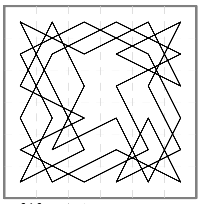

cf/fg (3)

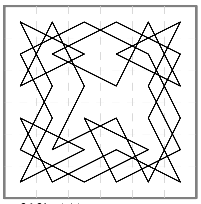

cf/fg (4)

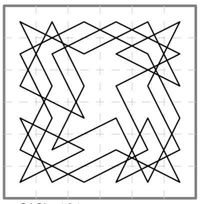

cf/fg (5) [8]

10 slants, cff type, 16 - 19 of 19:

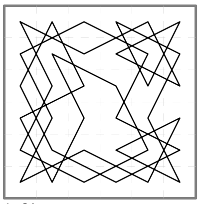

cf/fg (6)

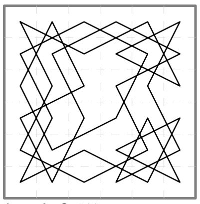

cf/fg (7) [7,10]

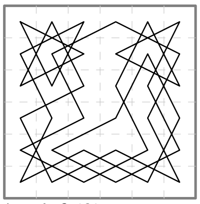

cf/fl (1)

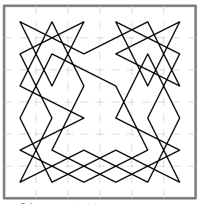

cf/fl (2)

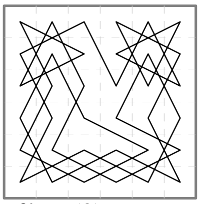
10 slants, cfm type, 6:

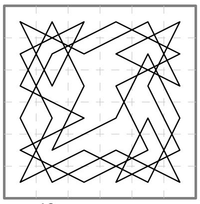

bf/cm


bm/cf (1)

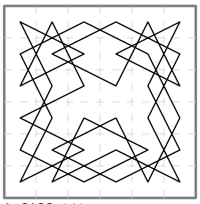

bm/cf (2)

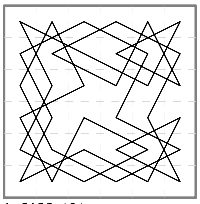

cf/gm (1)

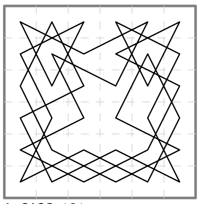

cf/gm (2)

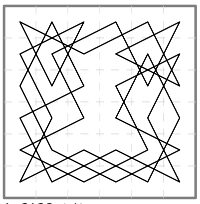

cm/fg

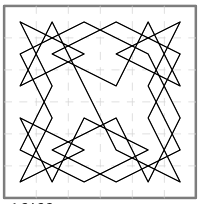
10 slants, fff type, 8:

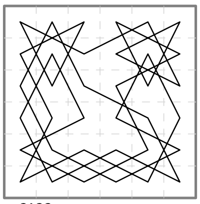

bf/ff (1)

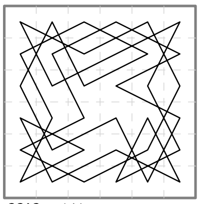

bf/ff (2)

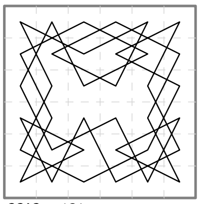

bf/ff (3)


bf/ff (4)

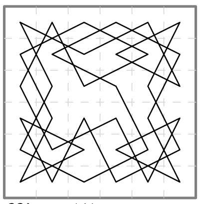

df/ff

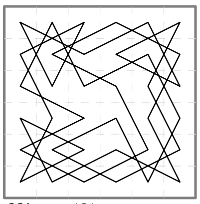

ef/ff

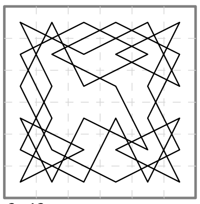

ff/fg (1)


ff/fg (2)

10 slants, ffm type, 3:


ff/gm (1)


ff/gm (2)

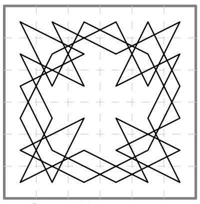

fg/fm

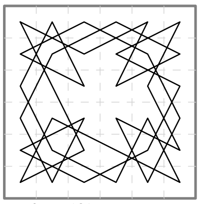
In the Knuth list the 10-slants tours are in classes 9, 10, 11 with 142, 130 and 16 members.

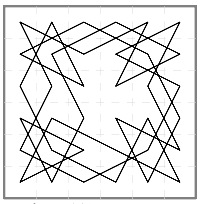
Asymmetric Closed Tours with 12 Slants

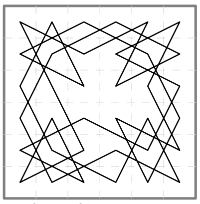
Total 44.
The two aa/aa tours are the only asymmetric tours with four central angles alike.
The first is also piece-wise symmetric and semi-magic (see the next page).

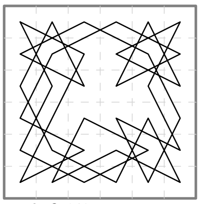
12 slants, 1-15 of 44:

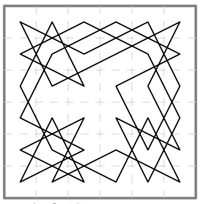

aa/aa (1)

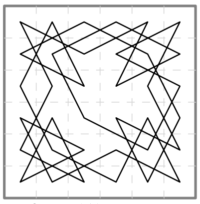

aa/aa (2)

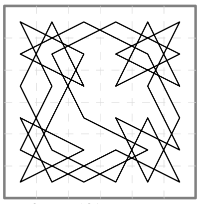

aa/ac (1)

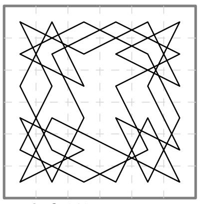

aa/ac (2)

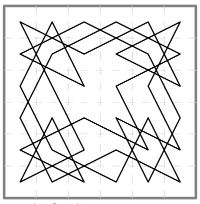

aa/af (1)

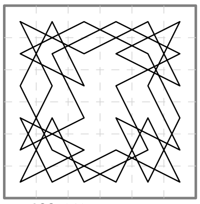

aa/af (2)

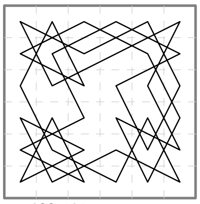

aa/am (1)

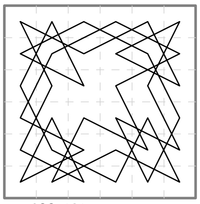

aa/am (2)

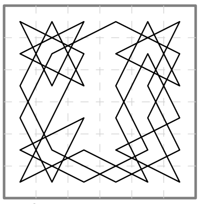

aa/cf (1)

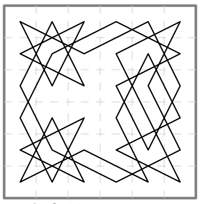

aa/cf (2)


aa/ff (1)

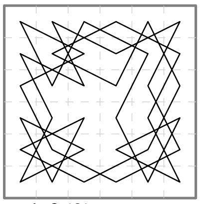

aa/ff (2)

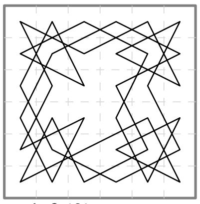

aa/ff (3)

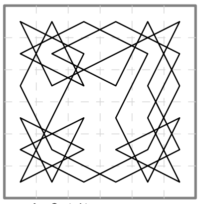

ac/ac

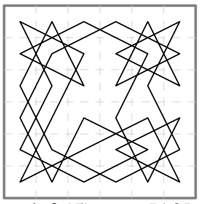

ac/af (1)

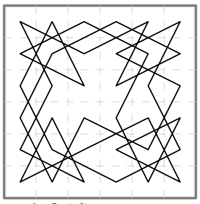
12 slants, 16-30 of 44. T = Takefuji.

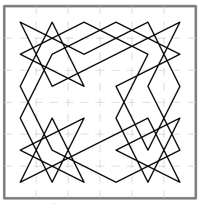

ac/af (2) 7x

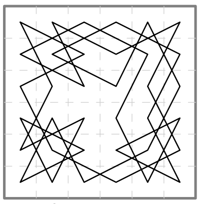

ac/af (3)

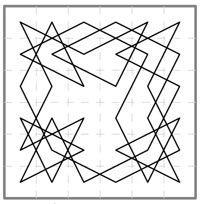

ac/af (4) [10]

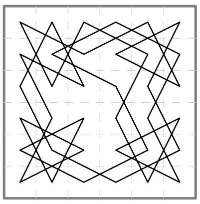

ac/af (5) [10]

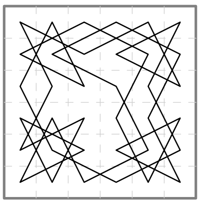

ac/af (6) [10]

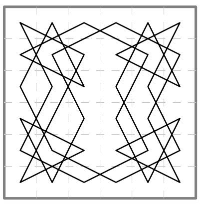

ac/af (7)

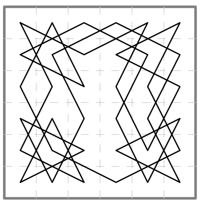

ac/af (8)

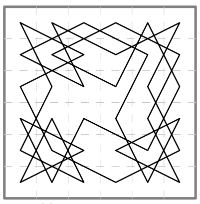

ac/af (9)

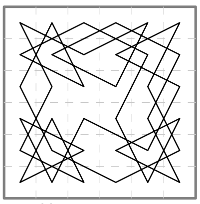

ac/am (1)

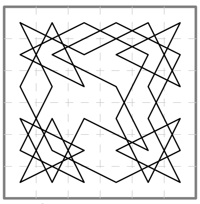

ac/am (2)


ac/cc (1)

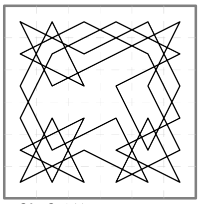

ac/cc (2)

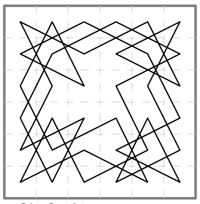

ac/ff (1) T

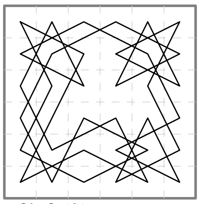

ac/ff (2)

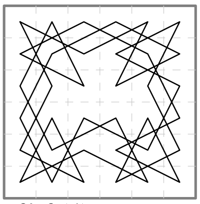

ac/fm

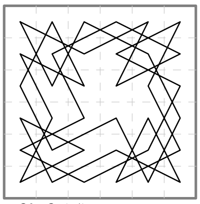
12 Slants 31-44 of 44:

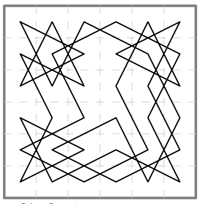

af/af (1)

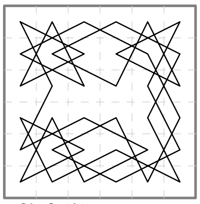

af/af (2)

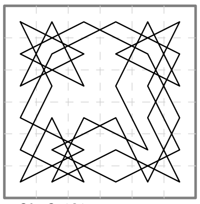

af/af (3)

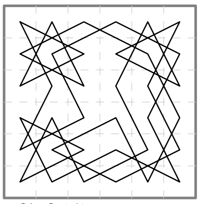

af/af (4)

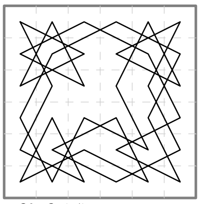

af/af (5)

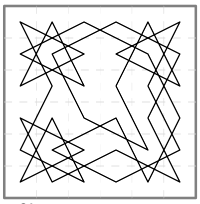

af/cf (1)

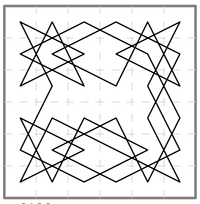

af/cf (2)

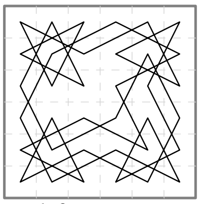

af/cf (3)

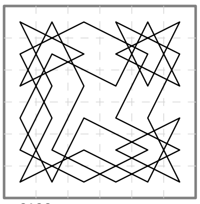

af/cf (4)


af/cf (5)


af/cm


af/ff


am/cf


cf/ff

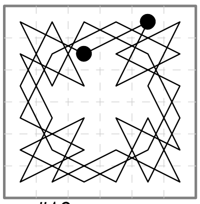
In the Knuth list the 12-slant tours (including symmetric) are in five classes 12 to 16 with tour numbers 16, 1, 24, 4 and 1 respectively. The singletons are 13.000 = 12af/ff (with nets divided 3:3) and 16.000 = 12ac/ac (with nets divided 2:4). The 28 cases with one 2:4 or one 3:3 are those including an odd number (1 or 3) of 'c' angles. Tour 12af/ff shows the maximum 11 lateral-acute angles (as does 12af/cf(2)) and the minimum 8 lateral obtuse angles (as does 12aa/aa).

Semimagic 6×6 Tours

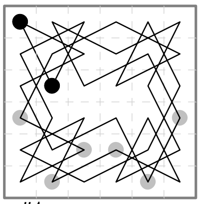
My 2003 Existence theorem (see ⌘ 1) implies that a magic knight tour is impossible on the 6×6 board. However, semi-magic tours (i.e. magic in the files but not the ranks) are possible.

Awani Kumar (May 2002) made a complete enumeration of semi-magic 6×6 knight tours, finding the total to be 88 consisting of 26 closed and 62 open. The tours occur in 44 pairs that are reverse numberings of each other. For a complete catalogue see the KTN website.

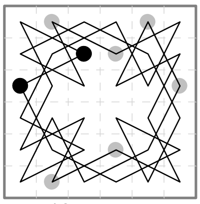
The 12-slant quaternary tour on the 6×6 board is semi-magic when numbered from a suitable point. This property was noted by Maurice Kraitchik (1927, p.36). The lines in the other direction add to two totals, 129 and 93, each three times, so the tour is quasi-magic or three-quarters magic. Its diagonals together add to 222. This is tour #16 (and reverse #69) in Kumar's list.


#16

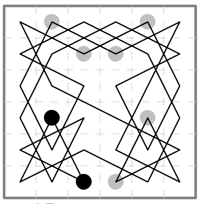
2	25	4	33	36	11
27	32	1	12	5	34
24	3	26	35	10	13
31	28	17	8	21	6
16	23	30	19	14	9
29	18	15	22	7	20


#1

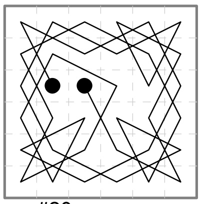
1	10	35	28	25	12
34	29	2	11	8	27
3	36	9	26	13	24
30	33	20	5	16	7
21	4	31	18	23	14
32	19	22	15	6	17


#19

2	25	28	21	12	23
27	20	1	24	29	10
36	3	26	11	22	13
19	16	5	32	9	30
4	35	18	7	14	33
17	6	15	34	31	8


#85

14	25	30	7	12	23
29	6	13	24	31	8
26	15	28	9	22	11
5	36	3	16	19	32
2	27	34	21	10	17
35	4	1	18	33	20


#20

2	27	18	31	4	29
17	12	3	28	19	32
26	1	36	13	30	5
11	16	9	22	33	20
8	25	14	35	6	23
15	10	7	24	21	34

The other twelve pairs of closed tours occur in three groups of four, corresponding to three geometrically congruent closed tours, numbered from different starting cells: (a) #1-53, 10-78, 18-83, 65-66; (b) #11-79, 17-52, 19-84, 55-58; (c) #12-72, 13-81, 14-59, 21-85. One example of each case is shown, the grey dots indicating the alternative numbering points.

There are 50 tours in which the diagonals add to 222, consisting of 24 closed and 26 open. This includes all the closed tours except #1 and its reverse #53, though others in group (a) do add to 222.

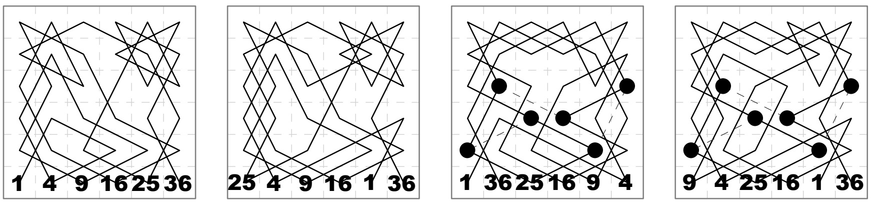
In case (b), where the underlying tour is piece-wise symmetric, there is a constant difference of 6 or 30 between numbers in diametrally opposite cells. Tour #19 was given in my note on ‘Piece-Wise Symmetric Tours’ (*Journal of Recreational Mathematics* Vol.28 No.1 1997). The first aa/ab tour in the 10-slant section is the only other of this type, though not semi-magic.

Group (c) shows approximate axial symmetry and is illustrated by #85 above.

The open tour shown #20 and its reverse #25, are the only ones where the diagonals add to 110 and 112, as near as possible to the magic constant.

Figured 6×6 Tours

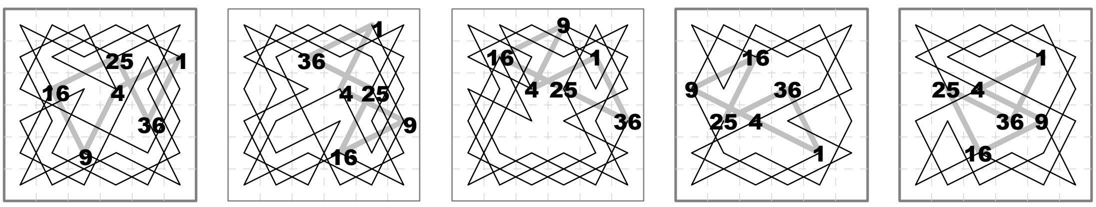
Following my work on ‘Wazir Tours with Squares in a Row’ (*Chessics* vol.2 #21 Spring 1985 p.56, reported in ⌘ 2) the unique result for the wazir on the 6×6 board naturally led me to look at the same problem for the knight on that board, and to find that there was again a unique tour. This was in ‘Notes on the Knight's Tour’ (Special Issue of *Chessics* vol.2 #22 Summer 1985 p.61). The first diagram shows the <u>only</u> solution with the squares in order of magnitude.



Other solutions are possible with the squares in different sequence on the first rank. Three cases are shown, two having alternative routes. A tour with squares along the second or third rank is impossible. These results were reported in the booklet *Figured Tours* (1979).

It only recently occurred to me to try the Dawson problem of a tour with the square numbers in a closed knight circuit on the 6×6 board. It has been known for a long time (apparently in *Chess Amateur* though I've not located the exact reference) that there are 25 such geometrically distinct circuits. One is too large to fit on the 6-board. The others can be placed on the board in various positions (81 at my last count), and the numbers can be placed on them in up to 12 ways.

However, many cases are easily eliminated. For instance in the first diagram (14 Apr 2017) the node at f5 must be an end-point (1 or 36) since there is only one other move available there. The second (15 Apr) is the only solution found with a symmetric circuit. The third and fourth (24 Apr) use the same asymmetric circuit but in a different position on the board, and numbered from a different point. The fourth and fifth diagrams (10 May) each represent four solutions differing only in the route taken from 25 to 36. Since my search has not been made by computer programming, it is possible some solutions may have been missed.

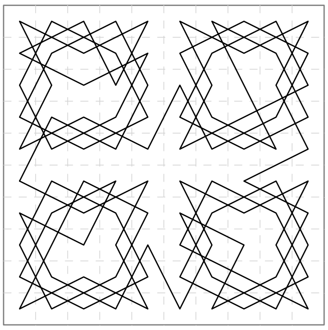


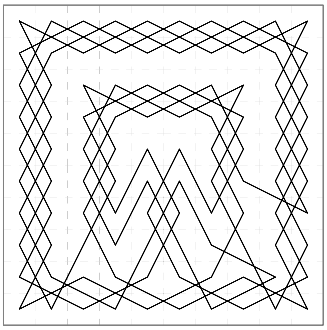
The 10×10 Board

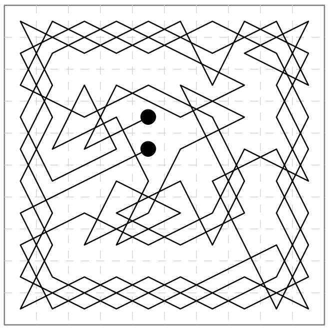
The earliest knight tour on the 10×10 board was one by Euler (1759) connecting four copies of a 5×5 tour (see the quaternary symmetry section below for the diagram). The next in historical sequence is one by Chapais (1780) of this type, but asymmetric. Three were given by Laisement (1782). We arrange the tours here according to the type of symmetry they show.

10×10 Asymmetry

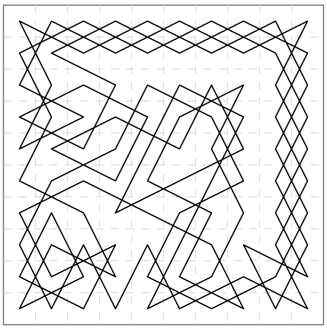
Chapais (1780, Fig.30) has an asymmetric tour by joining four 5×5 tours. A tour by Laisement (1782, Fig 27) has approximate axial symmetry deviating only at the links h5-j4 and g3-i2. Warnsdorf (1823) gives a much more irregular open tour.

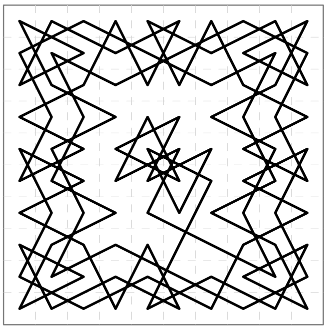

Chapais 1780

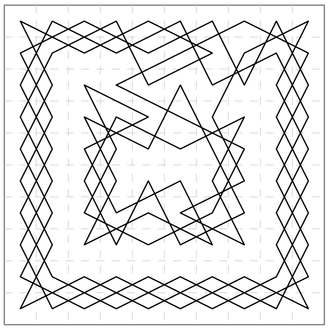

Laisement 1782


Warnsdorf 1823

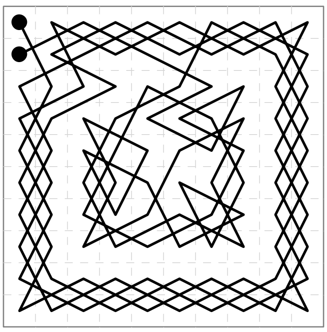
The next three diagrams show asymmetric closed tours by Ciccolini (1836), Troupenas (1842) with an Euler cross tour in the centre, and one by Slyvons (1856).

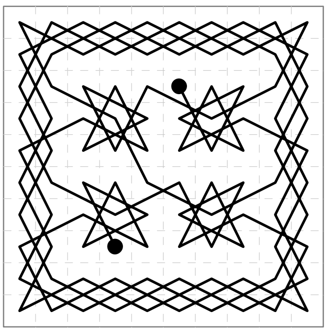

Ciccolini 1836

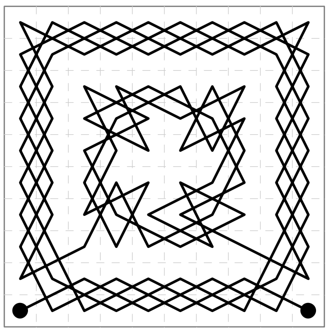

Troupenas 1842


Slyvons 1856

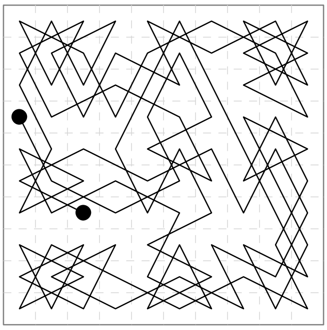
The next five examples are asymmetric open tours. The first is by Harikrishna (1871) shown as #82 in Iyer (1982). The second another 19th century example collected by Murray from a Prague newspaper, *Zlata Praha* (1895). The third is by Lucas (1894) after Delannoy (1886).

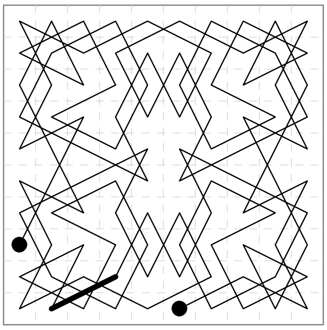

Harikrishna 1871

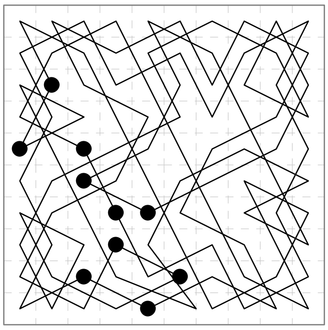

Zlata Praha 1895


Luxas 1894

The fourth is from *Essener Anzeiger* 13/4/1926, collected by Murray. The Bergholt tour is from *Queen* 1916, showing approximate biaxial symmetry, formed of two 50-move axial paths, one the reflection of the other, with one move deleted in each circuit and the ends connected.

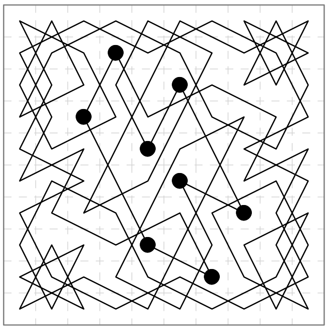

Essener Anzeiger 1926


Bergholt 1916


Engelhaupt 1998

The closed asymmetric piece-wise tour by Hans Engelhaupt given in the *Journal of Recreational Mathematics* 1998 shows a constant difference of 10 when numbered from any of the dotted cells.

The asymmetric tour below, shown in geometric and arithmetic forms shows maximal Bergholtian symmetry (Jelliss, *Jeepyjay Diary* 5 March 2016). The linkage used in this 10×10 example will not work on the 8×8 board (see ⌘ 7), since moves through two corners would be prevented. Diametrally opposite numbers add to 98 with the exception of the three pairs in bold and circled. The 00 underlined indicates 100. The crosslinks are 1-100-99-98-97 and 47-48-49-50-51.

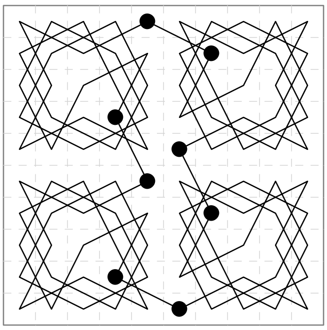


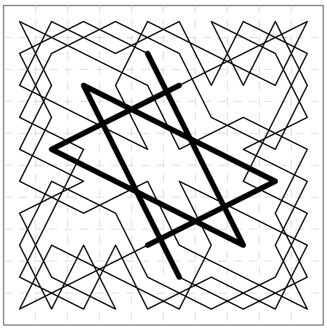
17	22	89	26	45	38	87	34	31	36
90	25	18	1	88	27	44	37	86	33
21	16	23	46	39	**48**	3	32	35	30
24	91	**00**	19	2	43	28	57	4	85
15	20	93	40	47	56	**49**	84	29	6
92	69	14	**99**	42	51	58	5	78	83
13	94	41	70	55	96	79	**50**	7	74
68	63	66	95	**98**	59	52	75	82	77
65	12	61	54	71	10	97	80	73	8
62	67	64	11	60	53	72	9	76	81

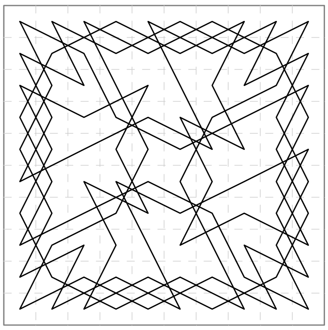
Jelliss 2016

10×10 Binary Symmetry

A 19th century example by Ciccolini (1836), though similar to Euler, only has binary symmetry. The star formation in the second tour (Jelliss 1980s) is impossible in a tour on an 8×8 board.

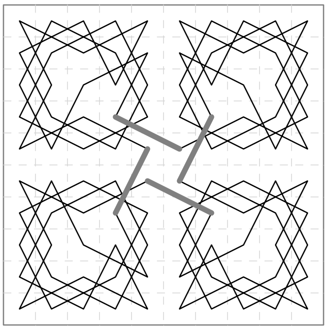

Ciccolini 1836

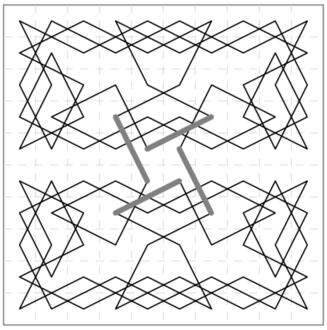

Jelliss 1980s

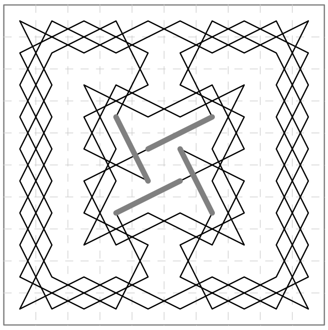

Jelliss 1985 no right angles

In the first issue of *Chessics* (1976) I proved that every 8×8 knight's tour contains a right angle, but in *Chessics* 22 (1985) I showed that this result does not hold true on the 10×10 board, by giving a tour, in binary symmetry, with no right angles, the third diagram above. Birotary tours without right angles are also possible. See below.

The next three tours with binary symmetry all show mixed quaternary symmetry with the same four moves in pure birotary symmetry, shown by the bolder line. A 19th century example by Slyvons (1856) is the earliest. The four identical 5×5 tours are arranged to show biaxial symmetry, and could be joined, vertically or horizontally, to form two axially symmetric half-board tours, but instead the four links between them are in birotary symmetry.


Slyvons 1856

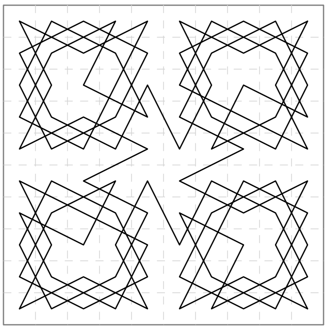

Bergholt 1916

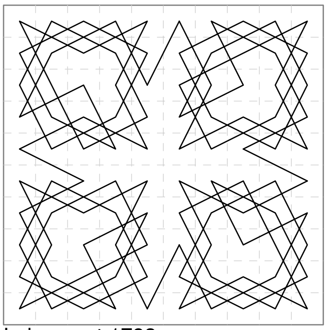

Jelliss 2015

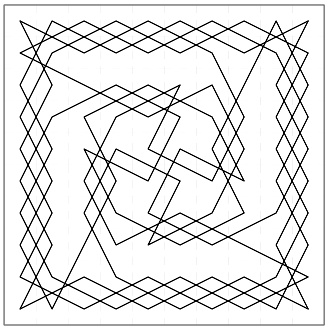
The tour in the second diagram is by Ernest Bergholt from the magazine *Queen* 1916. This uses a 50-move half-board axially symmetric tour of Sulian type, reflected in the lower half-board. By deleting one move in each circuit the loose ends are connected to form a closed tour with diametral symmetry. This was a precursor of his tours (on 8×8 and 12×12 boards) with mixed quaternary symmetry. The other diagram (Jelliss 2015) adds a border to one of the 6×6 mixed symmetry tours.

10×10 Quaternary Symmetry

Euler presented the first 10×10 knight's tour in 1759, it has birotary symmetry (i.e. is unchanged by 90° rotation) and is formed of four 5×5 tours linked together, as shown here. Denis Bailliere de Laisement (1782, Fig 26) has a similar birotary tour but with links a6-c5 etc. His Fig.25 (Plate IV) is another 10×10 tour with quaternary symmetry with a more elaborate and attractive design.


Euler 1759


Laisement 1782


Laisement 1782

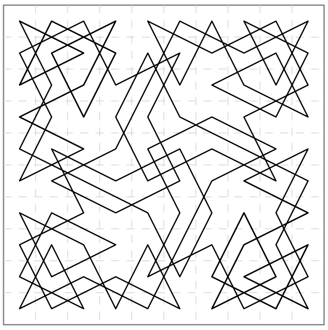
The *Kaleidoscope Echiquieen* by Carle Adam (1867) contains 100 tours on the 10×10 board but I have only seen the first two tours, which are like Euler's tour, joining four copies of a 5×5 tour in quaternary symmetry. Kraitchik (1927) also diagrams a tour of this type, as does S. Vatriquant in *L'Echiquier* (Jan 1929 ¶314) but these are all very easy to construct. H. J. R. Murray (ms ‘Note on the combination of tours on the 5² board to form symmetrical tours on the 10² board’ dated 2 Jan 1917) calculated that there are 28 tours of Euler's type, linking the quarters e1-f3, e1-g2, e5-f3 or e5-g4.

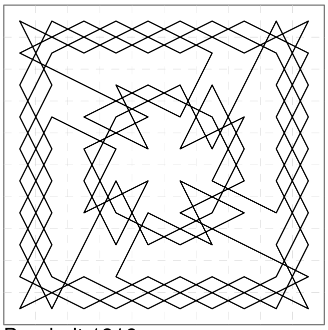
Tom Marlow in a letter to me dated 29 Dec 1997 (published in *Games and Puzzles Journal* #16, 1999, p.288) applied his computer to calculating the number of 10×10 knight's tours with quaternary symmetry, finding a total of **415,902** geometrically distinct solutions. This is more than twice W. H. Cozen’s estimate of 200,000 made in his *Mathematical Gazette* article in the 1960s.

The process of enumeration is described as follows: The method of search was to start a tour at c2, proceed to a1 and then to b3, numbering these squares as 1, 2, 3. The chain was then extended to a length of 25 cells and at each step the corresponding cells, after repeated 90° rotation, were marked as unavailable. At this stage a check was made as to whether a link was possible to h5, and if so a tour recorded. After this, or at any previous dead end, the programme systematically back-tracked and tried a new step forward until all steps from b3 had been tried. The total was then recorded. The actual total recorded was 831,804 but each geometrically distinct tour can be diagrammed in two forms in each case, one a reflection of the other in the vertical median. The routine described counts both versions, so the totals must be halved. (Because of the 180° rotational symmetry of the tours, reflection in the horizontal median has the same effect as reflection in the vertical median.) The same count was found by a second (more time-consuming) check as follows. The start was made at b1 and extended by the same method until a link to j2 could be sought. A link to a9 would produce a tour that is a 90° rotation of a tour linking at j2 so is not geometrically different and should not be counted. This method produced the same count.

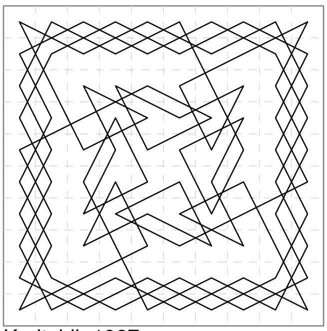
Marlow also noted: “Tours fall into two types. The first links b3 to b8 and then, via a10, c9 to h9, i8 to i3 and h2 to c2; call this a **circular tour**. The second type runs b3 to i3, h2 to h9, i8 t0 b8 and c7 to c2; call this a **loop tour** since it loops the loop at each corner. The count of the two sorts is as follows: circular tours 206,937, loop tours 209,065, total 415,902.”

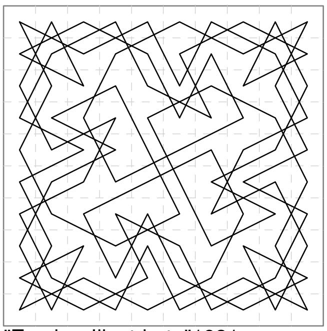
Continuing the historical examples, these two are from Bergholt's series in *Queen* 1916.

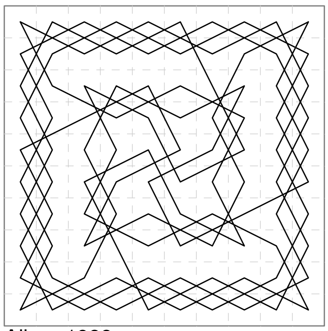

Bergholt 1916


Bergholt 1916

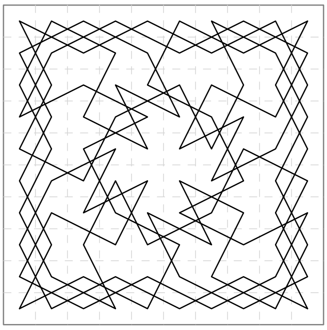
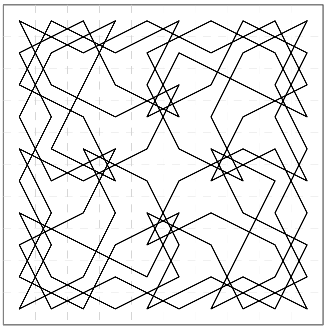
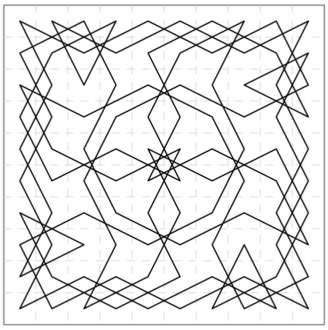
The diagrams below show a closed quaternary tour by Kraitchik 1927 and one from *Zurcher Illustrierte* 27/5/1931, collected by Murray. Also shown is another border braid quaternary example used by ‘Alban’ (Will Scotland) for a crossword in the magazine of the Crossword Club 1988.


Kraitchik 1927

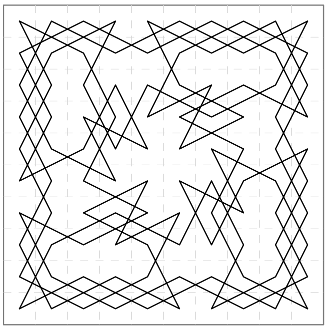
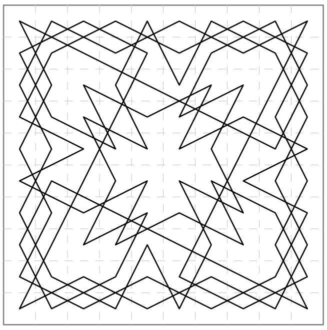
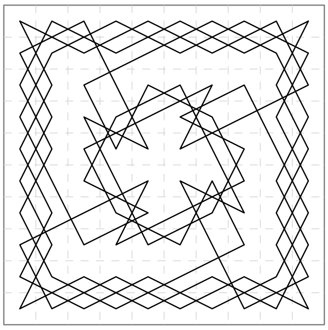

"Zurcher Illustrierte"1931


Alban 1988

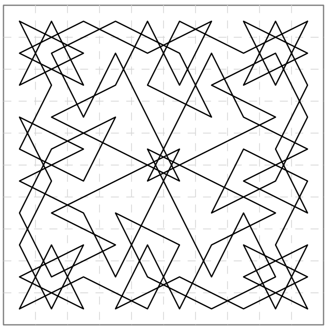
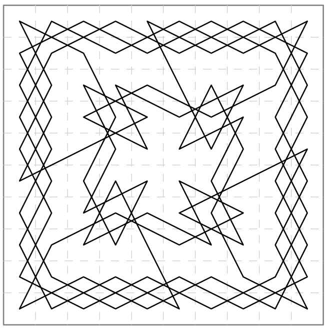
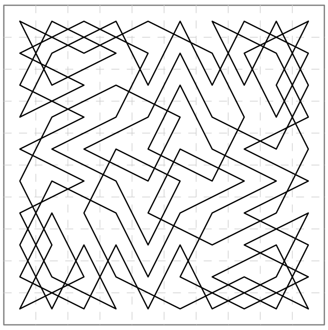
There follow six tours by Murray in quaternary symmetry from his 1942 manuscript on *The Knight's Problem*. His particularly nice third example has just four ‘slants’ (e.g. a2-c3).

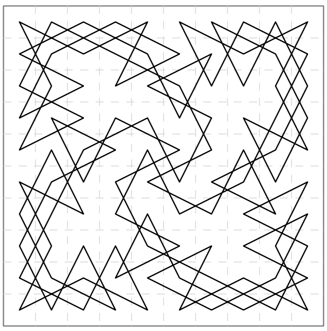
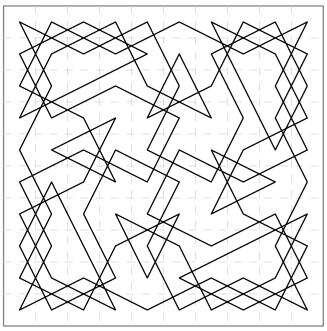
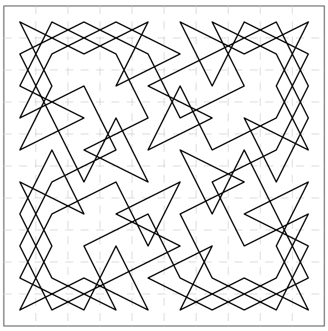
Murray continued.

W. H. Cozens published an article on ‘Cyclically Symmetric Knights Tours’ in *Mathematical Gazette* (1960). I quote three that he gave (his fourth was of the Euler type)

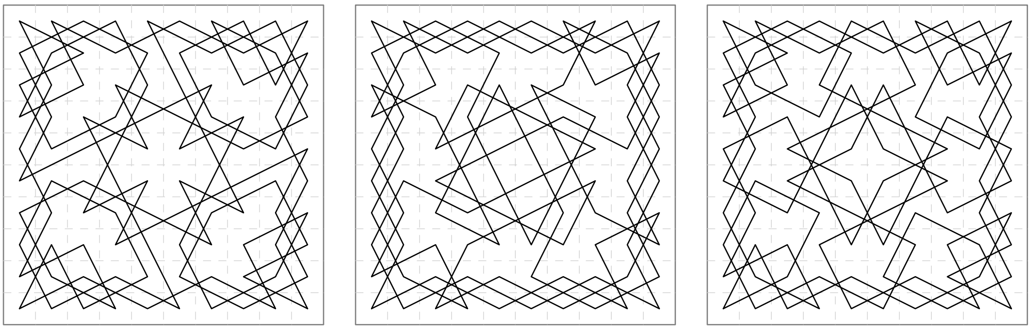
The next three are from a set of notebooks left by Mr Cozens, containing numerous diagrams of tours of this type, though many of them are repetitive since he seems to have been attempting to enumerate all tours with particular central configurations.

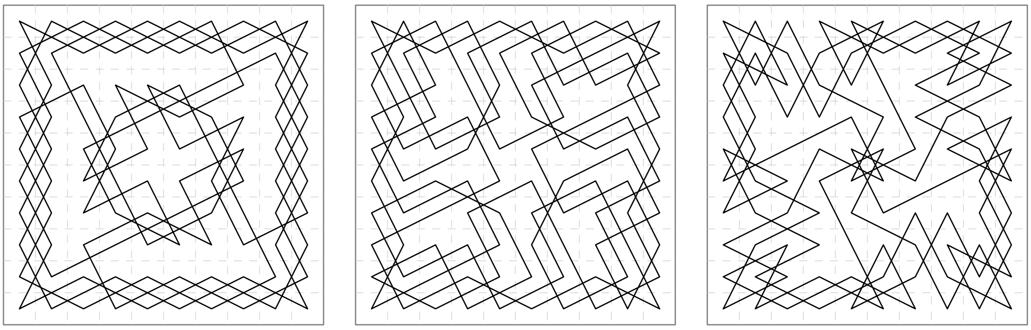
My Own Work on 10×10 Birotary Tours

The examples that follow are all results of my own research. The first examples were constructed just for the interest of their patterns. The first five date from the 1980s.

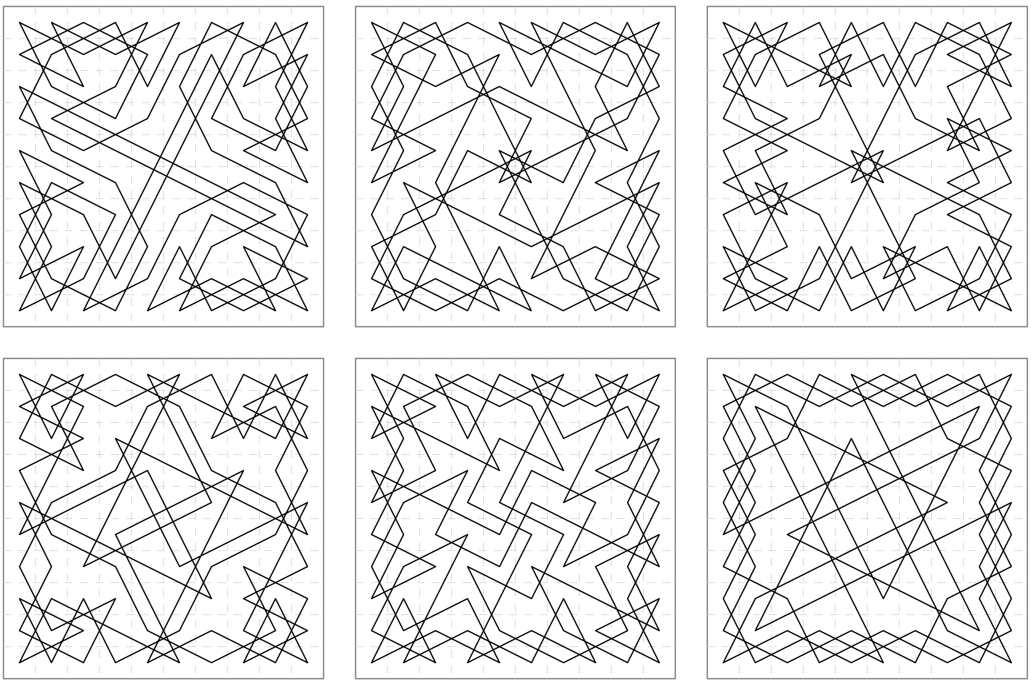
The central Maltese Cross and octagonal star (dated 10 Jan 1986) is impossible on the 8×8 board.



The fifth (dated 11 Feb 1986) is evidently an attempt to maximise the number of right angles. The sixth (8 Apr 1996) is an offshoot of my study of tours without right angles; it has eight right angles on the cells coded '1', forming a rotor fan design.

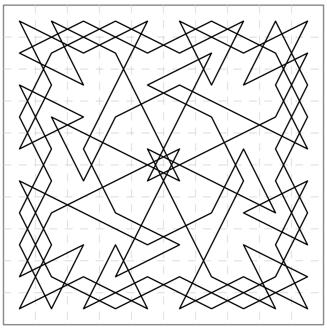
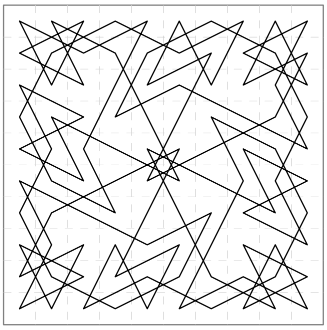


The next six date from the 1990s. The last is a quaternary tour showing the 3-square inside 7-square which is impossible in a closed tour on the 8×8 board.

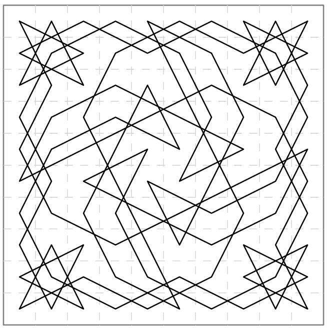
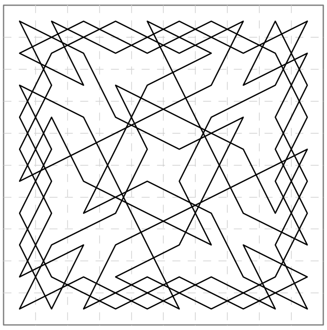
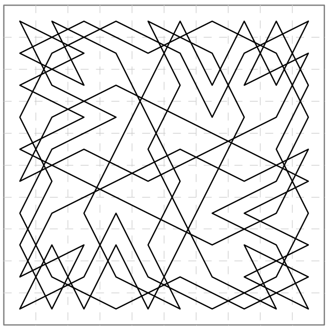
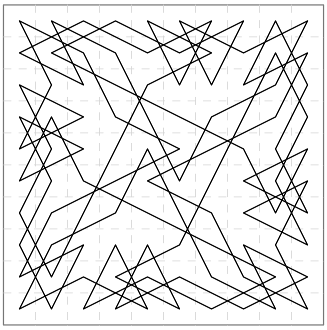
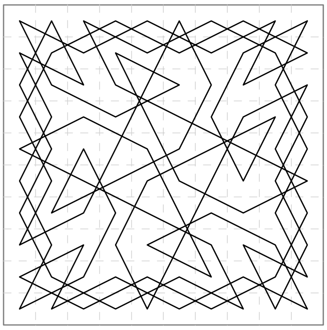
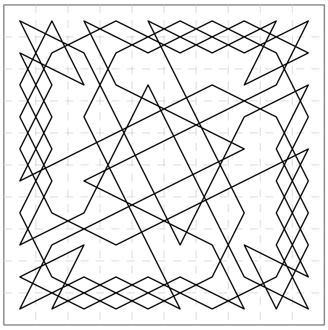


10×10 Birotary Tours Without Right Angles

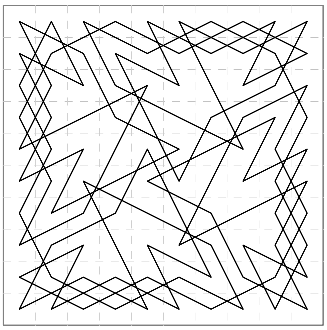
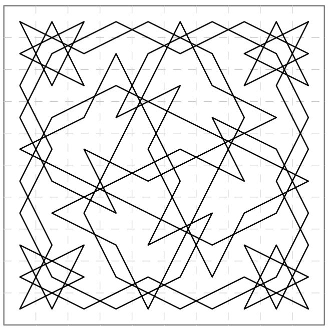
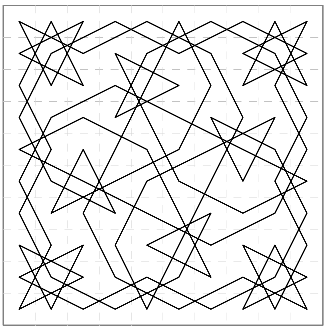
As noted in the binary section above tours on the 10×10 board without right angles can be constructed. I have enumerated 40 examples in quaternary symmetry, of which I give a selection. Classified by central angle the 40 are 'c' 3, 'e' 5, 'i' 4, 'j' 10, 'l' 5' 'o' 11, 'p' 2. There may be others. The two with 'p' centre are the only ones not including the d8-h6, h7-f3, g3-c5, c4-e8 square.

Apart from these I show one tour for each of the central angle formations.

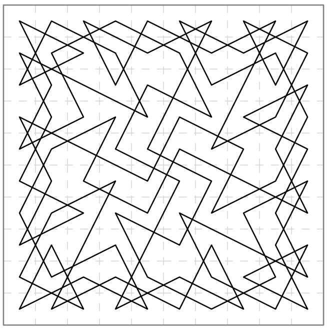
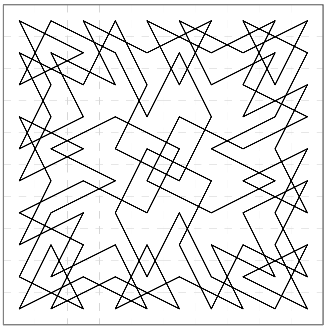
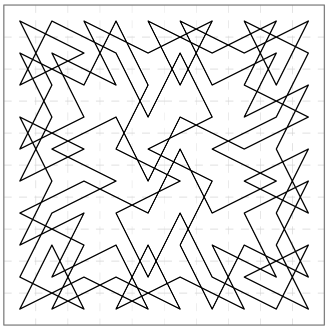







Here are three more of the tours without right angles.

10×10 Birotary Celtic Tours

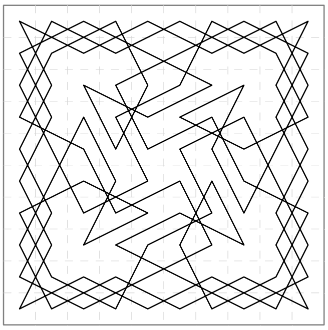
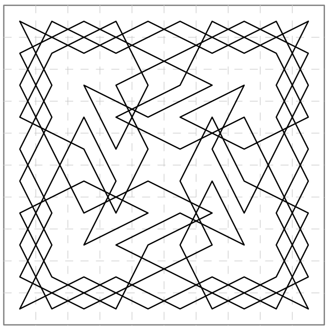
Here are three 10×10 Celtic tours with 90° symmetry that I found. These were published in the *Games and Puzzles Journal* (one in #16 1999 p.287, the others in #21 online Sep-Dec 2001).

Celtic tours, so named by Prof. D. E. Knuth, are tours in which no ‘size 1’ triangles occur. Size 1 triangles are the smallest possible formed by knight moves, say by a1-c2, b2-d1, b1-c3, and have an area 1/120 of a board-square (see ⌘ 1). The triangle a1-c2, b1-c3, c1-b3 size 2 has an area 1/30.

10×10 Birotary Bordered from 6×6 Centre Pattern

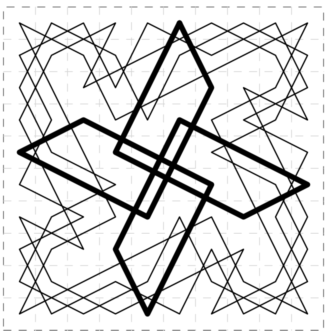
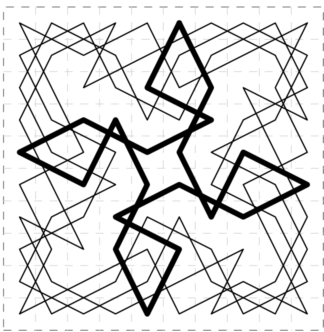
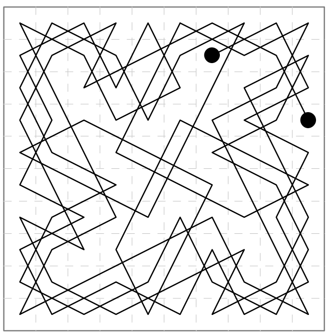
The following two (Nov 2009) use 20 birotary moves on the 6×6 board, which cannot form tours on that board, to form tours on the 10×10 board, other moves being in octonary symmetry.

On the 10×10 and larger boards the central angle patterns in birotary tours can be further classified according to the 15 patterns of angles on the cells coded '1' adjacent to the cenral 2×2. These angles are coded: 310, 410, 413, 510, 513, 514, 610, 613, 614, 615, 710, 713, 714, 715, 716.

10×10 Birotary Pseudo Tours

On the 10×10 board I find just two different ways of covering the board with five 20-move circuits, each in 90° rotary symmetry. The open tour (Jelliss 1999) is a piece-wise symmetric tour derived from the first pseudotour. If numbered, diametrally opposite cells differ by 10 throughout.

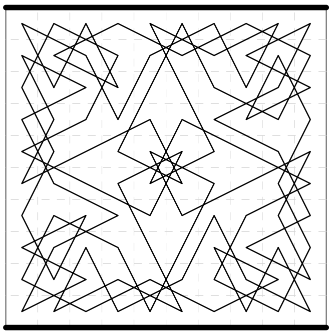




Jelliss 1999

Semimagic Tours on the 10×10 Board

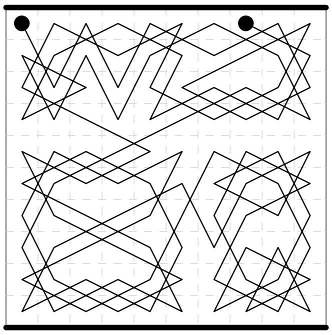
In 2002 Tom Marlow, at request of A. Kumar, set up his 10×10 symmetric tours programme to look for semi-magic tours, and concluded that there was only the one shown here. The files all total to 505. The rank sums are also regular, though not magic. Ranks 1-4 all total 655, rank 5 is 455, rank 6 is 555 and ranks 7-10 are 355. Tom also noticed that every vertical pair in the two middle ranks totals 101. The background pattern formed by the 4·x and 4·x + 1 numbers, shown grey, is the same as that formed by the 4·x + 2 and 4·x + 3 numbers but rotated a half-turn. This result was published in *The Games and Puzzles Journal* #25 (online) Jan-Feb 2003.

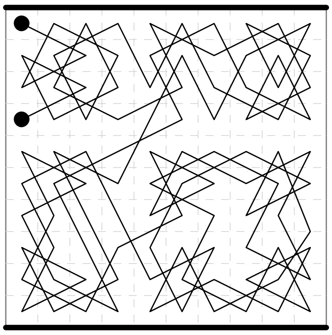
27	94	25	22	35	16	31	20	33	52
8	23	28	93	30	21	34	51	48	19
95	26	7	24	15	36	17	32	53	50
6	9	92	29	12	39	54	49	18	47
91	96	11	14	63	88	37	40	55	60
10	5	90	87	38	13	64	61	46	41
97	68	99	4	89	62	79	42	59	56
00	3	82	67	86	65	74	57	76	45
69	98	1	84	71	80	43	78	73	58
2	83	70	81	66	85	72	75	44	77



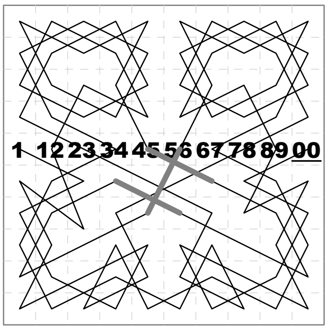
Tom wrote: “I was suprised that only one semi-magic tour appeared and also that its numbering started at the same point that my programme starts in its search for tours. This despite the fact that the programme cycles the numbering of each tour in the search for magic. However it does find a cycled version of the same tour and this and other checks that I've done make me reasonably sure that nothing is being missed.”

In 2008 Awani Kumar constructed 3343 semi-magic 10×10 tours, all apparently open tours of compartmental type, starting at the top left, these two examples being the first and last in the series:



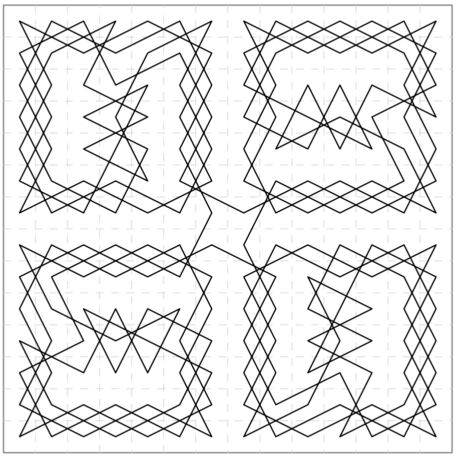
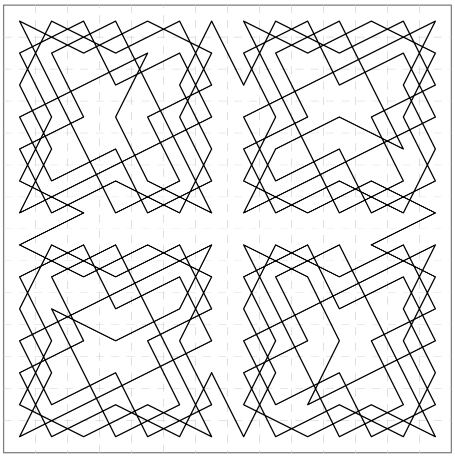


This 10×10 open figured tour with approximate axial symmetry was set as a puzzle in my booklet on *Figured Tours* (1997). It has the arithmetic progression 1 to 100 with common difference 11 along a rank and the segments alternate in the lower and upper ranks.

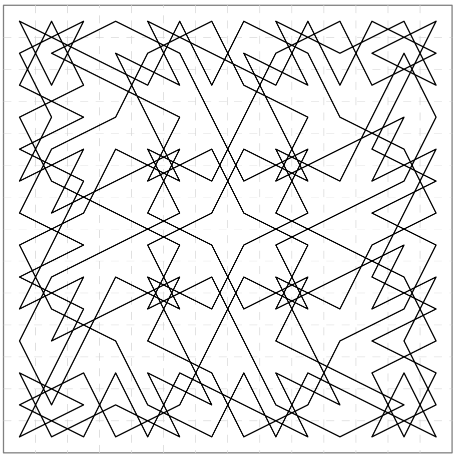
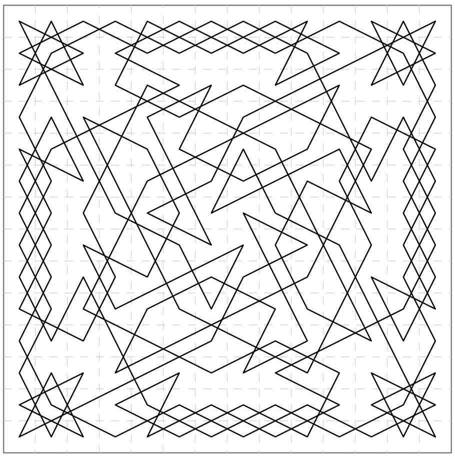


The 14×14 Board

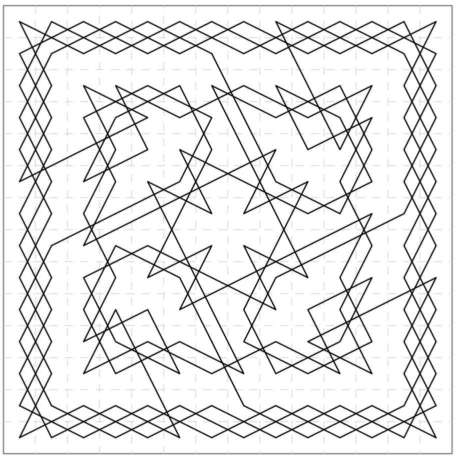
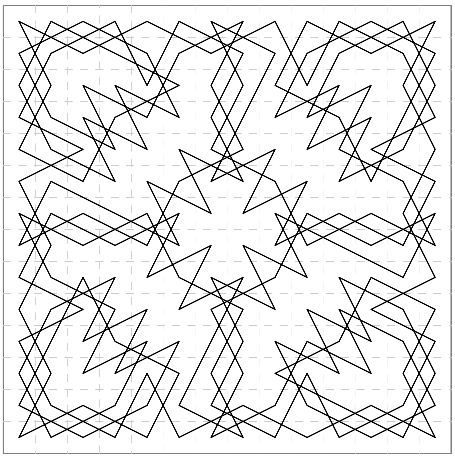
All shown are birotary. Kraitchik (1927) and Murray (1942) join four 7×7 open tours.

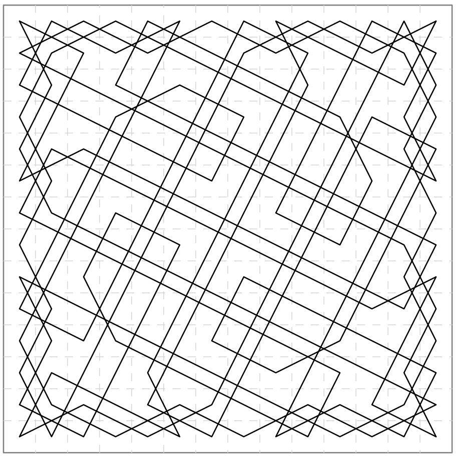
The earliest printed example I have come across is the first diagram below by Archibald Sharp from his book *Linaludo* (1925). The second example is from Kraitchik (1927).

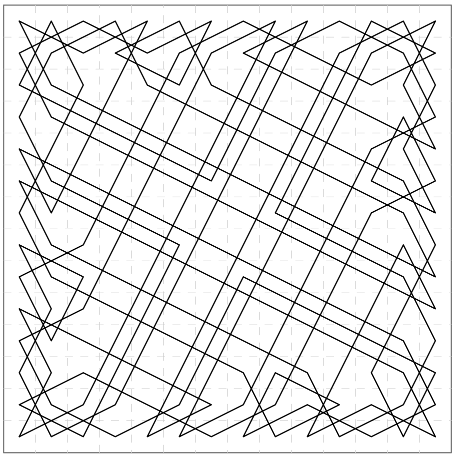



The next seven are my own. The first, from 1986, places my 10×10 tour incorporating a Maltese cross within a complete border braid as frame. Then comes another Maltese cross design.

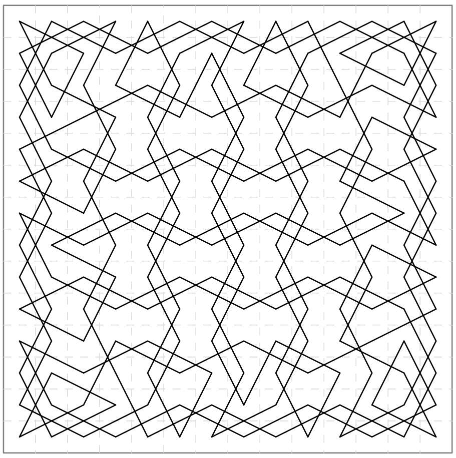



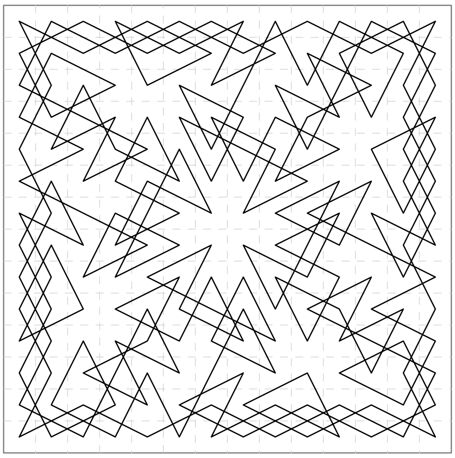
The next two attempt tartan plaid effects, based on the ‘35’ and ‘45’ arrangements of nightrider lines (where every 3rd and 5th, or every 4th and 5th line in a set of parallels is turned at right angles).



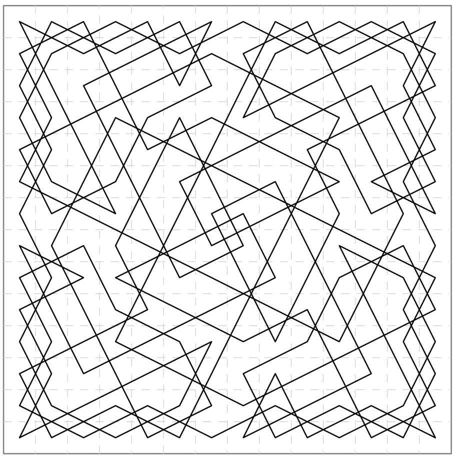


The next has a central mosaic pattern. The sixth example includes sequences of seven successive three-knight-move triangles in the central region.



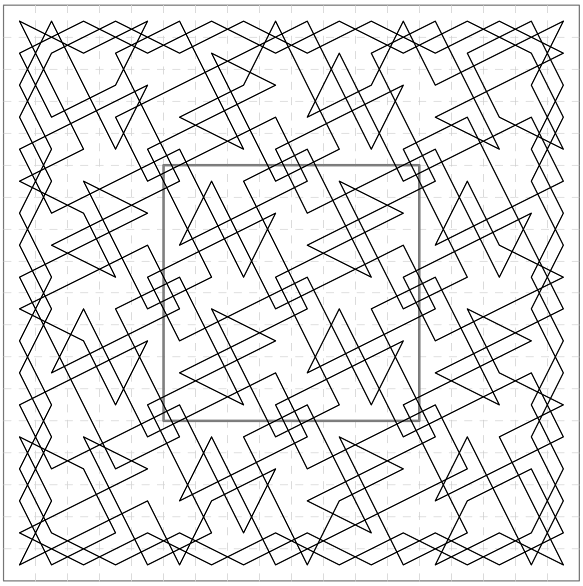


Here is an example incorporating 7-unit and 1-unit squares and Greek crosses.



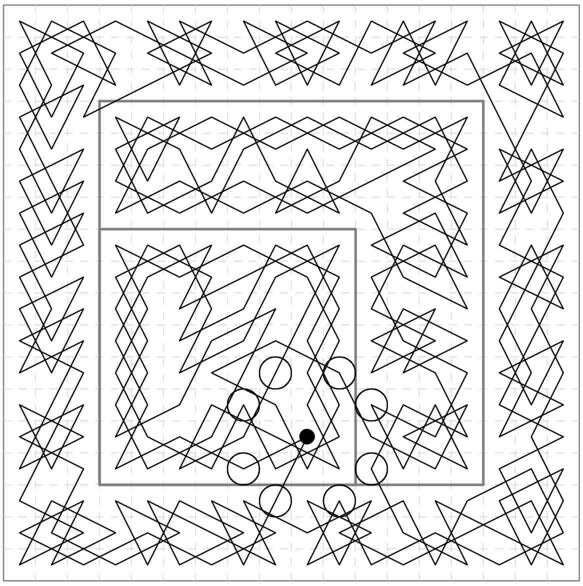
The 18×18 Board

This tour (Jelliss 1999) was the cover illustration on issue 16 of *The Games and Puzzles Journal*.



It shows oblique quaternary symmetry (unchanged by 90 degree rotation) with the central area using a repeating (wall-paper type) pattern borrowed from Archibald Sharp (1925).

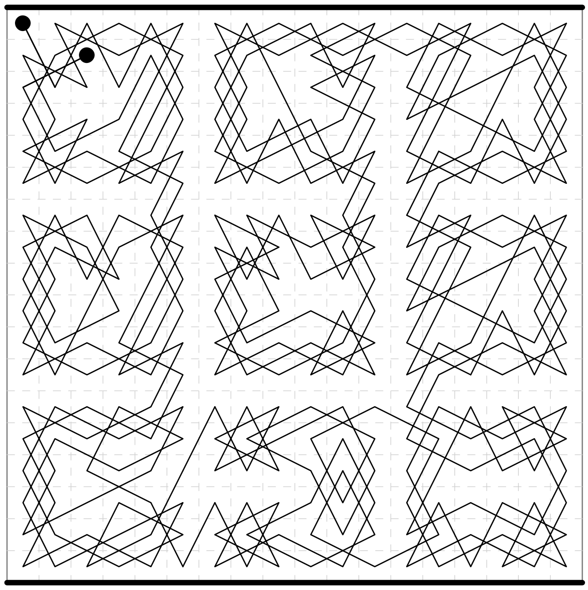
Figured tour by the Rajah of Mysore. The numbers round 1 are 2, 16, 32, 64, 72, 144, 162, 324. Thus the 8×8, 12×12 and 18×18 tours are all closed. The details of this tour were sent to me by Maria Schetelich a Sanskrit scholar at Leipzig University in October 2017. It is among work of Krishnaraja Wodeyar that was sent to Leipzig Museum in 1907.



Maria Schetelich says that works by the Rajah in the Leipzig Museum came in 1907 from Hugo Boltze (born 1864?), who was at that time in charge of the Calcutta branch of the London Antiquities Company, S.J. Tellery & Co. and a well-known figure among Europeans looking for Indian 'curiosities'. Maria also believes that there are other works in the British Library.

. KNIGHT'S TOUR NOTES

As on the 6×6 and 10×10 and 14×14 boards no 18×18 magic tour is possible, only semi-magic as Awani Kumar has done in this tour formed from nine 6×6 patterns. Numbered with 1 at top left and 324 at c13 the files add to 9×325 = 2925.



==

Larger Oddly Even Boards

==

30×30. Awani Kumar has constructed a semi-magic tour on the 30×30 board, magic being impossible. It is based on a 5×5 arrangement of 6×6 patterns joined in a spiral. The dots mark the cells numbered 1 and 900. The magic sum is 901×15 = 13515. Many adjacent cells add to 901.