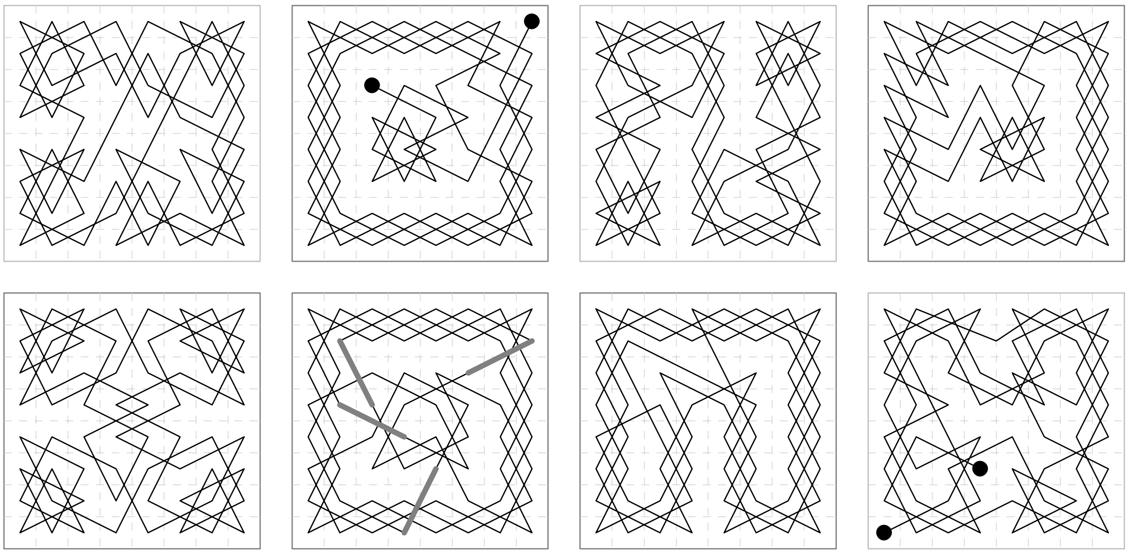
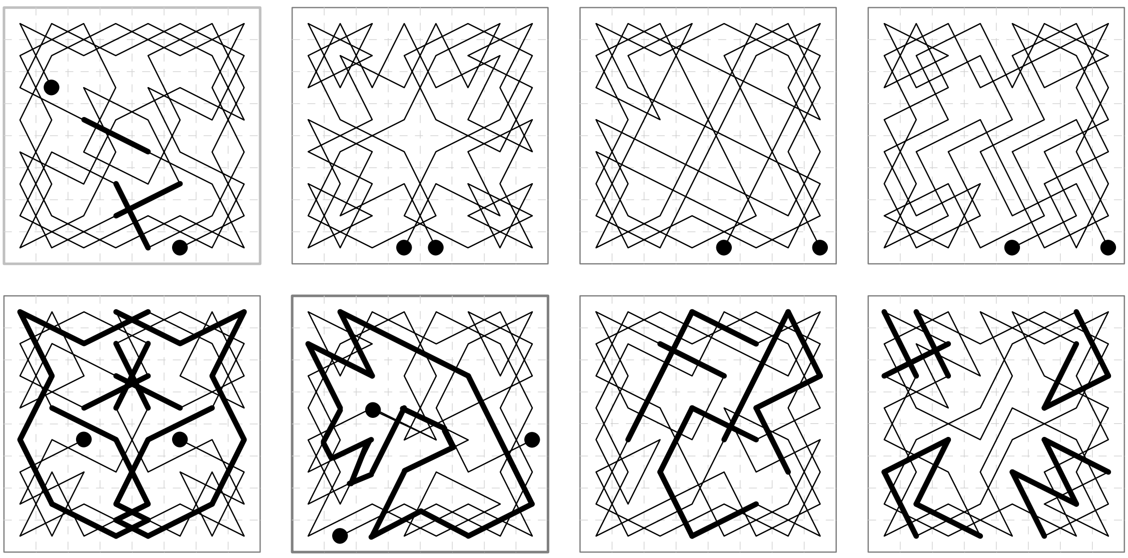
⌘ 6

Geometry of Chessboard Knight Tours



by G. P. Jelliss



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Title Page Illustrations: 16 selected chessboard knight tours.
Al-Adli c.850 first recorded closed tour. Ali ibn Mani c.1350 mediaeval open tour.
Nilakantha 1640 first recorded symmetric tour. Euler 1759 the most cited closed tour.
Vandermonde 1771 linking of four identical circuits.. Collini 1773 method of annuli.
Laisement 1782 simple linking of braid. Warnsdorf 1823 squares and diamonds.
Roget 1840 three slant tour. Falkener 1892 maximum slants.
Hogrefe 1924 ten three-move lines, Moricard 1982 maximum 40 right angles,
Diane A, of Orleans: 1878 Cat pictorial tour. P. C. Taylor: 1937 Knight pictorial tour.
H. D. Benjamin 1950 for *Fairy Chess Review*. G. P. Jelliss 1991 tribute to H. J. R. Murray.

Contents

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http://www.mayhematics.com/
Knight's Tour Notes, Volume 6, Geometry of Chessboard Knight Tours.

Some History of Knight Tours

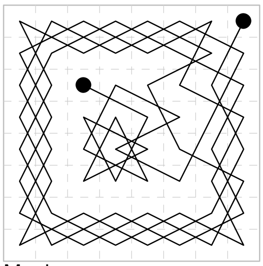
The Earliest Full-Board Knight Tours

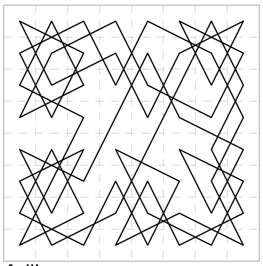
Many of the earliest knight tours found were on the 4×8 half chessboard. Their history is outlined in the section on 4-rank boards in ⌘ 4. The sources of information cited there also apply to the full-board tours. For much fuller details of the mediaeval manuscripts the reader should consult those more scholarly works. It is regrettable that the mediaeval work on knight tours, while well known to chess historians, has been largely ignored in the mathematical literature, in which accounts of the subject either make unsubstantiated claims of ancient origins or trace its history back no further than the start of the eighteenth century (Ozanam 1725).

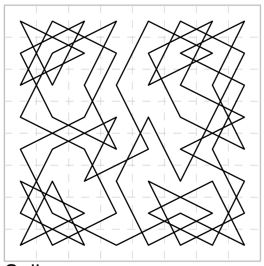
The discovery that a chess knight could visit every square of the 8×8 board in a series of moves without retracing its path or visiting any square twice, seems to have occurred around the year 800, somewhere in the Islamic empire which then extended from north Africa to northern India, and was ruled from Baghdad by the Abbasid dynasty. The discovery would soon have made its way to the capital Baghdad which was probably the largest city in the world by 814 [*Times Atlas* 1989].

It was in Baghdad that the first books on a form of chess, were written. In particular two authors, **al Adli ar Rumi** (flourished 840) and **Abu-Bakr Muhammad ben Yahya as Suli** (c.880-946) are known, from a bibliography by an Nadim (988) to have written books on chess that included knight tours. These books do not survive complete, but extracts are quoted in manuscripts compiled later, such as those by Baghdadi (1141), Othman (1221) and Hakim (c.1350). The form of chess these players and their colleagues wrote about, known as Shatranj, is the earliest for which accurate rules are known and for which moves based on actual games have been preserved as examples of strategy. We also have endgame studies, problem compositions, and tours.

Six full-board knight tours (not counting those formed of two half-board tours) survive from the middle ages. Three are from from Arabic sources:


Mani


Adli


Suli

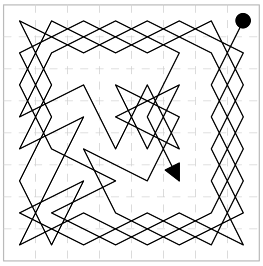
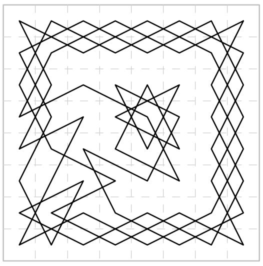
The most primitive of these full-board tours is in the form of an open path following the edges of the board and ending in the centre, a design typical of many later examples. This appears in the Hakim manuscript where it is ascribed to **Ali ibn Mani** an otherwise unknown player.

The tours are shown here in graphical form, though in the manuscripts various devices were used for their presentation (see Lettered Tours in ⌘ 11 for more on this). The al-Adli tour, given in the same manuscript, is the earliest to which we can ascribe a definite date, since Adli flourished c.840. It was presented there as a sequence of cells numbered (in the alphabetic system of numeration then in use) from 1 at h8 to 64 at f7. The use of coordinates to record positions and moves in chess also goes back at least to this time. One manuscript labels the files with arabic letters equivalent to y, k, l, m, n, r, sh, t; and the ranks: a, b, j, d, h, w, z, hh. [Murray 1902]

The later master of Shatranj as-Suli based his works on those of al-Adli, which he criticised. His tours are more structured, the one shown here (numbered from e5 to d3) incorporating an axially symmetric 30-cell tour within the left half-board (with the 'meteor' corner formation). His other tour, shown in the note on 4×8 tours, joins two half-board tours into a closed tour. Suli also gave two tours by combined pieces, knight + fers and knight + alfil (see Augmented Knights in ⌘ 10).

The next recorded full-board tour comes from an Indian work written in Sanskrit. This is described in an article by F. Bernhauer (1997) and is from a manuscript called the *Manasollasa*, which is translated as ‘Freude des Geistes’ (i.e. ‘Delight of the Spirit’) a traditional sobriquet for chess, and is described as a ‘Fürstenspiegel’ (‘Princely Mirror’) written for King Somesvara III of the Kalyani area in central India (c. 1150).

The tour in this source is described in the form of a list of two-letter coordinates, but the sequence of syllables is somewhat corrupt. Bernhauer gives an open tour as in the first diagram below. However, as listed on the third page of Bernhauer's article, the coding ends with the cell a knight's move from the corner. This suggested to me that the tour was intended to be closed, and led me to the version shown on the right which is the most likely interpretation in my view. This tour is remarkably similar to the much quoted first closed tour given by Euler (1759) over 600 years later.

In the coordinate system used, the files are lettered c, g, n, d, t, r, s, p and the ranks by short and long vowels, shown here as a, â, i, î, u, û, e, ê (since I am unable to find barred vowels). This system presents a tour as a sequence of syllables that is pronounceable and looks like Sanskrit. The tour (in the reentrant version), when split up into groups of four syllables for clarity, runs:

```
pa si pu se / tê ne cê gû / nî cu gi ca / nâ ta sâ pî
sû pê re dê / ge dû gu ci / ga dâ ra pâ / sî pû sê te
nê ce nû gê / cû gî câ na / tâ sa pi su / pe rê de nu
tû rî di tu / ri dî ru ti / du rû tî ni / cî gâ da râ
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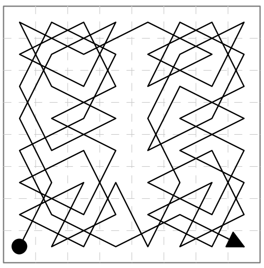
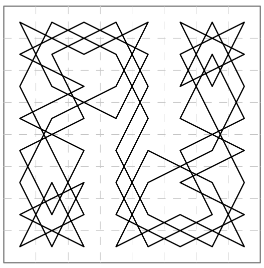
A fifth full-board tour appears in the Persian work of **Muhammad b. Mahmud al Amuli** (d. 1352) called the *Treasury of the Sciences*. The tour is from one corner to an adjacent corner.

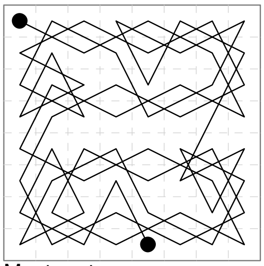



The sixth full-board tour (though 300 years later) is the remarkable symmetric tour from the work of **Bhatta Nilakantha** *Bhagavantabhaskara* (c. 1640). The tour is presented in three different ways, numbered from different points, the first two cases being attributed to Nilakantha's ancestors. This tour is a very remarkable isolated achievement, being a fully symmetric closed tour, before the work of Euler (1759). It includes 7-move stars in two corners and complete quadrangles (four moves intersecting each other) in the sides. However it should be noted that the Sanskrit scholar Maria Schetelich, whom I met in 2018, cites the *Yantracintamani* by **Damodara Bhatta** (c.1550) as a much earlier source for this tour. The tour first became known outside India in an article by Monneron (1776) and is also given in Shir Muhammad-Khan (1796), Harikrishna (1871), Weber (1873), Stenzler (1874), Hoffmann (1893), Naidu (1922), Murray (1913) and Iyer (1982). The name ‘Bhatta’ is a title signifying a scholar. For fuller biographical and reference details of works cited in these History pages see the Bibliography in ⌘ 12.

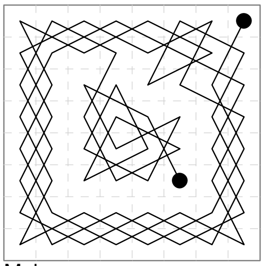
Rediscovery of the Problem 1725-1823

For several centuries the mediaeval work on the knight's tour seems to have been forgotten in Europe, except possibly among a few antiquarians. The subject was discovered anew early in the eighteenth century, appearing as a talking point in social gatherings where chess was played, and receiving publicity in popular collections of recreations and puzzles.

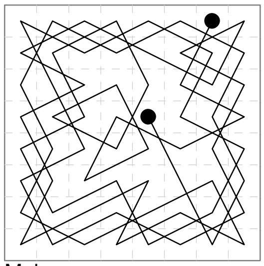
1725: Jacques Ozanam. Knight’s tours reappeared in the 1725 edition of the *Recreations Mathematiques et Physiques* ... founded by Jacques Ozanam (1640-1717). This title began in 1694 and Ozanam’s name continued to be associated with numerous editions until the mid-19th century.



Montmort



Moivre

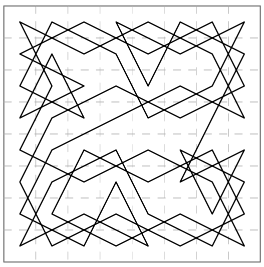


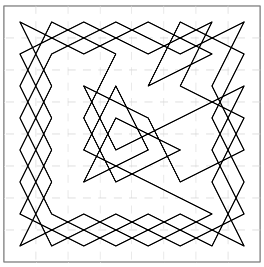
Mairan

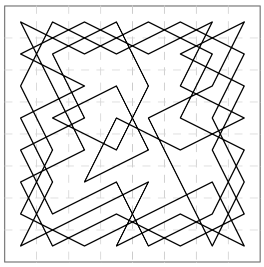
The edition of 1725, published in four volumes in Paris by Claude Jombert, is the first to contain the three knight's tours by the mathematicians **Pierre Rémond de Montmort** (1678-1719), **Abraham de Moivre** (1666-1754) and **Jean-Jacques d’Ortous de Mairan** (1678-1771). According to a marginal note it was Mairan, Director of the Académie Royale des Sciences, Paris, who supplied the tours to the editor in 1722.

Rouse Ball (1939) says the tours by Montmort and Moivre “were sent by their authors to Brook Taylor who seems to have previously suggested the problem”. Unfortunately he gives no reference to where he learnt of the involvement of Taylor (1685-1731) another well known mathematician and I have been unable to verify this. Similarly Kraitchik attaches the date 1708 to the Montmort tour, but it does not appear in Montmort's famous work *l'Essai d'analyse sur les jeux de hasard*, Paris 1708. A slight variation of the Moivre tour in which the last three moves are reflected is mentioned in the text and is sometimes diagrammed in later accounts. The tours are presented as numerical arrays, not in the geometrical form that we show here.

It is evident that these tours do not reach the same degree of development as was achieved by Adli and Suli 800 years earlier. All are open tours. The Moivre tour is on the same plan as the Mani tour in that it starts in a corner and skirts the edges of the board, as far as possible, before filling the centre. The Montmort tour is similar to the Amuli tour and others formed by connecting half-board tours. It is constructed on the ‘domino method’ which applies to all tours on the 4×8. The first half and second half of the tour cover alternate pairs of cells, with a single link a7-c6 connecting them.



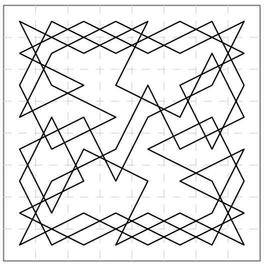




Later composers have constructed closed versions of these three open tours, shown here for comparison. The Montmort version is my own, a8-b6-d5-b4-c2-e1 is the series of insertions and deletions; an alternative is a8-b6-d5-f4-g2-e1. The Moivre version is from Willis (1821) h8-f7-h6-f5-e3-g2-h4-f3. The Mairan version is from Laisement (1782) deleting one move, e5-g6-e7-g8.

1759: Leonhard Euler. The first extensive mathematical study of knight's tours was presented in 1759 by the prolific and influential mathematician Leonhard Euler (1707-1783) though not printed until 1766. He had been thinking on related subjects since at least 1736 when he presented his paper on the geometry of situation: where he solved the famous problem of a tour of the Bridges of Königsberg, proving it to be impossible, which marked the beginnings of network theory.

The first mention of knight’s tours among his papers is a symmetric chessboard tour sent in a letter to Christian Goldbach (1690-1764) dated 26 April 1757. This however did not get wider circulation until published among the correspondence collected by Paul von Fuss in 1843. This tour was also diagrammed more recently by Miodrag Petkovic (2009).



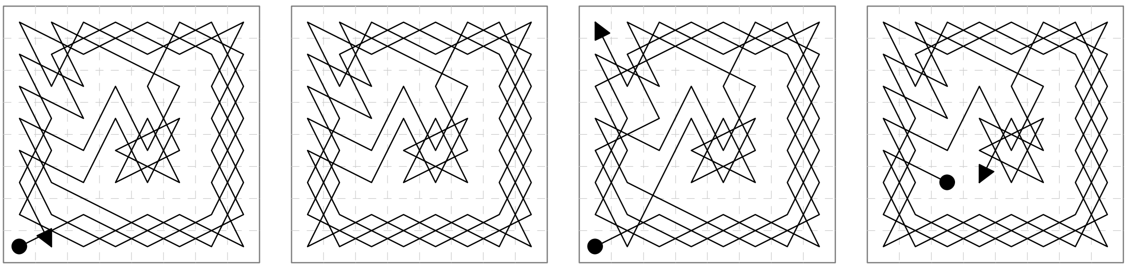
The title of the 1759 paper is: ‘Solution d’une Question Curieuse qui ne Paroit Soumise a Aucune Analyse’ {Solution of a curious question which does not seem to have been subject to any analysis} published by the Berlin Academy in 1766 (see Bibliography for fuller publication details). Kraitchik (1927) says that the Academy of Sciences of Berlin proposed in 1759 a prize of 4000 francs for the best memoir on the problem of the knight, but that the prize was never awarded. At that time Euler was himself Director of Mathematics at the Berlin Academy (resigning in 1766 in favour of Lagrange) so he was presumably ineligible for the prize.

He begins: §1: “I found myself one day in company where, on the occasion of a game of chess, someone proposed this question: to traverse with a knight all the cells of the chessboard, without ever arriving twice at the same, and commencing from a given cell.” He describes how the course of the knight was followed by placing counters on the cells and removing them one at a time as the cells were visited by the knight.

§2-4: Gives an example open tour shown, in the same way as in Ozanam, by numbering the successive cells occupied by the knight on an unchequered diagram, and notes that the numbering of any tour can be reversed. We show the tour below in graphical form.

§5: Acknowledges that he has been guided in his researches by an idea furnished by Monsieur Bertrand of Geneva (though no mention of this has been traced in Bertrand's later writings).

§6-8: Gives a diagram of a closed tour derived from the open tour (by deleting d2-b3 and joining the loose ends b1-d2, b3-a1). Euler indicates that this solves the stated problem of making a tour from any given cell; in fact that it provides two ways of doing it, since the circuit can be described in either direction (second diagram).



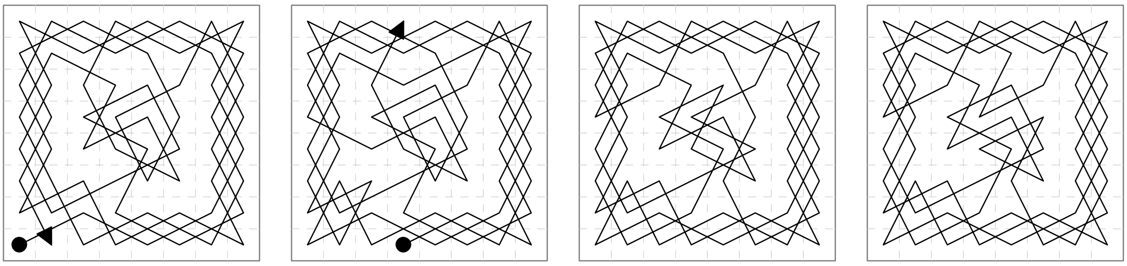
§9: Indicates that he will explain a sure method by means of which one can “discover as many satisfactory routes as one might wish: for though the number of these routes may not be infinite, it will always be so large that one would never exhaust them”. He distinguishes ‘*simple*’ (i.e. open) tours and ‘*rentrante*’ (reentrant) tours, where the last cell 64 is a knight move from the initial cell 1.

§10-14: Derives a series of 17 further tours from the initial two by applying his method. These are all very similar, five are closed; we show a corner-to corner example and one other which is the only open tour he gives that does not have one end on an edge cell. He did not attempt the task of making a tour between all possible pairs of cells of opposite colour, though his method could be used for that purpose.

The general rule Euler describes is simply to make repeated application of what Murray (1942) calls the ‘Bertrand Transformation’. This means locating a sequence of cells connected by knight moves, and numbered, say a,...,e,f,...,y, in which an end cell, y, is a knight move from an internal cell, say e; it can then be renumbered in the sequence a,...,e,y,...,f. If an unused cell z (or the end of another sequence of moves z,...) is a knight move from f then z,... can be incorporated into a,...,y by a.,..,e,y,...,f,z,.... Geometrically the move ef is deleted and the two moves ey and fz inserted.

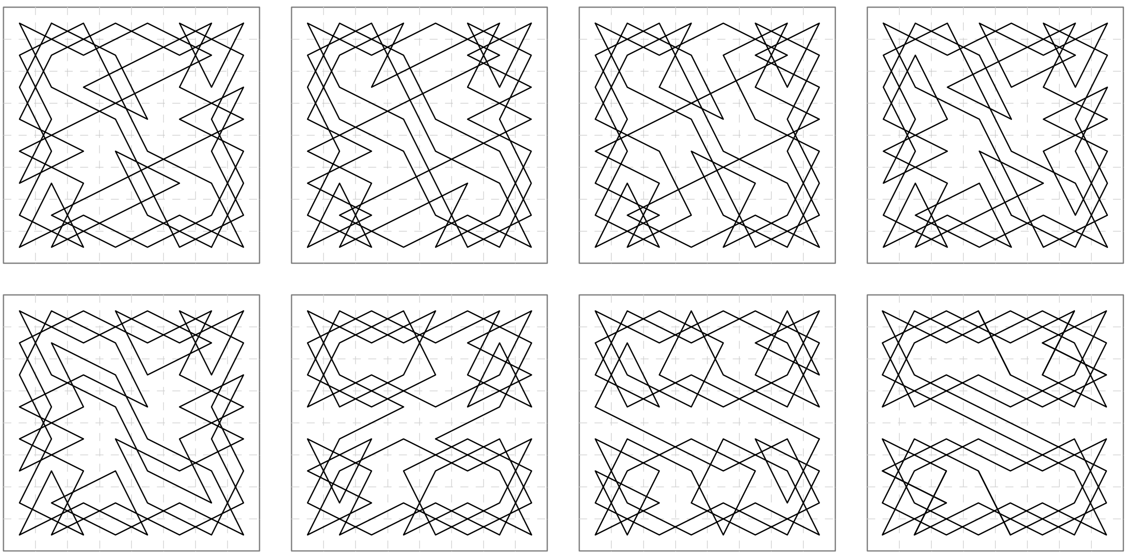
Using the numerical notation the first open tour, numbered 1,...,64, has cell 51 (d2) a knight's move from cell 64 (b1) and 52 (b3) is a knight's move from cell 1 (a1). This enables it to be converted to a reentrant tour 1,...,51,64,...,52 as shown in the second diagram. Using the same numbering, the third diagram has the formula 1,...,11,32,...,41,12,...,31,64,...,42.

§15-23:Starting from another randomly constructed partial tour Euler derives a further 35 irregular tours, four of which are closed tours, all again very similar to the initial try. We show four.



§24-30: Having explained a way of finding many tours he goes on to consider tours of special types using this method. First he constructs symmetric tours, invariant to 180° rotation, which he forms by constructing two diametrally opposite paths simultaneously, connecting the ends to form a circuit, and seeking to join in any unused cells by the Bertrand method. Five tours result.

These, and the one he sent to Goldbach, are the first fully symmetric tours to be composed since the Nilakant-ha tour over a century earlier (c.1640). Euler also points out that when a tour is presented in numerical form symmetry on the 8×8 board is characterised by the property that the numbers in diametrally opposite cells have the constant difference of 32.



§31-34: Next he turns to symmetric tours composed of two equal 4×8 half-board tours, constructed by the same methods, and presents five examples. We show three here. For diagrams of the other two see the section on simple linking of the halfboard crosspatch pattern, p.55.

§35-44: In the concluding sections of his paper Euler turns to tours on other square boards 5×5, 6×6, 10×10, and on small rectangular boards, 3×4, 3×7, 4×5, 4×6, 4×7, 5×6, which apart from the 4×8 board that was popular in mediaeval times, seem not to have been considered before. He concludes with four tours on cross-shaped boards, two of which show quaternary symmetry with diagonal axes, a type of symmetry not possible in tours of rectangular boards. See the volumes on Shaped Boards (⌘ 3), Oblong Boards (⌘ 4), and Odd and Oddly Even Boards (⌘ 5).

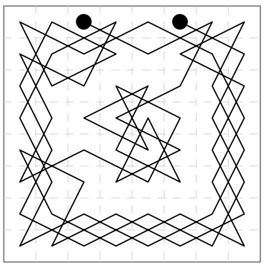
In §43 Euler made an error (repeated by many subsequent writers over the next century and a half) in stating that closed tours are not possible on boards of less than 5 ranks. This was not corrected until 1918 when Ernest Bergholt published closed tours on the 3×10 and 3×12 boards.

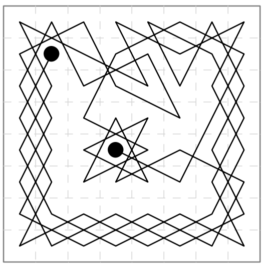
Another common mistake in the literature, almost impossible to eradicate, is the claim that Euler composed the first magic knight tour. This was achieved by William Beverley in 1848.

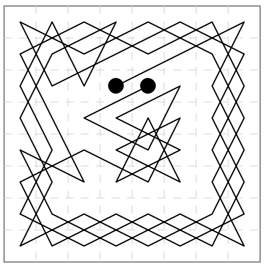
Euler’s method and tours have been the basis of numerous subsequent articles in magazines and encyclopedia entries, particularly the first closed tour in the 1759 paper. Unfortunately these articles rarely add anything new. The first review of it, quoting his first two tours, appeared in the *Journal Encyclopédique* in 1767. Further work on small boards was carried out by: Willis (1821), Warnsdorf (1823), Bergholt (1918), Papa (1920), Kraitchik (1927) and Murray (1942).

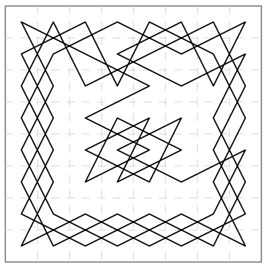
1766: Lelio dalla Volpe. As noted above the knight’s tours in Ozanam and Euler were presented in numerical form, not in the graphic geometrical form we use for most tours in this work. The first work to show knight’s tours in the form of line diagrams as well as in numerical form is a beautifully printed Italian book *Corsa del Cavallo per tutti scacchi della Scacchiere* Bologna 1766, published by Lelio dalla Volpe (1685-1749) and his son Petronio dalla Volpe (1721-1794). The border is shown as a doubled line, with finer lines dividing the board into squares, but the end-points of the tours are not specially marked. The book has 38 diagrams but only 19 distinct tours.

There are ten open tours numbered I to X, followed by the three tours from Ozanam, and then six closed tours numbered 1 to 6, the first of which is also used on the title page. These are all preceded by numerical forms of the same tours. The frontispiece is accompanied by a Latin caption: “O curas hominum! quantum est in rebus inane. Quandoque tamen et Sapientibus placent.” {Ah, Human cares! How much they are in empty matters. Though sometimes they please the wise.} There is also a Latin caption above and below the diagram: “Et semel a quovis, Cuncta attingit Equus.” {And one at a time moving whithersoever, the whole is touched by the Horse}. But my translations may be improvable!


open tour III


open tour V


open tour IX


closed tour 2

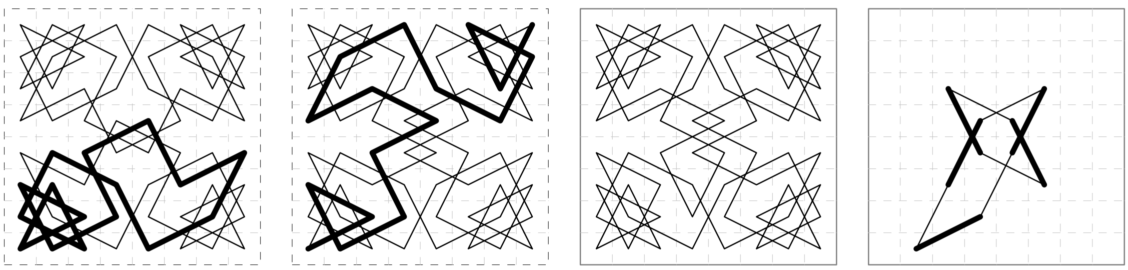
The ten open tours commence successively at the ten typical cells abcd8, bcd7, cd6, e5, thus solving the problem mentioned by Euler of showing an open tour commencing at any given cell. The start and finish points are not distinguished geometrically. The only source cited is Ozanam, and the tour I is Moivre's (in variant form), reflected to start at a8 and end at f6. We show four tours: tour III which incorporates a 14-cell path in the central 4×4, tour V which ends in the middle of a 7-move star, tour IX in which the initial and final moves are unintersected, and closed tour number 2 which includes a 3×4 compartment. The other closed tours are reminiscent of Euler's first closed tour.

1769: Farkas Kempelen (1734-1804) of Hungary, also known as Wolfgang von Kempelen, or de Kempelen, achieved fame as inventor of the first chess-playing ‘automaton’, called the ‘Turk’, which was built in 1769, and was exhibited, with much publicity and popular interest, throughout Europe and America until 1854. This was a period when there was much interest in the development of ingenious mechanisms, and their employment in lifelike automata. While the seated figure of the Turk had the mechanical ability to move chess pieces on the board in front of it, it was not of course capable of playing chess like a modern computer, but was a ‘cabinet illusion’, the box supporting the figure being shown apparently empty, or occupied by machinery, but in fact concealing an operator.

Besides playing chess with members of the public it was also able to demonstrate a knight's tour, from any square chosen by a spectator. For more on its history *The Oxford Companion to Chess* 1984 by D. Hooper and K. Whyld is a good source. The influence of the Automaton in promoting popular interest in the knight's tour was considerable; as much as reviews of Euler's 1759 paper over the same period in a wide range of publications. For further on the subject see sections below on Windisch (1783), Maelzel (1804), Egan (1820) and Willis (1821).

1771: A-T. Vandermonde. The second most cited work on tours after Euler, though he only gave the one 8×8 tour, was the article ‘Remarques sur les Problèmes de Situation’ presented by Alexandre-Théophile Vandermonde (1735-1796) in 1771 and published by the Paris Academy of Sciences in 1774. He begins by introducing a special coordinate notation whereby the cell in the y-th rank from the bottom and the x-th file from the left is denoted in a fractional style with y above x but no fraction line between them. Thus, using the more conveniently typed form y/x, the path shown in the first diagram below is presented as 5/5 4/3 2/4 1/2 3/1 2/3 1/1 3/2 1/3 2/1 4/2 3/4 1/5 2/7 4/8 3/6. From 3/6 the knight can continue to 5/5 making a closed circuit. Apart from its use by commentators on Vandermonde's work, this notation has never caught on in preference to the Cartesian (x,y).

By reflecting this circuit up-down and left-right (which is shown arithmetically by inverting the fractions or replacing the upper or lower terms by their complements 9–x or 9–y) he covers the board with four of these circuits which together form a pattern with biaxial symmetry. We use a dashed border to indicate that a diagram is a pseudotour.

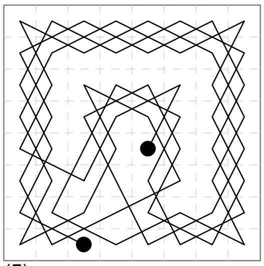


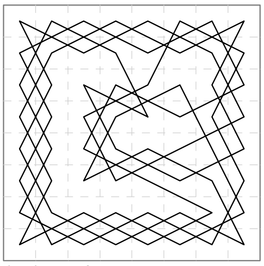
Vandermonde then links opposite pairs of these circuits together, by deleting a pair of parallel moves and joining the loose ends, to give two congruent circuits each with diametral symmetry, which together still form a pattern having biaxial symmetry. Finally he links together these two circuits by the same method to form a true tour, which however does not preserve the symmetry. The overall effect of the transformation from the four circuits to the tour is that there are five moves deleted and five new ones inserted. He presented the tour in the form of a geometrical diagram in which the squares of the board are omitted and the points where the knight moves meet are represented by small black and white circles.

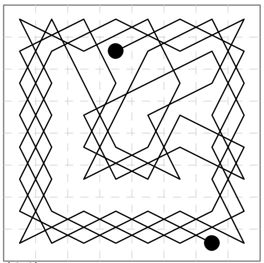
Wenzelides (1849) and Jaenisch (1862) showed that it is in fact possible to link Vandermonde's four circuits to form a symmetric tour with only four deletions and insertions. In fact I found this can be done in four ways, and there are also three ways leading to asymmetric closed tours. For more on this see Quaternary Pseudotours in ⌘ 8.

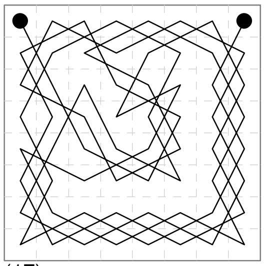
Vandermonde was also the first to construct a three-dimensional knight's tour. His tour is in a 4×4×4 cube (not 8×8×8 as some commentators appear to have assumed). See Space Chess in ⌘ 11 on Alternative Worlds.

1772: C. A. Collini. A method of constructing many tours from the pattern of eight concentric circuits on the 8×8 board was demonstrated by Cosimo Alessandro Collini (1727-1806) who was at various times Private Secretary to Voltaire and to the Elector Palatine. His first account, with one example tour, is spread over several issues of the *Journal Encyclopédique* in September and October 1772. He published a fuller account the next year as *Solution du Problème du Cavalier au Jeu des Echecs* (Mannheim 1773). A book of 60 pages, including 28 tables, 20 of which are tours. In the book his Part I prescribes the initial square, Part II the final square, Part III a closed tour, and Part IV different start and finish squares. Table (5) in the book is the tour shown in the Journal article. The tours are given in tabular numerical form. Table (11) is the first of four reentrant tours. These all use the minimum number of eight deletions and insertions. To solve the problem when the start and finish squares are on the same circuit and not a knight move apart takes more than eight deletions, as in example (14). The corner-to-corner tour (17) uses more deletions than necessary.


(5)

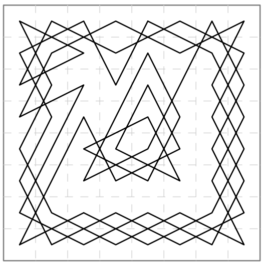

(11) e5-d7

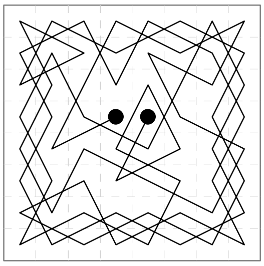

(14)


(17)

An Italian version appeared in 1774 in the *Magazzino Toscano*.
For further work on Collinian tours see p.58.

1773: Chevalier W—. A closed asymmetric tour much quoted in subsequent literature was sent in a letter from Prague on 20 Apr 1773 by Le Chevalier W— and published in the *Journal Encyclopédique* that year. The tour (numbered c2 to a1) also appears in the 1790 and later editions of Ozanam’s *Recreations*. He is described as ‘Capitaine au régiment de Kinski’ and his regiment as ‘dragons, au service de l’Impératrice-Reine’.


Chevalier W

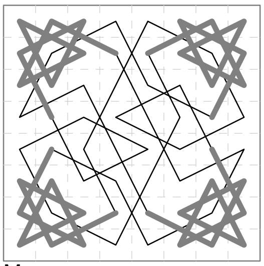

Malabar

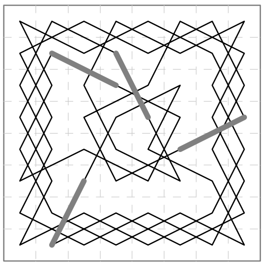
1776: Monneron. The open tour shown above is ascribed to a composer from Malabar in south-west India and was sent from the East Indies to the *Nouveau Dictionaire* edited by Pancoucke 1776. The article *Echecs*, in the volume covering the letters BO-EZ is signed Monneron. This is possibly Jean Louis Monneron (1742-1805) one of several brothers in a prominent French family. The article also shows the Nilakantha tour of 1640 (its first appearance outside India or Sri-Lanka).

Both Monneron tours are quoted in Laisement (1782) and in Hoffmann (1893) who gives this tour the heading ‘Du Malabar’ as if that was the name of its author.

This *Dictionaire* article may be the source of the statements by Lucas (1882) and Kraitchik (1927), that knight's tours were known in ancient times in India.

1780: G. Monge. The well known French mathematician Gaspard Monge (1746-1818) left some unpublished manuscripts with symmetric tour diagrams. This was recently reported by Herbert Bastian (*Schach* Oct 2017). These newly found tours anticipate the work of Wenzelides (1849) in showing a variety of centre formations with differing angles, though they all have the same 8-move 'meteor' corners as in Vandermonde (shown bold here). The linkages between the corners are formed of paths of 7 and 9 moves. Herbert Bastian hints that there may be other work on tours by Monge found by the historian Rene Taton (1915-2004). See under Symmetry in ⌘ 7 for more diagrams.


Monge


Chapais

1780: Chapais. Ten tours of Collini type (one being the same as Collini #15 but rotated) appear in a section at the end of a manuscript work on chess endgame theory, *Essais analytiques sur les Echecs* ms Paris 1780, by Chapais (first name unknown). This work also includes example tours on larger boards of sides 9, 10, 11 and 12 [details from H. Bastian 2017]

1782: Denis Ballière de Laisement (1729-1800) who was known as a music analyst was evidently inspired by the work of Vandermonde whose work he cites along with Monneron, Euler and Ozanam in his *Essai sur les Problèmes de Situation*... (Rouen, 1782). He gives a closed version of the Mairan tour (see 1725 above). His tours are shown geometrically as small circles joined by lines similar to Vandermonde. On the 4×4 board he diagrams pseudotours of two open circuits formed of squares and diamonds though not using these terms. On the 6×6 the quatersymmetric tour with lines through the centre cells delineating a Greek Cross (Fig.7 Plate II) is shown for the first time. His other small board examples are from Euler.

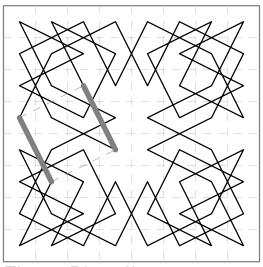


Fig.10 Plate II

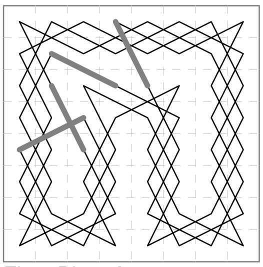


Fig.1 Plate A

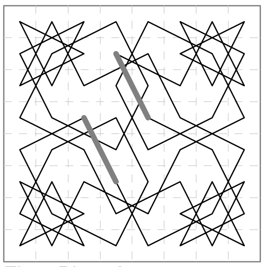


Fig.5 Plate A

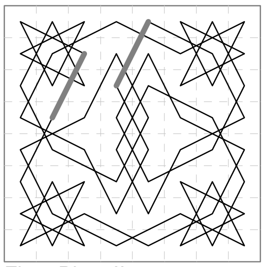


Fig.9 Plate II

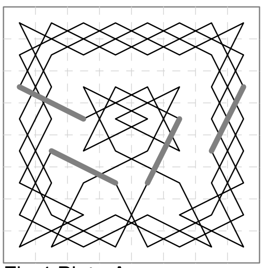


Fig.4 Plate A

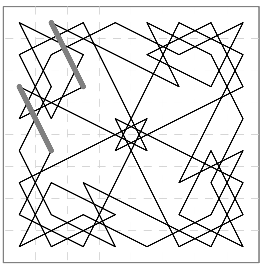


Fig.6 Plate A

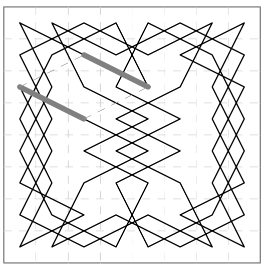


Fig.10 Plate A

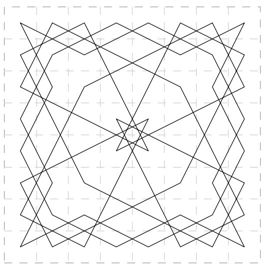


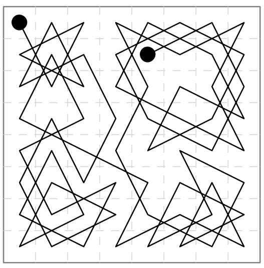
Fig.33 PlateVI

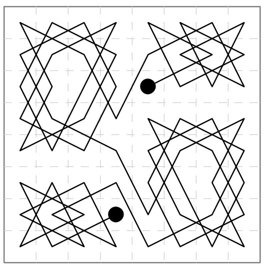
The 4×4 pseudotour consisting of a 4-cell and 12 cell path is arranged in direct symmetry in the quarters (Fig.10 Plate II) to form a pseudotour of squares and diamonds type. However the only way to join the two circuits, in the manner of Vandermonde, as described in his text, disrupts two of the diamonds. An 8×8 tour formed from the C-shaped crosspatch (Fig.1 Plate A) uses only 4 deletions and 4 insertions. Two tours (Fig.5 Plate A and Fig.9 Plate II have 7-move stars in the four corners (a task often attributed to Slyvons 1865) though in the latter I have chosen to make the linkage in a different place than Laisement indicates to avoid disrupting one of the stars. These tours and several others (Figs.4, 6, 10 Plate A) exhibit approximate axial or biaxial symmetry, with only a pair of moves (as in Vandermonde) needing to be replaced to give a symmetric pseudotour. A pseudotour with octonary symmetry (Fig.33 Plate VI) formed of two circuits of 12 and two of 20 moves cannot be linked to form a tour by the Vandermonde method. Laisement also gives tours on larger boards. A 10×10 tour with 90° rotational symmetry, the first of its kind after Euler, and with an attractive design (see ⌘ 5). Also tours 12×12 and 16×16 (see ⌘ 8).

This pioneering work has been unjustly neglected in my view.

1783: K. G. von Windisch. A regular visitor to exhibitions of Kempelen's automaton, in particular when it was shown at No.9 Savile-Row, Burlington Gardens, London, was Karl Gottlieb von Windisch (1725-1793). In a letter dated 18 Sep 1783 (in a book he published that year, and in an English translation in 1819) he gives an eye-witness account of how the knight's tour was exhibited: “The leap of the knight, which this machine makes traverse all over the board, is too remarkable not to be mentioned. It is this; as soon as all the chessmen are removed, one of the spectators places a knight on any one of the squares he thinks proper; the Automaton immediately takes it, and commencing from that square, and strictly observing the move of the knight, he makes it traverse the sixty-four squares of the chessboard, without missing one, and without touching any of them a second time; this is proved by the counter, which the spectator himself places on each square which the knight has touched, observing to put a white counter on the one from which he first begins and red counters on all those which he afterwards touches in succession. Try to do as much yourself with your chess-board, perhaps you will succeed better than I have done; all my attempts for that purpose have been unsuccessful.” [text from the late Ken Whyld]

1787: Richard Twiss. In his two-volume work on *Chess* (1787 and 1789) Richard Twiss (1747–1821) has three tours in Vol.1 and two in Vol.2. The first is Moivre's, the third is Euler's first closed tour. He writes: “The second on the annexed plate is without any regularity and was found only by repeated trials on a slate”. This tour, numbered a8 to e7, is partially compartmental. He expresses the view: “I believe it is not capable of a general solution.” At the end he says that in Guyot (1769) “is another solution; and others printed on cards are sold at the *Caffé de la Regence* in Paris.” One wonders if any of these survive in collections somewhere? The 8×8 tour in *Chess* Vol.2 is the King's Library tour (c.1275). He also gives one on a circular board. See our Bent Boards section p.720 for diagram. [Text details from K. Whyld]

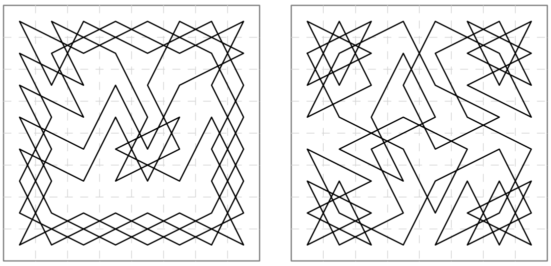

Twiss


Sachsen-Gotha

1797: Ernst von Sachsen-Gotha Three tours are given in an article in the early German newspaper *Reichs-Anzeiger* (18 Sep 1797). One (shown above) is a somewhat more regular compartmental tour that that by Twiss. The article, printed in German black letter type, is headed ‘Gelehrte Sachen’ {Scholarly Matters} and mentions the work of mathematicians including Euler and Vandermonde. The author is not apparent, but is named in Ahrens (1900) and other bibliographies to be (Herzog) Ernst II von Sachsen-Gotha (1745-1804). Herzog is a title akin to Duke. See p.55 for his symmetric double half-board tour. His work on Figured tours is in ⌘ 11.

1804: Maelzel Following the death of Kempelen in 1804 the showman and inventor Johan Nepomuk Maelzel (1772-1838) known for a version of the metronome, acquired the chessplaying automaton. It seems [from Wikipedia] that he then sold it but reacquired it in 1817. He exhibited the illusion in England and subsequently took the Turk to North America. **Dr S. Weir Mitchell** was one of a group who bought the automaton after the death of Maelzel in 1838 and exhibited it until it was destroyed by fire in 1854. A template used in the machine, now owned by the Library Company of Philadelphia, shows the first closed tour by Euler (1759).

1806: Joseph Dollinger in a work of 110 chess end-game compositions published in Vienna in 1806 (see the Bibliography for the long title) gives 24 knight tours at the end. The tours are shown by coordinates; a1-b3-a5-c4-d6-e4, etc. They are all very similar. No.5 differs in only two moves from the first closed tour given by Euler (1759). No.7 (shown below) is the only one without a two-move line. Six contain the double zigzag N in the centre, as here. None are symmetric. Six have a complete border braid from a1 to h1 to h8. Although the tours are all reentrant the initial squares are taken to be all different, following the sequence: abcdefgh1, abcdefgh8, cf3, cf6, de4, de5.

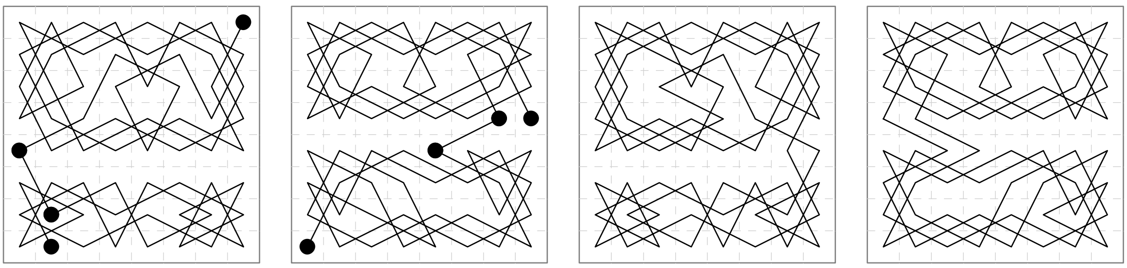


Dollinger Egan

1820: Pierce Egan (1772-1849) in his *Sporting anecdotes ... delineation of the sporting world* (London 1820, and Philadelphia 1822) includes a closed knight tour (shown above) with 7-move stars in the corners (p.188, US edition p.137) numbered a7 to b5. This is different from the examples shown by Laisement (1782). It has the comment that "Dr Hutton in his Mathematical Recreations [i.e. the English edition of Ozanam] gives three different methods to perform the same, but none of them like the above." This source also includes an account p.245-250 (or p.177-181) of Kempelen's automaton, similar to that given by Windisch. The automaton is recorded as appearing at Spring Gardens, London, 1819.

1821: Robert Willis (1800–1875) who became Professor of Applied Mechanics at Cambridge in 1837, made a mark much earlier when Kempelen's automaton was exhibited by Maelzel in London in 1821 with *An Attempt to Analyze the Automaton Chess Player of Mr DeKempelen. To which is added a copious collection of The Knight's Moves over the chessboard.* The diagrams numbered 1-18 are open tours of all boards smaller than 8×8, and closed tours 6×8 and 7×8. His 3×4, 3×7 and 5×5 examples are the same as given by Euler. (See ⌘ 4 for some diagrams of these.)

The tours 20-39 are 8×8, mainly from Euler and Ozanam. He gives six 8×8 irregular closed tours. One is a closed version of the Moivre tour. His closed version of the Mairan tour is the same as Laisement (1782). Here are his two open and two closed 8×8 compartmental tours.



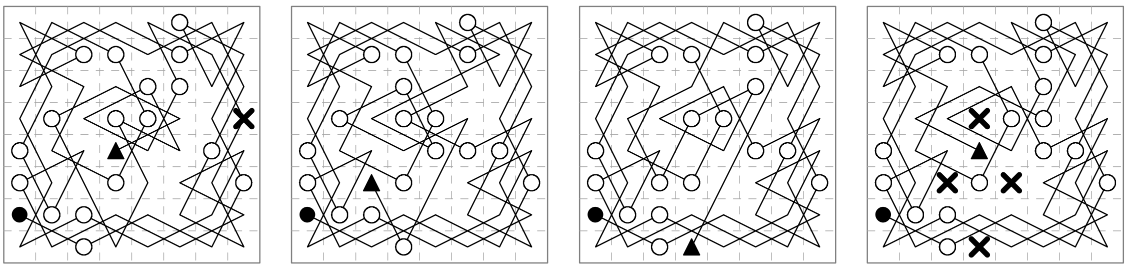
No authors name is stated, but Tomlinson (1845) identifies Willis as the author of the section on the automaton. The text suggests that the tours section was written by the same author: “Observing that the Automaton, under the direction of Mr Maelzel, occasionally traversed half the board, I was induced to pursue the subject, and found that the move might be performed on any parallelogram consisting of twelve squares and upwards with the exception of fifteen and eighteen squares”. (We would now say rectangle rather than parallelogram.)

Two reviews of the work of Willis are given in the *Edinburgh Philosophical Journal* 1821 and 1823. This reports “The Automaton Chess Player of M. de Kempelen was introduced into England by its inventor in 1783 and has during the last two years been exhibited in various parts of England and Scotland, under the direction of M. Maelzel”. It copies the 3×8 and 7×8 tours and the 20 tours 8×8.

1823: **Heinrich Christian von Warnsdorf** (1780-1858) provided several new stimuli to work on tours in a book that I take to mark the conclusion of this phase in the History, with the title *Des Rösselsprunges einfachste und allgemeinste Lösung* {Knight's tours simple and general solution} (Schmalkalden 1823). However, much of the first part of the book up to Figure 31 merely goes over earlier work from Ozanam and Euler.

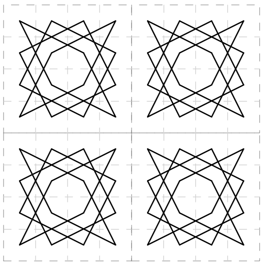
The first innovation of note is his famous Rule “Play the knight to a square where it commands the fewest cells not yet used”. This has become well known, and often cited in the mathematics literature (where his name often ends in ff unaccountably, which suggests very few have actually consulted the original source). The diagrams relating to his Rule are numbers 32 to 39 in Table 5. Somewhat surprisingly he begins by applying the rule to the 6×6 board with four examples (see ⌘ 5) then goes on to the 6×7, 6×8, 7×8 (see ⌘ 4) and 8×8 boards, giving one example for each. This section of the book concludes at #40 with a 10×10 tour (see ⌘ 5).

He only gives one example of the rule on the 8×8 board, shown here in geometrical form, but in this there is a deviation from the rule at h5. The correct move should be to f4 which has access to only two exits whereas g7 and f6 have three. This error only goes to show the robustness of the rule, since it still goes on to complete a tour. The white dots mark cells where the next move is not completely determined. In the second diagram I indicate how the correct continuation could have led Warnsdorf to a reentrant solution. It also leads to 17 other complete tours. One ending with a three-unit line is shown. It also leads to two cases where a dead end is reached, leaving four cells unvisited (the other case ends f5-d4-b5-d6) showing that the rule does not always produce a tour.

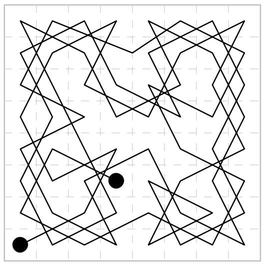


W. W. Rouse Ball's *Mathematical Recreations and Essays* (11th edition 1939 or earlier) states (p.181): “Warnsdorff [sic] added that when, by the rule, two or more cells are open to the knight, it may be moved to either or any of them indifferently. This is not so, and with great ingenuity two or three cases of failure have been constructed, but it would require exceptionally bad luck to happen accidentally on such a route.” However he gives no reference to where work showing this was done, and diagrams no example of it. Hence my own work on the topic (see under Synthetic Tours in Volume 7). When applied strictly the rule falls far short of producing a completely determined tour. If the knight can first be placed anywhere on the board then the only cells within the a1-d1-d4 octant from which the first move is fixed by the rule are b1, c1, d1, c2, d2. If the first move is given then the best it gives, before it reaches a position where it is undecided on the next move, is 18 moves, starting b3–c1. Nevertheless, much has been written about this rule, and recently it has been the subject of computer studies. Under the loosely applied rule when there is a choice of moves any of them can be taken. With this freedom the rule more often than not leads to a complete tour.

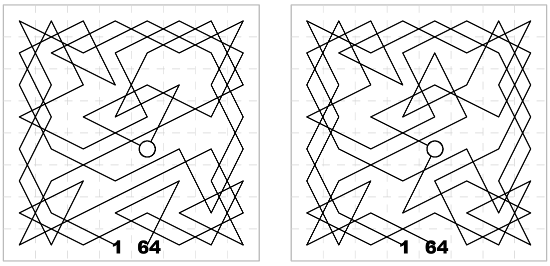
Warnsdorf's final notable contribution to the subject, and the first thing I noticed on seeing the Figures is that Fig.96 is a tour of squares and diamonds. This is at present the earliest known tour of this type. He may however have obtained the tour from another unknown source. It seems to have been added as an afterthought. Murray (1930) stated: “The first composer to give the figure which shows that the cells of the quadrant could be filled by four closed quartes was v. Warnsdorf (1823)”. This is correct, apart from the anticipations by Laisement (p.11) and Addison (p.18). The figure is not in graphic form but shows the squares and diamonds lettered A, B, C, D using a different type style in each quarter (more elaborate than I have use here). Strangely Murray makes no mention of the accompanying tour! Here it is in graphic form, together with a simulacrum of his chart.



A	B	C	D	a	b	c	d
C	D	A	B	c	d	a	b
B	A	D	C	b	a	d	c
D	C	B	A	d	c	b	a
α	β	χ	δ	*A*	*B*	*C*	*D*
χ	δ	α	β	*C*	*D*	*A*	*B*
β	α	δ	χ	*B*	*A*	*D*	*C*
δ	χ	β	α	*D*	*C*	*B*	*A*

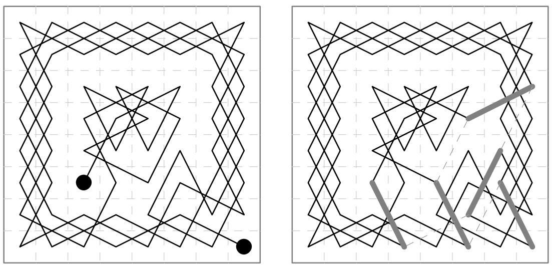


Over 30 years after his 1823 book Warnsdorf reappeared with a short article in *Schachzeitung* ‘Zur Theorie des Rösselsprungs’ (vol.13 Dec 1858 p.489-492). This is in response to Slyvons (1856). It includes two 8×8 tours with 1 and 64 at d1 and e1**.** Subject to the condition that the cell numbered 64 not be entered until the end, these tours are completely determined by the Warnsdorf Rule, apart from the choice at e4.



The same article includes a 6×7 tour and a symmetric 5×6 tour (see ⌘ 4).

1824: — von Müllner: in *Literatur-Blatt* (1824) and in *Algemeiner Lit. Zeitung* (1825) .The *ALZ* article is supposedly a review under the heading ‘Mathematik’ of Warnsdorf (1823) but the original contributions by Warnsdorf are in fact passed over. The Rule is only mentioned in the concluding section and is described as being plausible ‘und weiter nichts’ {and nothing more} and no diagrams of tours by the method are given. Müllner outlines the work of mathematicians who have contributed to the subject In the second part a tour very similar to Moivre's is given in numerical form and is transformed into a closed tour by Euler's method, as in the diagrams shown here. I show the linkage polygon h1-g3-f1-e3-f5-h6-g4-f2-d1-c3 formed by successive inserted (dark line) and deleted (broken line) moves.



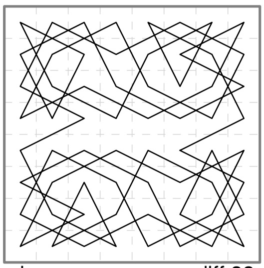
Squares and Diamonds 1823-1847

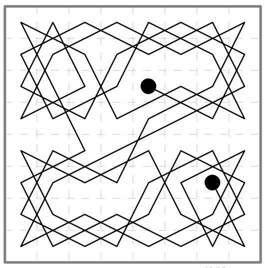
The description ‘squares and diamonds’ is applied to the pattern of four 4-move circuits that can be drawn to fill a 4×4 area, and used to cover the four quarters of the 8×8 board, or the h·k such areas on a board of sides 4·h × 4·k. It also refers to the method of forming tours from these circuits by deleting one side of each rhomb and seeking to connect up their loose ends. It should be noted that I apply the term ‘squares and diamonds method’ strictly to tours formed from the circuits by minimal deletions; the term is not applicable if more than one move in any circuit is deleted.

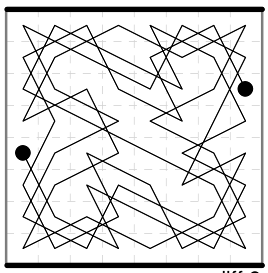
In retrospect al-Adli's tour of c.840 can be seen to use 8/16 rhombs, while as-Suli's have 6/16 and 9/16. Several of the mediaeval 4×8 tours can be seen to use 4/4 squares and diamonds at one end. The Amuli and Montmort tours have 10/16. One of Euler's symmetric double-half-board tours has 14/16. So the germ of the idea was about for a long time before it took hold. The method has a special place in the history of the search for a simple systematic method of construction.

Denis Balliere de Laisement (1782) came very close to the method, since he gave diagrams of open two-path pseudotours on the 4×4 board using squares and diamonds, and used one of these in each quarter to construct a closed pseudotour of two 32-move circuits of squares and diamonds, but connecting the two circuits by Vandermonde's method destroys two of the rhombs. Haldeman (1864) conjectures that the first squares and diamonds tour might have been exhibited by Kempelen's automaton, with a date as early as 1783, and the Laisement tour makes this feasible, but the only tour definitely associated with the automaton, the first closed tour in Euler (1759) is not of S&D type.

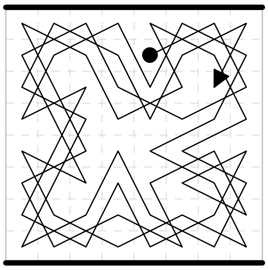
1825: F. P. H— Appendix to *Studies of Chess* (1825). This appendix quotes an Euler tour and gives three originals, with diametrally opposite numbers differing by 32, 16 and 8 respectively. The first is a double half-board tour, but not one of Euler's examples, and not quite of S&D (14/16). The last two tours are stipulated to satisfy the condition: “If the board were divided into quarters … the difference between any two squares through whose centres a line drawn would cut that quarter in half, to be 2”. This numerical condition is equivalent to the geometrical one that the quarters be formed on the squares and diamonds plan. The third solution also adds to 260 in all the files, though the text does not mention this fact. It is the first known example with this semi-magic property, which we show by the black lines along top and bottom of the diagram. The rank sums vary widely.

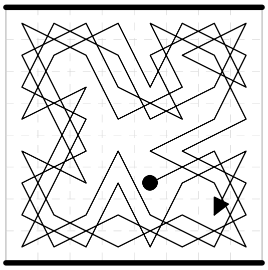

bc= diff 32

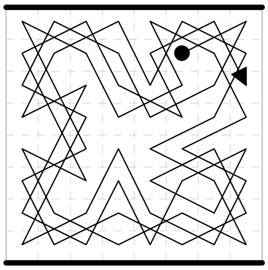

diff 16

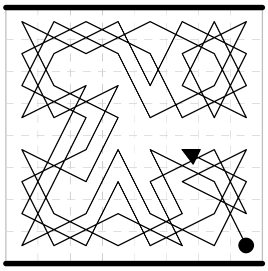

diff 8

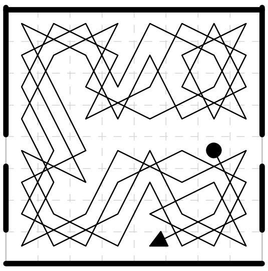
1826: Clemens Rudolph Ritter von Schinnern: *Ein Dutzend mathematischer Betrachtungen* (1826). This curious little 36-page booklet [copied to me by Prof D. Singmaster] gives the first full exposition of the ‘squares and diamonds’ method. There are 8 tour diagrams (numbered 4 to 11) but only 4 geometrically distinct tours, the others being reflections or reversals, or in one case a cyclic renumbering. All four tours are semi-magic, that is with the ranks or files all adding to the magic constant 260. They are oriented in the diagrams below so that the files are magic. Three tours, (7) and (9) and (10), are reentrant. Tour (10) is a numerical variation of (9), numbered from f7 to h6. The other two, (8) and (11), are open tours. The final tour (11) is particularly notable, since it has all 8 files and 6 ranks magic. The first and fourth ranks add to 260±4. It is impossible to get any closer than this to constructing a magic tour, short of the real thing, which took another 22 years! Tours shown by Wenzelides (*Schachzeitung* 1850) as Fig.93 and Fig.109 are identical to tours (11) and (7) in von Schinnern (1825) but with the numbering cycled by 16 steps in the latter case, so Wenzelides seems to have found these independently.


(7) rotated +90 deg


(9)

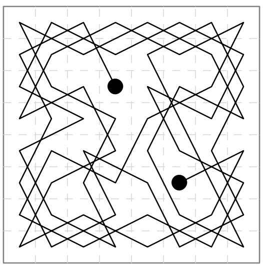

(10) inverted

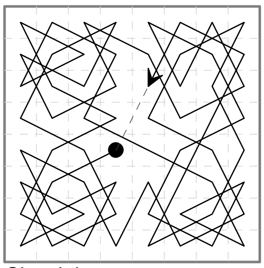

(8)

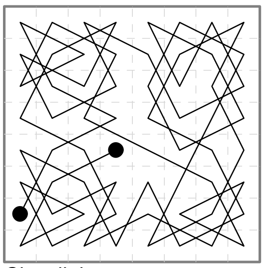

(11)

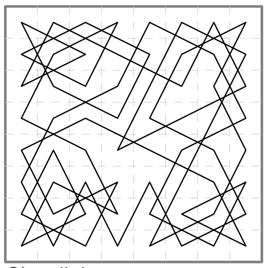
From this time on the Squares and Diamonds method was seen as a key to the possible construction of knight's tours that might also be magic squares. Here we look at work by various authors in the period leading upto the first solutions by Beverley and Wenzelides in 1848-9.

1827: Friedrich Wilhelm von Mauvillon (1774-1851) *Anweisung zur Erlernung des Schachspiels* ... (1827). The tours are in Chapter 6 §12 Rösselsprung p.239-240 and Tab X Figs 1-5. The first tour diagrammed is Euler's first closed tour. The second, third and fourth tours are not visible on the Google digitization. The fifth tour (shown below) is an open tour of squares and diamonds (also a three-slant Rogetian tour for explanation of this term).


Mauvillon


Ciccolni


Ciccolini


Ciccolini

1836: Teodoro Ciccolini: *Del Cavallo degli Scacchi* (1836). In this book Ciccolini begins with the diagram of the 16 four-move circuits and one closed tour of squares and diamonds type, as shown above, but goes on to derive a large number of other tours from it by Euler's method, which destroys the squares and diamonds structure.

Ciccolini's purpose seems to be to solve the Collini problem of finding a route between any two squares of opposite colour by providing a complete catalogue of such tours. For example the first tour derived from the master tour is presented, using the same method as Euler, as 1–25, 64–43, 26–42, indicating that one follows the given tour as far as cell 25 (f4) then instead of 26 go to 64 (e6) then back to 43 (c1) and from there to 26 (e2), ending at 42 (a2). Diagram above.

This catalogue seems to have been misinterpreted as a list of squares and diamonds tours by later influential writers such as Lucas (1895) and Ahrens (1901), possibly based on the short account given by Poirson-Prugnaux (1849), so that Ciccolini was taken to be originator of the method.

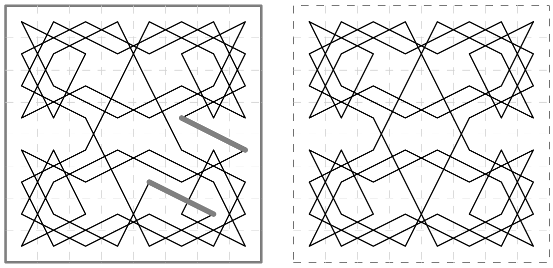
Tabula 21 shows the original 8×8 tour presented on a circular board formed by identifying the a and h sides of the board. Tabulae 19-20 show a modification of the original tour (diagram above) and this is used in Tabula 22 where a braid is added along two edges to form a 10×10 tour.

Tabulae 24–25 show a 10×10 symmetric tour formed by joining four 5×5 tours (like Euler's but not quatersymmetric). See ⌘ 5 for diagram.

1837: George Augustus Addison (1792-1814) ‘General Solution to the Knight's Trick at Chess’ *Indian Reminiscences* (1837). This article published posthumously provided one of the earliest in a new phase of interest in tours showing approximate biaxial symmetry. Addison's method of construction is described as “A very convenient practical solution of the general problem on the ordinary board” by ‘M. J.’ editor of *Cambridge Mathematical Journal* (in footnote to Moon 1843).

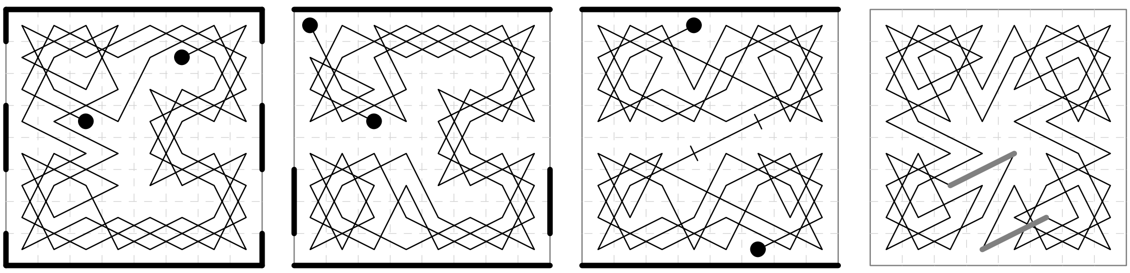
He was aware of the squares and diamonds pattern since he gives a chart of the form shown here (except that he prints the numbers below the letters rather than alongside them) where the rhombs are lettered A to P and the cells numbered along each circuit (thus A1-A2-A3-A4 is a diamond).

A1	M1	I1	E1	G1	K1	O1	B1
I2	E2	A2	M2	O2	B2	G2	K2
M4	A4	E4	I4	K4	G4	B4	O4
E3	I3	M3	A3	B3	O3	K3	G3
F3	J3	N3	D3	C3	P3	L3	H3
N4	D4	F4	J4	L4	H4	C4	P4
J2	F2	D2	N2	P2	C2	H2	L2
D1	N1	J1	F1	H1	L1	P1	C1



His method of constructing a tour from them however results in more than one deletion in some of the rhombs so his tour is not of ‘squares and diamonds’ type. If the links e3-g2 and f5-h4 are replaced by e3-f5 and g2-h4 the pattern becomes a biaxially symmetric pseudotour of two 32-move circuits (shown in the third diagram, a broken line border indicating it is a pseudotour). The tour was arrived at by starting from the squares and diamonds, linking them in pairs to form 8-move circuits, then linking these in pairs to form 16-move circuits, then linking these to form 32-move circuits.

1838: Christoph Wilhelm Zuckermandel (1767-1839) *Rösselsprung* (1838) gives an account of the squares and diamonds method, with 112 tours in numerical form. There is also a plate of line-drawn figures. The final pages 87-88 have three semi-magic tours. Two use the same braid in the right half as the Beverley magic tour of ten years later.



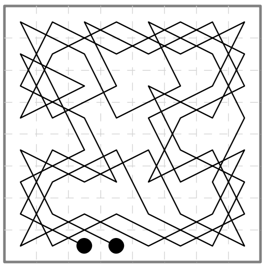
The tour visible on the plate, the fourth diagram here, is approximately birotary.

All the tours but two are asymmetric and of squares and diamonds with only occasional slight deviations. In the first batch of 27 tours, pages 62-72, tour #20 is Euler's cross-shaped tour, #21 is one of the 6×6 tours with quaternary symmetry, #22 is on a shaped board omitting three cells at each corner, #23 omits the four centre cells, all others are 8×8. Tours 24-27 are of double half-board type, but surprisingly none is entirely of squares and diamonds, though 26 and 27 are symmetric.

For more details see the Bibliography in ⌘ 12.

1839: J. E. Thomas de Lavernède ‘Problème de Situation’ (1839). The project of constructing tours of squares and diamonds type that solve the Collini problem of making a tour between any two cells of opposite colour was successfully carried out by this author, more systematically than in the work of Zuckermandel the previous year. He calls the elements ‘quarrés’ and ‘losanges’ and uses an elaborate notation for the cells. For example d6 is B'_3 indicating that it is on the secondary upward diagonal (the main diagonal being labelled A) first rank up (labelled '), of the upper left quadrant (numbered 3). The downward diagonals being labelled with lower case a and b. In his Figures 4 and 5 he gives the tour shown here in numbered board form. The same tour is given in all eight orientations in Table 6, and as tour 17 in Table 7. The notational form of the tour (first orientation in Table 6) appears as shown on the right here. B - b" denoting the two ends of the first rhomb c1 - d3, and so on.

3	a'''	B'''	b'''	A'''	a'''	B'''	b'''	A'''	2
	B''	a''	A''	b''	B''	a''	A''	b''	
	b'	A'	a'	B'	b'	A'	a'	B'	
	A	b	B	a	A	b	B	a	
	a'''	B'''	b'''	A'''	a'''	B'''	b'''	A'''	
	B''	a''	A''	b''	B''	a''	A''	b''	
	b'	A'	a'	B'	b'	A'	a'	B'	1
	A	b	B	a	A	b	B	a	



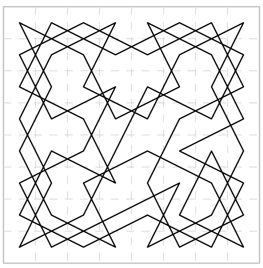
B - b" : B3 - b"3 : B'''2 - b'2 : B'''1 - b'1 :
A''' - a' : A1 - a"1 : A2 - a"2 : A'''3 - a'3 :
B"3 - b'''3 : B"2 - b2 : B"1 - b1 : B' - b''' :
A'3 - a3 : A'2 - a2 : A"1 - a'''1 : A" - a.

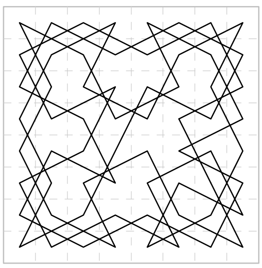
He then gives, in Table 7 covering six pages, a laborious list of 256 tours of squares and diamonds. This number arises since there are 64 choices of the initial square and 32 of the final square (of the other colour), but each tour can be shown in 8 orientations by rotation or reflection, thus giving 64×32/8 = 256 cases. The first part of the list, numbered 1-128, shows tours of the type 4D-4S-4D-4S or the reverse (where D = diamond, S = square).

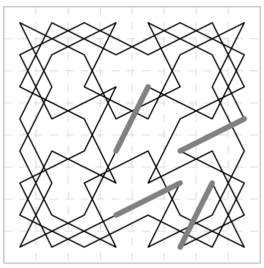
The other tours in the list cover cases where one or two of the sequences of four diamonds or four squares are broken into two or more parts, and 16 cases where the start and end points are in the same square or diamond. This work thus goes some way towards enumerating tours of squares and diamonds type, though it is difficult to extract tours with particular properties.

1840 Roget. Describes a method related to squares and diamonds. See the next section (p.21).

1842: Eugéne-Théodore Troupenas (1799-1850) in *Le Palamède* (1842) gives an account of the Euler and Vandermonde methods as usual, and expounds the squares and diamonds method in some detail (p.273–6) with original example tours. His work was stimulated by an example sent to him from England by a Mr Anderson (first diagram here, numbered a1-c2).





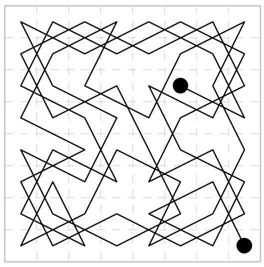
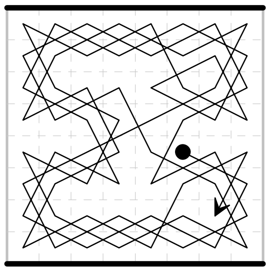


42	59	24	7	46	61	26	11
23	6	43	60	25	10	47	62
58	41	8	21	64	45	12	27
5	22	57	44	9	28	63	48
40	55	20	1	36	49	30	13
19	4	37	56	29	16	33	50
54	39	2	17	52	35	14	31
3	18	53	38	15	32	51	34

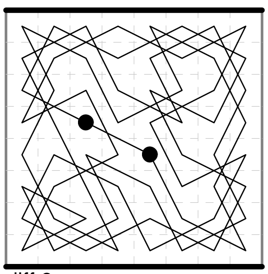
Troupenas then gives two similar tours of his own. It is pointed out by Jaenisch (1862) that one of the Troupenas tours, can be regarded as formed from a biaxially symmetric pattern of four circuits by deleting one move in each circuit and joining the ends, as shown by the darker lines. This is also the first example of what I call a ‘demi-magic’ tour, ranks and files adding to two different values (260±16). The lower ranks and left-hand files all add to 244 while the other ranks and files add to 276. This Troupénas tour is quoted in Rouse Ball (1939) misleadingly as ‘Roget’s Solution’. Both Troupenas tours found wider circulation in *Beauties of Chess* by Aaron Alexandre (1846).

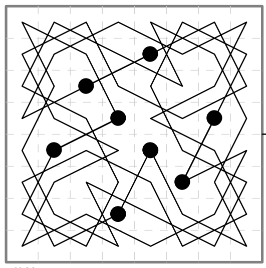
1842: L. Perenyi in *Mnemonik des schachspieles:* Tafel 30 is the squares and diamonds pattern, but no tours of this type are shown.

1845: Charles Tomlinson *Amusements in Chess* (1845) expounds Roget's method (see the next section) but conflates it with the squares and diamonds. He concludes with a semi-magic tour, which is both of squares and diamonds type and Rogetian. The rank sums are far from magic however.

1847: R. Franz: ‘Rösselsprung’ *Schachzeitung* (vol.2, p.341-343, 1847). Closed asymmetric tours of squares and diamonds showing differences of 8 or 16 (cf F. P. H— *Studies of Chess* 1825) in diametrally opposite cells. One quasimagic (files adding to 260, ranks to 260±4), the other demimagic (adding to 260±2 in rows and columns, cf Troupenas 1842). I show the cells numbered 1, 16, 17, 32, 33, 48, 49, 64 by dots, which make evident a degree of symmetry about the a1-h8 diagonal. [A][M]

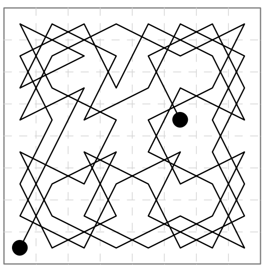

diff 8

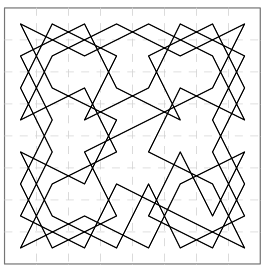

diff 16

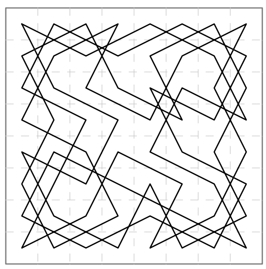
46	55	22	13	58	35	18	11	-
23	14	47	56	17	12	59	34	+
54	45	16	21	36	57	10	19	-
15	24	53	48	9	20	33	60	+
44	49	4	25	64	37	8	31	+
3	26	41	52	5	32	61	38	-
50	43	28	1	40	63	30	7	+
27	2	51	42	29	6	39	62	-
+	-	+	-	-	+	-	+	

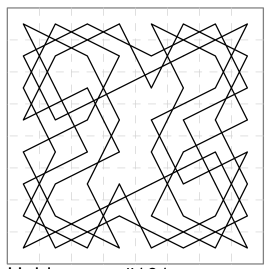
According to Murray “Franz said that his tours were the nearest approach he had been able to make to a tour with equal-summed rows and columns, and he expressed doubts whether it was possible to do better. It was not then known in Germany that Beverley had already constructed a tour with both rows and columns equal-summed”.

Here are three tours of squares and diamonds from Haldemann (1864). The open tour he cites is by J. H. Alexander. Haldemann's tour #101 is symmetric (see p.73 in our catalogue of such tours).


Alexander


Haldemann #98


Haldemann #100 (= #97)


Haldemann #101

For further on squares and diamonds see the notes on Roget that follow and the works by Wenzelides (1849), Scheidius (1850), Mann (1859), Haldeman (1864) and others.

We tell the history of the discovery of magic knight tours in monograph ⌘ 9. In all there are 34 of squares and diamonds. We list their catalogue names here: 12a, 12b, 12m by Wenzelides (1849), 00b by Mysore, 12o, 12n, 00a by Jaenisch, 27i, 34e by Exner, 05b by Caldwell, 00i by Unknown composer, 00c, 05f, 05c, 23a, 27l by Francony, 27p by Reuss, 05g, 34d, 14b, 23l, 23g, 23d, 23c, 23m, 23n, 34f, 03a by Ligondes, 14d by Feisthamel (1884). No more were discovered until 12p, 01d by Murray (1939), then 23q, 01h, 03b by Marlow (1988) who completed the enumeration.

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Rogetian Tours

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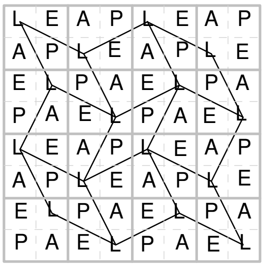
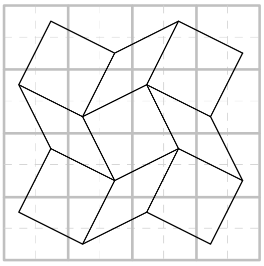
Roget's Nets 1840

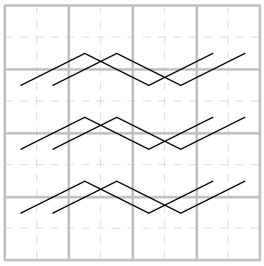
A significant advance on methods of constructing and analysing knight's tours on the 8×8 board was made in 1840 by **Peter Mark Roget** in the *Philosophical Magazine*. He believed that the specification of both the start and finish cells was a new condition, but it had been considered by Collini in 1773. He claims: “A great many years ago I contrived a method by which the problem in this new and extended form may be resolved with the greatest ease.”

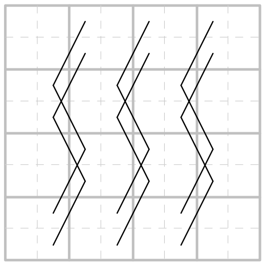
Roget divides the board into ‘four separate systems, of 16 squares each’ lettered L, E, A, P, as in the diagram (actually he used lower case for the vowels). Knight's moves are then either of the types LL, EE, AA, PP forming four 4×4 nets, each of 24 moves or of the types EL, AL, EP, AP, (or their reverse) these each consist of six strands of three moves. Moves LP or AE are not possible.



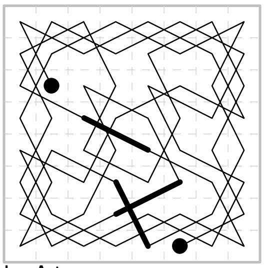
LL net

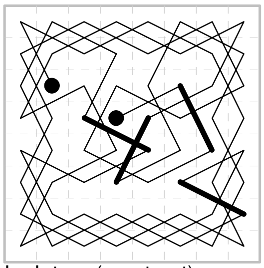

EE net

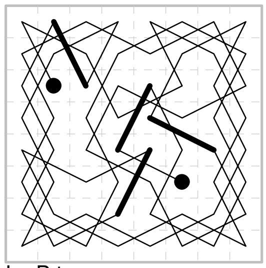

LE


LA

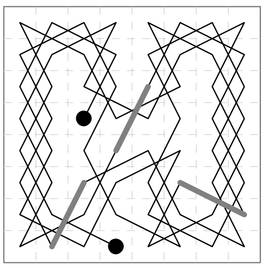
Roget's method is then to traverse each of the four nets, L, E, A, P, separately, as far as possible, so that the tour is in four parts, except when the start and finish points are in the same net or when the ends of the tour are L and P, or A and E, when it is necessary to traverse one of the nets in two parts. Roget gave three example tours, as shown. The vowel-consonant moves are emphasised here.

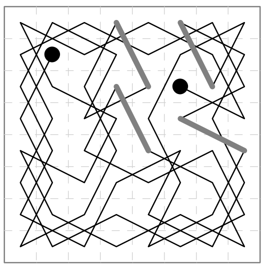

L...A tour

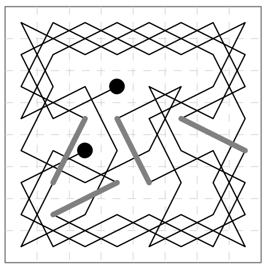

L...L tour (reentrant)


L...P tour

Eugéne Pelletier de Chambure. seems to have discovered the same method independently in a presentation given in 1861 to *l`Institut Egyptien, Cairo* published in Paris in 1862. Instead of L e a P notation he uses r B b R indicating red and blue colouring in light and dark shades. As in Roget he gives tours that solve the three cases where the end-points are on compatible nets, on incompatible nets, or on the same net.







A. C. Crétaine *Etudes sur le Problème de la Marche du Cavalier au Jeu des Echecs* ... Paris 1865. Also has an account of the method. Plate D includes an indexed diagram using the letters PUOL, OLPU, UPLO, LOUP (meaning wolf) instead of Roget's LEAP. Both Roget and Lavernède are cited in the introduction, together with most of the usual suspects.

William Hand Browne in *New Eclectic Magazine* 1870 gives another account similar to Roget's method, though the article is stated to be ‘Chiefly founded upon A. Crétaine's *Etudes* Paris 1865’. Instead of labelling the quartes LEAP as in Roget he uses LION, and instead of using Roget's more flexible nets he restricts the linkages to those that follow the squares or the diamonds.

R. C. Read in *Eureka* #22, 1959 is a much later account of the method, with minor refinements, in which he shows how Roget's method satisfactorily solves the problem of presenting a tour as a conjuring trick. With practice, a tour can readily be drawn between any two cells of opposite colour quite quickly, even blindfold!

H. J. R. Murray valuably extended Roget's analysis to even×even boards in general (see ⌘ 1) by calling the two types of knight moves ‘slants’ and ‘straights’ (which I spell as ‘straits’), although less memorably he used the lettering A, C, B, D (ms 1942 and *British Chess Magazine* 1949).

Enumeration of Rogetian Open Tours

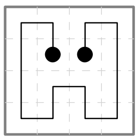
Here we seek to enumerate all 8×8 tours of the Rogetian type in which each net is toured in one piece. The basic idea of the method used here is due to H. J. R. Murray, who gives an account of his attempt at the enumeration in *The Knight's Problem* ms (1942) and in his *FCR* article (1949) but with varying results. There appear to be some errors in the data he worked from, and his account is difficult to follow. My results reported below need to be independently checked.

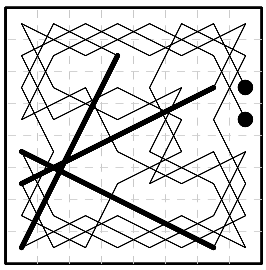
A Rogetian tour of the type we are interested in is either an open tour with three slants or a closed tour with four slants. We deal with the more general open case first.

If we divide the board into blocks of 2×2 cells and number them 1 to 16 then this numbering in conjunction with Roget's LEAP indexing gives a name to every cell on the board; for example L1 is the L-cell in block 1. If we number the blocks according to a wazir tour (I use the H-shaped tour, since it has maximum symmetry) then straits take the knight to a block of opposite parity (even-to-odd or odd-to-even) while slants do not alter the parity.

L	E	A	P	L	E	A	P
A	P	L	E	A	P	L	E
E	L	P	A	E	L	P	A
P	A	E	L	P	A	E	L
L	E	A	P	L	E	A	P
A	P	L	E	A	P	L	E
E	L	P	A	E	L	P	A
P	A	E	L	P	A	E	L

3	2	15	14
4	1	16	13
5	8	9	12
6	7	10	11





The four nets formed by straits can be seen to be distorted versions of the moves of a wazir on a 4×4 board. The number of ways a knight can then tour one of Roget's nets, from one given block to another, is thus the same as the number of ways a wazir can tour a corresponding 4×4 board whose cells represent the blocks. The three-slant tour (showing the letter K) illustrated is one of 25 that include three three-unit lines; the others differ only in the paths of the end sections LL and AA (five choices each: end blocks 1, 3, 9, 11 or 13). I dedicate this tour to Donald Knuth. The three lines of the K enclose a size 2 triangle. A closed four-slant tour with four three-unit lines is impossible. Two of the 25 tours are reentrant, but I have preferred the non-reentrant tour shown since it is unique among the 25 in that its four knight-paths are equivalent to the same non-reentrant wazir tour.

The LEAP index diagram on the 8×8 board has the properties that: (a) 180° rotation does not alter any of the indices, (b) 90° rotation interchanges L and P but leaves E and A unchanged, (c) reflection in a diagonal interchanges E and A but leaves L and P unchanged, (d) reflection in a median interchanges both pairs (since such a reflection is equivalent to reflection in diagonal combined with 90° rotation).

. KNIGHT'S TOUR NOTES

THEOREM: Any directed three-slant open 8×8 tour can be arranged, by rotation, reflection or reversal of direction of description (if necessary) so that it tours the four nets in the sequence LEPA.

Proof: There are eight possible sequences in which the four nets may be taken: LEPA, LAPE, PELA, PALE, APEL, EPAL, ALEP, ELAP. Those beginning with a vowel are reversals of those beginning with a consonant. LAPE is converted to LEPA by reflection in a diagonal. PELA is converted to LEPA by 90° rotation. PALE is converted to LEPA by reflection in a median. QED

If a tour is presented in LEPA form it remains in LEPA form when rotated 180°, but the middle EP slant is altered from (a, b) to (a ± 8, b ± 8) where a and b are the numbers of the blocks it connects. So we only need to calculate the number of tours with the EP slant starting on a low-numbered block (1 to 8).

A complete LEPA tour is formed of four paths, LL, EE, PP, AA joined by three slants LE, EP, PA. If the middle slant EP is odd (i.e. connects odd-numbered blocks) then slants LE and PA must be even, and vice versa. The list of blocks used by the end cells and slants gives a sort of ‘formula’ for a Rogetian tour. For example the illustrated tour (K) has formula 13(8,10)(1,5)(16,8)13. If we ensure the middle E < 9 then there are $8\cdot 12\cdot 6\cdot 6\cdot 8 = 432\cdot 64$ formulae with odd EP (there being 8 choices for the end-points namely all the odd blocks, 12 for even LE, 6 for odd EP, 6 for even PA) and there are $8\cdot 6\cdot 3\cdot 12\cdot 8 = 216\cdot 64$ formulae with even EP; total $(432 + 216)\cdot 64 = 648\cdot 64 = 41{,}472$.

The numbers of wazir tours from corner, middle and edge cells to other cells are summarised in these three charts: The start cell is marked X.

A corner (52)	B middle (36)	C edge (25)
8 0 4 0	8 0 4 0	6 0 2 0
0 8 0 4	0 4 0 4	X 2 0 1
6 0 8 0	2 X 4 0	4 0 4 0
X 6 0 8	0 2 0 8	0 2 0 4

If we take the median value of 4 choices for each of the four connecting routes LL, EE, PP, AA, this provides us with a rough estimate for the total number of 3-slant tours of $41472\cdot 256 = 10{,}616{,}832 \cong 10$ million, which proves to be close to the actual figure found.

To express the calculation of the number of three-slant knight tours mathematically denote by LE(i,j) an array which takes the value 1 if cell Li joins to cell Ej and 0 otherwise. This represents all the LE slants. Arrays EP(i,j) and PA(i,j) are similarly defined. Then let C(i,j) denote the number of wazir routes connecting block i to block j. This will be 0 when i and j are both odd or both even. The values of C(i,j) for i odd and j even are shown below. This table can be read row-column or column-row, since C(i,j) = C(j,i). Also shown are the even values of LE(i.j); the six odd-value 1s occur at {1,5}, {1,9} and {9,13}.

C	**2**	**4**	**6**	**8**	**10**	**12**	**13**	**16**
1	2	2	8	4	4	4	8	4
3	6	6	8	8	4	4	8	8
5	2	4	6	2	2	1	4	4
7	1	2	6	2	4	2	4	4
9	4	4	8	4	2	2	8	4
11	4	4	8	8	6	6	8	8
13	2	1	4	4	2	4	6	2
15	4	2	4	4	1	2	6	2

eLE	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**
2	0	1	0	0	0	0	0	1
4	1	0	0	0	0	0	0	0
6	0	0	0	1	0	0	0	0
8	0	0	1	0	1	0	0	0
10	0	0	0	1	0	1	0	0
12	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	0	1
16	1	0	0	0	0	0	1	0

We can now calculate the number of LEPA tours between blocks a and h from the formula: [C(a,b)LE(b,c)C(c,d)]EP(d,e)[C(e,f)PA(f,g)C(g,h)] summed over the repeated suffixes (Einstein's convention). Since LE for example is either 1 or 0 the first bracket only gives a nonzero value when the LE value is 1: it merely acts as a ‘permitter’. This calculation can be reduced to the smaller 8×8 arrays instead of 16×16 by treating the cases with EP even or odd separately.

The above calculations, done by hand using a small electronic calculator in the early 1990s, led to the results summarised here (not yet independently checked). The LE column gives the number of LL-EE routes ending at x. The EP column is the middle slant (x,y). The PA column gives the number of PP-AA routes starting at y. The last column is the product of these two numbers.

LE	EP	PA	Product
1708	(1,3)	1336	2,281,888
1708	(1,5)	546	932,568
2668	(3,1)	704	1,878,272
1191	(5,1)	704	838,464
1191	(5,7)	510	607,410
1227	(7,5)	546	669,942
510	(2,16)	1708	871,080
704	(8,10)	1227	863,808
704	(8,16)	1708	1,202,432
Total tours with 3 slants			**10,145,864**

Enumeration of Rogetian Closed Tours

The above count of tours with three slants includes reentrant tours, where the initial L-cell and the final A-cell are a knight's move apart. These AL moves are slants. Thus every reentrant open tour with three slants determines one closed tour with four slants. However, from one closed tour with four slants we can derive eight reentrant open tours with three slants, by deleting the AL slant in each of the eight orientations of the tour, which are all distinct since all Rogetian closed tours are asymmetric (symmetric tours contain at least eight slants).

The same formulae can be used for the calculation as given for three-slant tours, except that the values inserted for the first and last factors have to be reduced to apply to individual initial and final cells. Note that the total of reentrant tours with odd middle-slant is the same as the total with even middle-slant. This follows since from each closed tour we derive two even-middle and two odd-middle reentrant tours. Since this enumeration has not been independently checked only a summary of the totals is given here.

L1-A9	205,232		
L1-A15	148,812		
L7-A9	147,756	odd subtotal	501,800
L2-A4	42,316		
L4-A2	46,428		
L4-A8	64,360		
L6-A8	157,616		
L8-A4	58,344		
L8-A6	132,736	even subtotal	501,800
Total reentrant tours with 3 slants			**1,003,600**
Total closed tours with 4 slants			**125,450**
(this is an eighth of the reentrant total)			

Rogetian Tours of Squares and Diamonds Type

Roget's method is often confused with the squares and diamonds method. This is because there are some tours that are of both these types. The Rogetian tours with three slants that are also squares and diamonds tours can be enumerated, by adapting the methods used for Rogetian tours outlined above. The number of wazir tours connecting the blocks is considerably reduced under this condition. There is only ever one possible connection, or none. I find the results (Aug 2003):

Total 3-slant open tours of squares and diamonds type = **2688**.

Total of these reentrant = **368**.

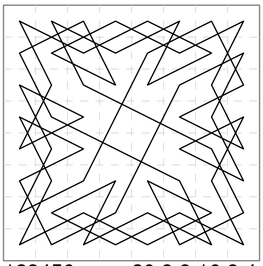
Total 4-slant closed tours of squares and diamonds type = 368/8 = **46**.

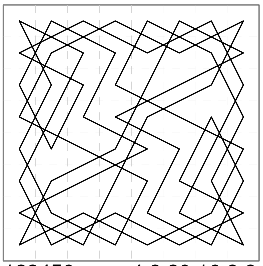
As a check on these results I have actually constructed these tours by a graphical method. All these tours are necessarily asymmetric. See the KTN website for complete diagrams.

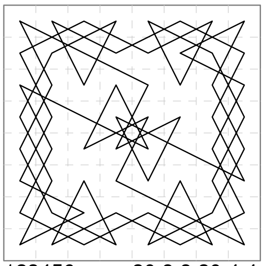
Angles in 8×8 Knight Tours

Tours With All Six Angles

Two-move journeys of a knight, including the switchback, are of seven types when classified in terms of the angle between the two lines, as we have seen in the notes on *Theory* (⌘ 1). We number the angles 0 to 6 so that the number measures the angle to the nearest multiple of 30°. An alternative terminology is to describe the non-null angles as diagonal acute (DA), lateral acute (LA), right (R), lateral obtuse (LO), diagonal obtuse (DO) and straight (S). These symmetric tours (Jelliss 1980s) show all six angles, each a multiple of 4 times (centres ii~, eo=. pp).


123456 20,8,8,16,8,4


123456 4,8,20,16,8,8

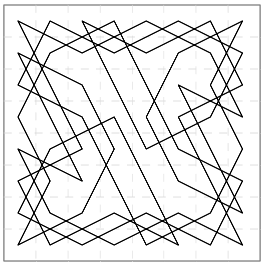

123456 20,8,8,20,4,4

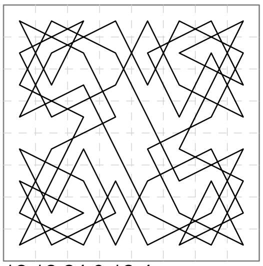
Minimum Angle Tasks

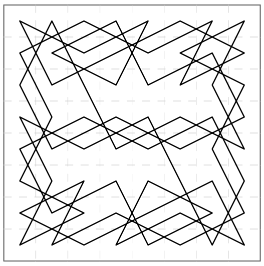
Every closed tour must contain 4 DA (1) angles at the corners; this minimum can easily be achieved, as in the middle tour above. The minimum for LA (2), LO (4), DO (5) and S (6) angles is zero, and these are also easily achieved. Examples follow. However the right angle case is different.

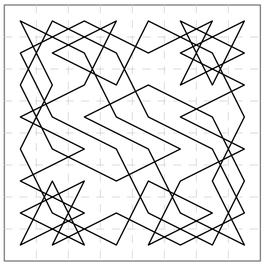
Single Minima Tasks

These symmetric tours show no LA, LO, DO or S angles respectively, the other angles each occurring a multiple of four times. (Jelliss 2018, but the LO solution is from *Chessics* 1978). The DO case was difficult to solve under this condition (centres ck~, bi=. jn~, eh).


12,0,16,16,12,8 0LA


12,12,24,0,12,4 0 LO


12,8,20,20,0,4 0DO

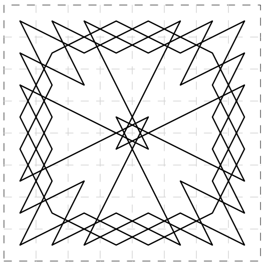

12,16,8,12,16, 0 0S

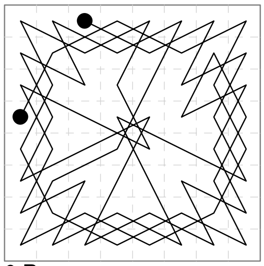
Minimum Right Angles

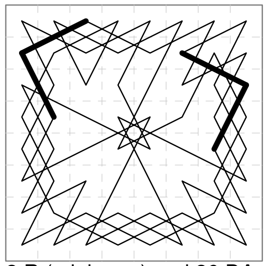
THEOREM: A closed knight tour of the 8×8 must contain a right angle. *Proof*: Take a diagram and draw in the two knight moves forced at each corner. Now consider the cells next to a corner (e.g. b1). At each of these there are three moves and two are at right angles, so we must take the third (e.g. b1-c3). The same three-move principle can now be applied at the mid-edge squares (e.g. d1, where d1-f2 is forced, since the path through c3 has already been fixed). Then at the other edge squares (e.g. c1, where c1-e2 is forced, since the path through b3 is now fixed). Next consider the centre cells. Moves like d4-e2 are out because of formation of a right angle at e2, so the paths must be e6-d4-f5 etc. Now take in turn the cells d2, c1, d1 and their like. We have now drawn a path through every cell, but the result is not a tour; it consists of four separate circuits superimposed, as shown in the first diagram below. QED.

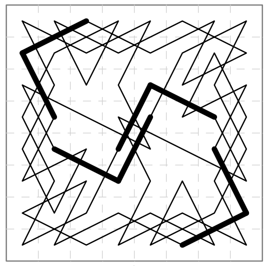
This theorem was one of my earliest tour discoveries (*Chessics* #1 1976 p.2). The same is true on the 6×6 board: consider a1, b1, d1, b2, forming 12-move circuits. But not on the 10×10.

The pattern formed in the course of the above proof is one of the pseudotours with octonary symmetry (see ⌘ 8). The circuits can be linked to form an open tour without a right angle, as illustrated (a 7-move path of alternately deleted and inserted moves links the end points).


0 R pseudotour


0 R open 28 DA


2 R (minimum) and 26 DA


4 R symmetric 22 DA

THEOREM: A closed knight tour of the 8×8 must contain at least two right angles. *Proof outline*: The same type of argument can be used to show that two right angles must occur, by trying to draw a tour with one right angle, but nine separate diagrams are needed, one for each basic position of the right angle (bcd1, bcd2, cd3, d4). This is left to interested readers to follow through.

A closed tour above with two right angles (Jelliss 1978, *Chessics* #5 p.5) shows this minimum can be achieved. Also shown above is a symmetric tour with four right angles (Jelliss undated).

THEOREM: An 8×8 closed knight tour with alternating right and non-right angles is impossible. *Proof*: Consider the restrictions involved at a1, b3, a3, a2. This leads to the linkages b4-c2-a1-b3-d2 and c4-a3-b1-c3-a2-c1-d3. But these include three sucessive non-right angles. QED

This result is also implied by the following more general one.

THEOREM: No 8×8 closed knight tour can have a right angle at an end of every move. *Proof*: Consider the restrictions involved in constructing such a tour at a1, b3, a3, a4, a4 and cognate cells sucessively. We cannot have a4-c3 since then we must have a4-b2 and then b2-d3 and b1-c3 and then a2-b4 and b1-d2, forming a circuit. The two cells a2, b1 cannot both go to c3. If neither go to c3, and similarly in the other corners, then circuits of less than 64 moves are formed. If one is assumed to go to c3 then by consideration of b1, c3, a7, b8, d5, d4, f6, f3 we reach a diagram with loose ends that cannot be joined into a tour. QED (This theorem remains true on the 10×10 board.)

Minimum Obtuse Angles

THEOREM: Every closed 8×8 knight`s tour contains at least one obtuse angle. *Proof*: (By obtuse here we mean LO or DO, not counting straights.) As before, we try to draw an obtuseless tour. Draw in the DAs at the corners. Now at a4 the path cannot go to b6 (forms LO there), also the path cannot be c3-a4-b2, since this would leave no route through d1 (d1-b2, d1-c3, d1-f2 all forming obtuse angles), so we must take a4-c5, and similarly a5-c4 etc. Now at d4 the path cannot go to b3, c2, e6 or f5 (forms obtuse angles there), also the path through d4 cannot be c6-d4-f3 or b5-d4-e2 (DOs at d4), nor can it be c6-d4-b5 since this leaves no path available through a7, nor f3-d4-e2 (no path at g1), so we must have one of the straights c6-d4-e2 or b5-d4-f3 (and similarly c6-e5-g4 or d7-e5-f3). There is no loss of generality if we choose c6-d4-e2. This forces f3-g1-h3, d8-b7 and c8-a7-b5 (otherwise g1-e2, d8-c6, a7-c6 form obtuse angles), and this in turn forces a5-c6 (else a5-b7 or a5-b3 give obtuses at b7, b3) and at e5 this forces the straight d7-e5-f3, which in turn forces e1-g2 and now there is no move available at h4 (obtuse at g2 or g6). QED. (This shows no pseudotour is possible either.)

Mimimum Lateral Angles

THEOREM: Every closed 8×8 knight's tour contains at least one lateral angle. *Proof*: Take a diagram and try to draw a tour without lateral acute or lateral obtuse angles. Draw in the corner moves. At a3 the move to c2 forms a LA angle at c2; and the two moves to b1 abd b5 form a LO angle, therefore we must draw in a3-c4 etc. Now at a4 the move to b6 forms a LO angle, and the move to c5 forms a LA, thus we must take c3-a4-b2. But applying this at the similarly placed squares, d1 etc, we form short circuits, such as a4-c3-d1-b2. QED

(Two octonary pseudotours can be made however.)

Minimum Different Angles

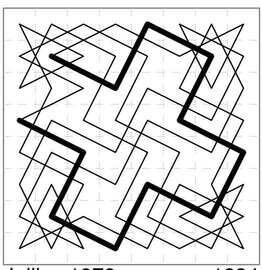
THEOREM: Every closed 8×8 knight`s tour contains at least four different angles. *Proof outline*: We have proved that DA and R angles must occur. Suppose a tour containing only three different angles is possible, then there are four cases to consider: with DO, LA, LO or S as the third angle. The DA, R, DO case is ruled out because it has no lateral angles; the DA, R, LA case is ruled out because it has no obtuses; the DA, R, S case is ruled out for both these reasons. This leaves the case DA, R, LO. Try to construct such a tour. First insert the corner moves. At a2 two of the moves form a LA angle (not allowed) so we must take the third move a2-b4, and b1-d2 etc. Now at b3 the only move that can be taken is to a5 (LO) since the other choices form S, LA or DO angles. Draw in b3-a5, b6-a4 etc. At b2 the pat a4-b2-d1 forms a DO and the paths a4-b2-c4, d1-b2-d3 form LA, while c4-b2-d3 forces a DO at e5 (c6-e5-f3)! Therefore the path through b2 must be a right angle, and that through e5 must then also be a right angle. here we must split the process up into four diagrams showing the four geometrically different ways of arranging these pairs of right angles. The resulting diagrams look promising but in fact they only lead to dead ends and short circuits (by the same arguments as have been used previously).

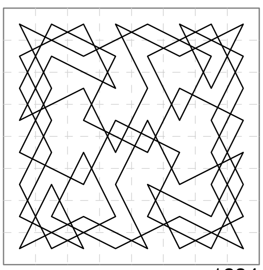
This theorem first appeared, in two parts, in *Chessics* #3 May 1977 and #5 July 1978.

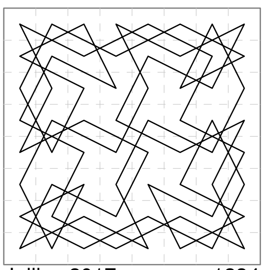
Double Minima Tasks

The above theorems lead to this task: To construct tours that contain only four different types of angle. Or in other words to construct tours in which two angle types do not appear. There are four cases: (1234, No DO or S; 1235, No LO or S; 1345, No LA or S and; 1346, No LA or DO). We can also seek to minimise or maximise one of the other angles. I gave solutions to these cases in *Chessics* #5 1978, diagrams C, I, J, H, which are the first shown in the following sets.

Case 1234 (without 56). No straight or diagonally obtuse angles. The first, asymmetric example, includes an 11-move cross and 36 right angles. The others are symmetric (no=, kk~, kn~).


Jelliss 1978 1234


Jelliss 2017 1234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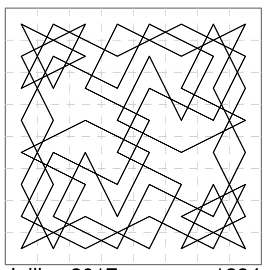
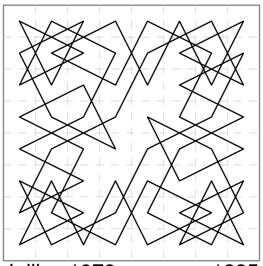
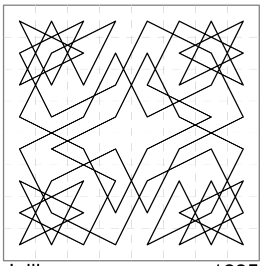

Jelliss 2017 1234

Case 1235 (without 46). No straight or laterally obtuse angles. Symmetric tours (hl, bh, fh, gg~). The second is a simple linking of an octonary pseudotour. The third remains a tour of the required type if the moves a5-c6, b7-d8, e1-g2, f3-h4 are replaced by the other sides of their rhombs (a5-b7, c6-d8 etc). The fourth is a double half-board tour of squares and diamonds type.


Jelliss 1978 1235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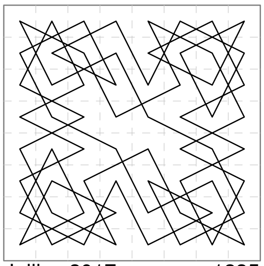
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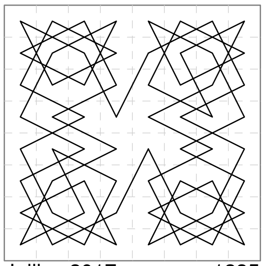
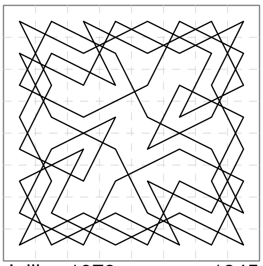
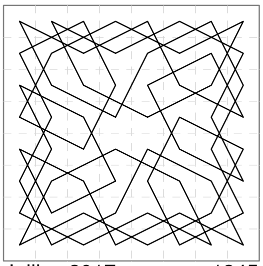

Jelliss 2017 1235

Case 1345 (without 26) No straight or laterally acute angles. Symmetric tours (hl, bc=, be~, cl~).


Jelliss 1978 1345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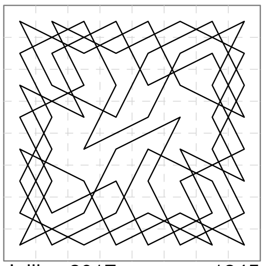
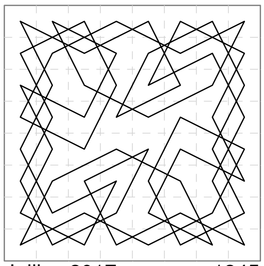
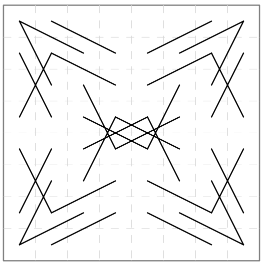
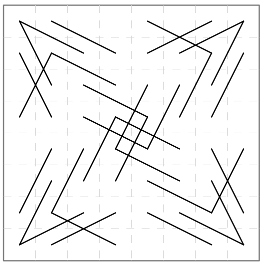

Jelliss 2017 1345

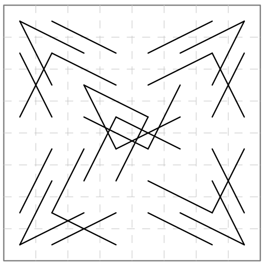

Jelliss 2017 1345

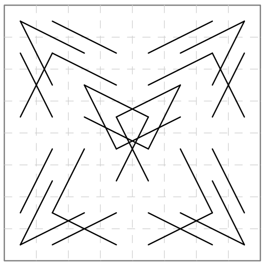
Case 1346 (without 25). No laterally acute or diagonally obtuse angles.

THEOREM: A symmetric solution is impossible. *Proof*: We draw in the corner and next to corner moves like a2-b4 and b1-d2 which are forced. We then consder moves through the centre cells and observe that they cannot be e6-d4-f5 and the like because there is then no route available via g7. This means the central angles are all right angles, and they can occur in four formations. These determine also the moves through g7 and the like. The first two can produce symmetric closed-path patterns but they prove to be pseudotours. The other two are asymmetric and axial so cannot produce symmetric tours. QED.

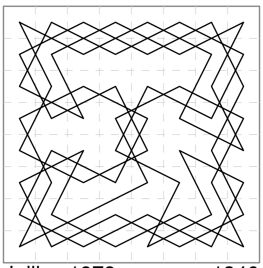


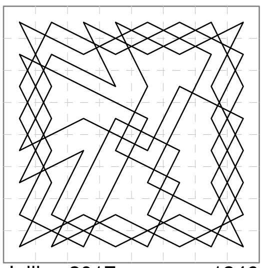


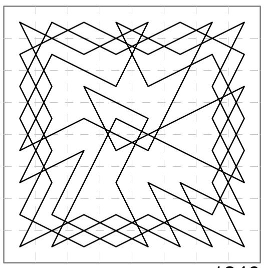


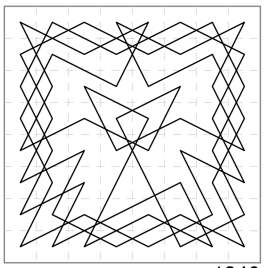


The following asymmetric solutions use the above configurations. The first shows that a three-angle tour is only just impossible, having only one straight. The last has maximum axial symmetry.


Jelliss 1978 1346


Jelliss 2017 1346


Jelliss 2017 1346

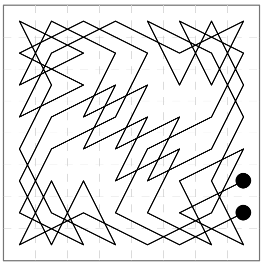

jelliss 2017 1346

Maximum Angle Tasks

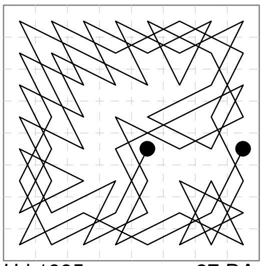
Maximising can be combined with minimising, so there is an overlap with the preceding section.

Maximum Diagonal Acute Angles

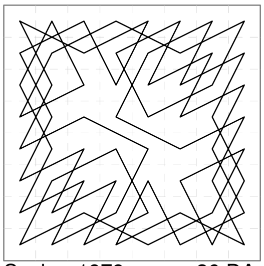
Zigzags formed of successive diagonally acute angles are a common feature in tours Käfer (1842) showed two intermeshed 6-move diagonal zigzags with 22 DA angles in an open tour. The record for diagonal acute angles stands at 28 in an open tour as shown in the open tour without right angles above. An open tour with 27 DA was shown by Dario Uri *J. Rec Math* 1995, diagram below. The best found in a closed tour is 26 DA as in the asymmetric tour with two right angles shown above and 26 DA is also the best found in a symmetric tour (J. J. Secker and Jelliss *Chessics* #7 1979, bk~, bk=). These are the only symmetric tours with the '6Vs' in a corner.



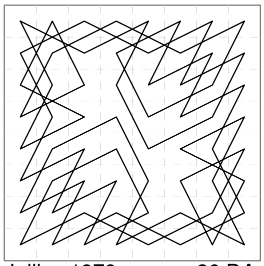
Kafer 1842 22DA



Uri 1995 27 DA



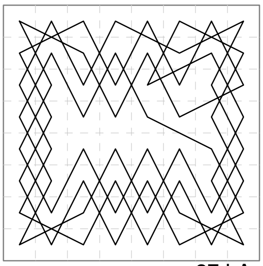
Secker 1979 26 DA



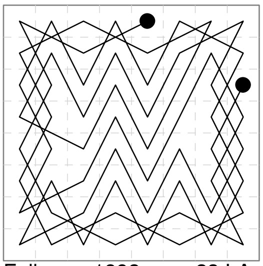
Jelliss 1979 26 DA

Maximum Lateral Acute Angles

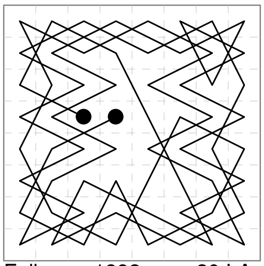
Zigzags of lateral-acute angles are referred to as 'herring-bone' patterns. These examples from Kafer (1842) and Falkener (1892) show various LA patterns. The maximum of lateral acute angles achieved is 30 (Murray 1942) in a closed asymmetric tour designed to show the maximum of single moves in two directions.



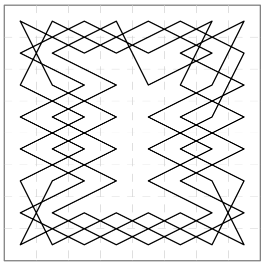
Kafer 1842 27 LA



Falkener 1892 23 LA

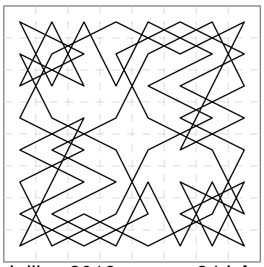


Falkener 1892 20 LA

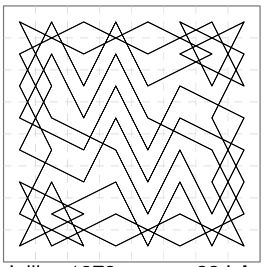


Murray 1942 30 LA

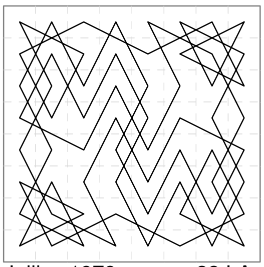
The best in a symmetric tour was increased to 24 from 22 while preparing this note.



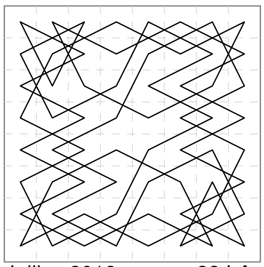
Jelliss 2018 24 LA



Jelliss 1978 22 LA



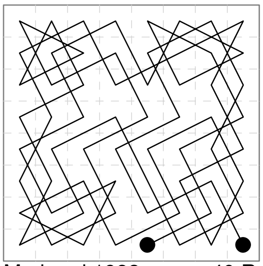
Jelliss 1978 22 LA

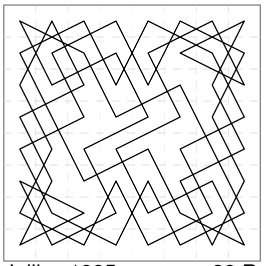


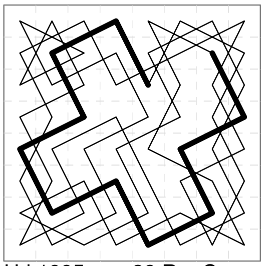
Jelliss 2018 22 LA

Maximum Right Angles

The maximum right angles achieved are 40 in an open tour by Bernard Moricard (in Berloquin *Jeux et Strategie* 1982) and 38 in a closed tour, maybe symmetric (Jelliss, *J. Rec Math.* 1995, fk=).


Moricard 1982 40 R


Jelliss 1995 38 R

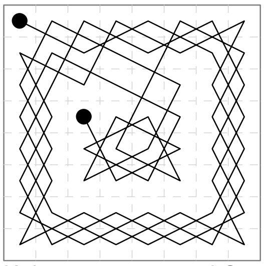

Uri 1995 38 R + Cross

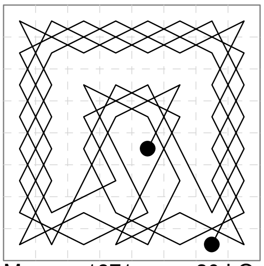
The first to show a Greek Cross formed of 11 successive knight moves in a tour was E. Slyvons (1856) see the shapes section below, and this has been done many times since. The tour by Dario Uri (*J. Rec Math.* 1995) with a Greek cross has the break in its outline at a different place from Slyvons.

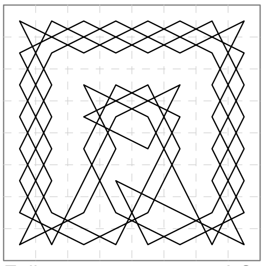
Maximum Lateral Obtuse Angles

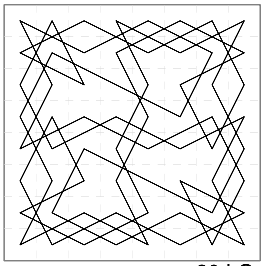
Successive knight's moves at lateral obtuse angles produce narrow zigzags, which take the form of ‘strands’ (when two zigzags are combined) or ‘braids’ (when four zigzags are combined). Lateral braids occur in the outer parts of the Mani, Somesvara, and Moivre tours. The form of these tours is similar to many others composed later, e.g. the Kafer (1842) below. Haldeman (1864) called such tours ‘Fillet and Field’ tours. Many of the earliest tours of the modern era, particularly those found by Collini's method, tend to have a lot of moves forming a braid round the edges.

The Rajah of Mysore shows an open tour with 30 LO and Falkener (1892) a closed tour with 29 LO. The last here shows 30 LO in a symmetric closed tour (Jelliss *Chessics* #5 1978, jj~).

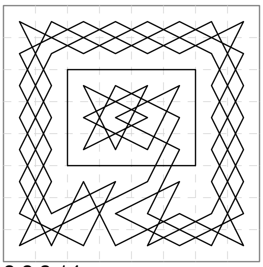

Kafer 1842 26 LO

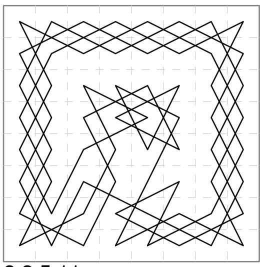

Mysore 1871 30 LO

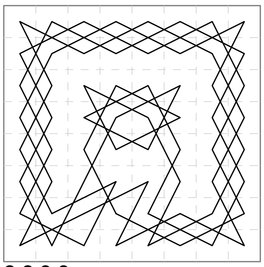

Falkener 1892 29 LO

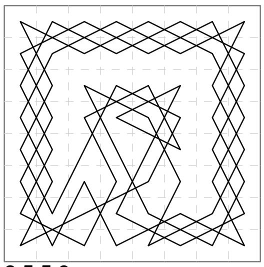

Jelliss 1978 30 LO

It occurred to me, for the record, to investigate closed tours with maximum length border braid, and found 19 tours (Jul 1990). They all consist of 44 moves in a braid of four 11-move strands extending from bc12 to de12. They differ in the way the eight loose ends of the braids are connected, and can be classified by the numbers of moves in these connections. The four types are 2,2,2,14 (four), 2,2,5,11 (six); 2,2,8,8 (seven), 2,5,5,8 (two). One of each is shown. In the first example the 14-move connection includes a 3×4 tour on the inner cells cdef345. In the third example the central formation is axially symmetric. These have 26, 26, 29, 28 LO respectively.


2,2,2,14


2,2,5,11

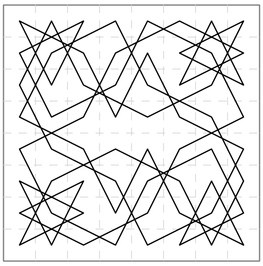

2,2,8,8

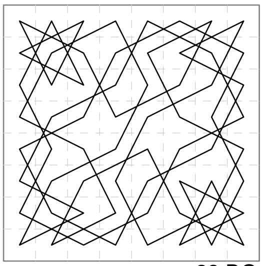

2,5,5,8

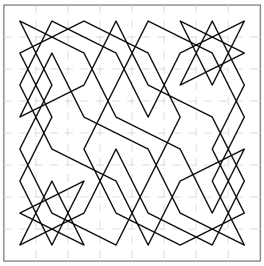
Arrangements with the eight ends on cdef12 only produce pseudotours.

Maximum Diagonally Obtuse Angles

The maximum DO I have found is 22 in *Chessics* (#5 1978) and in a more recently found example (2017), both symmetric, and another try with only 20 DO.

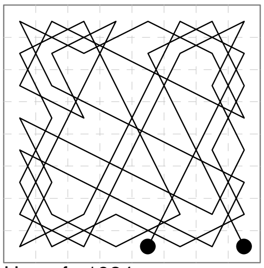

Jelliss 1978 22 DO

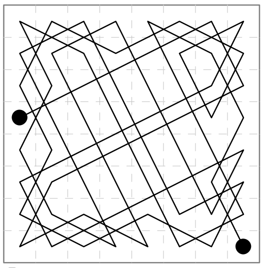

Jelliss 2017 22 DO

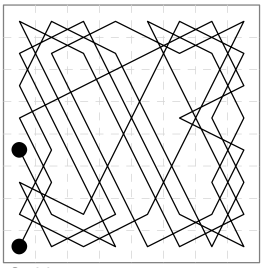

Jelliss 2018 20 DO

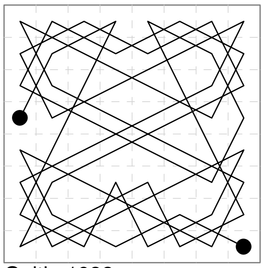
Maximum Straights

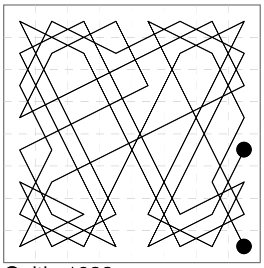
The problem of showing straight lines of moves has been one of the most intensively studied on the 8×8 board. The maximum of ten three-unit lines in an open tour was first achieved by Dr Max Hogrefe (in *Weser Zeitung* 13 Jul 1924). T. R. Dawson gave another example (in *Problemist Fairy Chess Supplement* Feb 1932) and Valeriu Onitiu reported examining all possible arrangements of ten lines on the 8×8 board, 1330 in all, and found that only six of them admit tours (*PFCS* Jun 1932).

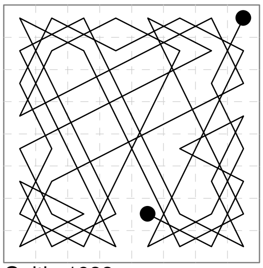

Hogrefe 1924

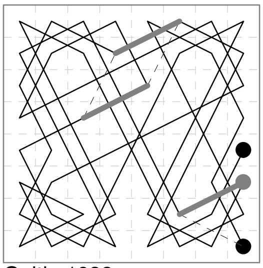

Dawson 1932

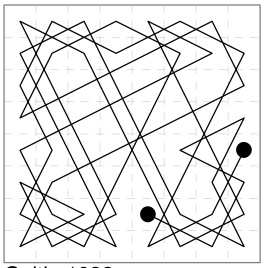

Onitiu 1932


Onitiu 1932

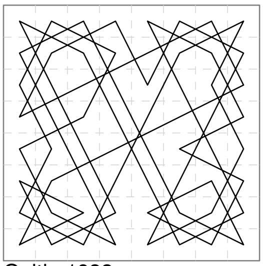

Onitiu 1932

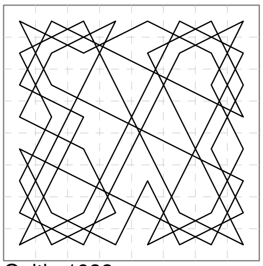

Onitiu 1932

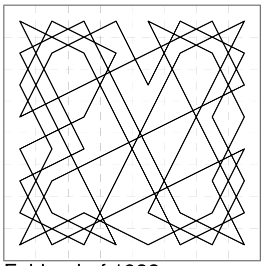

Onitiu 1932


Onitiu 1932

These six cases admit slight variations (two shown above), making a total of 12 geometrically distinct solutions. A variation of Hogrefe's solution, adding a 2-move line a1-c5, is: (a) delete h4-g2, b3-a5, c5-b7 (b) insert a5-b7, b3-c5, e1-g2. The maximum ‘straight angles’ in an open tour is 21.

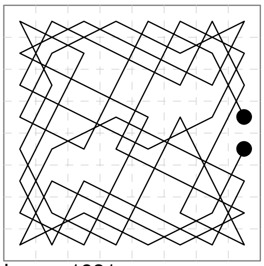

Onitiu 1932


Onitiu 1932

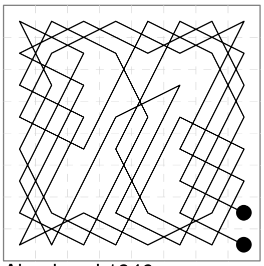

Fuhlendorf 1932

The maximum straight angles in a closed tour is 19, and the three-unit lines possible reduces to nine. Three examples from *PFCS* are shown here; two by V. Onitiu and one by G. Fuhlendorf.

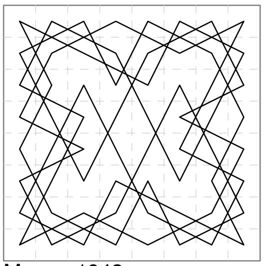
The maximum numbers of two-unit lines achieved (excluding three-unit lines) are 15 in an open tour by (E. Lange, *Sphinx* Aug 1931), 13 in parallel in an open tour (J. Akenhead *FCR* Oct 1946).



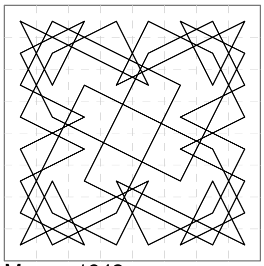
Lange 1931
15 two-unit lines



Akenhead 1946
13 two-unit lines, parallel



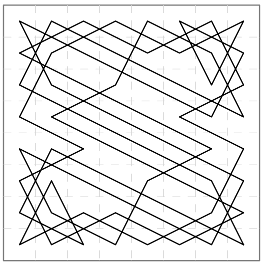
Murray 1942
12 two-unit lines



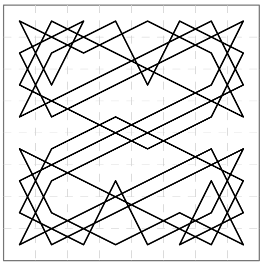
Murray 1942
12 two-unit lines

And 12 in a closed tour, which can be symmetric (Murray 1942). The above tours by Murray are also in mixed quaternary symmetry.

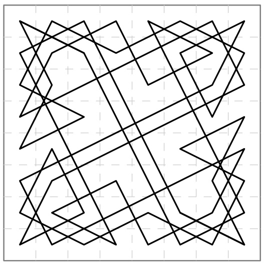
The maximum number of ‘straight angles’ in a symmetric tour is 18, and the maximum number of three-move straights is eight. This was first shown by Ernest Bergholt (*British Chess Magazine*, Mar 1918) with all lines parallel. As mentioned there, but not diagrammed, G. L. Moore found two other solutions. I show the diagrams from the manuscript Moore sent to Murray in 1920.



Bergholt 1918



Moore 1920

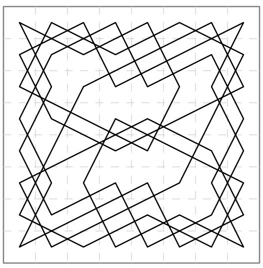


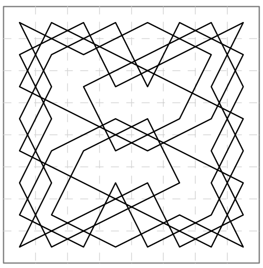
Moore 1920

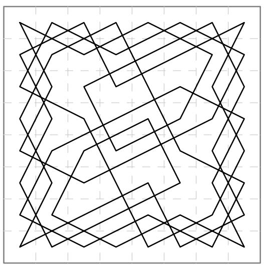
Other composers have subsequently found these three eight-line tours independently. Moore's first example also includes two two-move lines. Moore also explored the number of symmetric tours with six three-unit lines and reported finding 173.

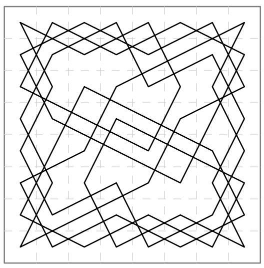
Circulation

E. W. Bennett *Fairy Chess Review* (vol.6 #10 Feb 1947 p.72, sol #11 Apr 1947 p.82, ¶7159). Proposed this interesting problem: “To construct a knight's tour so that the join of the mid-point of the board to the moving knight always rotates in the positive direction (anticlockwise).” An open tour was given as solution with the comment: “Curiously, no one obtained a closed tour; there seems no reason why not.” This was answered almost exactly 50 years later.









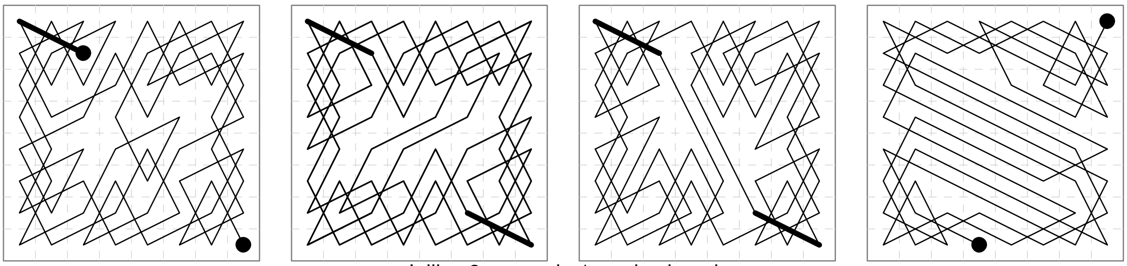
G. P. Jelliss *Journal of Recreational Math*ematics (‘Circular or Bennettian Tours’ Problems and Conjectures 2258, vol 27, #3, p.219, 1995; solution vol 28, #3, p.234, 1996-97). These are all possible closed symmetric solutions to the problem.

Directions: Modes and Senses

The chess knight moves in four different **line-modes** (a term I have invented to describe sets of lines parallel to each other). It also moves in eight different **line-senses** (taking account of the sense of movement along the lines). Confusingly the term **direction** is used with either meaning.

In a closed tour on a rectangular board all four modes must occur, since the eight moves through the corner cells provide two in each mode. This is also the case in an open tour on any even by even rectangle since although by starting in a corner an open tour can eliminate one of the modes there, the parallel line in the opposite corner cannot be removed, because the tour must end on a cell of opposite colour, and so cannot end in the opposite corner. Tridirectional tours are however possible on certain boards with one or both sides odd. (See ⌘ 4 and ⌘ 5).

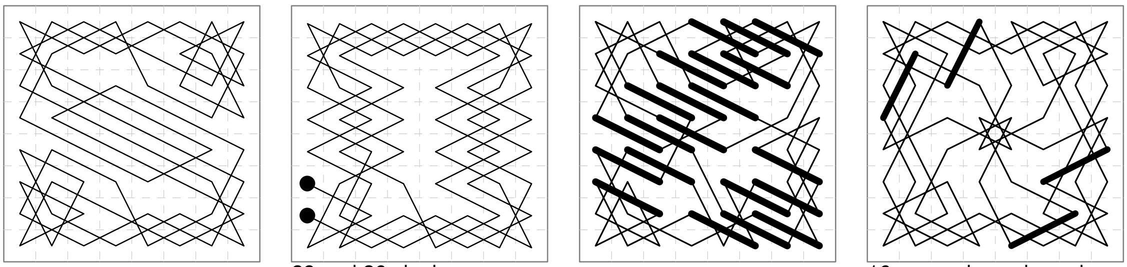
An 8×8 tour with only one move in a given mode is possible. Kraitchik (1927 Fig.79 p.32) gave an example, which I have improved on by making the odd move the last (or first) move in the tour. In a closed 8×8 tour there must be at least two moves in each mode. The minimum of two moves is shown in the next two diagrams (Jelliss 2017), which are also symmetric (eg=, dg=). The maximum moves in one line-mode appears to be 37 in an open tour, by E. Lange (*Sphinx* Jun 1931, which improves on earlier examples of 36 by Parmentier, and 35 by Käfer 1842).



Jelliss after Kraitchik — Jelliss 2 moves in 1 mode closed — Lange 1931

The Lange 1931 example with 37 moves in one direction also shows the maximum number of 16 non-intersected moves in an open tour.

In a closed tour 34 in one direction is possible, which may be symmetric (tour below, Jelliss undated). If we require the moves we count not to be joined up into lines the maximum in a given line-mode reduces to 28. Two sets of 28 are impossible (Kraitchik) but 28 in one direction and 26 in another can be shown as in the tour by T. B. and F. F. Rowlands (*Chess Fruits* 1884 used for a cryptotour). It thus shows 54 moves in two line-modes. This was converted by H. J. R. Murray into a closed tour with 27 in each of two line-modes (see the 30LA tour above).



34 in 1 mode closed Jelliss — 28 and 26 single moves Rowlands — 21 generic moves Jelliss — 16 moves in each mode Lange

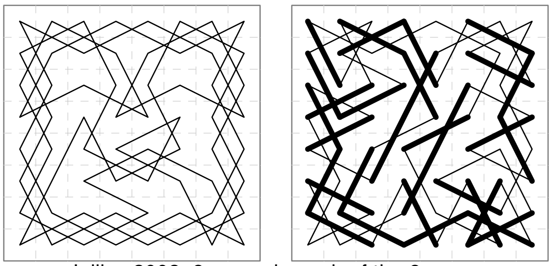
How many geometrically different positions are there for the end-points of an 8×8 open knight's tour? This question was discussed at length in *The Leisure Hour* (1873, p.813-5). Heinrich Meyer gave the correct answer, after numerous abortive attempts by others. The answer is 136, consisting of 21 reentrant and 115 non-reentrant. The third tour above shows my asymmetric tour showing all 21 generic positions of the knight's move with respect to the 8×8 board (*Chessics*, #22, p.66, 1985). It may be possible to find a more memorable arrangement of the 21 moves, but unfortunately the regular arrangements such as having ends in a 3×7 rectangle or a 1+2+3+4+5+6 triangle do not admit a tour. Also it is not possible to form any symmetric tour with the property. Memorising this tour would enable a conjuror to perform the trick of having black and white knights placed on the board anywhere, at a knight move apart, and then moving one knight to capture the other, but first visiting all the other cells en route.

Table of numbers of relative positions of end-points in 8×8 open knight's tours: {0,1} 16, {0,3} 12, {0,5} 8, {0,7} 4, {1,4} 14, {1,6} 7, {2,3} 15, {2,5} 9, {2,7} 3, {3,4} 10, {3,6} 5, {4,5} 6, {4,7} 2, {5,6} 3, {6,7} 1. Total 115. These totals can be calculated from formulas: With ends {r,s} apart there are $(8-r)\cdot(8-s)/2$ relative positions when $0 < r$ and $2 < s$, otherwise there are $18 - 2\cdot s$.

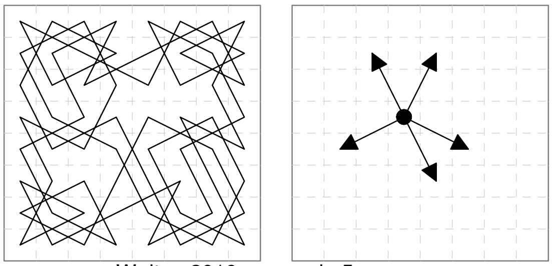
The fourth diagram above is another result by E. Lange (*Sphinx* 1931) and solves the problem of a tour with 16 moves in each of the four line-modes. This is not quite symmetric, as the four darker moves indicate; in fact we can prove that a symmetric solution of this problem is impossible.

THEOREM: <u>A symmetric tour with 16 moves in each line-mode is impossible.</u>

Proof: A symmetric tour can be split into two equal halves of 32 moves joining a corner to an opposite corner, say a8-h1. This journey is equivalent, over all, to a move of type {7,7}. If this half-tour contains m moves (1,2) then to meet the required equality it must have $8-m$ opposite but parallel moves $(-1,-2)$ and these combine to give a resultant move $(m-(8-m))\cdot(1,2) = (2\cdot m - 8, 4\cdot m - 16)$ in which both coordinates are even. Four such even moves cannot combine to give a move with odd coordinates.QED.



Jelliss 2002: 8 movesin each of the 8 senses



Welton 2010: moves in 5 senses

From this theorem it follows that it is also impossible to construct a symmetric tour with 8 moves in each of the eight line-senses, since removing the arrows would leave a tour with 16 moves in each line-mode. However a solution of this long-standing problem (mentioned by Murray in his 1942 ms) is possible in an asymmetric tour. The example above was constructed from Roget's four nets using a linkage polygon of alternate straights and slants (b3c5d7e5f3g1 e2c3d5e7f5d4b3) in which the deleted straights and the inserted slants occur in similarly oriented pairs. *Games and Puzzles Journal* issue 23 (2002) has a colourful rendering of this tour which shows the moves as arrowed lines. In the alternative solution, found about the same time, the bold lines are taken in the down and right directions, the others in the up and left directions. The Lange tour with 16 in each line-mode has the 16 divided 11:5, 10:6 or 9:7 as regards line-senses. My tours have them divided 8:8 in each case.

Jonathan Welton has found a unique tour on the 8×8 board that uses moves in only five line senses (sent to me by email on 20 Nov 2010). The 5 senses which lead to a solution can be described as 1, 2, 4, 5, 7. There is only one other way of selecting 5 moves from the 8, which is 2, 3, 4, 5, 7 but this will not yield a closed knight's tour. He notes that the 5-direction knight can only complete a closed tour of a rectangular board if at least one of the sides is a multiple of 4. The 8×8 board is the smallest board with a solution.

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Shapes in 8×8 Tours

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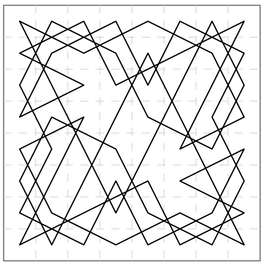
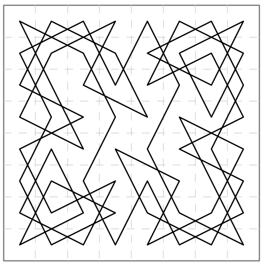
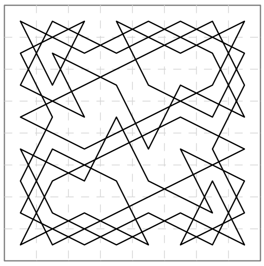
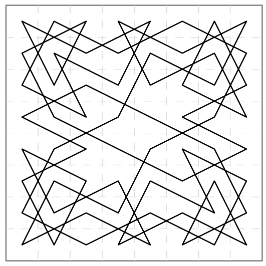
Triangles

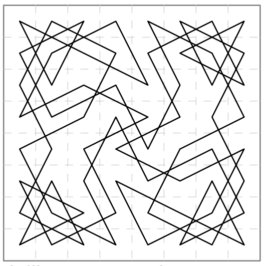
We now look at various shapes and configurations that can be seen in knight tours formed by whole or partial knight moves. In Knight-Move Geometry (⌘ 1) it is shown that knight-move triangles are all of the 3:4:5 shape and when numbered from the smallest upwards the area of the kth size is $k^2/120$. The first tour below (Jelliss 2019) shows triangles of all sizes 1 to 12 plus 15 and 21.

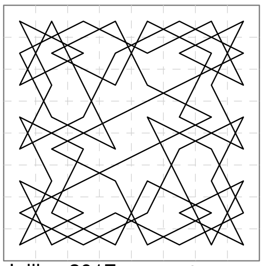
A ‘Celtic Tour’ as defined by D. E. Knuth is one with no size 1 triangle. He has a computer program which draws knight tours attractively in the form of celtic knots in which alternate intersections cross above and below each other. In ‘Five Notes on Celtic Tours’ *The Games and Puzzles Journal* #21 (online) Sep-Dec 2001. I proved that any 8×8 symmetric tour with a pair of moves of the type a2-c3-b1 must also contain a size 1 triangle, so is not Celtic. Here are three symmetric Celtic tours (centres cl~, do~, hi) from 50 I constructed: The Victor Gorgias (1871) tour with two octangles (see ⌘ 7) has 27 size 1 triangles, which may be near the maximum possible.

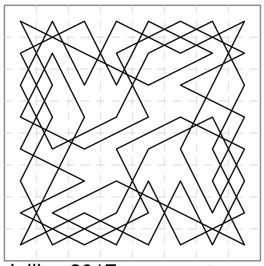





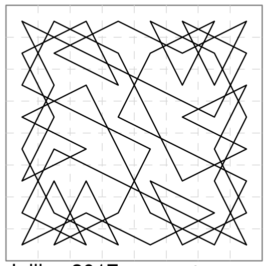
The first tour below is symmetric and shows **uncrossed** triangles of all sizes 1 to 6. How many of each can you see? I make it 8 of size 1, 14 of size 2, 8 of size 3, 8 of size 4, 4 of size 5 and 2 of size 6. There are of course other triangles, including some of size 5, that are crossed by other lines.

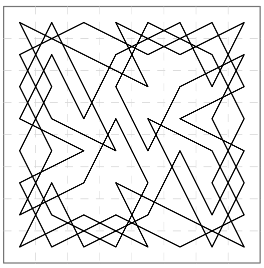
The other three tours each show four uncrossed triangles of sizes 7, 8 and 10. The 7-case also has two triangles of size 6 in the middle. The 8-triangle also has sizes 4 and 2 alongside it in the top left and bottom right quarters. The 10-case also has triangles of sizes 6, 5, 4 and 3 outwards in the edges.

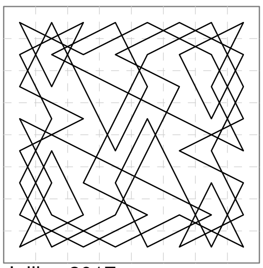

Jelliss 2017 sizes 1 - 6

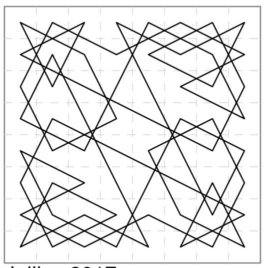

Jelliss 2017 size 7

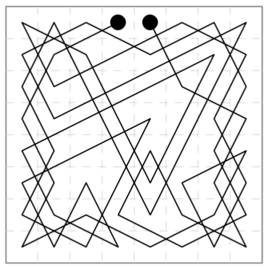

Jelliss 2017 size 8


Jelliss 2017 size 10


Jelliss 2017 size 11


Jelliss 2017 size 13


Jelliss 2017 size 14

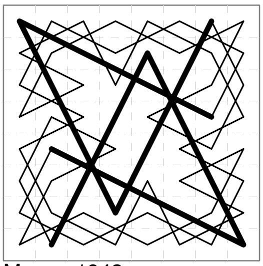

Jelliss 2001 size 24, 36

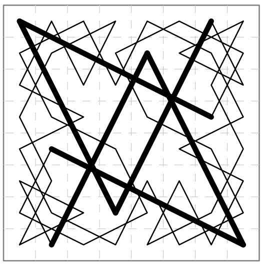
Three further symmetric tours each showing two uncrossed triangles of sizes 11, 13 and 14 follow. Size 14 is the maximum, since all larger triangles enclose a cell centre, as does the frequently occurring size 12 which consists of three successive knight moves.

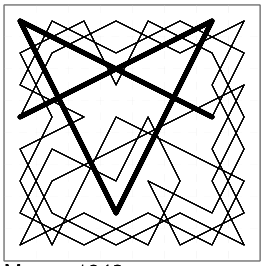
The open tour above shows two large triangles, of sizes 36 (the maximum on the 8×8) and 24, one inside the other (Jelliss *Games and Puzzles Journal* #18 March 2001 p.346).

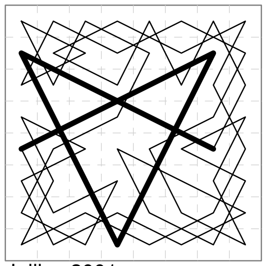
The reader is invited to try to construct tours improving on these or showing the missing cases. Some can already be found among examples showing other results.

A related problem of interest is the construction of tours with consecutive three-unit or two-unit lines. Murray (1942) gives an example showing three consecutive three-unit lines (twice) in a symmetric tour, and one showing four consecutive three-unit lines in an asymmetric closed tour. My own work shows alternative solutions, the 4-line formation being placed in a different position.


Murray 1942


Jelliss 2017

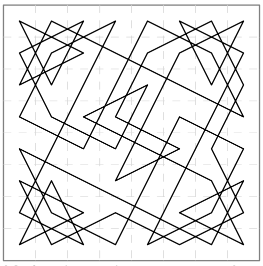

Murray 1942

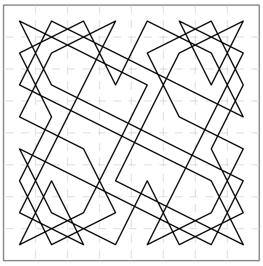

Jelliss 2001

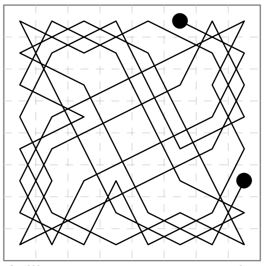
Quadrilaterals

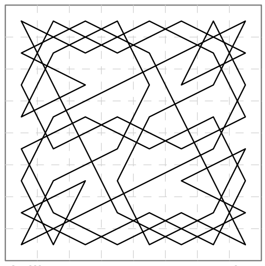
Squares. The first tour I have noticed to contain a square of minimum size (let us denote it 1×1, the unit being √5/5) is the one by Mairan (1725) and it contains six. A popular scheme is to show squares centred on the board. These are necessarily of odd dimensions, 1×1, 3×3, 7×7 or 9×9. Not 5×5 since this would consist of four knight moves forming a circuit. The recently discovered symmetric tours 7, 8, 9 and 11 by Monge (1780) are the first to have a central 1×1 square. Käfer (1842) shows all four cases combined in pairs (see ⌘.7 Symmetry).

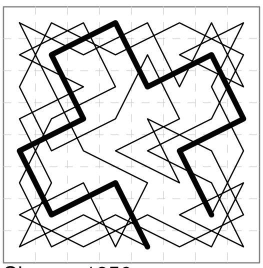
Here is an asymmetric example by Kafer, and more recent symmetric examples. The design by Bergholt (1918) includes size 4×4 and 5×5 squares, in the corners of the 9×9. My tours (Jelliss 1986) show a size 7 square dissected into a size 3 square and four size 2 squares (and four 2×3 rectangles), and a size 1 with size 7, skewed relative to each other.

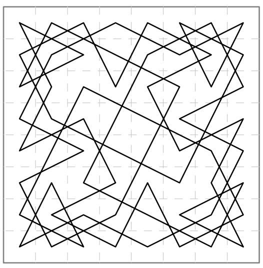

Kafer (1842) 1 and 9

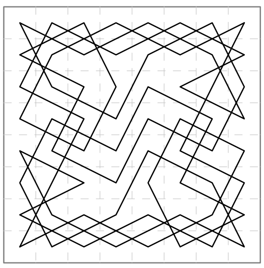

Bergholt 1918 1 and 9

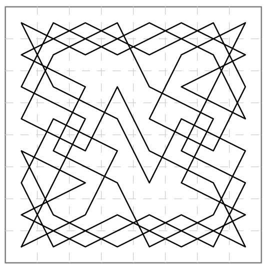

Jelliss 1986 3 and 7


Jelliss 1986 1 and 7


Slyvons 1856


Jelliss 1986 3

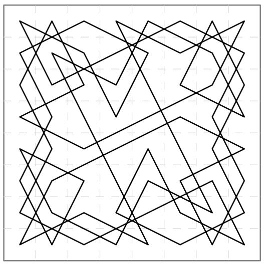
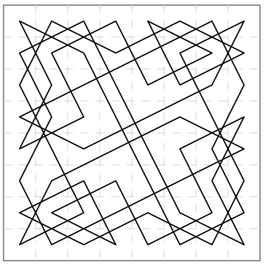
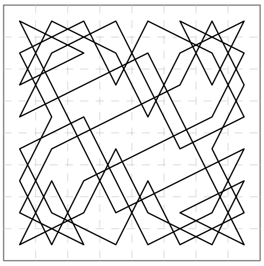
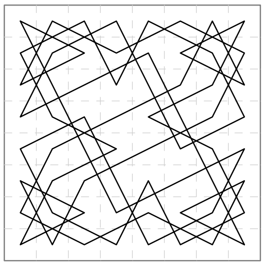

Jelliss 1986


Jelliss 1986

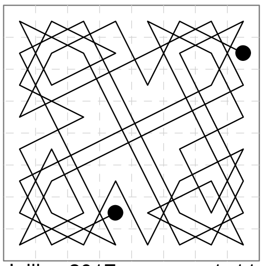
Slyvons (1856) was the first to show size 5 Greek cross (less one side). The size 3 square is included in the tour above with a central Greek cross (Jelliss 1986, ii~).

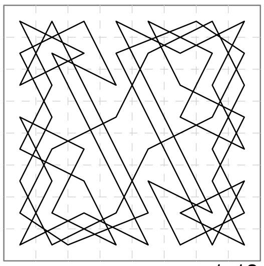
Three tours with a cross of this type occur among the mixed quaternary tours (type 023, see ⌘ 7). Two tours (Jelliss 1986, bk=, be~) contain a pair of size 1 Greek crosses (see also fh and bf= symmetric tours ⌘ 7).

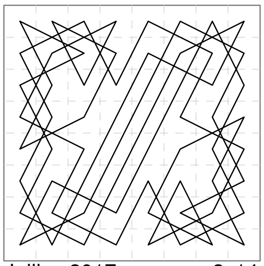
A square of knight moves has area equal to that of 5 cells (since the length of a knight move is √5) but it cannot occur in a tour since it is itself a circuit. Squares of this size 5 can however be formed by nightrider lines. The first two examples below (Jelliss 1986) include a pair of size 5 squares, and the other two include a pair of size 6 squares, overlapping in the central 3×5 rectangle, and also forming rectangles 3×7 and 6×7 among others.

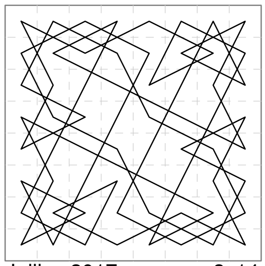





Oblongs. Rectangles 1×2 occur readily. Earliest I have noticed without is one of Euler's (1759).


Jelliss 2017 1x11


Jelliss 2017 1x12


Jelliss 2017 2x14

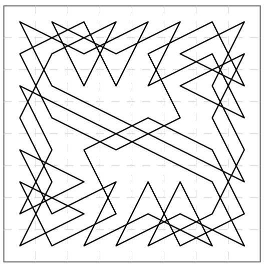

Jelliss 2017 3x14

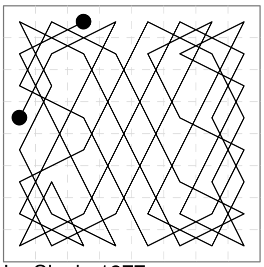
These tours show long rectangles not included in the straight line and squares examples.

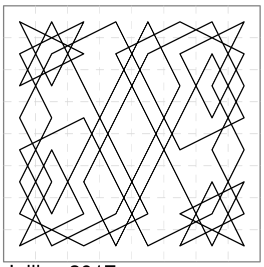
The open tour above shows two rectangles 1×11 crossing at the centre. Pseudotours can be shown with this design but not a closed tour. The other three tours here are symmetric. Second tour shows 1×12. Third tour 2×14, 2×13, 2×11 as well as 3×11 and 1×11 three times uncut. The fourth shows 3× 14, 12, 11, 9, 8, 7, 6, 5, 4, 3, 2, 1, among others.

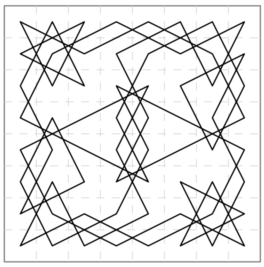
As with the squares a popular scheme is to show rectangles centred on the centre of the board. These are necessarily of odd dimensions, as with the 1×11 and 3×5 and 5×9 in the above tours .Magic tours include the cases 1×3, 1×5 and 1×7.

Lozenges. Besides rectangles we can also recognise paralellograms with lateral angles, which we call lozenges (particularly when the sides are equal). The two tours shown earlier with three consecutive three-unit lines doubled both include a 3×7 lozenge centred on the board.


Le Siecle 1879


Le Siecle 1877


Jelliss 2017

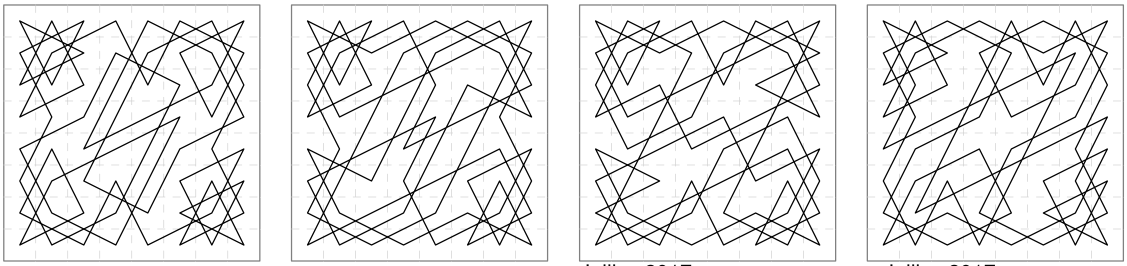

Jelliss 2017

The 1×1 lozenges occur in great numbers in any tour wth border braids. Central j angles produce a 1×3 lozenge, first shown in the Nilakantha tour of 1640 (p.4).

In the first example above the d-straights cut the 1×3 into three unit lozenges. This is from *Le Siècle* (¶688 17 Jan 1879). One of Euler's 1759 double half-board tours includes a 1×7 lozenge. The open tour from *Le Siècle* (¶220 13 Jul 1877) has crossing parallels that form six size 2 lozenges. Among Käfer's 1842 open tours there is an example with 3×3 and 5×5 lozenges centred on the board (see ⌘ 7). The third tour here shows this but in a symmetric closed tour. It also includes 1×1 and 2×2 lozenges. The fourth shows the maximum 5×5 and mimimum 1×1 lozenges centred on the board.

Diamonds. Rectangles or parallelograms with diagonal angles we call diamonds. The two tours with three consecutive three-unit lines doubled both include a maximum 7×7 diamond centred on the board. My solution also includes two 2×2 diamonds in the obtuse corners of the large diamond.

The next tour here, derived from a pseudotour in *Le Siècle* (¶16, 17/24 Nov 1876) by rotating the 1-1 links, shows a 3×9 rectangle with a 1×3 diamond stripe, centred on the board. A 3×3 diamond centred on the board cannot be shown in a tour since the two m angles form a circuit. Central p angles produce a unit diamond. The second tour shows maximum size 7 and minimum size 1 diamonds centred on the board, and also two unintersected 2×3 diagonal parallelograms in the obtuse corners of the large diamond. The third tour shows size 2 diamonds in the obtuse.corners of the size 7 diamond. This pattern also includes rectangles 1×2, 1×3, 3×4, 4×10. The fourth tour shows a size 5 diamond centred on the board with size 1 diamonds in its obtuse angles,

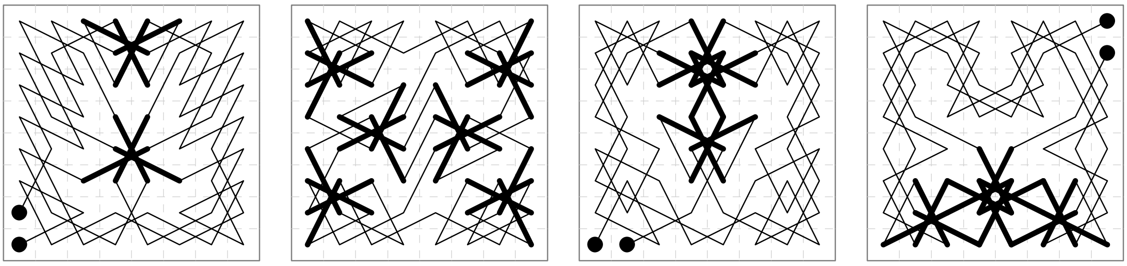


Le Siecle 1876 | Jelliss 2017 | Jelliss 2017 | Jelliss 2017

Of course many other shapes can be recognised and tours constructed to show them prominently as has been done in the examples given above. I leave this as a recreation for readers.

Intersections in Tours

Quadrangles and Octangles. These next tours show cases of four moves mutually intersecting (i.e. a 'complete quadrangle').



Le Siecle 27 Apr 1877 | R. Inwards 1925? | Le Siecle 10 Aug 1877 | Le Siecle 5 Jul 1878

The peacock design is from the column by A. Feisthamel (¶154 on 27 Apr 1877 in *Le Siecle*) and shows two quadrangles, one having all four arms extended to two-move length. The other diagram shows six and is from a manuscript written around 1925 by **R. Inwards** held at the Hague library. This quadrangle formation can be seen in many other tours (e.g. Nilakantha).

Two quadrangles can combine to make an **octangle**, often called a 'star' though different from the 7-move corner formations given that description. The octangle occurs frequently in the centre of symmetric tours, being the 'pp/pp' central angle formation.

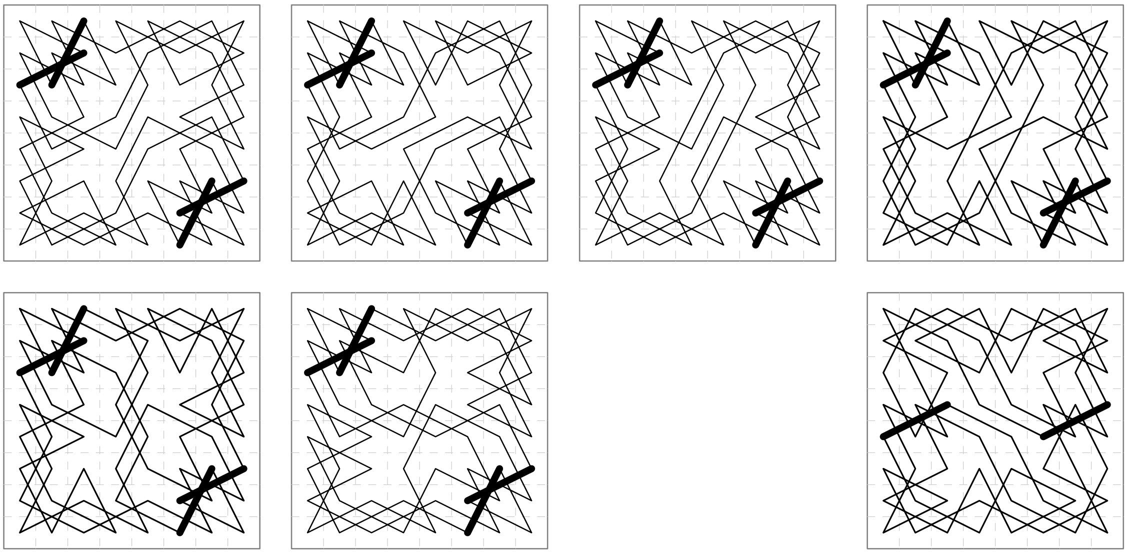
THEOREM: In a closed 8×8 knight's tour a move intersected at least four times must occur

Proof: Consider the move patterns possible at a corner. If an a2-c1 type move occurs in a tour then it must be cut 4 times, by moves through a1 and b1. Suppose no such moves occur, then we must have a VW type tour (b3-a1-c2 is the V and b4-a2-c3-b1-d2 is the W). This pattern must occur in every corner. Moves of type a3-c2 are now intersected 4 times. Suppose such moves do not occur, then we must have b5-a3-c4 and d3-c1-e2. Now moves of type b2-a4 are intersected 5 times. Suppose such moves do not occur, then we must have c4-b2-d3. We now have a pattern of 6Vs at each corner. In this pattern the triple Vs and double Vs at alternate corners join up to form closed circuits of 20 moves, and so cannot be part of a tour of the whole board. QED. (For tours with the 6Vs in opposite corners see the section on acute angles.)

Seven-fold Intersections

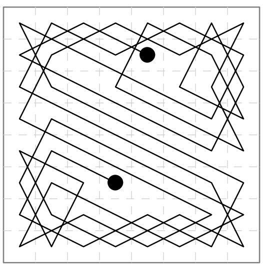
The maximum number of intersections that can occur on one move of a knight's tour is seven. Symmetric tours incorporating seven-fold intersected moves are shown below. The maximum that can be achieved on the 8×8 board is four seven-fold intersected moves, and they must be in the positions shown here, crossing one another in pairs.

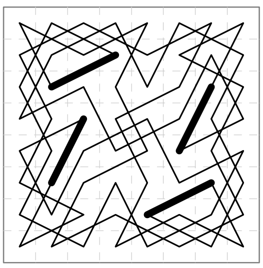
The six tours are all that are possible under these conditions. The other example shows two seven-fold intersected moves in a different position. [Jelliss *Chessics* #19, 1984].

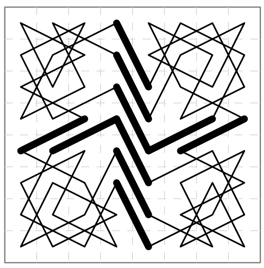


Uncrossed Moves

Kafer 1842 tour with 15 uncrossed moves (Lange 1931 shown earlier has 16.). Two tours (Jelliss 1984) with non-intersected slants. The first shows four non-intersected moves across the ‘border line’ (between the central 4×4 and the surrounding border), the second ten non-intersected moves connecting different quarters.


Kafer 1842

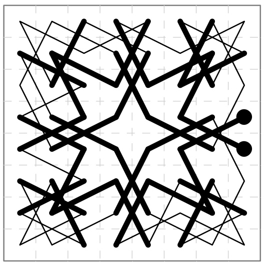

Jelliss 1984


Jelliss 1984

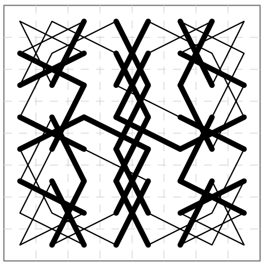
Slants

The first two examples below show the maximum slants in the open and closed cases (34 and 32). H. J. R. Murray (*Fairy Chess Review* #8368, November 1949, p.71–2, solution 1950, p.103) notes that closed tours have 32 slants maximum and cites an example he published in *BCM* 1917 and gives this new example. He claims there are 1811 of these symmetric tours. He notes there can be 34 slants in an open tour and gives this example by Falkener (1892).

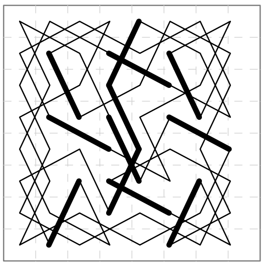
Falkener 1892



Murray 1949



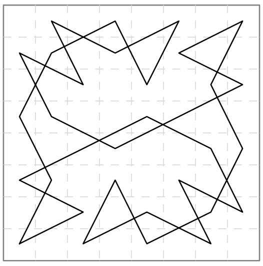
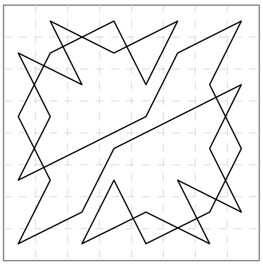
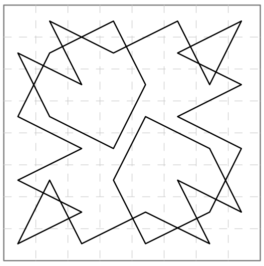
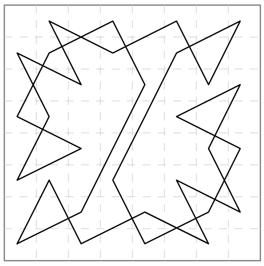
Akenhead 1949



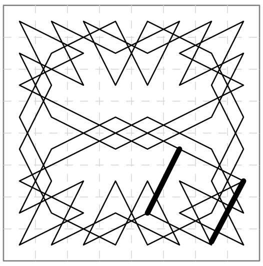
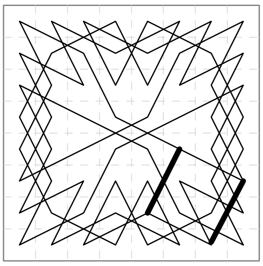
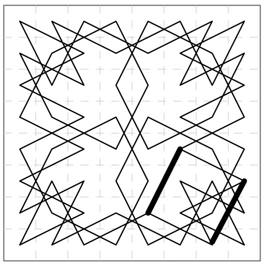
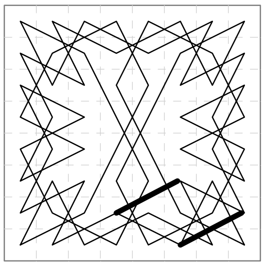
Akenhead's tour with 12 slants (*FCR* #8146, June 1949 p.4, solution August p.53) solves the problem of constructing a tour to show the nine fundamentally different slants, and a minimum number of extra slants.

Eccentrics and Conformals

An **eccentric knight** [G. P. Jelliss, *Chessics*, #8, 1979] is restricted to making edge-to-centre moves. It follows that eccentric knights starting on b1 and g1 can never meet; they each have access to half the cells of the board. Eccentric knights are able to tour the 32 cells which they are able to reach. There are 12 geometrically distinct closed tours, four being symmetric.

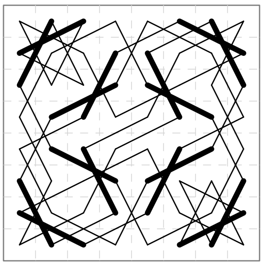
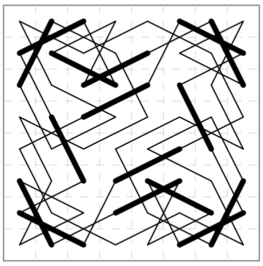
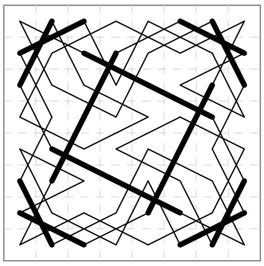





It is thus possible to place two eccentric knight tours on one board to form a pseudotour. Four near-symmetric tours can be derived from them by simple linking. The tour uses the darker moves.

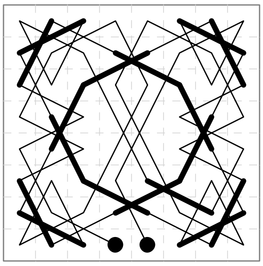
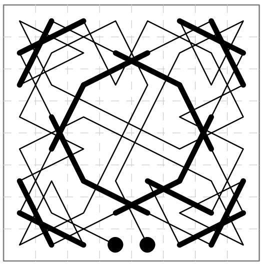
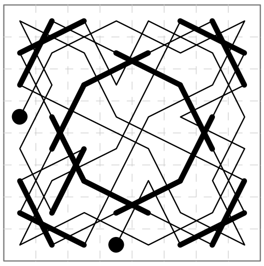





Every knight closed tour must contain at least 48 eccentric moves (one at each of the 16 cells like b1 c1, and two at all the other edge and centre cells) so the maximum 'conformal' moves is 16.

The following symmetric examples show this maximum. Eight must be the edge-to-edge connections.

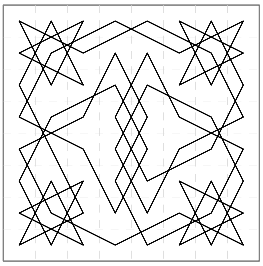
These open tours (Jelliss undated) showing approximate direct or oblique quaterary symmetry also show 16 symmetrically arranged conformals. A fully symmetric tour with this formation is not possible. Each example includes a 17th conformal which is not part of the pattern. Falkener 1892 gave an example anticipating these results, but not so symmetric and with one end on a central cell.

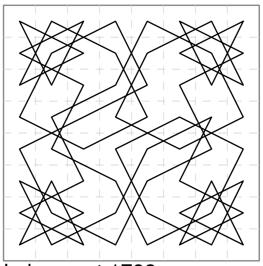
  

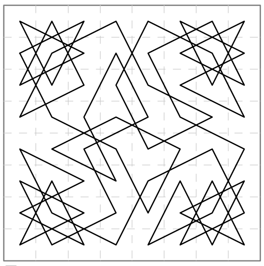
Stars in Tours

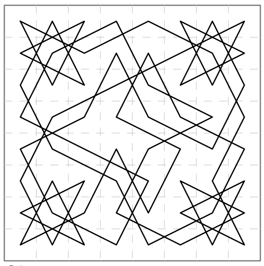
If we alternate acute angles of the two types we get a circuit of 8 moves in a 3×3 area. In a tour we cannot complete the circuit but have to stop one move short. These 7-move formations are usually called ‘stars’, though the Indian literature refers to them as ‘ponds’. The tour by Mani (before 1350) has a star in the centre, and the symmetric tour by Nilakantha (1640) has stars in opposite corners.

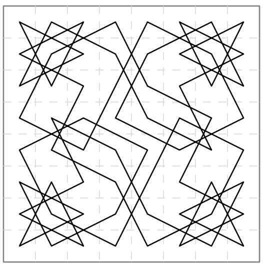
The task of showing a star in each corner has been a popular problem. We begin with some closed tour examples of this. The earliest two are from Laisement (1782), with others by Egan (1820), Slyvons (1856), Jaenisch (1862) who gives a symmetric solution, and Kraitchik (1927) derived from an open tour by Mme Parmentier.

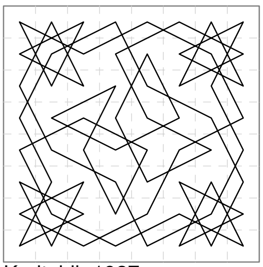

Laisement 1782

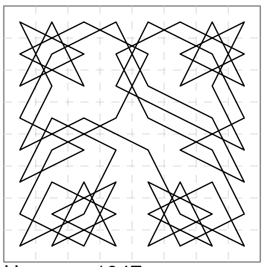

Laisement 1782

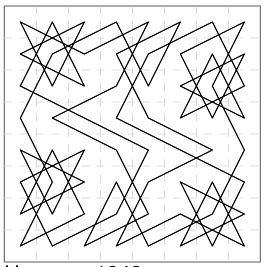

Egan 1820


Slyvons 1856


Jaenisch 1862 sym

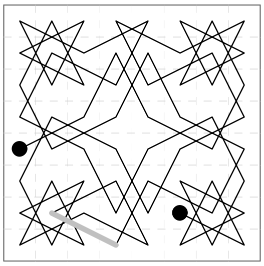

Kraitchik 1927

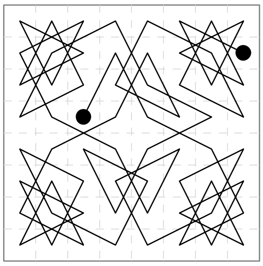

Hansson 1947

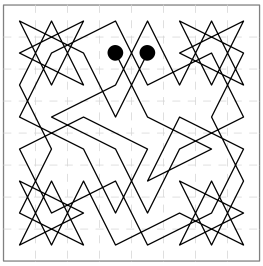

Hansson 1948 sym

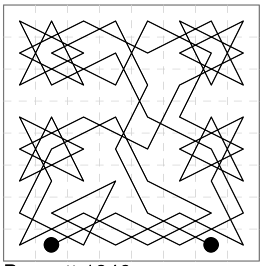
E. W. Bennett (*Fairy Chess Review*, October 1947, ¶7462) rediscovered the Slyvons tour (or one very like it) and raised the related problem of forming a tour with the four stars in other positions on the board. This led to a considerable amount of work by solvers, as reported by T. R. Dawson two years later (*FCR* November 1949). Two by Frans Hansson are shown above, one symmetric.

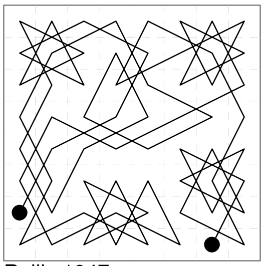
We now show some open tour examples. The first is from Slyvons (1856).

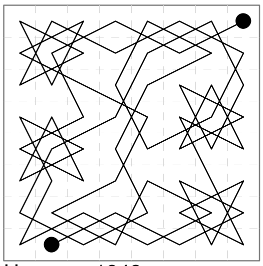

Slyvons 1856

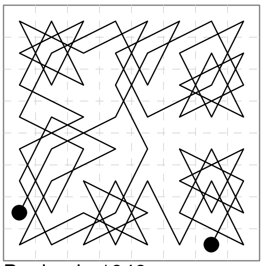

German c.1860

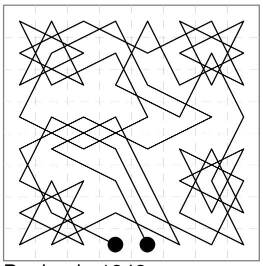

Falkener 1892


Bennett 1946


Reilly 1947


Hansson 1948


Benjamin 1948


Benjamin 1948

The second tour above is the solution to two cryptotours (#8 and #11) in a German newspaper cutting (c.1860) but one end is too central, spoiling the symmetry. The third is from Falkener (1892). The examples from *FCR* by Bennett, Reilly, Hansson and Benjamin have the stars in other positions.

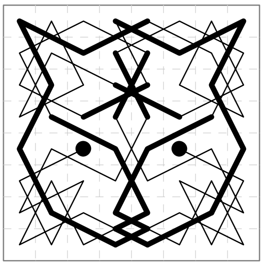
Much more work was done on this topic by the *FCR* solvers. Nineteen tours, all with the stars in different formations are mentioned, but only the positions of the star centres are given, as follows: b2,b7,g2,g7 (Slyvons); b2,b7,g2,f7; b2,b7,g2,f6; b2,b7,g2,e7; b2,b7,g2,e6; b2,b5,g2,g5; b2,g2,c7,f7; b2,b6,g2,e7; b2,b5,g2,f5; b2,b5,g2,f7; b2,b5,g2,g6; b2,b6,g3,g7; b2,b5,g4,g7; b2,b6,g4,g7; b2,g4,g7,c5; b2,g4,g7,c7; b2,g4,g7,d5; b2,g4,g7,d6; b2,c7,f7,g4. (*FCR* Nov 1949, Vol. 6 p.106, 116, 124; Vol. 7 p.85-6, 98, 157; Vol. 8, p.16.)

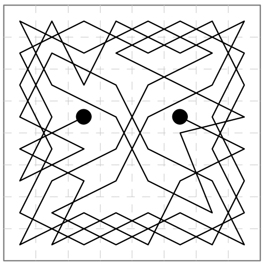
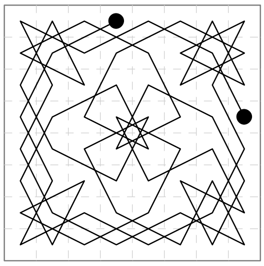
More stars in the corners will be found in the study of tours from Octonary pseudotours in ⌘ 8.

Graphic Tours

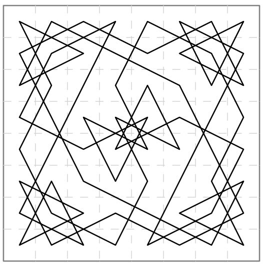
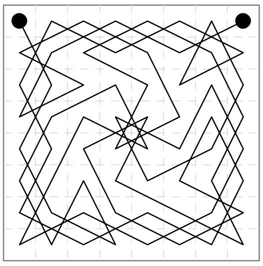
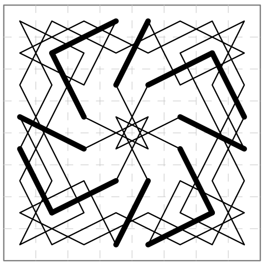
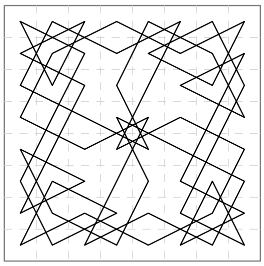
Pictorial

Probably the finest picture tours ever composed are the first two here, though they depend in part on partial moves. The cat portrait is by Mme Diane A of Orleans ¶616 (25 Oct 1878) *Le Siècle*. The emphasised lines are not essential. The asterisk on the cat's forehead is a complete quadrangle. The horses head depicting a chess knight is by P. C. Taylor (*British Chess Magazine* vol 57, 1937, p.620 ¶4546 Solution vol 58, 1938 p.93). “The problem - to find the knight! - A real chess picture puzzle”. The third which I call a mask is by Mlle Caroline Fox ¶394 (1 Feb 1878) in *Le Siècle*. The fourth, a flower. is from Falkener 1892 (#6 p.348).

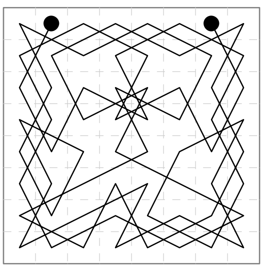


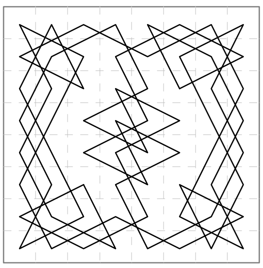



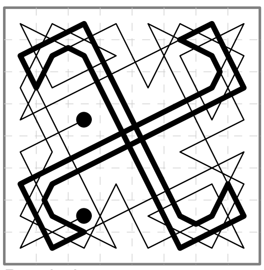
The other designs shown in this section are more abstract. Tours with the (pp) centre are popular for depicting rotating pieces of engineering like windmills or turbine fans. The first two here are ¶280 (21 Sep 1877) and ¶304 (19 Oct 1877) from the Feisthamel column in *Le Siècle*. The third is a tour with mixed quaternary symmetry by Murray (1942), with the 12 oblique moves marked (one of the 49 of type 156). The windmill design (Jelliss 1986) is after Pearson 1907 who gave an open tour. I have since noted that 538 (26 Jul 1878) in *Le Siècle* is similar, but a pseudotour.

More examples. The first is by A. Béligne *Le Siècle* 28 Sep 1877 ¶286.. Second *Le Siècle* 15 Feb 1878 ¶406. Many tours exhibit crosses in the centre. See p.475 for two by Ernest Bergholt.


Le Siecle 28 Sep 1877

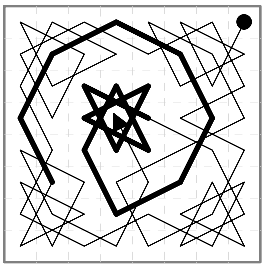
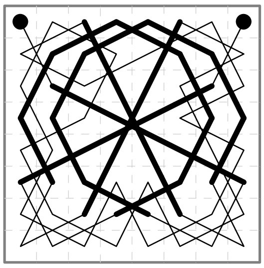

Le Siecle 15 Feb 1878


Bergholt 1918

The tour depicting a swastika is by Ernest Bergholt and appeared in the magazine *Queen* (5 Jan 1918) and wished readers ‘Good Luck!’ The darkened moves and part-moves are emphasised in the diagram. A handwritten note above the tour, among Murrray's notes, states: “Victor Kaefer type. The numbers in each pair of conjugate cells differ by 16”.

This was composed when the swastika was still an old pagan good luck symbol and did not have the adverse associations it has since acquired.

The next three are my own work. ‘Salute to Halley’s Comet’ composed upon it being visited by the Giotto probe (Jelliss *The Problemist* vol.12 #9 May 1986 p.159) showing obtuse and acute octagons. ‘Water Wheel’ showing two octagons, composed about the same date, but not published.

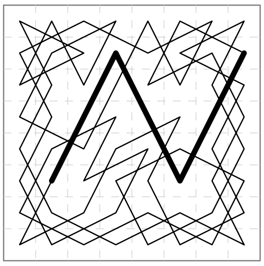
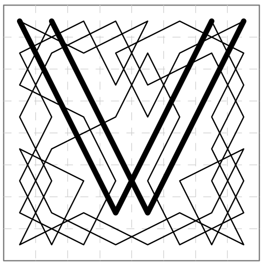
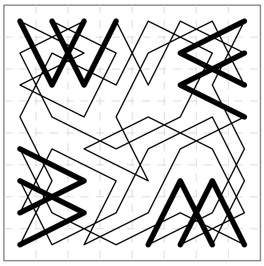
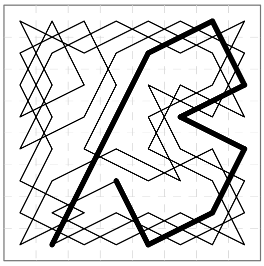




‘Starburst’ using the angle sequence 132313231... was published in *Figured Tours* (1997). When numbered it has eight successive odd or even numbers aroud the central point d4. See p.259 for a fuller star on a 9×9 board.

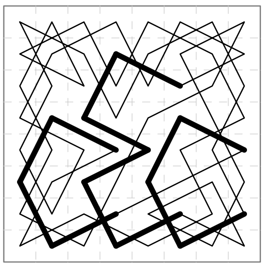
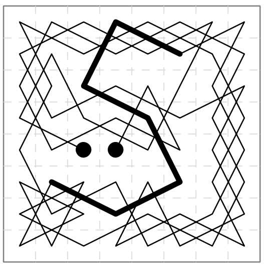
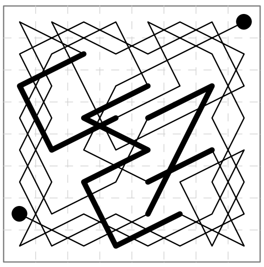
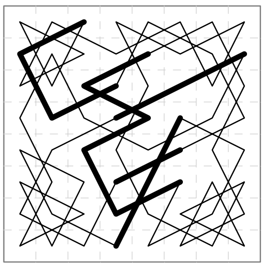
Monogram

A persistent tradition is the construction of tours in which certain of the move lines are emphasised to supposedly delineate the initials of the dedicatee or to celebrate some anniversary. I say ‘supposedly’ because many of these constructions seem to depend unduly on the artistic licence admitted to the draughtsman in selecting the lines that are to stand out, or in curving the knight's moves to make recognition of the symbols possible. But perhaps I may be accused of a lack of imagination or of artistic sensitivity.

We begin with some 19th century examples. Slyvons (1856) has these three tours, one showing N (which he used as a frontispiece), a doubled V (with two smaller ones between), and a pattern that can be regarded as showing WEBM in the corners. The tour showing a letter ‘B’ is by A. L. Maczuski in *Boy's Own Paper* 1883.

The next set of diagrams show monogram tours from the early 20th century. Alexander Fraser *British Chess Magazine* vol.31 (Dec 1911) p.465 gave monogram tours showing MD and ECC (shown here) for Edinburgh Chess Club. He also mentions earlier examples in *Strand* magazine.

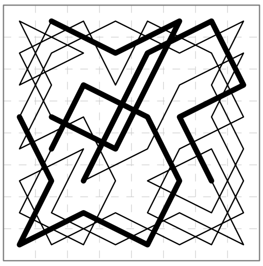
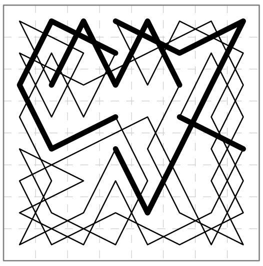
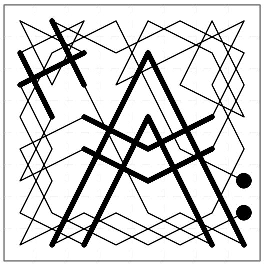
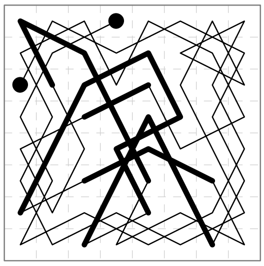





He had further examples in *BCM* vol.44 (1924) p.173, showing NM (for N. Munro). The *Chess Amateur* reports a tourney for ‘S’ compositions, arranged by the Scheveningen Chess Society in 1922 on its 10th anniversary, won by T. R. Dawson with a simple ‘S’ monogram.

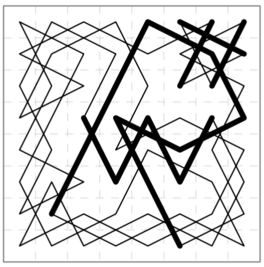
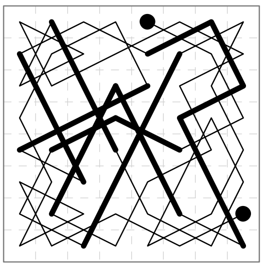
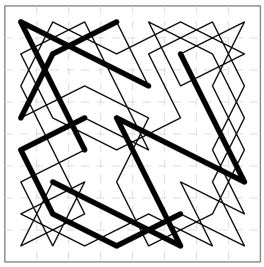
In *Cahiers de l’Echiquier Francaise* 1928 ‘Le Problème du Cavalier’ (p.166-168) Dawson gave an historical account of tours showing letters (citing some of those above) with an open ‘CEF’ monogram tour p.167. Another ‘CEF’ monogram, in a closed tour by H. Rohr p.388 is similar. The tours diagrammed above depend on curving the lines to form script style letters. The straight line forms are unconvincing. There is a footnote that the style of writing F differs in England and France.

The distinctive tours that follow mainly show letters that are intended to honour particular dedicatees. A flurry of these monogram tours appeared in the pages of the *Fairy Chess Review* 1932-1939. P. C. Taylor in particular seems to have set off the trend, having presented a whole series at a lecture to the British Chess Problem Society on 26 Feb 1932. These subsequently appeared in *FCR*. The dedicatees were regular contributors to the magazine. This series continued in Dawson's column in the *British Chess Magazine* during the war, and resumed in *FCR* in 1945.

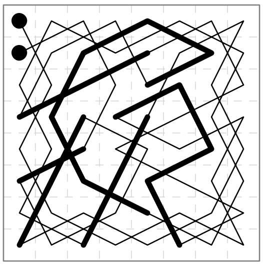
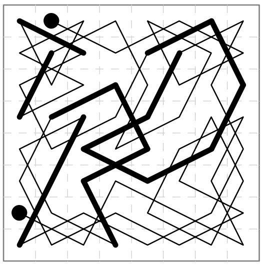
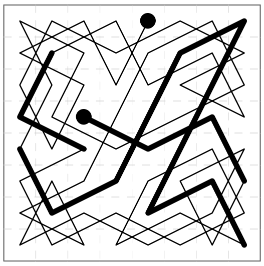
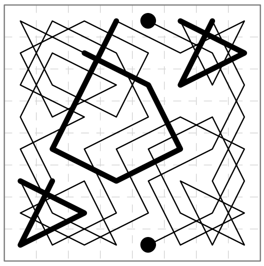
The following four monograms are by P. C. Taylor, apparently from his 1932 lecture to the BCPS and dedicated to various of its members. *Fairy Chess Review* (vol.3 #11 Apr 1938 p.120 ¶3181) for T. R. Dawson. (vol.3 #12 Jun 1938 p.130 ¶3252) for C. M. Fox. (vol.3 #13 Aug 1938 p.141 ¶3325) for H. A. Adamson (the H specified used the lines b7-e2 and d8-f4 but I much prefer the small H a7-b5, b8-c6, a6-c7). (vol.3.#14 Oct 1938 p.150 ¶3389) for F. R. Adcock. I also have a note of HAA monograms by P. C. Taylor in *BCM* (vol.62, 1942, p.168, sol p.218, ¶5819) and (p.272, sol p.26 1945, ¶6548) but they appear to be repetitions of this 1938 tour.

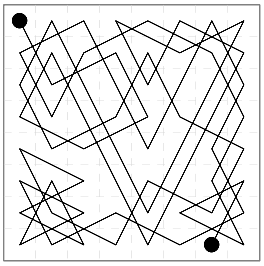
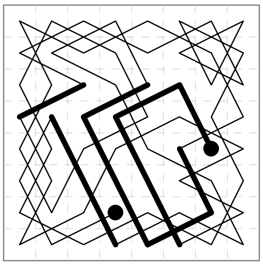
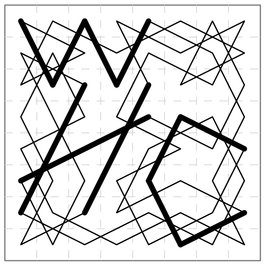
Three more by P. C. Taylor *Fairy Chess Review* (vol.3 #15 Dec 1938 p.159 ¶3466), for W. H. Reilly. (#16 Feb 1939 p.175 ¶3590), for H. A. Russell. (#18 Jun 1939 p.199 ¶3764), for A. C. White (the text specifies C from c8 to e7, but the lower one is clearer).

Four more: A. Lapierre of Wattrelos (*FCR* vol.3 #16 Feb 1939 p.175 ¶3591-2), for Fairy Chess Review and T. R. Dawson. (vol.3 #18 Jun 1939 p.199 ¶3762), for AL (the text specifies the path from c7 to c1 as a script L, but this path bears no resemblance to an L in my experience, though the A is amusing). The three D's are by Miss Dorothy R. Dawson *FCR* vol.4, #3 Dec 1939 p.43 ¶4131).

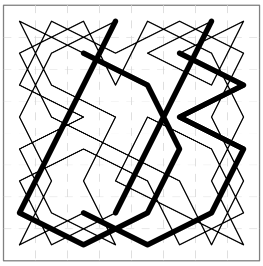
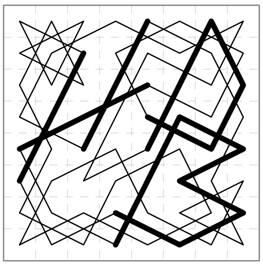
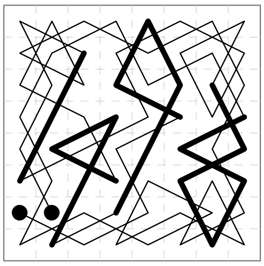
Marking the end of the war T. G. Pollard gave a tour delineating a Victory Vee (5 V's in fact) in *FCR* (vol.6 #9 Dec 1946 p.65, sol #10 Feb 1947 p.74 in text, ¶7094).

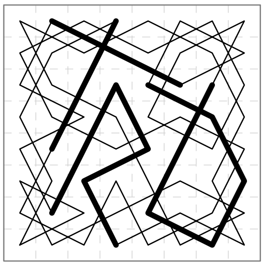
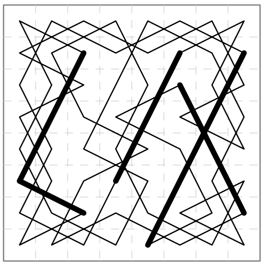
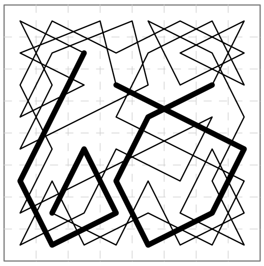
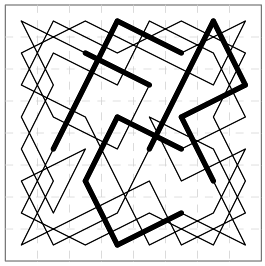
There followed a series of mutual dedications, exemplified by a W. H. Cozens tour (*FCR* vol.6 #5 Apr 1946 p.33 ¶6768) dedicated to TGP and a tour by T. G. Pollard (*FCR* Oct 1945 p.7, misnumbered 147, sol Dec p.18 ¶6540) dedicated to WHC.

Others by P. C. Taylor in *BCM* not shown here are: (vol 63, 1943 p.216, sol p.261, ¶6147) showing PCT and (vol.64, 1944 p.24, sol p.70, ¶6280) showing QED. One in *BCM* by T. G. Pollard (vol.65 p.79, sol p.129, ¶6649) a TRD monogram, and two by T. R. Dawson (vol.64 1944 p220/269) MMMM tour and (vol.65 1945 p.155 ¶6754 sol p.201) FWM monogram for F. W. Markwick.

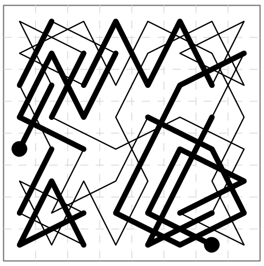
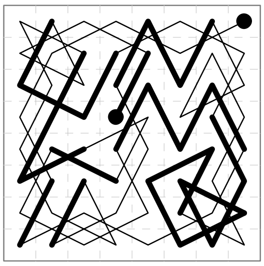
A final flurry of monograms tours in FCR: T. R. Dawson (*FCR* vol.6 #7 Aug 1946 p.46 ¶6880-6881, sol #8 Oct 1946 p.57), tours showing DB (for Mrs Daisy Benjamin) and HDB (for H. D. Benjamin). The second upright of the H could be taken as c3-e7, or the small H at a6-b8, b4-c7, a7-c6. A tour by W. H. Reilly (*FCR* vol.7 #2 Oct 1948 p.13, misnumbered 105, sol #3 Dec 1948 p.22 in text, ¶7868): Complete a knight tour delineating 1948 (looks more like 1498 to me!)

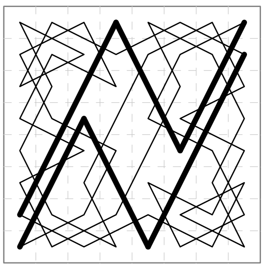
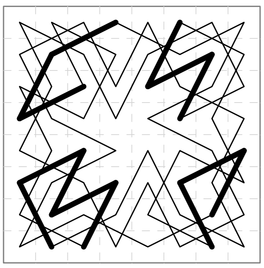
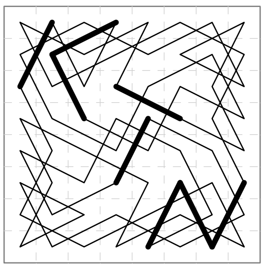
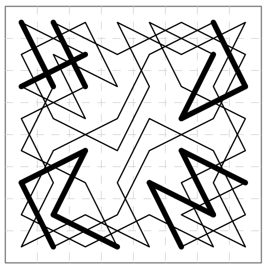
H. D. Benjamin and Mrs Daisy Benjamin. Monogram tours in *Fairy Chess Review* (vol.6 #8 Oct 1946 p.56 ¶7005 by DB and HDB, sol #9 Dec 1946 p.66) showing TRD. Or the small T of a8-c7, b7-a5. (vol.7 #4 Feb 1949 p.27 ¶7975 by HDB, solution #5 Apr 1949 p.37) showing LIX for Dawson's 59th birthday. (#9 Nov 1949 p.86 ¶8550 by DB, sol #14 Oct 1950 p.122) showing 60. This tour is shown as a series of move directions, based on Dawson's mode notation. The series ended with H. D. Benjamin's particularly nice ‘FCR’ (#14 Oct 1950, in obituary p.128, ¶8850 by HDB composed Jun 1947, sol #16 Feb 1951 p.140).

E. W. Bennett (*FCR* 1948 ¶7715 and ¶7778). tried to elaborate on the monogram tours by showing longer words, but this was stretching the idea too far, allowing multiple solutions. ¶7715: “On the line-drawn tour read a word of 10 letters describing Fairy Chess, all letters in order left to right and down the board. You do not use a3-b1-c3-d1-e3 which comes in by chance AND SO is left out.” Solution INIMITABLE as shown. [But RJF finds DELIGHTFUL, taking part only of full knight moves.] ¶7778: “What possibilities are there in Fairy Chess?” Solution UNLIMITED as shown. [Dawson wrote: The license I allowed ... is too much of a good thing. Infinite, endless, many, lots, all you want, are all easily formed this way. By limiting all letters to full S moves the intended word is not so obvious. One might even try for inexhaustible, titani(a)c or phantasmagorical says AWB!]

Finally some original monogram tours of my own composed 1991. Unlike earlier examples they are all symmetric. The first is ‘N’ for Nightrider, in nightrider moves. In the others the lettering is asymmetric: ‘BCPS’ for the British Chess Problem Society and ‘KTN’ for Knights Tour Notes (composed 16 August 1991). Finally ‘HJRM’ for H. J. R. Murray (composed 25 December 1991).

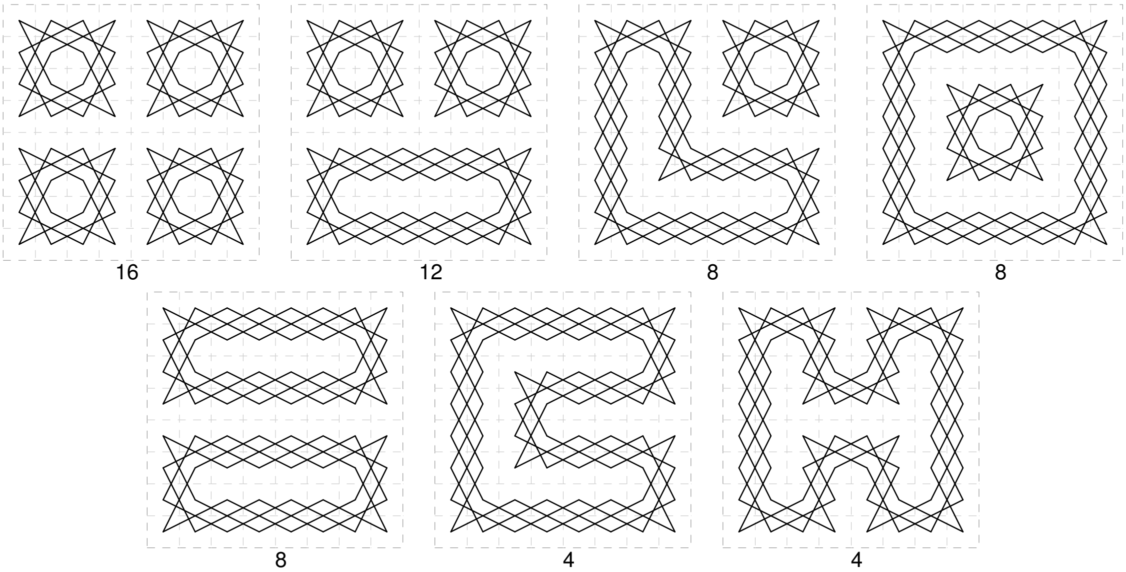
Magic tour (12g) by Jolivald 1882 is also a symmetric tour showing a striking N or Z, when suitably oriented.

Crosspatch Patterns & Compartmental Tours

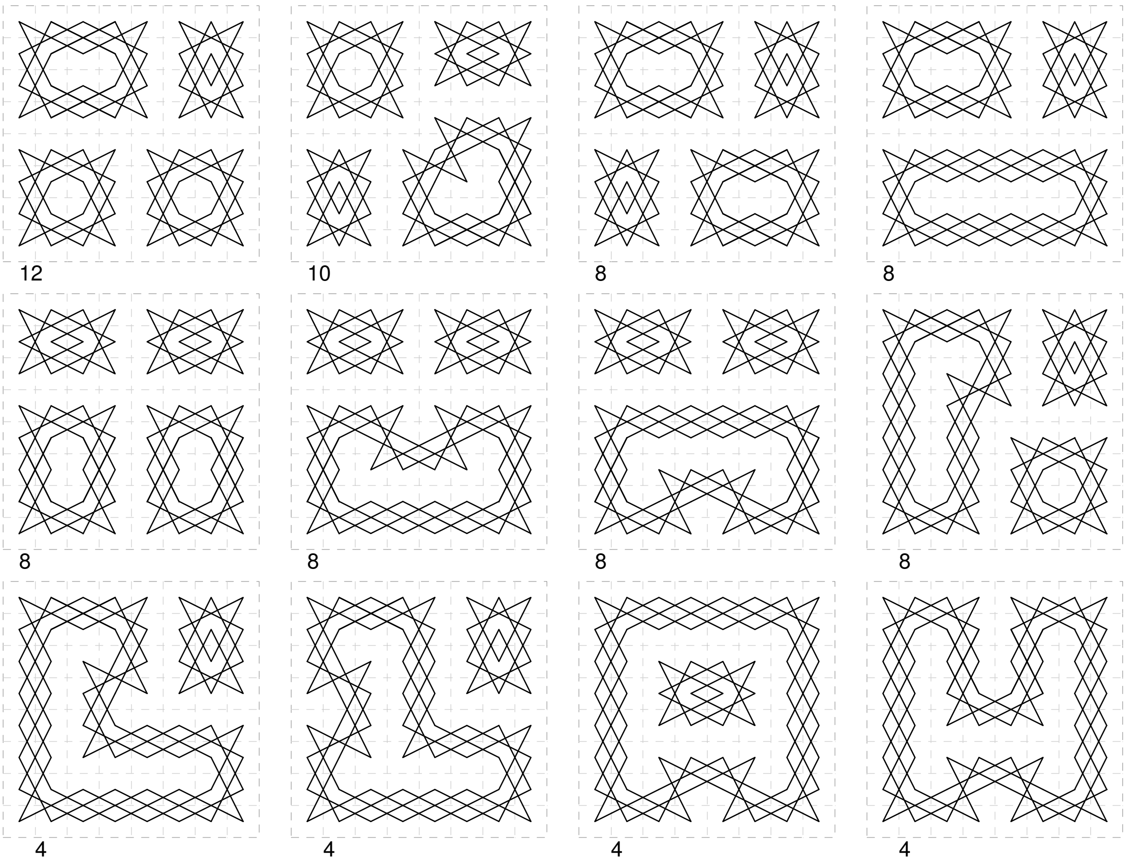
An interesting group of knight pseudotours are those that have the property that every move is centrally crossed by another knight move. I call them **crosspatches** (*Chessics* #20 Winter 1984).

There are 19 geometrically distinct crosspatch patterns on the 8×8 board.

There are 7 solely of ‘straits’



and 12 contain ‘slants’. The 3×4 pseudotour is a component in 11.



This enumeration implies that no actual knight tour, of one circuit, is possible in which each move is centrally crossed by another. A rigorous mathematical proof of this, for any rectangular board, was recently published by Nikolai Beluhov (2013).

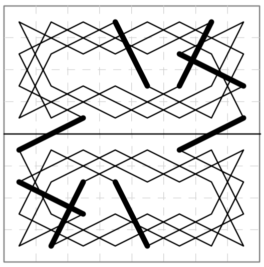
An obvious way to construct tours on larger boards is by joining together tours on smaller boards, we call such tours **compartmental**. Strictly speaking, in this definition, a tour is only compartmental if removing the links between the compartments leaves a complete open tour in each of the parts. If one part has a complete tour but the other does not (for example it may consist of two separate paths) it can be called **semi-compartmental**, but it is difficult to separate the cases. A further sub-division in a compartment may make one or both partial tours semi-compartmental. Many tours derived from the crosspatches will be of this type.

Tours from the Strait Crosspatches

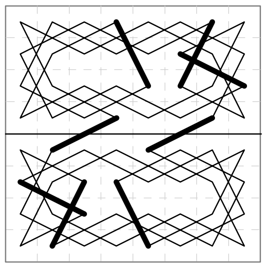
The 'strait' cases correspond to the ways of covering the 'reduced' 4×4 board (where each 2×2 block is represented by a single cell) with wazir-move circuits.

The two octonary crosspatch patterns, namely the Collinian (or Annular) and the Squares-and-Diamonds types lead to numerous tours by simple linking which we study in detail in later sections, but first we look at tours derived f rom the other crosspatches.

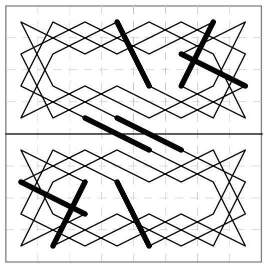
From the pattern consisting of two 4×8 boards filled with edge-hugging circuits, nine symmetric tours can be formed by simple linking. Three of double half-board type were given by Euler (1759).



Euler 1759 cc=

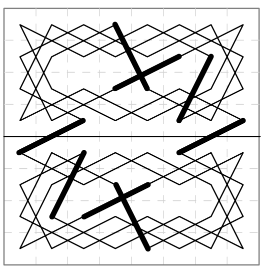


Euler 1759 cg=

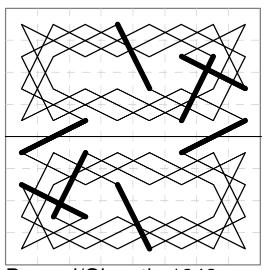


Euler 1759 cd=

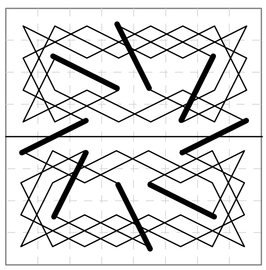
Three others of double half-board type are possible. One was given by von Sachsen-Gotha (1797). The half-board tour published by Gianutio (1597) can be joined to a copy of itself on upper and lower halves of the 8×8 board to form a symmetric tour. This seems to have been first noted by L. Perenyi (1842). However D. E. Knuth points out that the tour is in fact diagrammed with ends at the top of an 8×8 board, so the possibility of forming a symmetric double half-board tour was evidently missed by Gianutio himself. The sixth solution, found by Bergholt (1918), has the slants in a regular rotary pattern. It also shows mixed quaternary symmetry, type (156) see ⌘ 7.



Von Sachsen-Gotha 1797

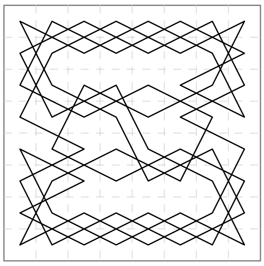
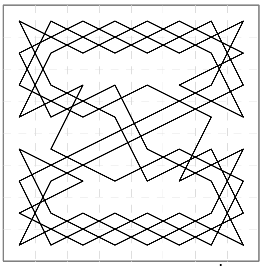
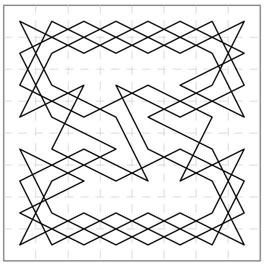


Perenyi/Gianutio 1842 cc=



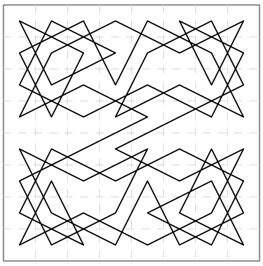
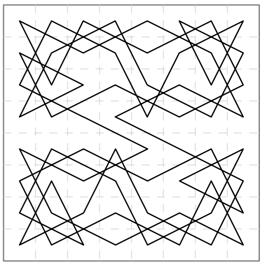
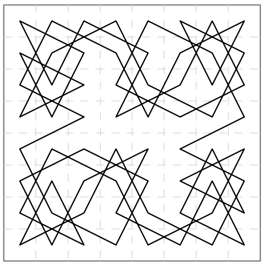
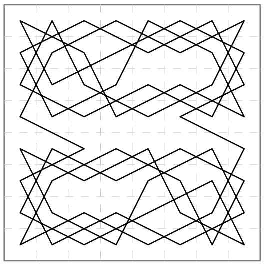
Bergholt 1918 cc=

The other three cases are not of double halfboard type. They have centre angles (ce=), (de=), (eg=), with six slant links crossing the horizontal median, and one slant in each half.

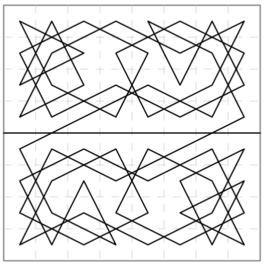
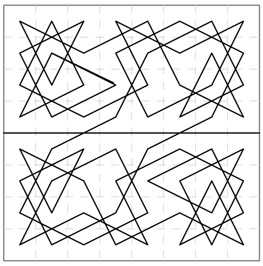
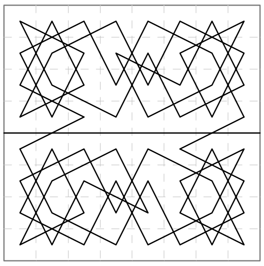
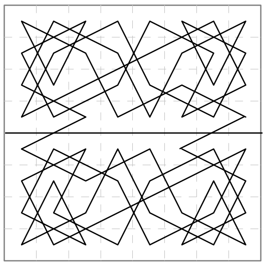
Jelliss 1985 ce= Jelliss 1985 de= Jelliss 1985 eg=

Like the Gianutio tour three other mediaeval 4×8 tours (see ⌘ 4) can also be joined to copies of themselves to form a symmetric fullboard tour. One half of the Suli double half-board tour, the Somesvara variant of Rudrata, and the Florentine half-board tour.

Suli doubled Somesvara doubled Florentine doubled Hone 1832

One from Hone (1832) probably by Walker is also shown above. Here are diagrams of some later symmetric double halfboard tours, some of which have partial magic properties, two from Carl Wenzelides (1849). Two from Jaenisch (1862 §93 p.40-45).

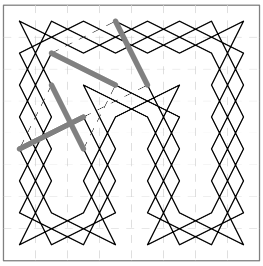
bb= bh bb= fg=

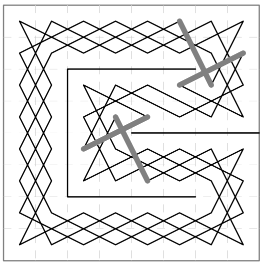
We return to this subject and the enumeration of double half-board tours in later sections.

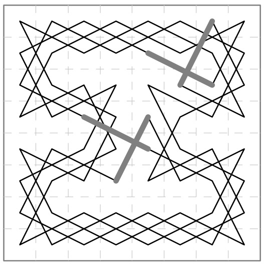
In *Chessics* I reported having enumerated all geometrically different tours derivable from the C and H pseudotours by simple linking, finding 41 from C and 15 from H.

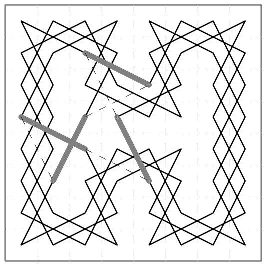
Recently I found that a solution of the C pattern was given by Denis Bailliere de Laisement (1782). I show it below in the original orientation. His diagram clearly shows the linkage polygon of deleted and inserted moves. My own C solution, which I sent out on a 1985 New Year Card, and was published in the January 1985 issue of the *Problemist*, is the only one of the 41 solutions that keeps within the C shape. In fact it is the only tour, formed by any method, that will keep within the C-shaped track.

The H pseudotour is #53 in Harikrisha (1871) and in Naidu (1922) but without a related tour. A closed tour given by J. B. D—. in the *Leisure Hour* (1873) is a simple linking of the H-pattern (oriented on its side). His other six tours are also related to the H-pattern and two of the open tours have the minimum of three links. There is no tour that will keep within the H-shaped track.

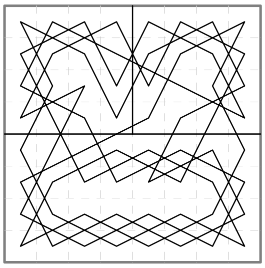

Laisement 1782

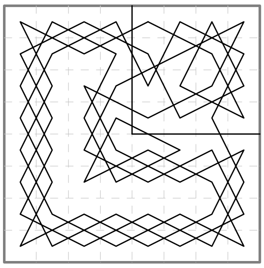

Jellliss 1985


J. B. D. 1873


Jelliss 1985

My solution given in *Chessics* (fourth diagram here) is the only one that has a symmetric linkage polygon. This leaves two other strait crosspatches to consider (Jelliss 1985):

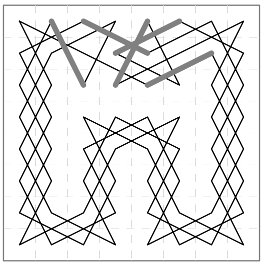


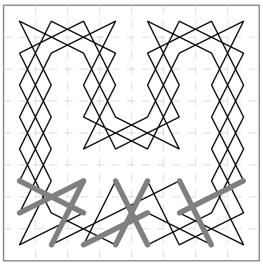


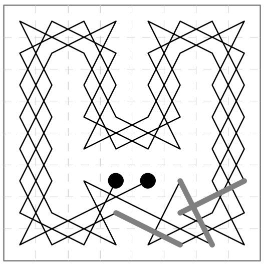
Tours from the Slant Crosspatches

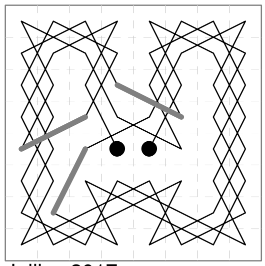
In *Chessics* 21 Spring 1985 I gave one example tour derived from each of the crosspatch patterns by simple linking, except for one case, the H with shifted crossbar, which necessitates one further deletion in order to join the pieces into a closed tour.

A tour of this type was given by Pearson (1907) under the title ‘Marble Arch’. A similar one in Cretaine (1865) uses 8 deletions and insertions. Open tours with only four deletions are possible.

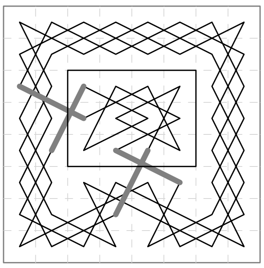

Pearson 1907

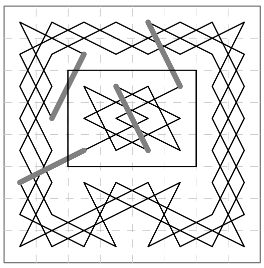

Cretaine 1865


Jelliss 2017

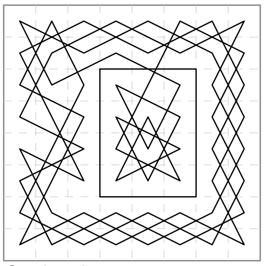

Jelliss 2017

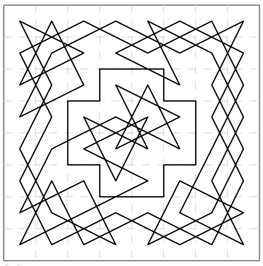
The crosspatch pattern with an internal 3×4 can be linked with only four deletions and insertions as shown in my solutions here. The 3×4 area can be a tour as in the second of these.

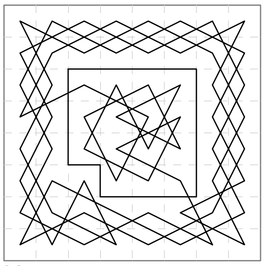

Jelliss 1985

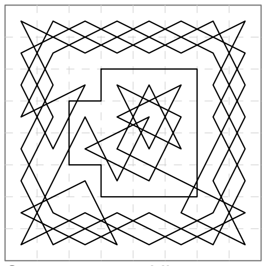

Jelliss 2017

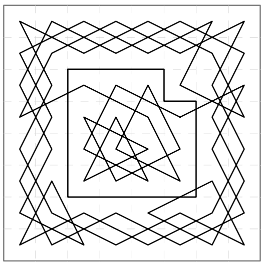
Many other tours have been constructed having one compartment inside another. Examples follow. The earliest, by Giuseppe Gasbarri (1836) incorporates an internal 3×4 tour. Two more examples come from the Rajah of Mysore (1871) his #66 has the Euler cross at centre and his #68 has a 15-cell partial 4×4 tour at centre. The next two examples are tours used by Staunton for the cryptotours VI and X published in his column in 1871 and 1872. The open tour with cross-shaped centre by Diane A of Orleans ¶670 in *Le Siécle* 27 Dec 1878, a stop-press addition, shows approximate axial symmetry.

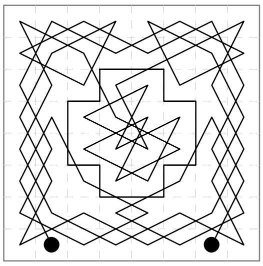

Gasbarri 1836


Mysore 1871 #66


Mysore 1871 #68

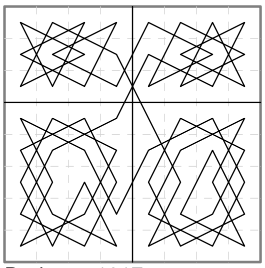

Staunton 1871 VI

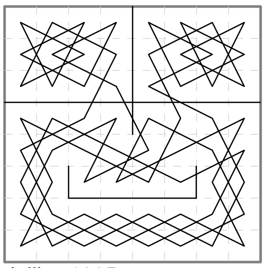

Staunton 1872 X

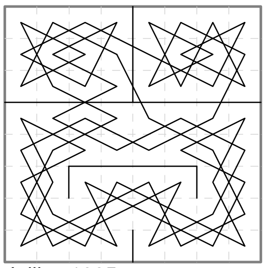

Diane A 1878

See also the examples with maximum border braid in the Angles section p.30.

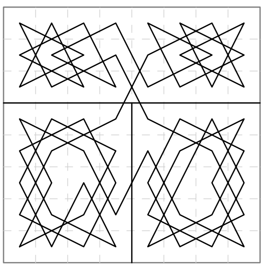
The other three axial crosspatch patterns have two 3×4 components. A tour that solves one case was shown by Dudeney (1917). The other two diagrams are my own solutions (Jelliss 1985).

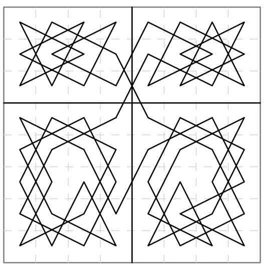

Dudeney 1917


Jelliss 1985

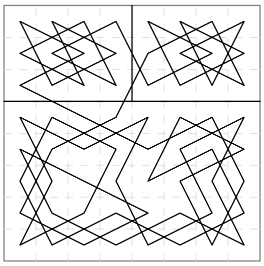

Jelliss 1985

Other compartmental tours of this type can be found in the literature, though they usually use more deviation from the crosspatches. Here is a selection. Murrray cites the first as by Hirsch Silberschmidt in *Das Gambit* 1829. Another is from Slyvons (1856).

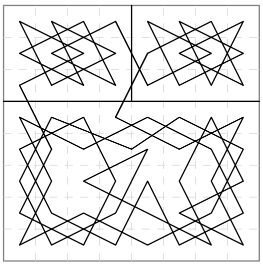

Silberschmidt 1829


Slyvons 1856

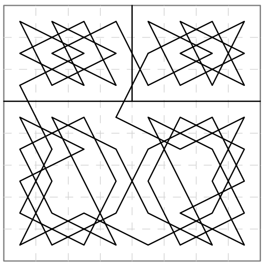
In the other three below (which I have inverted) all have the same 3×8 section, but the 5×8 is less structured. One by the Rajah of Mysore (his #65), one from Kraitchik (1927) who also gave two similar but open tours, and one from Murray 1942. They all have an AP of difference 3 along the third rank down when numbered from any cell in the lower part.



Mysore < 1871



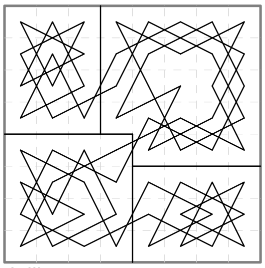
Kraitchik 1927



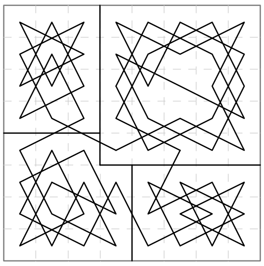
Murray 1942

Other less structured compartmental tours of the 3×8 + 5×8 type were shown by Willis (1821), Hoffmann (1893), and Murray (1942).

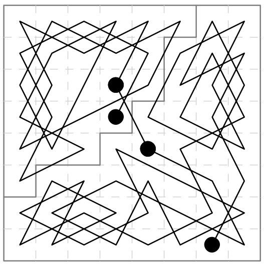
One of the slant crosspatch patterns has a diagonal axis of symmetry. Here is my solution from *Chessics* (Jelliss 1985). The second diagram is a tour by Slyvons (1856) of si,ilar structure. One of the components is a 15-cell partial 4×4 tour. The open tour from *Le Siecle* (20 Dec 1878 ¶664) divides the board into two near-triangular compartments of 24 and 40 cells.



Jelliss 1985



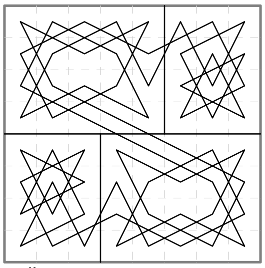
Slyvons 1856



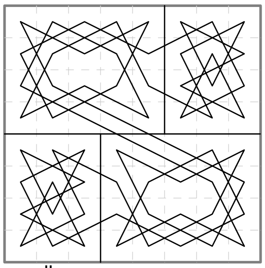
Le Siecle 20 Dec 1878

For other tours with approximate diagonal symmetry see ⌘ 7.

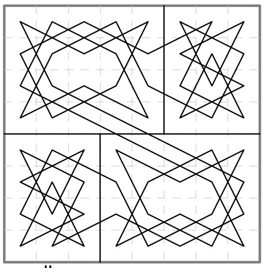
The centrosymmetric slant crosspatch generates 13 symmetric tours by simple linking, of which eight are of double half-board type. In three the 4×8 can be further divided into 3×4 and 5×4.



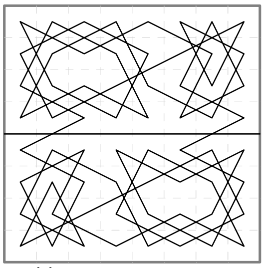
dl~



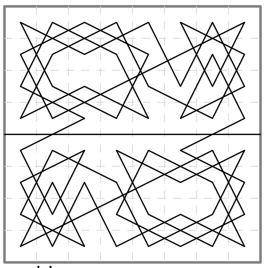
dl~



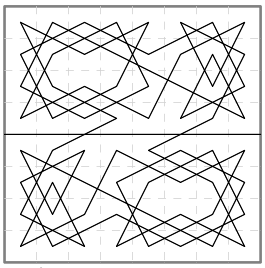
dl~



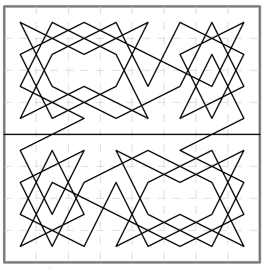
bl~



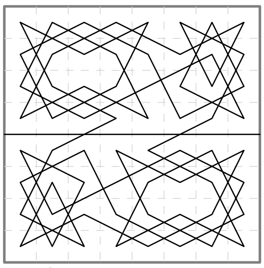
bl~



fg~

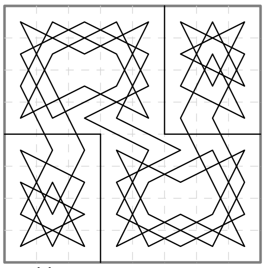


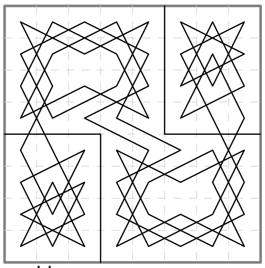
cl~

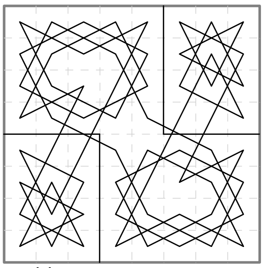


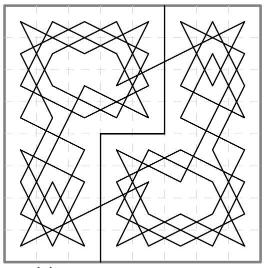
gl~

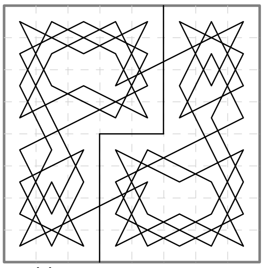
Here are the five remaining symmetric tours formed by simple linking from the centrosymmetric slant crosspatch. They all have shaped half-board components. As noted in the History pages Sachsen-Gotha (1797) gave an open tour based on this pattern.


kl=

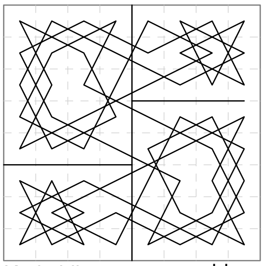

kl=

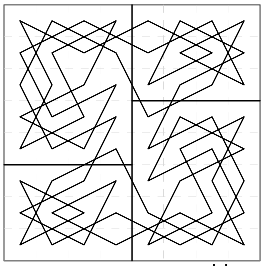

bh


bl~

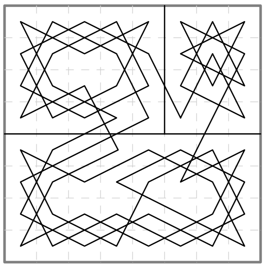

bl~

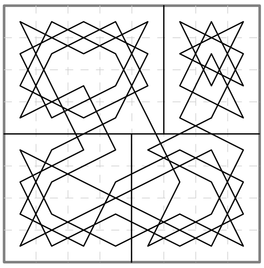
Two examples of similar structure from Maurice Kraitchik 1927.

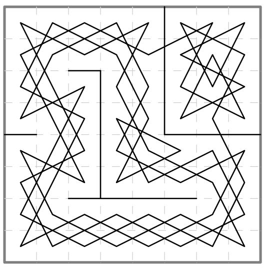

Kraitchik 1927 bi~

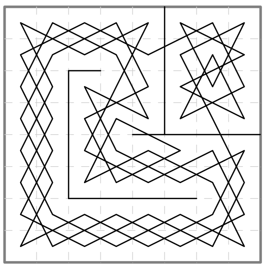

Kraitchik 1927 bl=

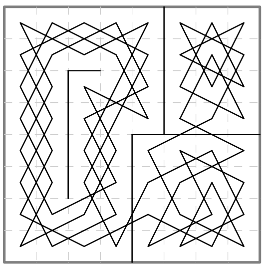
Here are tours formed by simple linking from the five remaining slant crosspatches. Three with L-shaped components (Jelliss 1985).











Some More Enumerations

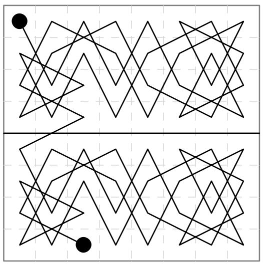
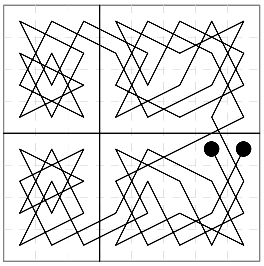
Enumeration of Double-Halfboard Tours

Here we look at ways two half-board tours 4×8 can be combined to form tours on the 8×8, and how to enumerate them. In the section on tours of 4-rank boards (⌘ 4) we report how Sainte-Marie (1877) correctly counted the tours on the half-chessboard. This he did by counting the half-tours on the white inner and black outer cells of the 4×8 board, finding 118 with inner end on the a file, 32 on the c file, 54 on the e file, 42 on the g file, from which the total of the full 4×8 tours is calculated as 118×32 + 32×54 + 54×42 = 3776 + 1728 + 2268 = **7772** geometrically distinct.

Single-Link Open Tours We can use the Sainte-Marie results to calculate the number of 8×8 tours formed from two 4×8 tours with a single join between the ends. There are three cases: when the joining move is a4-c5, b4-d5, c4-e5. Other cases are rotations or reflections of these.

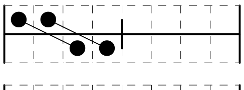
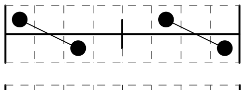
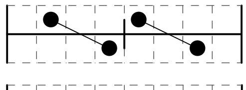
As shown in the note on 4-rank tours the numbers of 4×8 tours with an end at a4, b4, c4, d4 are respectively 7630, 2740, 2066, 3108 (adding to twice the above total). So the numbers of 8×8 tours joining a4-c5 is 7630×2066 = 15,763,580, and those b4-d5 are 2740×3108 = 8,515,920, and those c4-e5 are 2066×3108 = 6,421,128. Adding these three totals we get, for the number of geometrically distinct 8×8 tours formed of two 4×8 tours connected by a single link, the grand total: **30,700,628**.

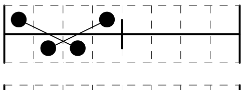
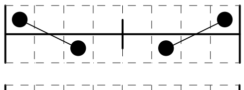
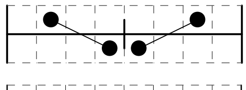
Tours of this type were called by M. Kraitchik (1927) ‘parcours bi-semiaréolaires’ {double halfboard tours} and he calculated their number as this figure multiplied by 4, that is 122,802,512, counting the number of different diagrams formed if the division line is horizontal. This number is stated incorrectly in Rouse Ball (1939) to be “the number of reentrant paths of a particular type”.

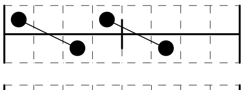
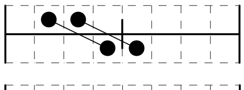
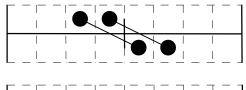



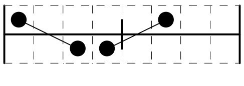
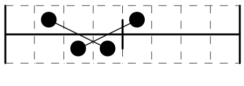
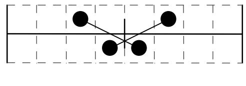
It should be noted that there are other tours that might be called double halfboard tours that are not included in the above enumeration. Above right is a semicompartmental example I composed.

Double Link Closed Tours Probably of more interest are the 8×8 tours formed by joining two 4×8 tours together by two links to make a closed tour. There are 12 geometrically distinct methods of joining. Note of course that the two ends of the 4×8 tour must be on opposite coloured cells, i.e. one in an odd-numbered file and the other even. We code the linkages according to the files used.

12,34  16,38  25,47 

14,23  18,36  27,45 

14,36  23,45  34,56 

16,34  25,34  36,45 

The numbers of 4×8 tours G(i,j) with ends in the i and j files can be calculated from the H(i,j) half-tour numbers (given in ⌘ 4 p.40). For example G(1,2) is the sum of products of the types H(1,1)·H(32) = 22·3 = 66 where the central link is 1-3 and H(1,2)·H(4,2) = 8·5 = 40 where the central link is 2-4 and so on for the twelve possible positions of the middle link.

The results of these calculations with both terminals in the same side are:

G(1,2) = 672, G(1,4) = 772, G(1,6) = 502, G(1,8) = 1872/2 = 936, G(2,3) = 180, G2,5) = 276, G(2,7) = 240/2 = 120, G(3,4) = 208, G(3,6) = 136/2 = 68, G(4,5) = 304/2 = 152. Total 3886.

The results for terminals in opposite sides are:

G(1,3) = 518, G(1,5) = 752, G(1,7) = 682, G(2,4) = 284, G(2,6) = 174, G(3,5) = 214, G(1,1) = 1860/2 = 930, G(2,2) = 232/2 = 116, G(3,3) = 134/2 = 67, G(4,4) = 298/2 = 149.Total 3886.

Using these numbers of geometrically distinct 4×8 tours with ends on given files, calculated as G(i,j) from the analysis of 4×8 tours, we find the following totals:

Linkage	Formula	Calculation	Totals
12,34	G(1,2)×G(3,4)	672×208	139,776
14,23	G(1,4)×G(2,3)	772×180	138,960
14,36	G(1,4)×2G(3,6)	772×136	104,992
16,34	G(1,6)×G(3,4)	502×208	104,416
16,38	[G(1,6)/2]×[G(1,6)+1]	251×503	126,253
18,36	G(1,8)×2G(3,6)	936×136	127,296
23,45	G(2,3)×G(4,5)	180×304	54,720
25,34	G(2,5)×G(3,4)	276×208	57,408
25,47	[G(2,5)/2]×[G(2,5)+1]	138×277	38,226
27,45	G(2,7)×2G(4,5)	120×304	36,480
34,56	[G(3,4)/2]×[G(3,4)+1]	104×209	21,736
36,45	G(3,6)×2G(4,5)	68×304	20,672
	total of closed double-half-board tours		**970,935**

Kraitchik gives the total of closed tours as 7,763,536 = 8×970,442, which is less than I have found, a difference of 493 (this appears to be related to the number of symmetric cases).

Symmetric Double-Halfboard Tours Among these closed tours we find from the centro-symmetric connections (16,38) and (25,47) and (34,56) respectively only 502 + 276 + 208 = **986** symmetric tours. This is a quarter of the total 3944 cited by Kraitchik so our figures agree here.

We can now calculate the number of tour diagrams, assuming the division line to be horizontal, since under this restriction each symmetric tour has two orientations and each asymmetric tour four orientations: T = 2.S + 4.(G–S) = 4.G – 2.S = 3,881,772.

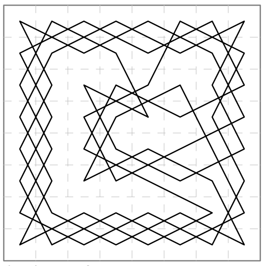
We can also calculate how many reentrant tours are included among the single-linked open tours. By deleting one of the two links from the closed tours, each symmetric tour gives one reentrant tour and each asymmetric tour gives two, hence: S + 2.(G – S) = 2.G – S = R **= 1,940,884**.

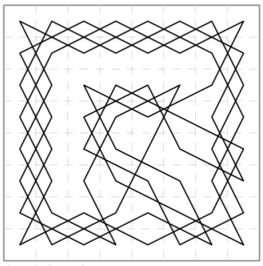
This figure, multiplied by 16 so as to count the tours in all 8 orientations and numbered from either end, i.e. 31,054,144, was given by **Emmanuel M. Laquière** (1881). Thus almost 1/16 of the double-half-board tours are reentrant.

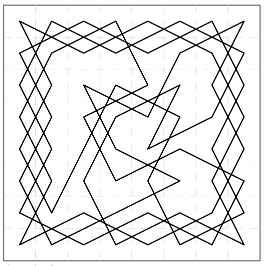
Collinian Tours

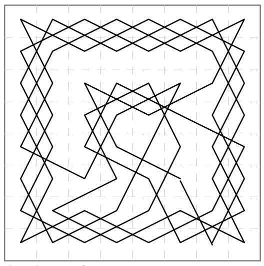
Cosimo Alessandro Collini (1727-1806) published an article over several issues of the *Journal Encyclopédique* in 1772 that included one tour, and the next year published a 60-page book *Solution du Problème du Cavalier au Jeu des Echecs* including 28 figures 20 of which are tours (Tables 5, 7–9, 11, 13–21, 23–28). Part I of the book prescribes the initial square, Part II the final square, Part III a closed tour, and Part IV different start and finish squares. Table (5) in the book is the tour shown in the 1772 Journal article. He also published an account in Italian in 1774. (See the Bibliography for further details).

Collini's method is to form tours from the pattern of eight knight circuits centred on the board, consisting of the squares and diamonds in the central 4×4 surrounded by a border braid. He seeks to construct tours beginning and ending at any given cells of opposite colour, by joining them up with minimal deletion of links. All tours produced by this method will be asymmetric. Of the 20 tours only four are reentrant (11, 13, 15, 25). These all use the minmum number of eight deletions and insertions. In the original articles by Collini the tours are given in tabular numerical form, but we show them graphically here.

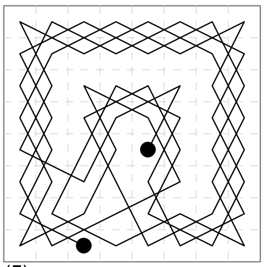

(11) e5-d7

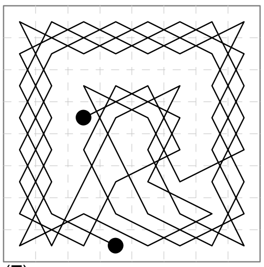

(13) h1-f2

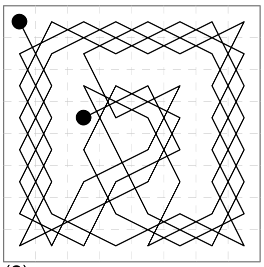

(15) f6-e4

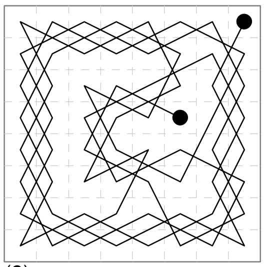

(25) g1-f3

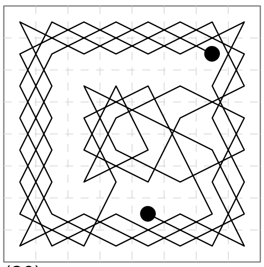
Of the open tours 8 use the minimal number of 8 deletions and 7 insertions. Only in one tour (20) does he connect two inner circuits in succession; the others all alternate outer and inner circuits.

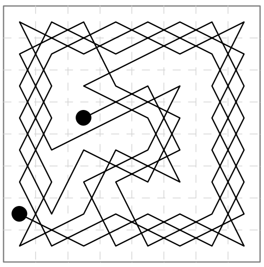

(5)

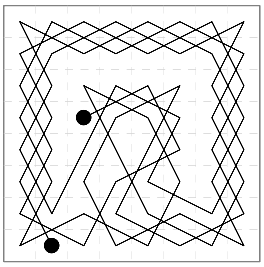

(7)

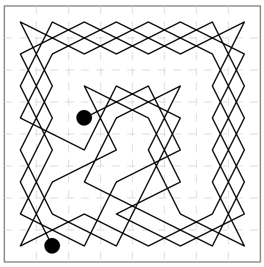

(8)


(9)

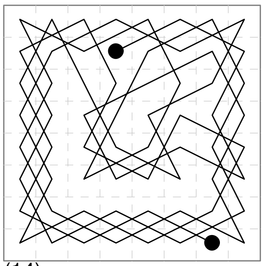

(20)

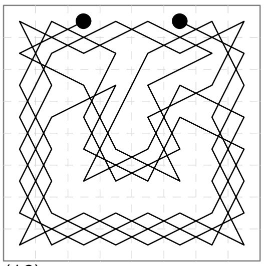

(24)

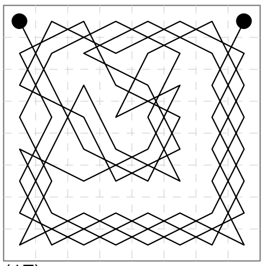

(26)

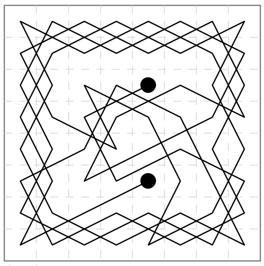

(27)

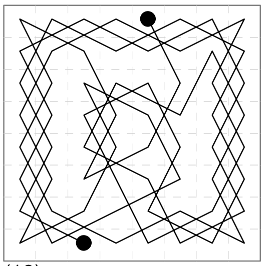
To solve the problem when the start and finish squares are on the same circuit and not a knight move apart takes more than eight deletions, as in his example (14). The corner-to-corner tour (17) is one where he makes more deletions than necessary.

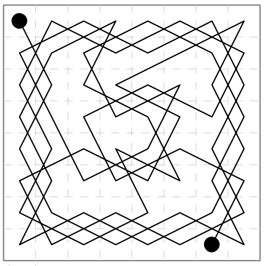

(14)

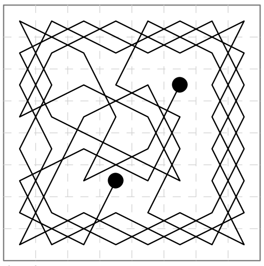

(16)

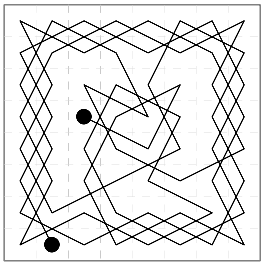

(17)


(18)

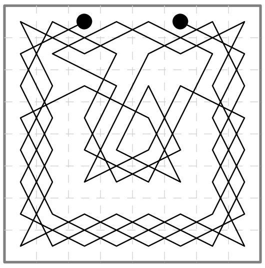

(19)

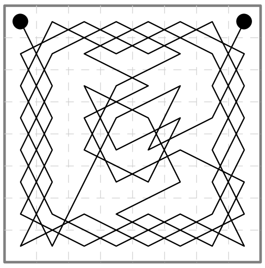

(21)

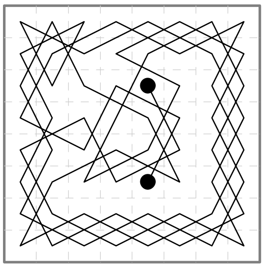

(23)

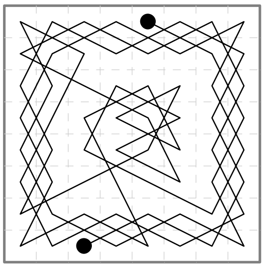

(28)

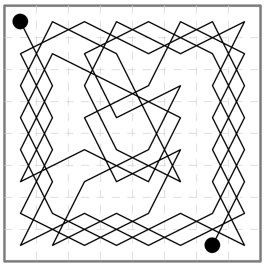
The following Collinian open tours are my own work apart from two. The first seven tours are improvements on Collini's examples (16, 17, 18, 19, 21, 23, 28) using fewer deletions.

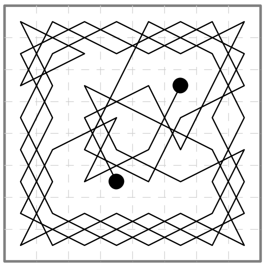


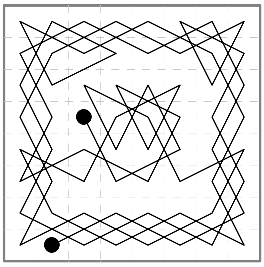


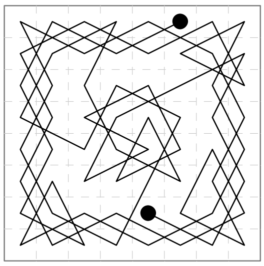


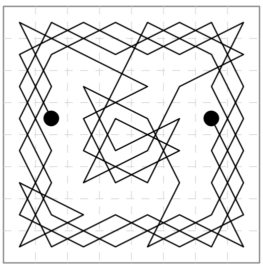


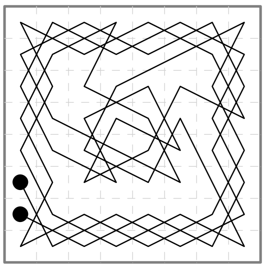


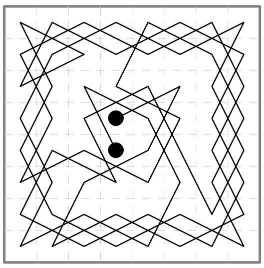


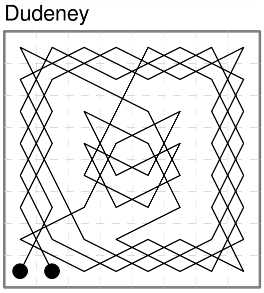



Dudeney


J. W. Brown 1919

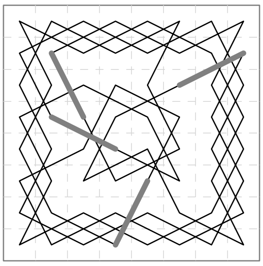


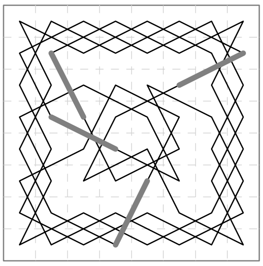


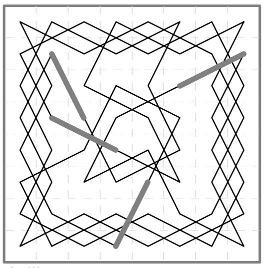


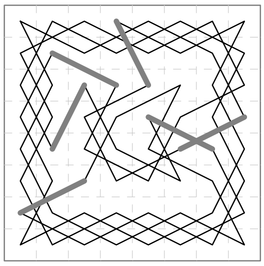
The tour by H. E. Dudeney was supplied for the Eschwege (1896) book. The John W. Brown tour is in *Notes & Queries* and was constructed apparently without being aware of the previous work of Collini. The three with ends on adjacent cells are cases not shown by Collini.

We now show some closed Collinian tours.


Chapais 1780


Merklein 1863


Jelliss 1970


Frost 1876

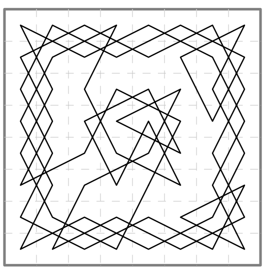
The first three diagrams are almost identical The Chapais (1780) example, is the same as one in Laquière 1880. The Mercklein tour (1863) was shown as four separate paths spread over four boards. My own tour is one of the earliest tours that I constructed (recorded as Mar 1970, though not published until *Chessics* #21 Spring 1985). I have reflected the Chapais tour, and rotated my own so that the slants are in the same positions in each tour. These all use alternating inner and outer circuits (type IOIOIOIO) as do many of Collini's examples, and are also Rogetian, using only four slants, which form straight two-move lines where they meet the inner circuits. (See also my example with approximate diagonal symmetry in ⌘ 7). The fourth diagram which has six slants is used by A. H. Frost (1876) to construct a 12×12 tour by adding a border.

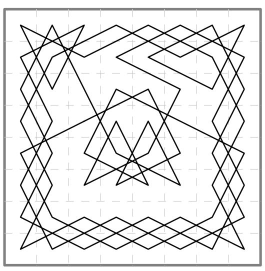
I have made some attempt at enumerating closed tours of Collini type, by dividing them into classes according to the sequence in which the inner and outer circuits are visited.

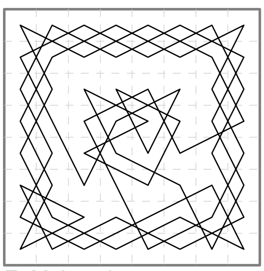
THEOREM: To show all possible sequences of inner and outer circuits requires 33 open tours, or 7 closed. *Proof:* (a) The number of arrangements of 4 Is and 4 Os in sequence is 8!/4!4! = 70, but from this we must remove the 8 that contain IIII or OOOO which are impossible in tours. There are thus 62 possible sequences of 4 Is and 4 Os in simply linked open tours. (b) We must now take account of symmetry. The number of arrangements of 2 Is and 2 Os in one half of the tour is 4!/2!2! = 6, but two of these are the impossible OOII-IIOO and IIOO-OOII. Thus of the 62 sequences only four are symmetric; 58 are asymmetric. (c) The reversal of a tour has the reverse ‘IO’ code, so to show all possible codes we need to construct 58/2 + 4 = 33 open tours. (d) For closed tours we need only count the number of different cyclic arrangements of 4 Is and 4 Os which is 8, from which we exclude the impossible case IIIIOOOO, giving 7. QED

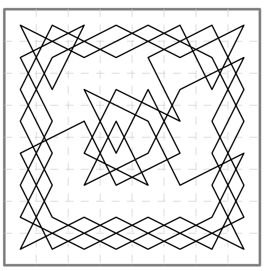
Here is a list of the seven cyclic sequences. The first number in brackets after the code is the number of non-cyclic arrangements formed by breaking the cycle or reversing its sequence, or both (these total 62), the second is the number of derivable open cases not counting reversals as different (total 33): IIIOOOIO (16, 8); OOOIIOII (8, 4); IIIOOIOO (8, 4) IIOOIIOO (4, 3); IIOOIOIO (16, 8); IIOIOOIO (8, 5); IOIOIOIO (2, 1).

The first diagram below is one of 24 tours of type IIIOOOIO. The second tour is one of the 16 of type OOOIIOII, and has an axially symmetric centre. The third tour, by E. M. Laquière (1880) is one of 16 of type IIIOOIOO. A closed tour in which each circuit is linked to one of the same type must be of pattern IIOOIIOO. The fourth diagram shows one of 16 such tours.



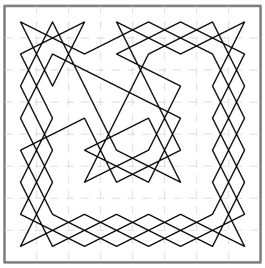
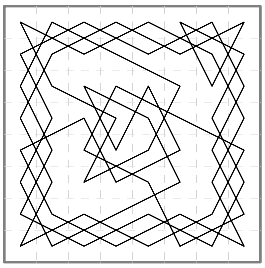
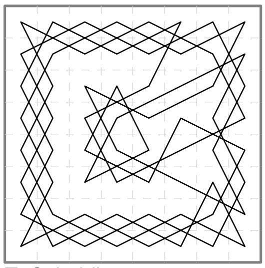



E. M. Laquiere 1880



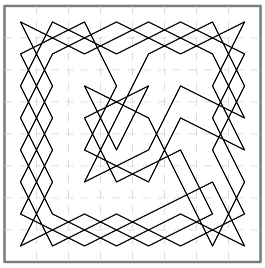
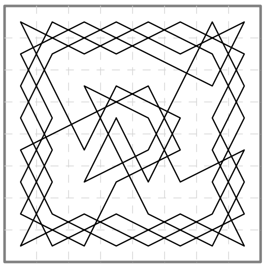
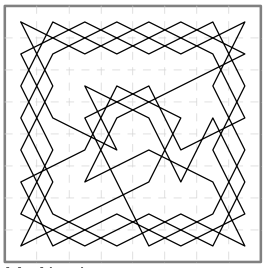
There are considerably more tours in the remaining three classes than in those above. The enumeration remains to be completed.

Here are three of type IIOIOOIO. The last diagram is from T. Scheidius *Sissa* 1850.

T. Scheidius 1850

and two of type IIOOIOIO, and another of type IOIOIOIO sent to me some years ago by Michael Abraham. Many examples of this alternating type were shown above.

M. Abraham

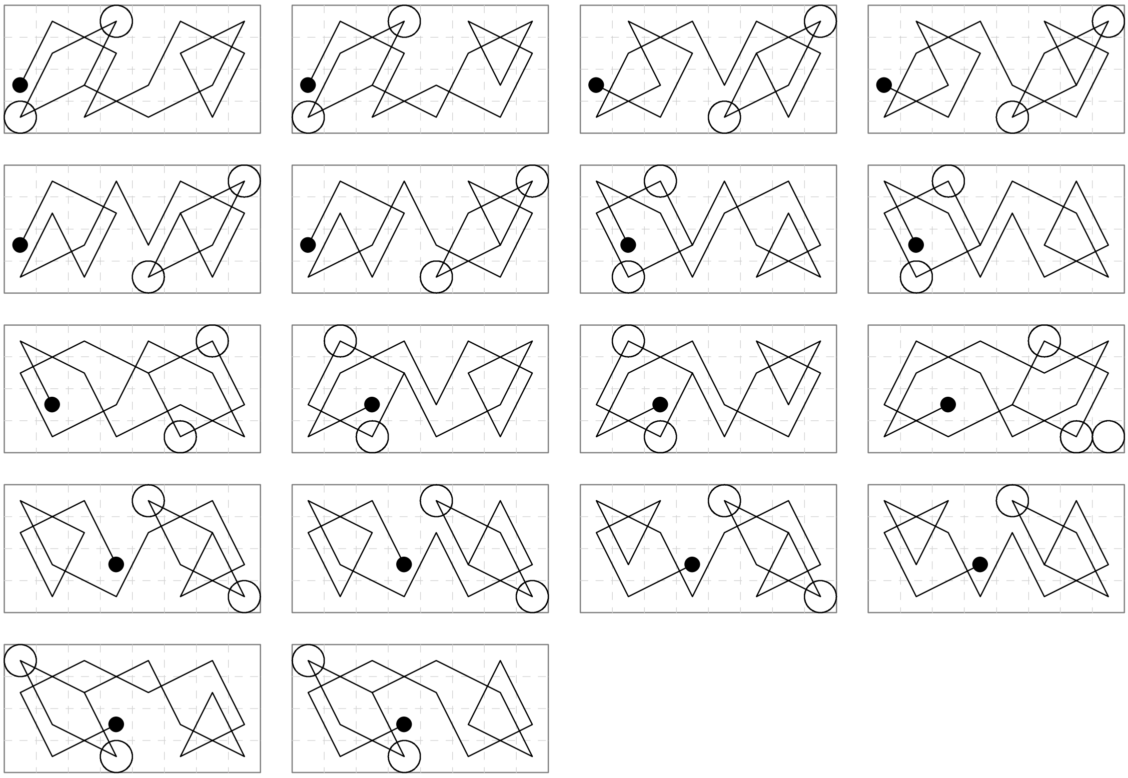
By the nature of Collini's method none of the tours it produces can be symmetric. However, symmetric tours can be produced by ‘double linking’. We can derive four symmetric doubly linked tours from any singly linked closed tour by using the same 16-move linkage polygon rotated to the other side of the board (i.e by a half-turn) and then joining up the pseudotour thus formed (consisting of two 32-move circuits) by reversing one of the central pairs of linkages. This is similar to the process of Symmetrization. Some distinctive tours found by this method are those with centres (no=), (no~), (fi~), (cl~) in my collection of symmetric tours showing central angles (see ⌘ 7).

Symmetric Squares and Diamonds Tours

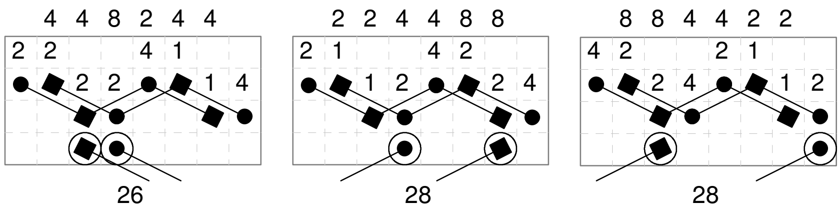
We now turn to the other octonary pseudotour that, like Collini's circuits, produces a large number of tours by simple linking, namely the pattern of squares and diamonds. The enumeration of all possible tours of this rhombic type is an unsolved problem. Here we attempt an enumeration of the squares and diamonds symmetric tours of double half-board type.

Double-Halfboard Rhombic Tours

There are 36 partial half-board tours of squares and diamonds type: Internal end on: a-file 12, b-file 6, c-file 6, d-file 12, External ends in top or bottom rank, shown here as two to a panel.



To form a 4×8 half-board tour from two of these requires a single link. The internal link in the upper half-board tour can be any of the six slants marked with a round spot at one end and a square spot at the other in the following chart. There is then one or more 15-move paths connecting each end of this link to the similarly marked circled cell in the bottom rank. The number of paths available (1, 2 or 4) is shown above the spot.



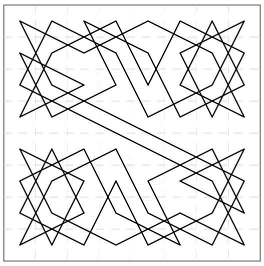
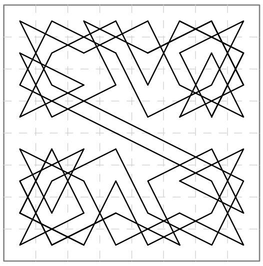
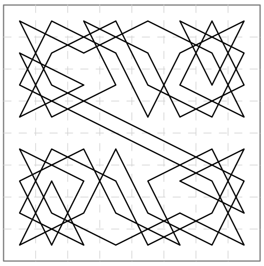
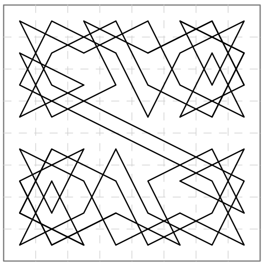
The number of tours using the link (2, 4 or 8) is the product of these two numbers and is shown at the top, above the link. These add to 26 in the first case and to 28 in the other two cases, totalling 26 + 28 + 28 = **82 tours**. I am reasonably sure that this is the correct total.

There are three symmetric linkages between the half-boards, Narrow, Medium and Wide, according as the links are separated by 1, 3 or 5 units.

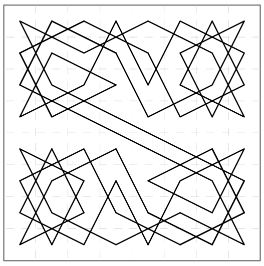
It seems somewhat anomalous that the links in the Narrow case slope down to the right while the wider cases slope in the opposite direction. A tour with half-board linkage a-c, b-d, c-e can of course be reflected left to right to show respectively linkages f-h, e-g, d-f instead, but sloping the other way, but the links between the two half-boards are also reflected and slope the other way.

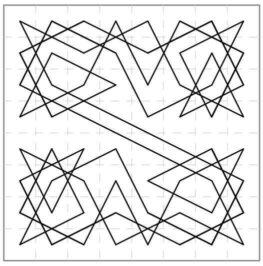
Diagrams of all these tours follow. Within each panel the tours are grouped according to central angles, and secondarily according to the angles at b7 and g7. N = Narrow, M = Medium, W = Wide.

N: 4 a-c

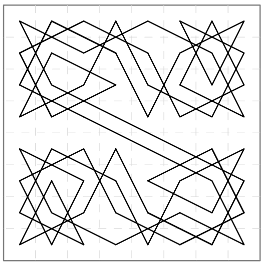





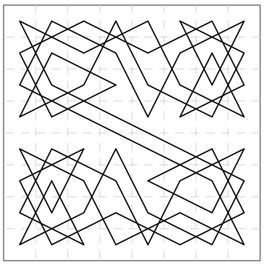
N: 4 b-d





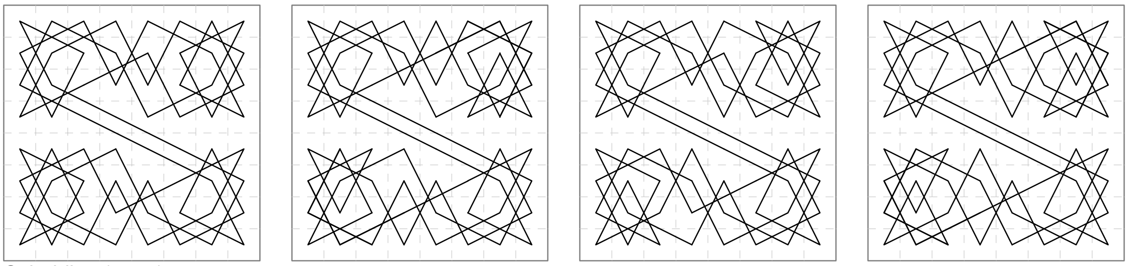
Wenzelides (1849)



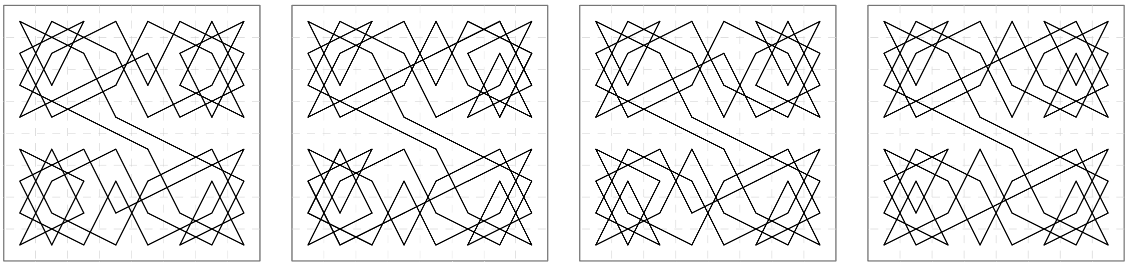


Wenzelides (1849)

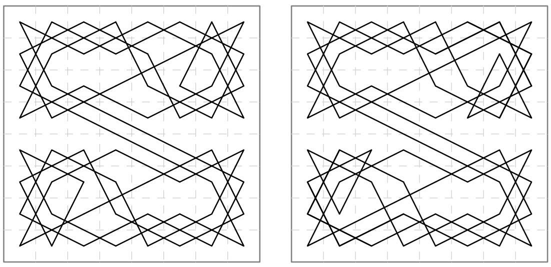
N: 8 c-e



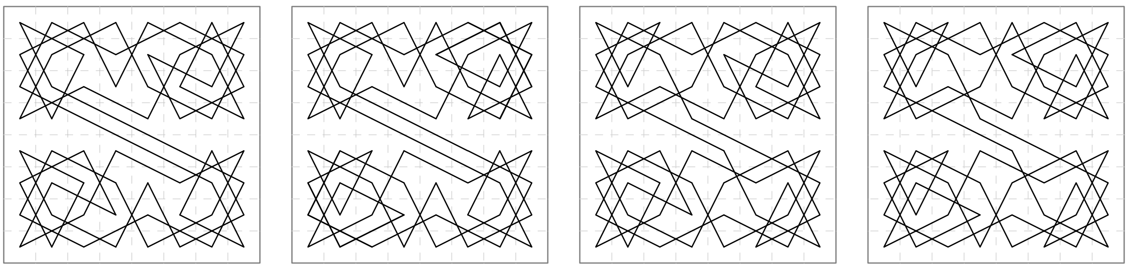
Scheidius (1850)



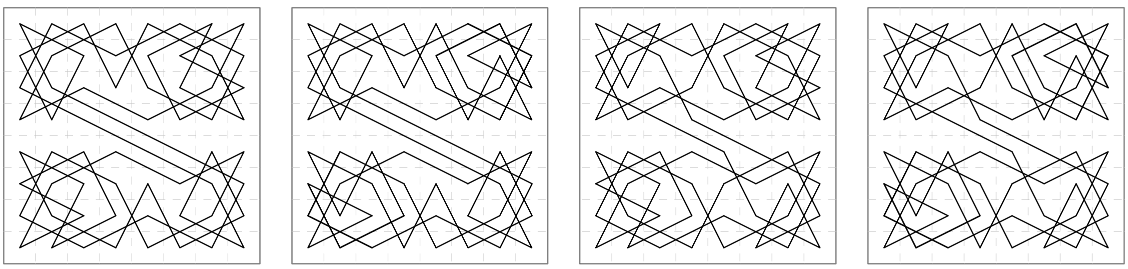
N: 2 d-f



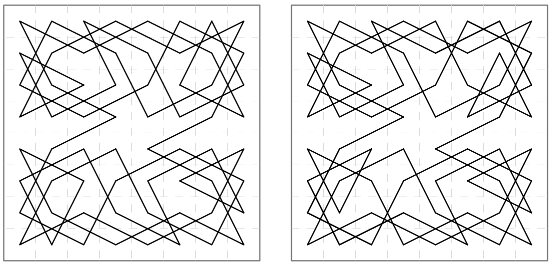
N: 4 e-g



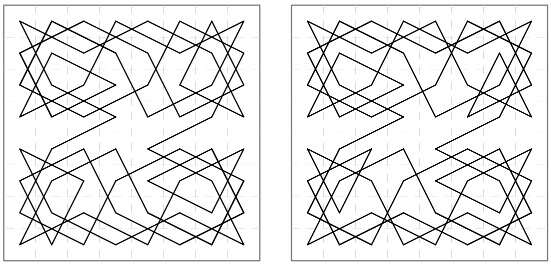
N: 4 f-h



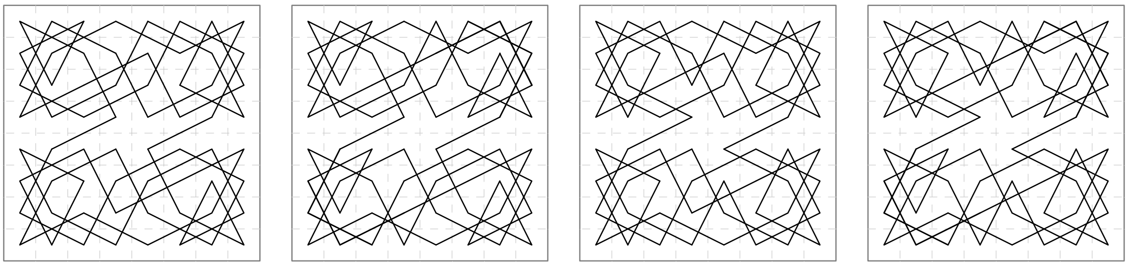
M: 2 a-c



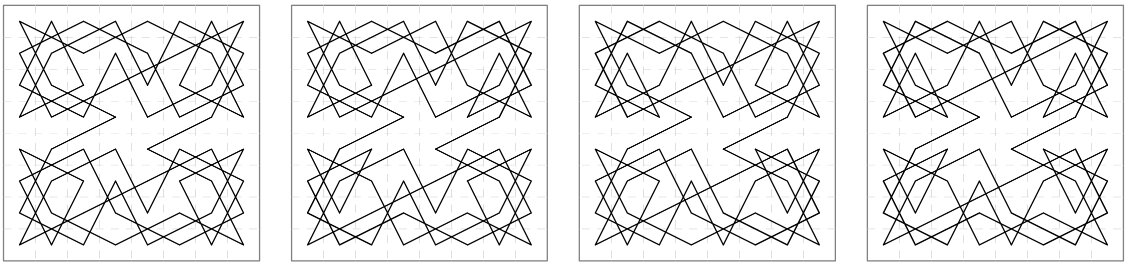
M: 2 b-d



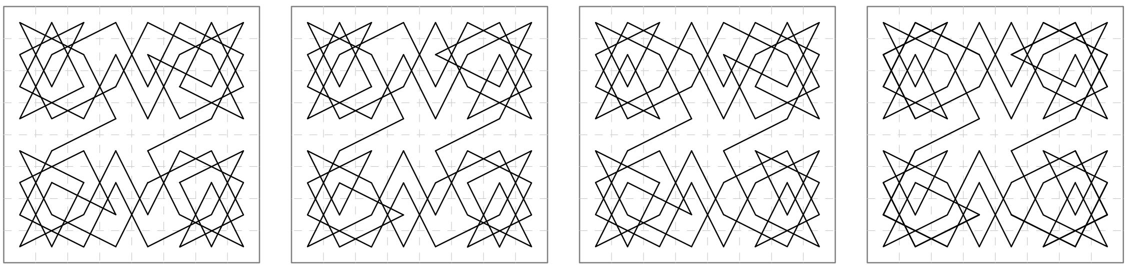
M: 4 c-e



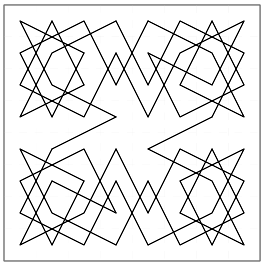
M: 4 d-f

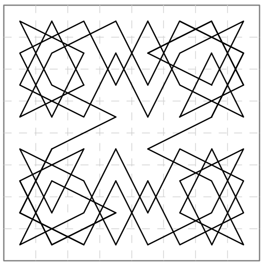


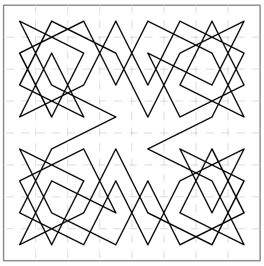
M: 8 e-g.

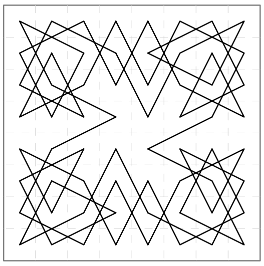


M: 8 e-g. continued

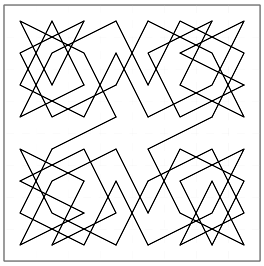

Scheidius (1850)

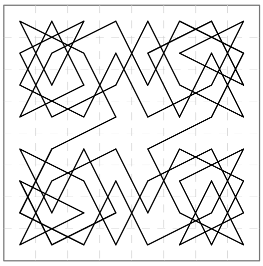


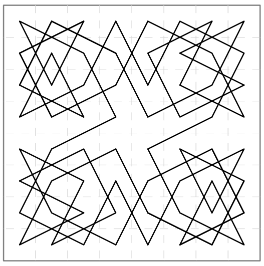

Ball (1956) "Roget method"

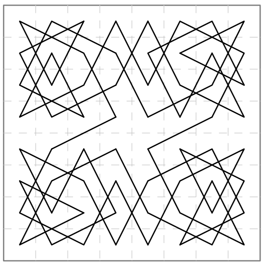


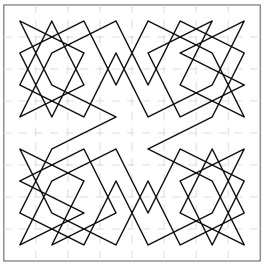
M: 8 f-h

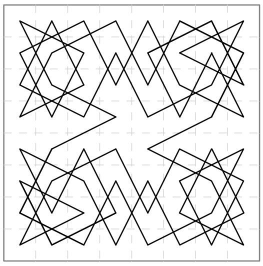


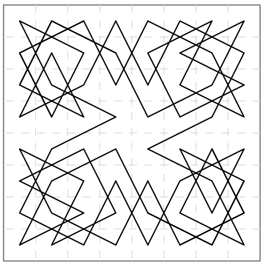


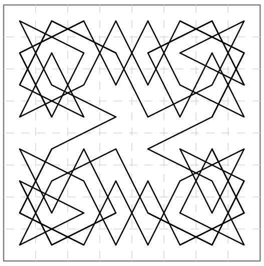




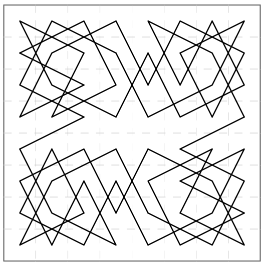

Jaenisch (1862)

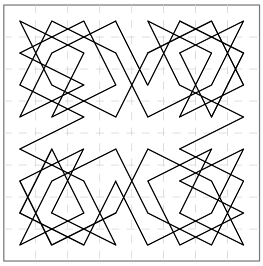


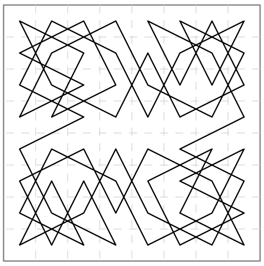


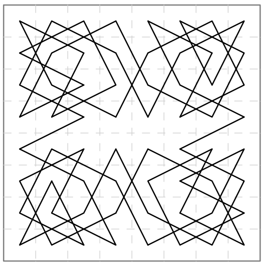


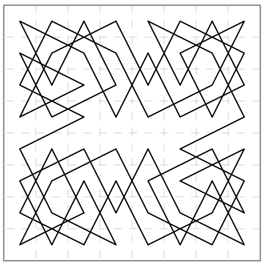
W: 8 a-c

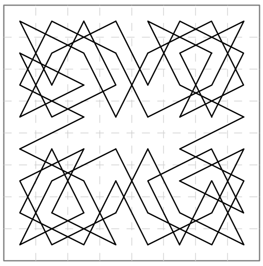


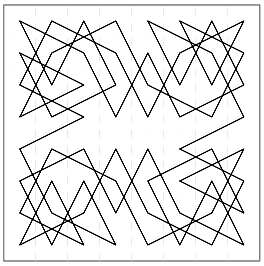

Jaenisch (1862)

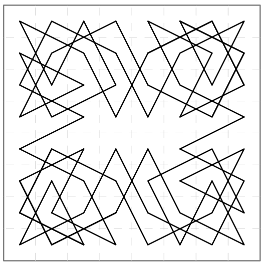




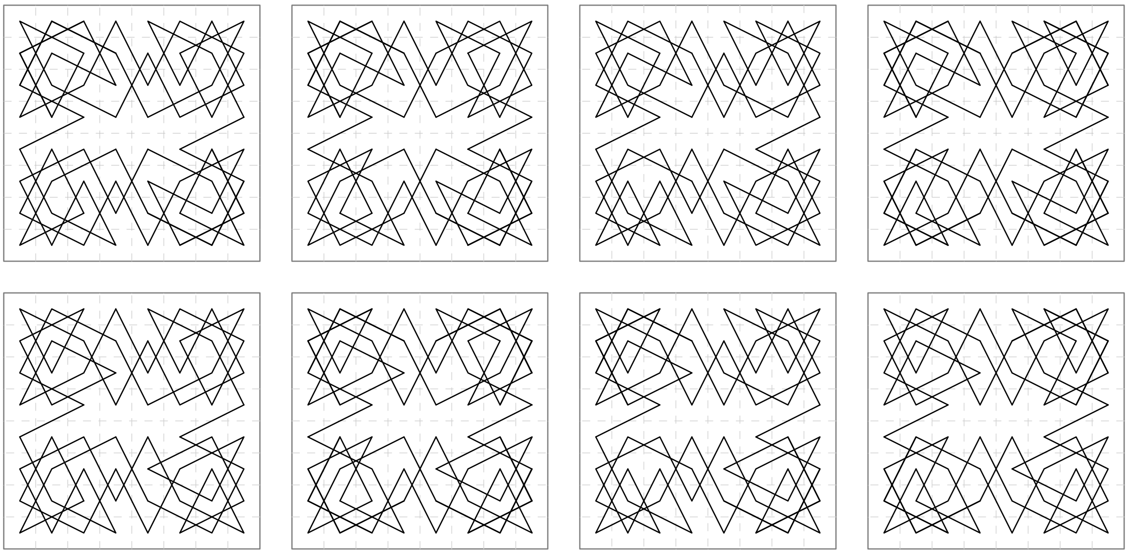




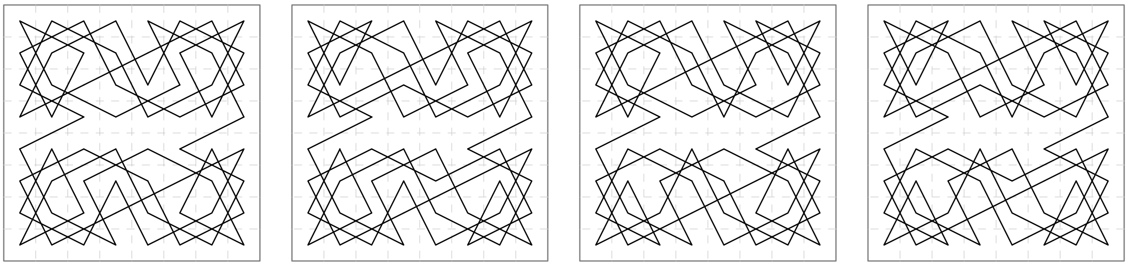




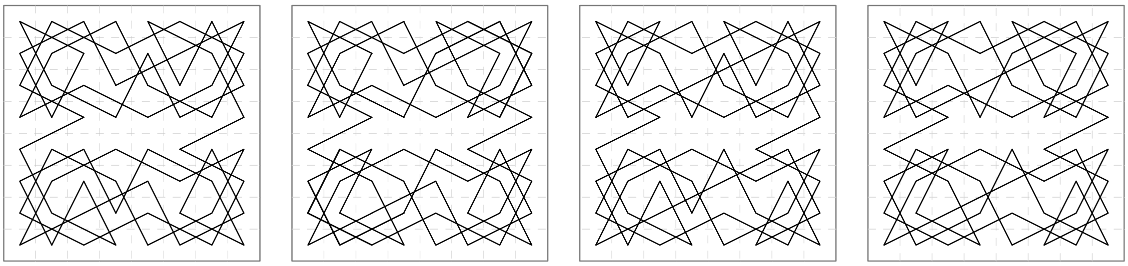
W: 8 b-d



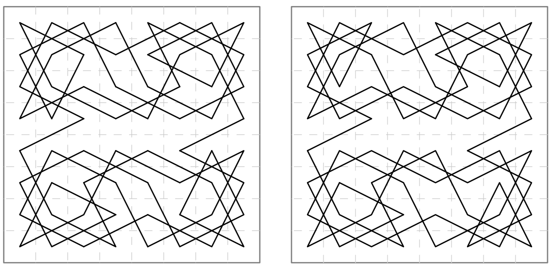
W: 4 c-e



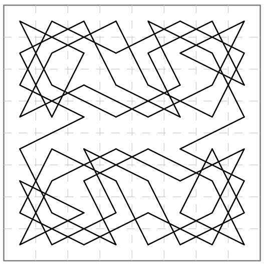
W: 4 d-f

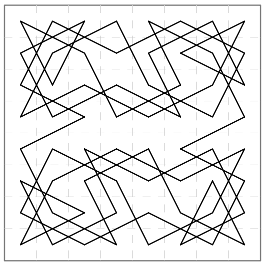


W: 2 e-g



W: 2 f-h



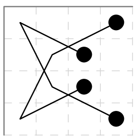


This completes the catalogue of symmetric double half-board tours of squares and diamonds type.

Full-Board Rhombic Tours

An enumeration of the squares and diamonds symmetric tours of double half-board type was given above. Here we try to enumerate the symmetric full-board tours of this type.

My first method of enumerating these tours was to note that, by reflection if necessary, we can always arrange for the diamond passing through the top left corner to be in the same orientation. There are then four ways of placing the second diamond: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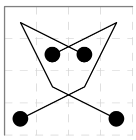
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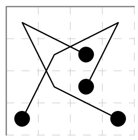
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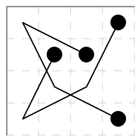

4

For a symmetric tour the arrangement of the diamonds in the diametrally opposite quarter is then just the same rotated 180 degrees. The pattern in the other two quarters can be any of the four, giving ten cases: 11, 12, 13, 14, 22, 23, 24, 33, 34, 44. But the pattern can either be reflected left to right (denoted =) or rotated 90 degrees (denoted ~) making 20 separate cases.

In the resulting diagrams there are many forced moves that can be put in due to other choices being blocked or forming short circuits. However the rest of the enumeration, done by hand, using the drawing facility in Lotus WordPro, not by computer programming, proved to be incomplete and six missing cases were later found, mainly by comparison of groups of similar tours. The numbers found in each case are now as follows (H = double half-board tours, X = other cases, T = totals):

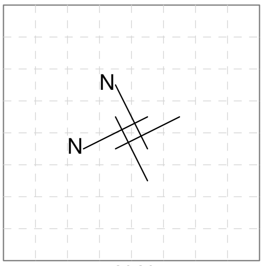
=	11	12	13	14	22	23	24	33	34	44	T
H	8	3	7	12	1	4	2	3	2	4	46
X	20	11	29	14	0	2	2	5	4	4	91

~	11	12	13	14	22	23	24	33	34	44	
H	3	5	8	2	1	3	3	4	6	1	36
X	5	4	16	1	7	25	1	38	6	0	103

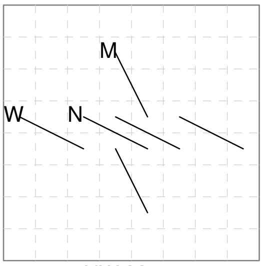
These numbers form no obvious pattern, so it is not clear if some cases might still be missing. A separate enumeration of the half-board cases (see the preceding notes) confirmed the total of 46 + 36 = 82.

However when the diagrams are rearranged according to the patterns of slants crossing the medians (as used for the half-board enumeration), the resulting numbers show more regularity, falling mostly into groups of 2, 4, 8 and 16, which gives me more confidence in the numbers found.

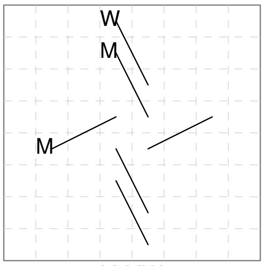
In the catalogue that follows the pairs of slants along the medians are conveniently named, as in the half-board case, narrow (N), medium (M) and wide (W). These diagrams illustrate how the tours are classified by the pairs of medial slants.



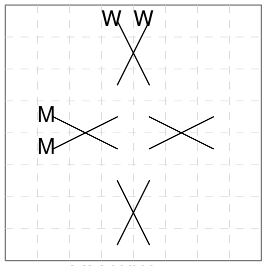
N-N



NW-M



M-MW

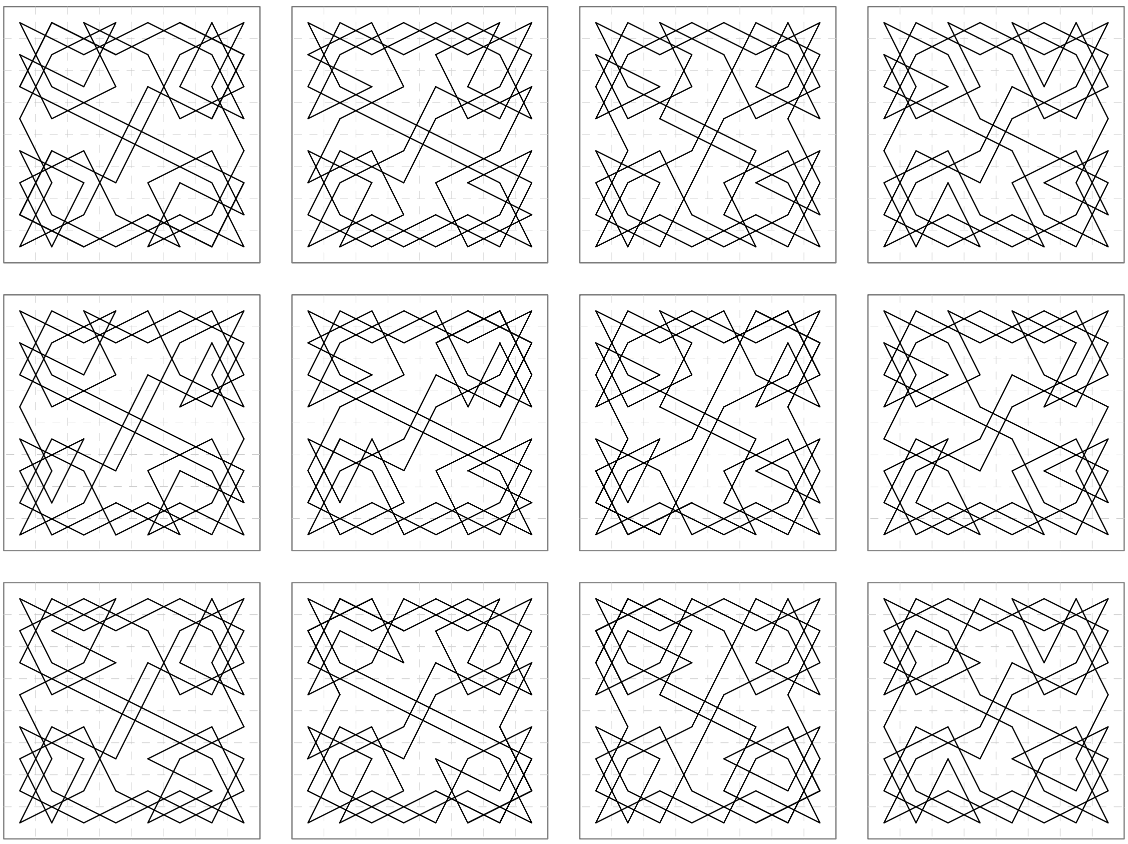


MM-WW

Within each section the diagrams are arranged according to the pattern of the secondary slants on lines bisecting the horizontal and vertical half-boards. These are labelled a, b, c indicating that they start in the first second or third file or rank from either end of the half-board. Where the above conventions do not fix the order the tours are arranged according to their central angles, or simply according to the similarity of their patterns.

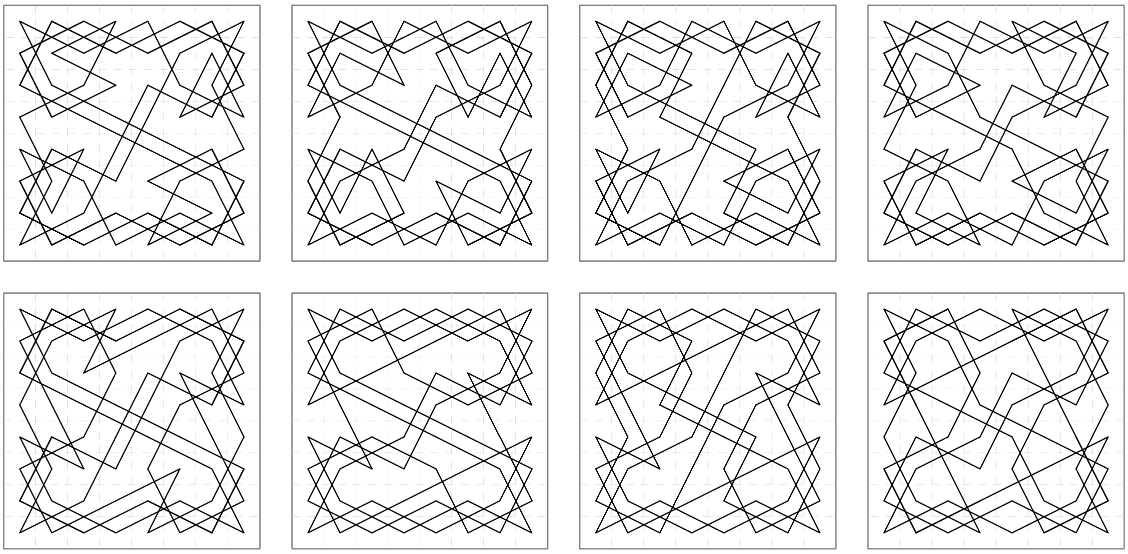
DOUBLES: two pairs of medial slants = 82 tours (the same as for the halfboard case).

20 N-N These all include a central minimal square.
(4 each of the a-a, a-b, b-a, b-b, and c-c types)

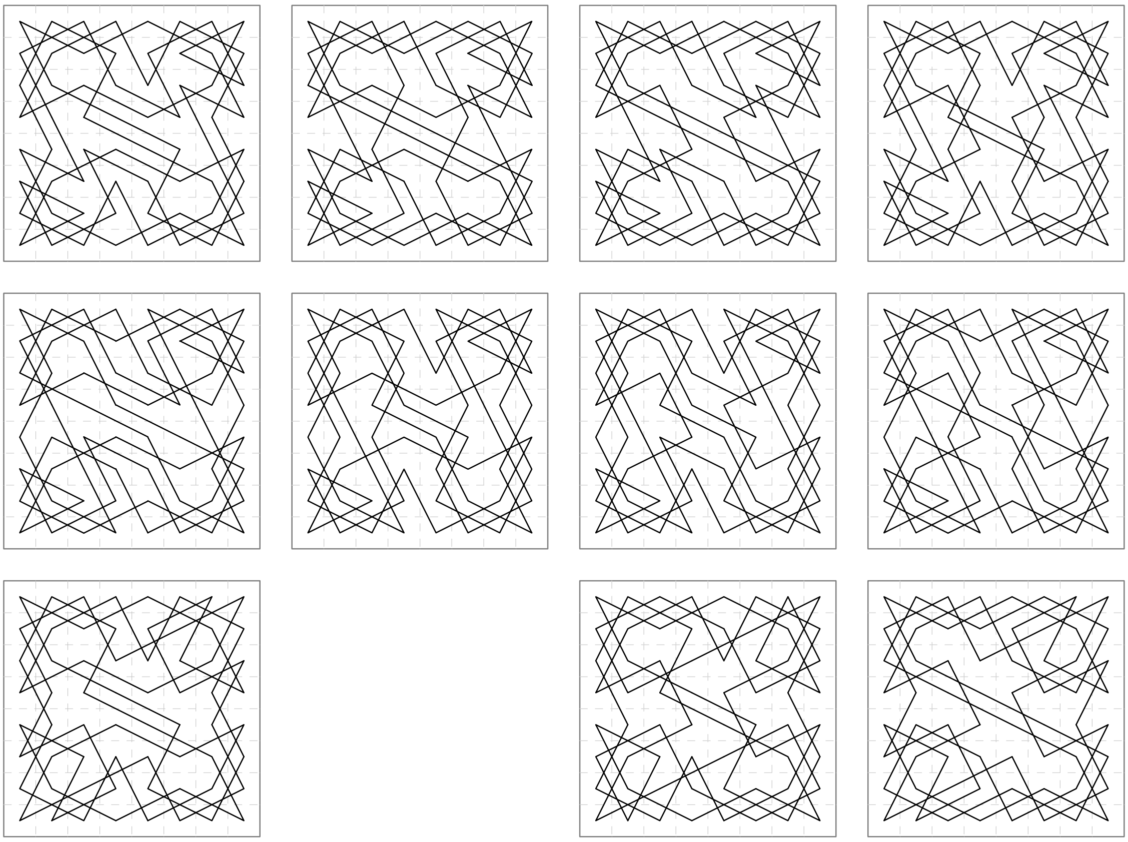


continued

continuation



30 N-W: These fall into two groups of 15. N-W first batch of 15: (8 a-c, 7 c-a).
The missing 16th tour in each pattern proves to be one of the double half-board tours.



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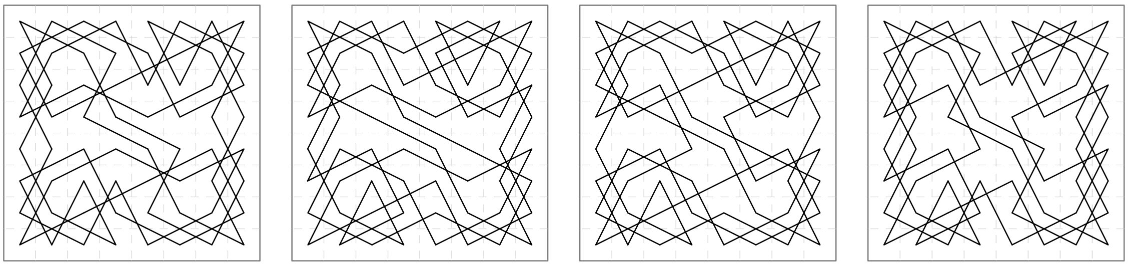
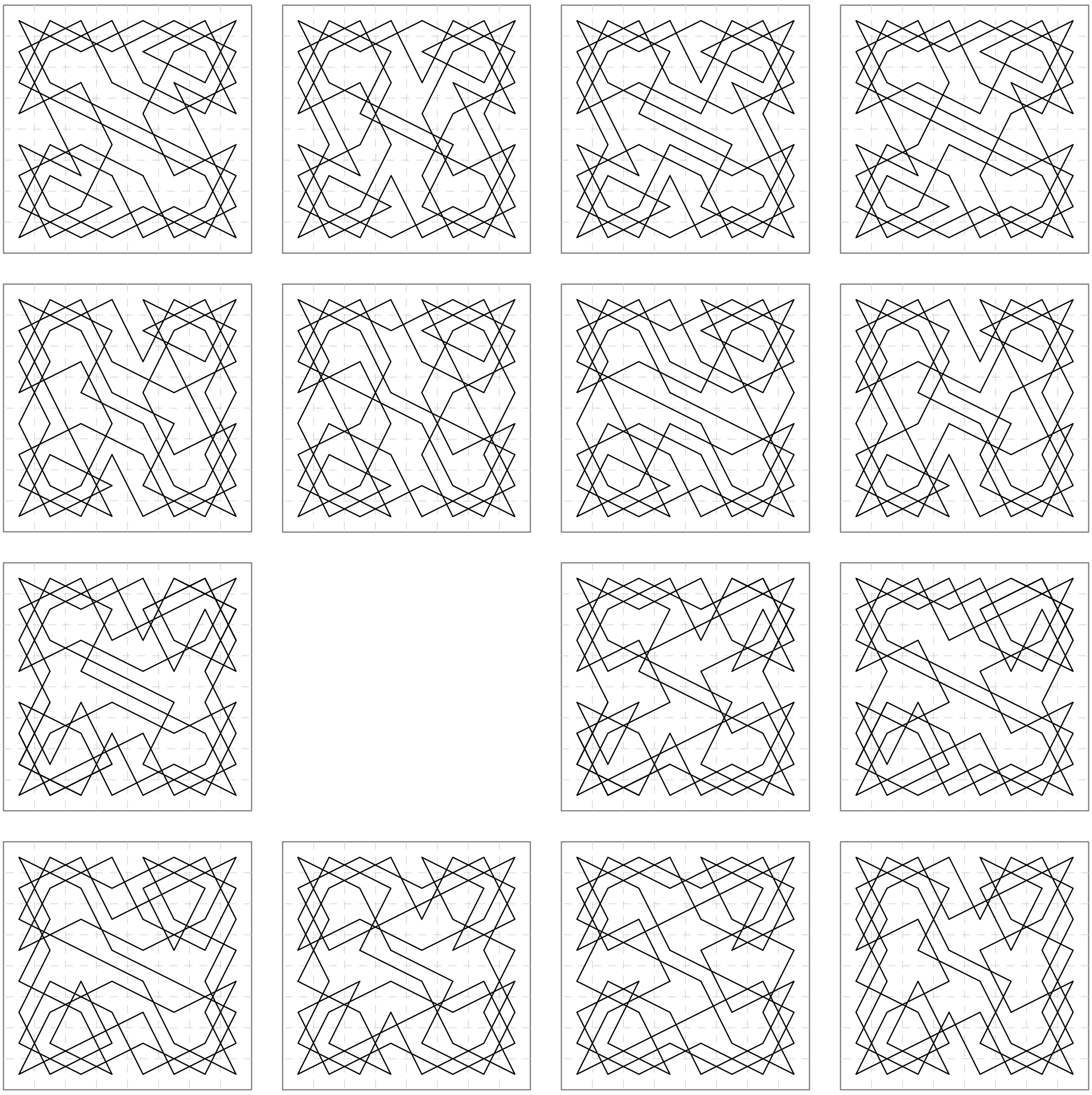
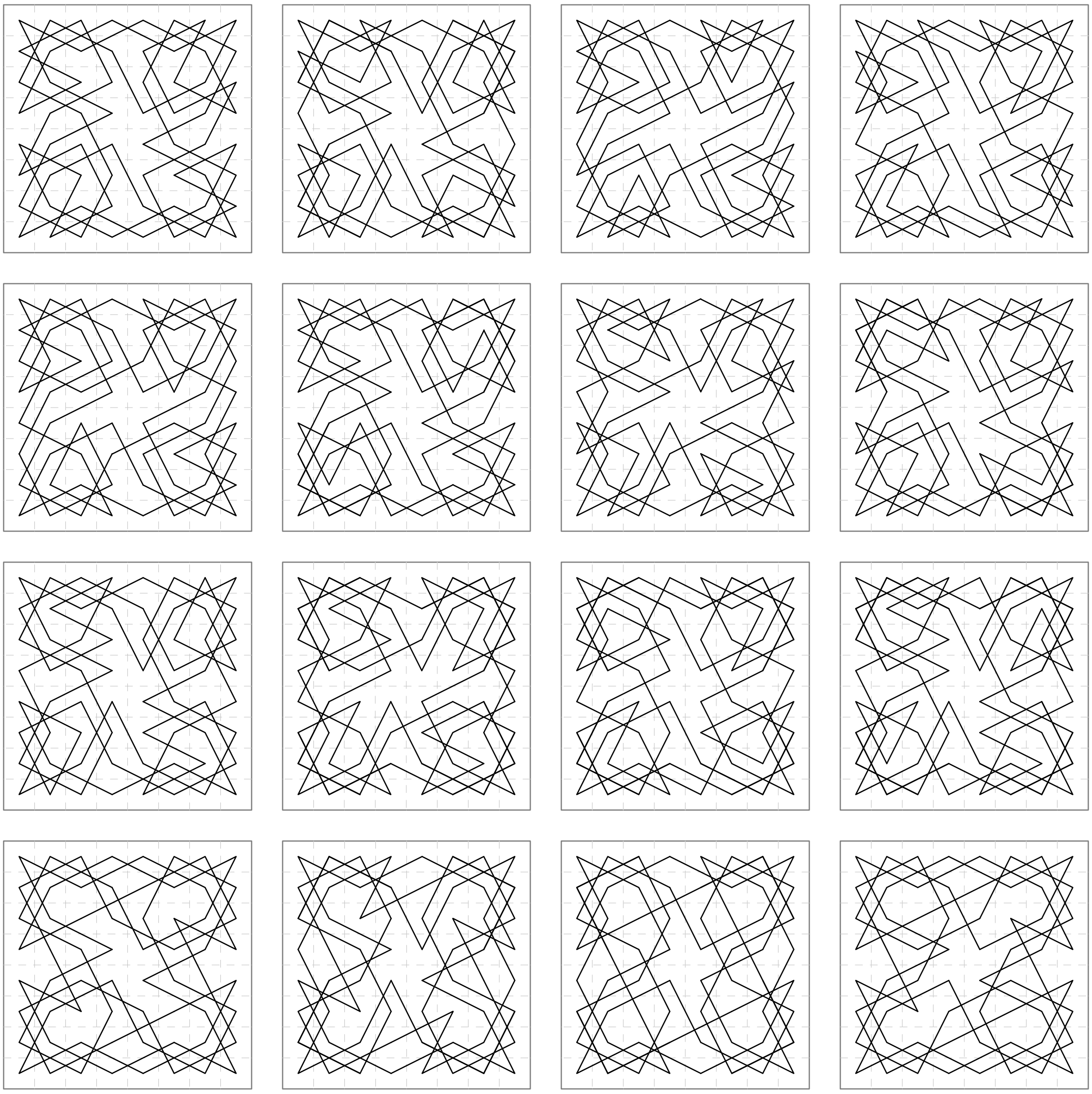


N-W second batch of 15 (8 b-c, 7 c-b type)

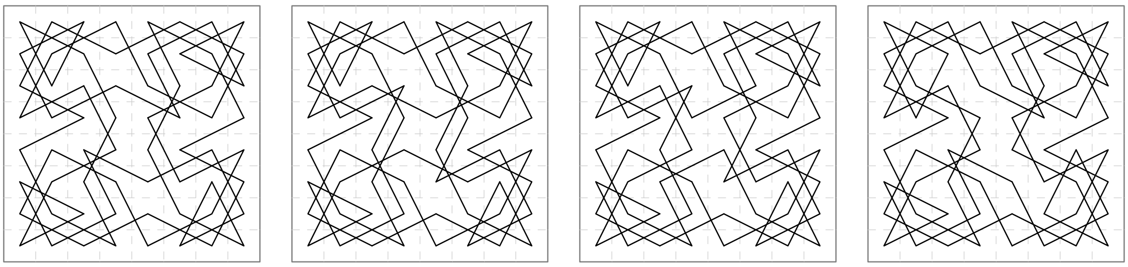


16 M-M:

(3 a-a, 3 a-b, 3 b-a, 3 b-b and 4 c-c). These groups of three tours can each be made up to four, all of similar pattern, by including a double half-board tour, except in the a-a case where the fourth in the set proves to be a 90 degree rotation of the first in the group. This seems somewhat anomalous.



16 W-W:. (3 a-a, 3 a-b, 3 b-a, 3 b-b and 4 c-c) As before the groups of three tours can each be made up to four, all of similar pattern, by including a double half-board tour.



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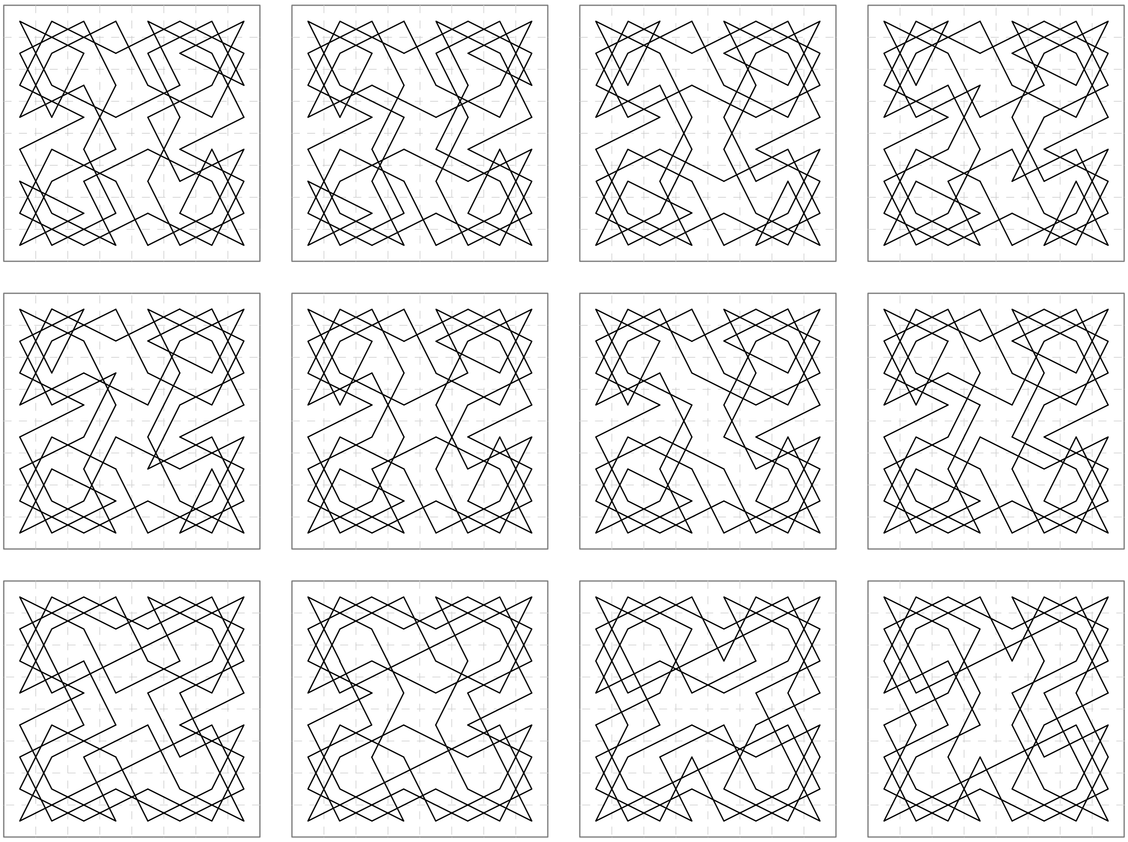
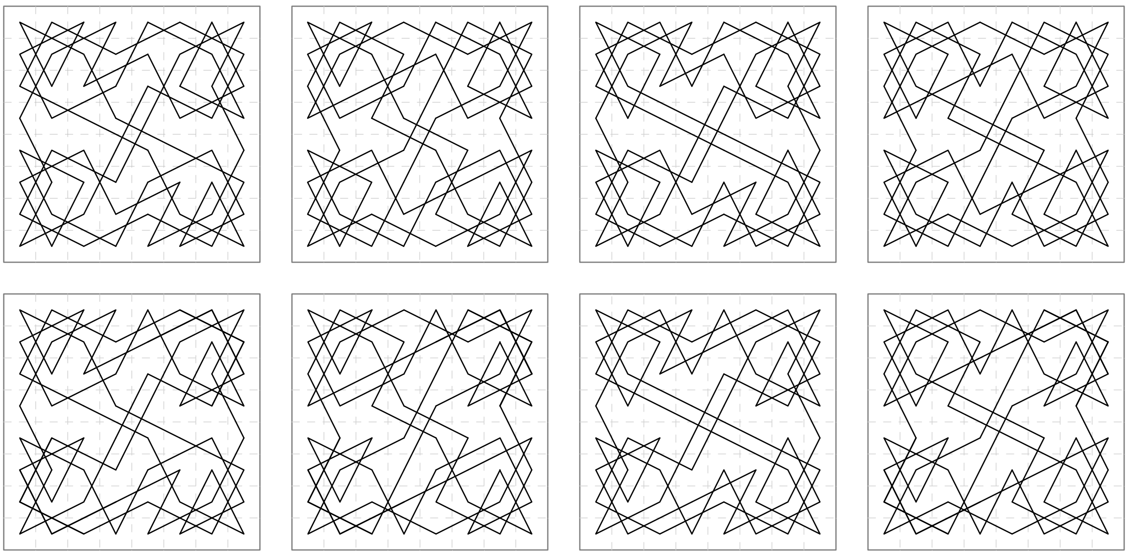


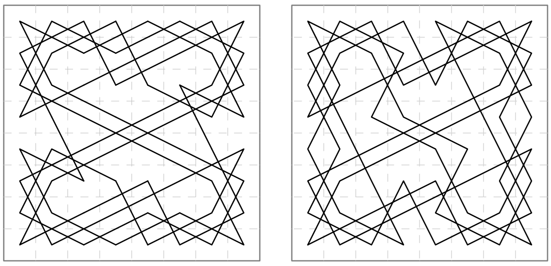
Haldeman 1864 #101

TRIPLES: three pairs of medial slants = 98 tours.

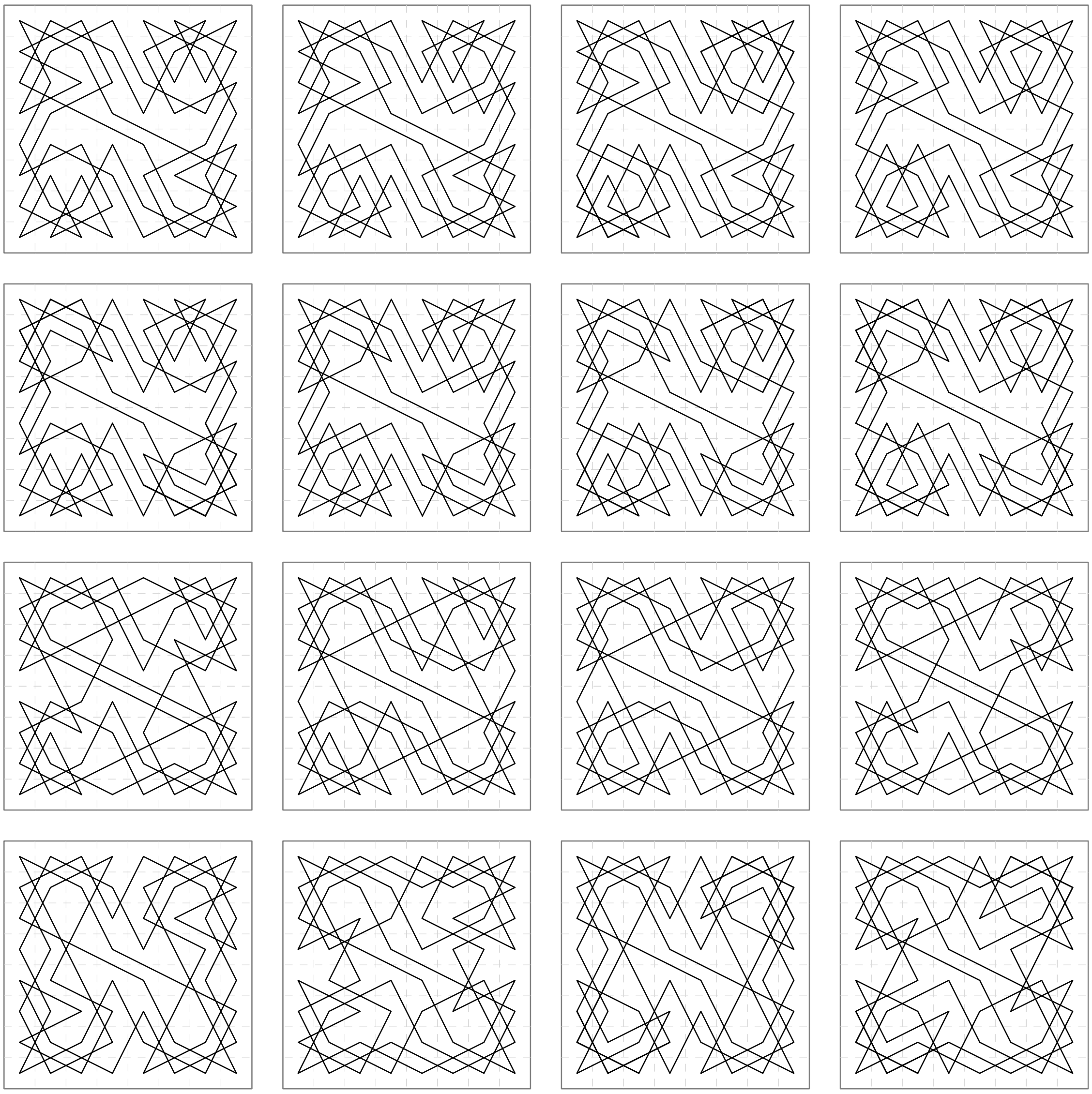
8 N-NW (2 c-aa, 2 c-ab, 2 c-ba, 2 c-bb).



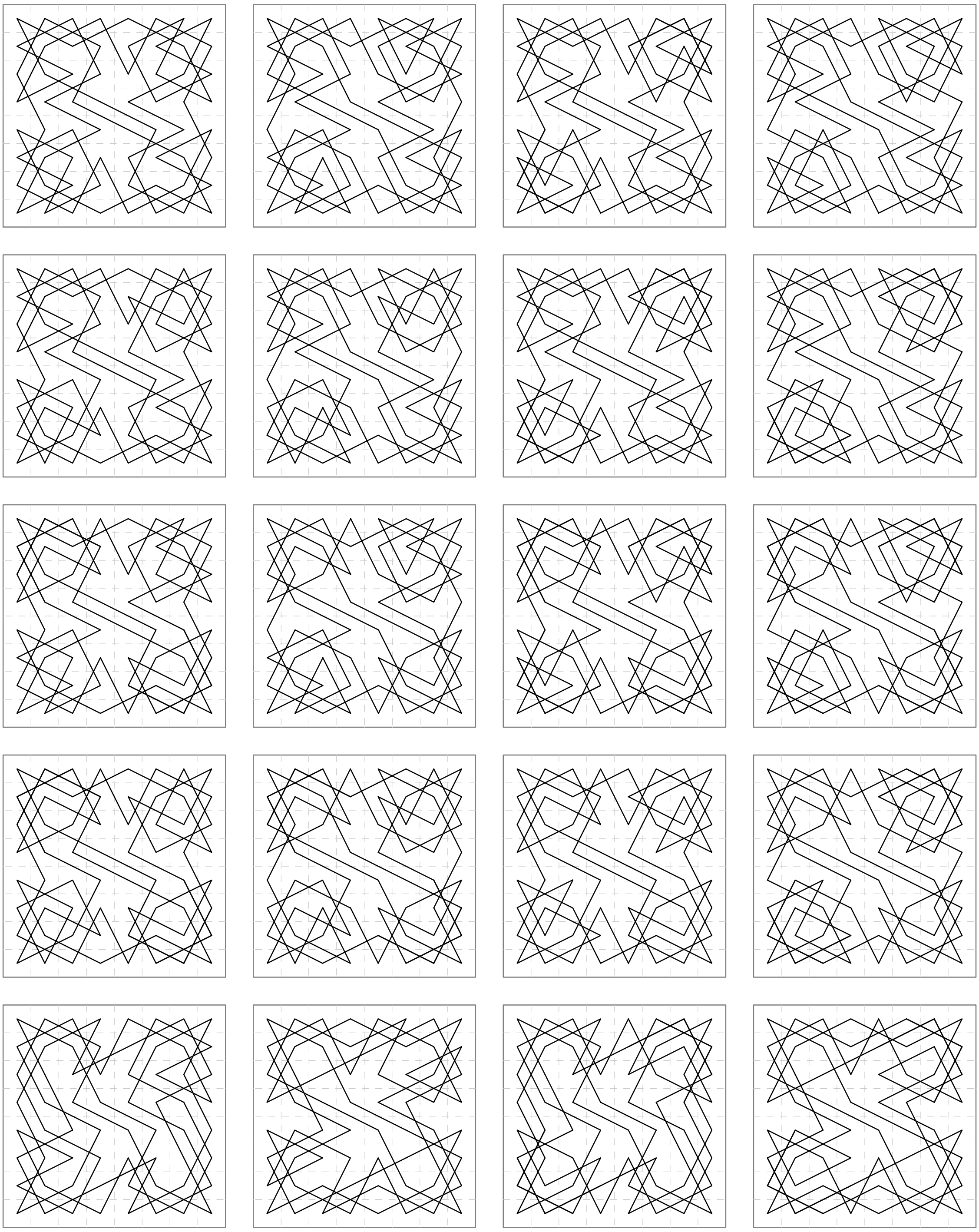
2 NN-W (2 cc-c). These are the only tours with a central lozenge.



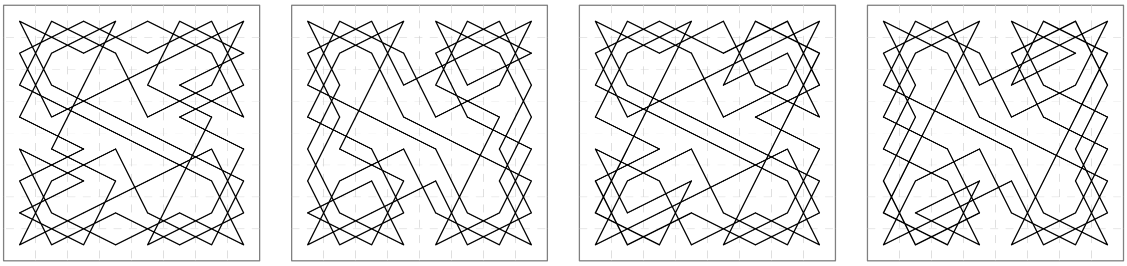
16 N-MW: (a-aa, a-ab, a-ba, a-bb), (b-aa, b-ab, b-ba, b-bb), (2 c-ac, 2 c-bc), (2 a-cc, 2 b-cc).



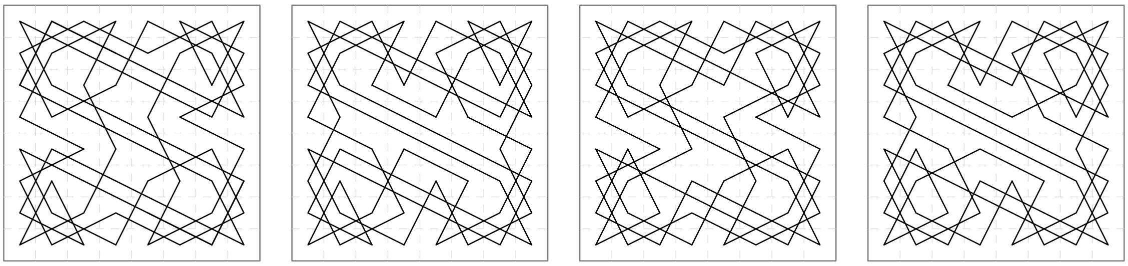
20 NM-W: (2 aa-a, 2 aa-b) (2 ab-a, 2 ab-b), (2 ba-a, 2 ba-b), (2 bb-a, 2 bb-b), (2 ca-c, 2 cb-c).



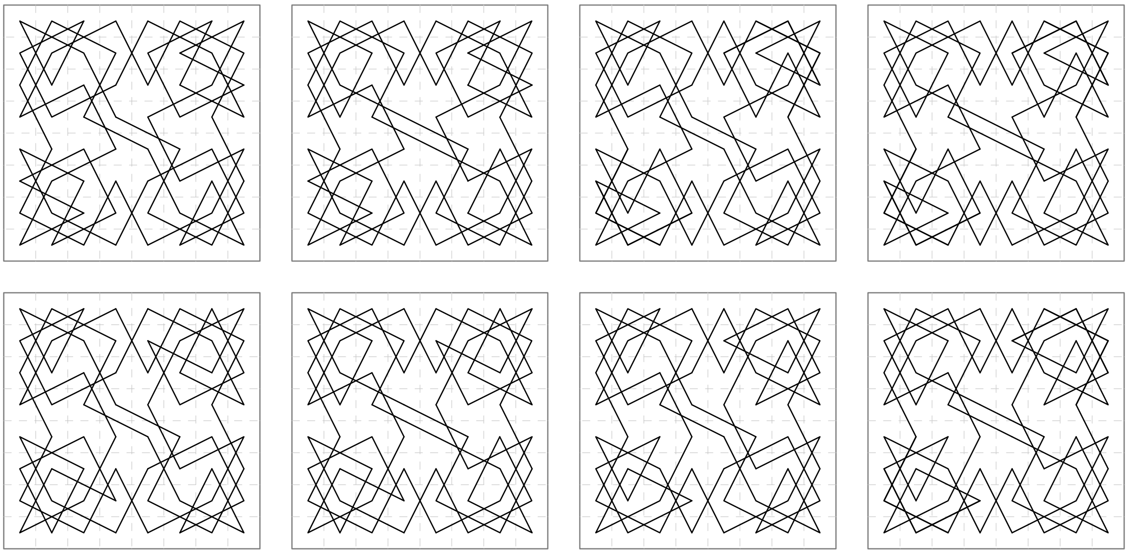
4 NW-M: (2 ca-c, 2 cb-c).



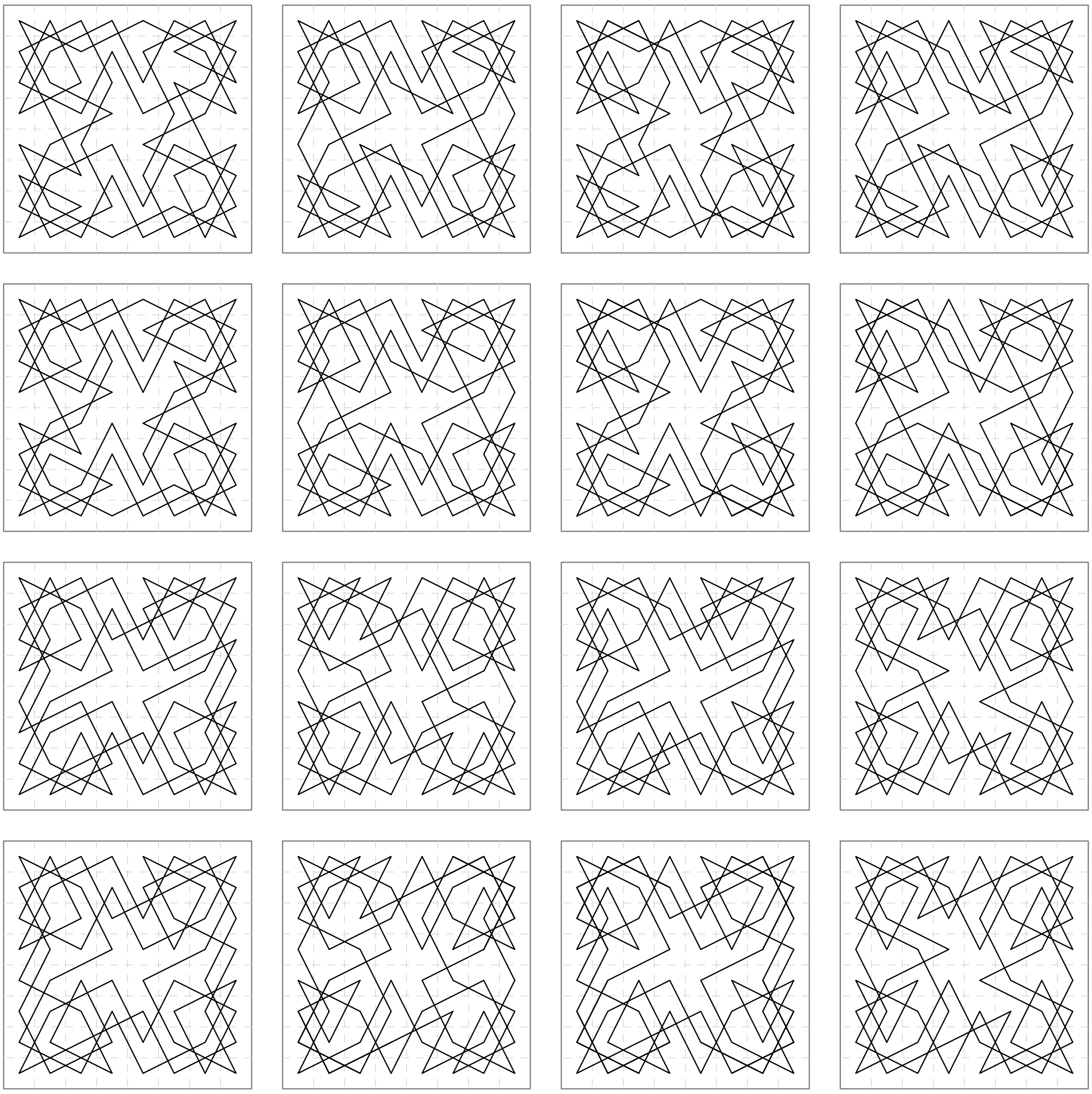
4 NW-W: (2 cc-a, 2 cc-b).



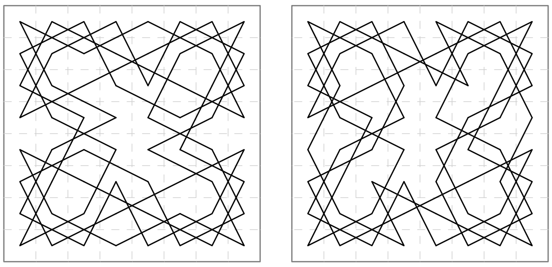
8 N-WW: (a-aa, a-ab, a-ba, a-bb, b-aa, b-ab, b-ba, b-bb).



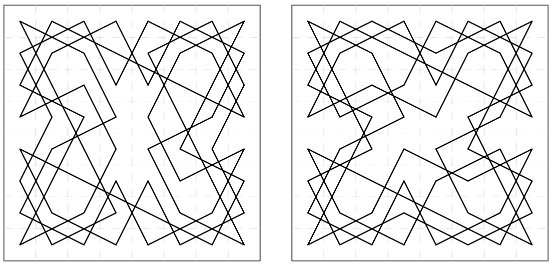
16 M-MW:
(2 a-ca, 2 a-cb, 2 b-ca, 2 b-cb) (2 c-aa, 2 c-ab, 2 c-ba, 2 c-bb).



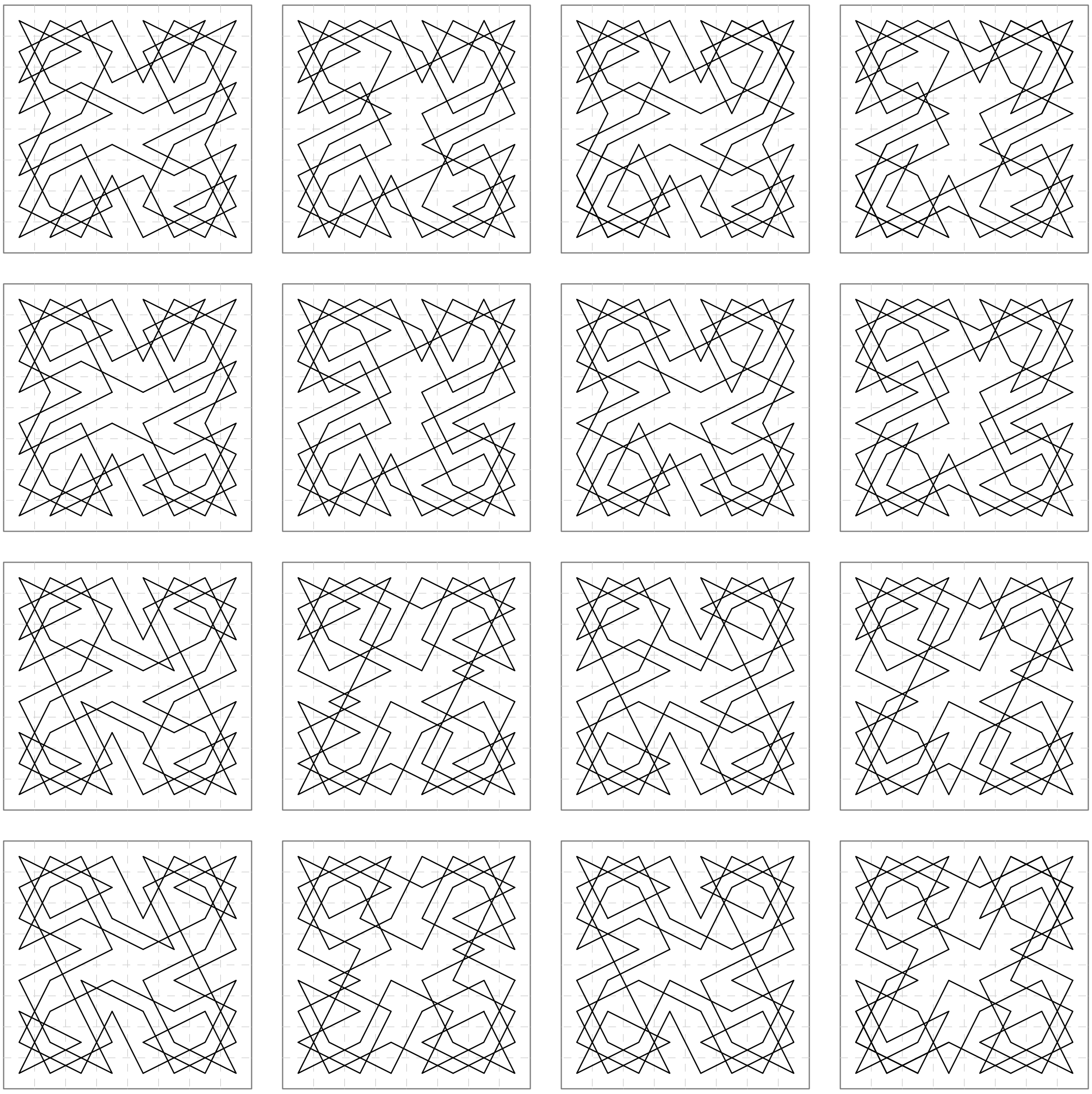
2 MM-W (cc-c)



2 M-WW (c-cc)

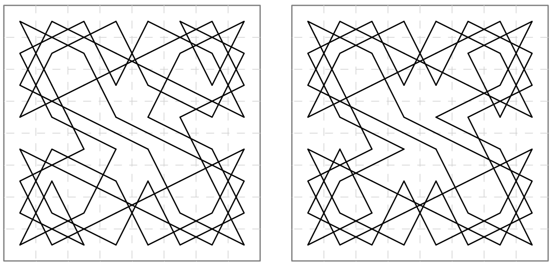


16 MW-W: (2 ac-a, 2 ac-b, 2 bc-a, 2 bc-b) (2 aa-c, 2 ab-c, 2 ba-c, 2 bb-c).

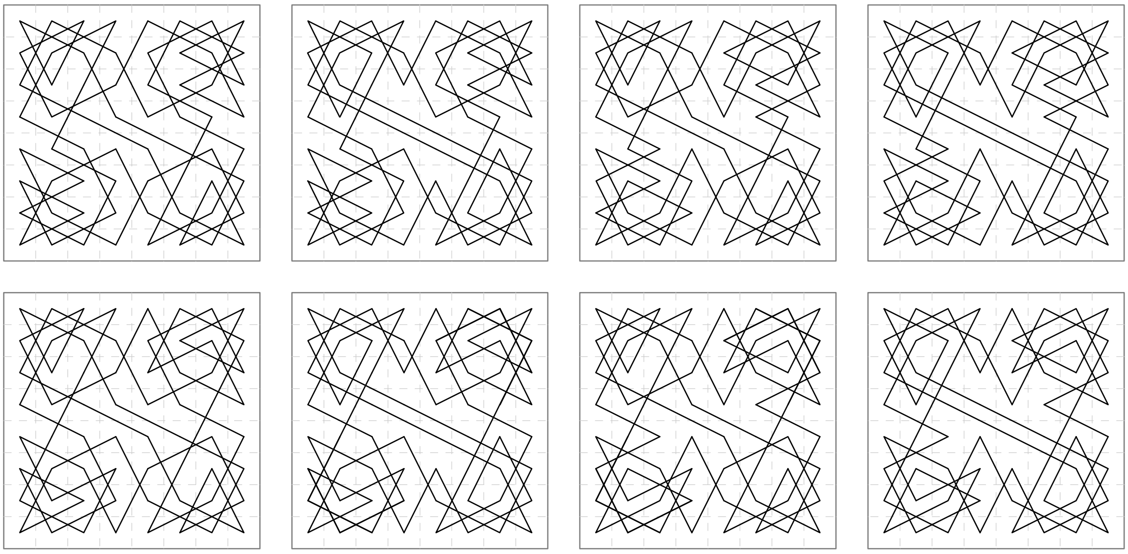


QUADRUPLES: four pairs of medial slants = 14 tours.

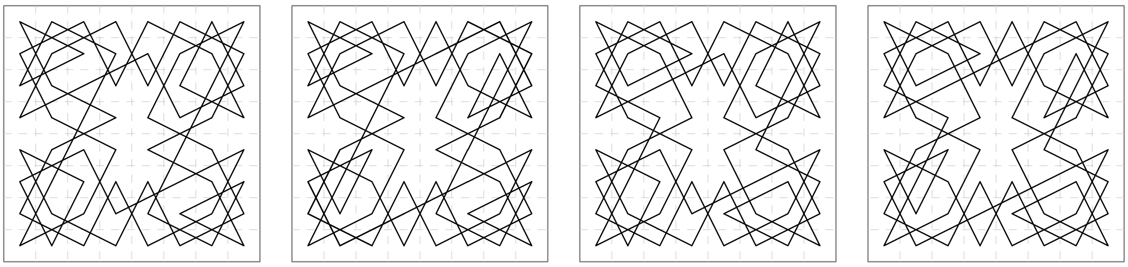
2 NM-WW (cc-ac, cc-bc).



8 NW-MW: (aa-ca, aa-cb, ab-ca, ab-cb, ba-ca, ba-cb, bb-ca, bb-cb).



4 MM-WW: (ac-ac, ac-bc, bc-ac, bc-bc).



Enumeration of All 8×8 Knight Tours

On the 8×8 board all **G** geometrically distinct **open tours** are asymmetric so the number of **open tour diagrams** is **T** = 8·**G**. For the **closed tours** on the 8×8 board the situation is more complicated. Let **C** be the number of geometrically distinct closed tours (these are geometrical paths with no end points). Of these a certain number **B** are **symmetric closed tours** having binary symmetry (unchanged by 180 degree rotation). Thus the number of asymmetric closed tours is **A** = **C** − **B**, and since asymmetric tours can be diagrammed in 8 orientations and symmetric tours in 4 orientations, the number of **closed tour diagrams** is **D** = 8·**A** + 4·**B** = 8·**C** − 4·**B**. I find it difficult to understand why the number of diagrams is the total most writers quote. To my way of thinking the number of geometrically distinct tours is by far the most significant. Counting all the orientations of a tour as different is a bit like counting a mixed herd of zebras and ostriches by the number of legs!

Brave attempts were made to calculate the number of tours in the precomputer age. In the first mathematical paper devoted to tours Leonhard Euler (1759) merely noted that the number of tours possible was very great. In his *Treatise* on chess and mathematics C. F. de Jaenisch (1862, vol.2, p.268) in possibly the earliest attempt to quantify the matter argued that there are 168 knight's moves in the complete net of moves on the 8×8 board (42 in each of the 4 directions) and an open knight's tour uses 63 of these, so an upper bound on the number of open tours is the number of ways of choosing 63 objects from a set of 168 which is 168C63 = 168!/105!63! T. R. Dawson in ‘The Problem of the Knight’s Tour’ in *Chess Amateur*, Half Hours section (1922-3) attributed this upper bound to Dudeney and calculated it to be < **122·(10^45)**. Since this count includes un-tourlike patterns such as all moves incident with de3456 (plus 3) it is clearly way over the mark.

Some other original enumeration work was done by various other contributors to that series. F. Douglas found an approach to the problem by considering each cell separately, noting that there are 4 cells (the corners) where 2 moves are available, 8 where there are 3 moves, 20 with 4 moves, 16 with 6 moves and 16 with 8 moves. The number of ways a knight path can pass through a cell where there are n moves is nC2 (and 2C2 = 1, 3C2 = 3, 4C2 = 6, 6C2 = 15, 8C2 = 28). Thus if we choose the moves through the black cells we get (1^2)·(3^4)·(6^10)·(15^8)·(28^8) patterns of 64 moves. This formula simplifies to (2^26)·(3^22)·(5^8)·(7^8) which is about **4.74·(10^30)**. This is better than Jaenisch but still way out, since it includes many arrangements where some white cells are not used and others are multiply joined. H. A. Adamson in the same source made some minor refinements to this count but the total is still in the region of 10^28.

O. T. Blathy (*Chess Amateur* 1923) was inspired by the above method to try a statistical approach, which I've simplified here. He takes the geometric mean of the number of moves at each cell: namely the 64th root of [(2^4)·(3^8)·(4^20)·(6^16)·(8^16)] which I work out as 5.256. Then we choose a start cell (64 choices) make our first move (5.256 choices) then our second move (4.256 choices) then down to the 63rd move (which I take as 1 choice), reducing the factor 4.256 by 0.07 (i.e. 4.256/61) at each step. Finally we divide this total (**N**) by 16 (to give **G**). If my calculations are right this gives **G ≅ 19.065·(10^15)**. Blathy's original method was more complicated and arrived at a total 5.517·(10^11), which, if the most recent results for closed tours are correct, is too low a figure.

Computers have made possible the enumeration of tours on the 8×8 board, so we can now give exact figures for some of the quantities listed above. Mario Velucchi very kindly sent me a copy of a Technical Report, TR-CS-97-03 dated Feb 1997, by Brendan D. McKay of the Computer Science Department, Australian National University, Canberra, ACT 0200, Australia. Dr McKay reports the total number of geometrically distinct closed tours of the 8×8 board (or in his jargon “equivalence classes under rotation and reflection of the board”) to be: **C = 1,658,420,855,433**.

Prof Donald E. Knuth, in a letter to me on 5 Jan 1993, stated the number of symmetric closed knight tours on the 8×8 board to be **B = 608,233**. The above results combined with this total for **B** give the number of asymmetric closed tours **A = 1,658,420,247,200** and the number of closed tour diagrams (in McKay's jargon ‘undirected tours’) **D = 13,267,364,410,532**. This work doesn't seem to have received any independent confirmation yet.