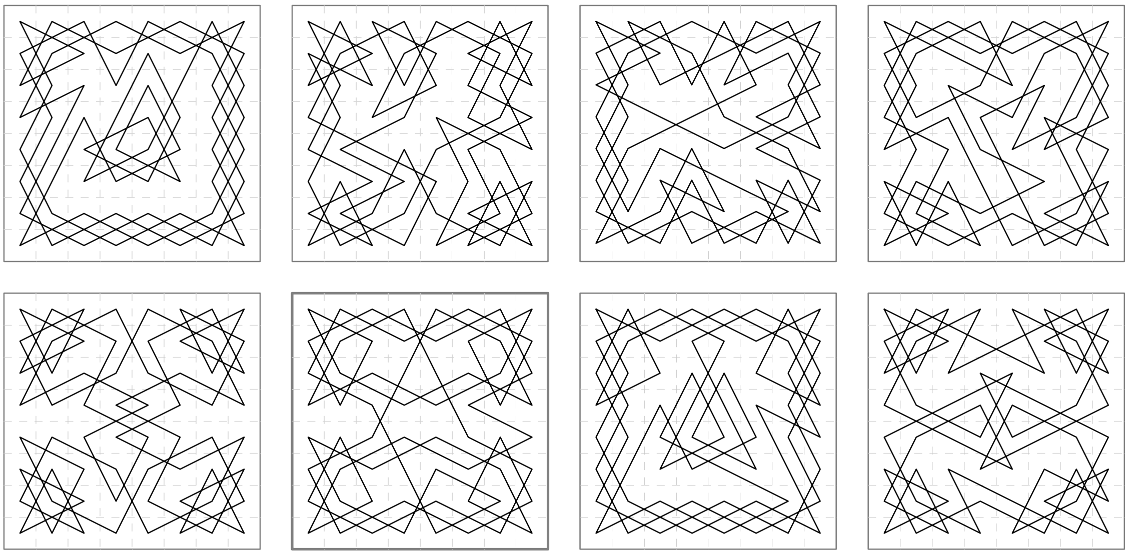
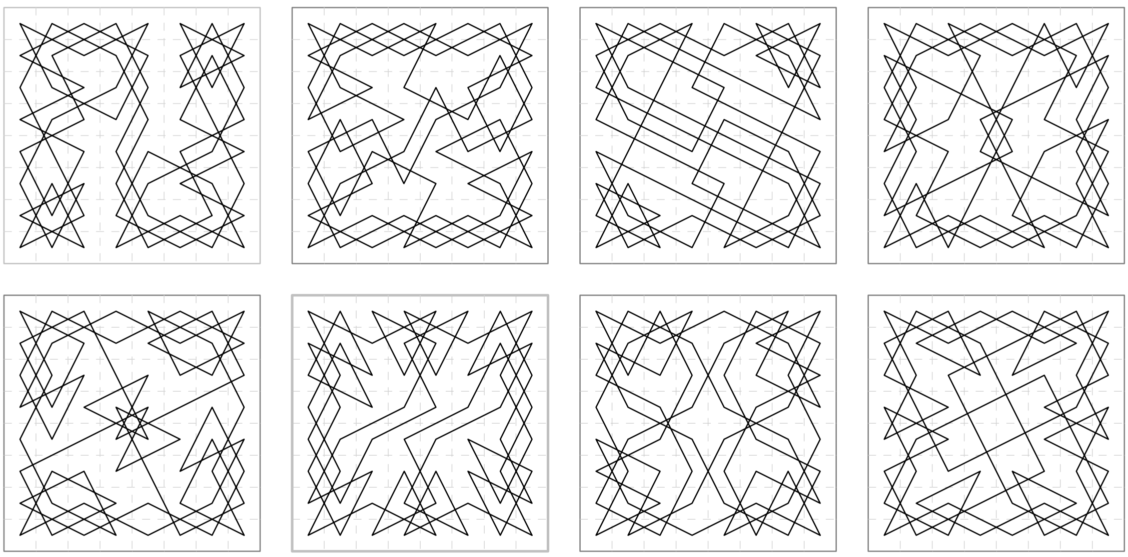
⌘ 7

Symmetry in Chessboard Knight Tours



by G. P. Jelliss



2019

Title Page Illustrations:
Irregularity: Chevalier W 1773, Legendre 1830, Rothe 1841, Jelliss 2014.
Approx Axial: Vandermonde 1771, Addison 1837, Wenzelides 1849, Chambure 1862.
Exact Rotary. Nilakantha 1640, Euler 1757, Kafer 1842, Brede 1844.
Wenzelides 1849, Pongracz 1855, Bergholt 1918. Murray 1842.

Contents

http://www.mayhematics.com/
Knight's Tour Notes, Volume 7, Symmetry in Chessboard Knight Tours.

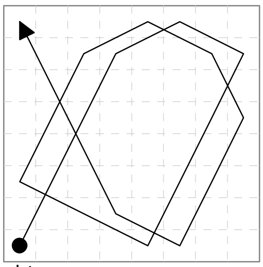
Asymmetry

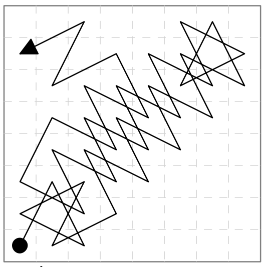
Synthetic Tours

It is natural to try to devise some simple rule governing movement of the knight which would lead to the completion of a knight's tour, making it possible to demonstrate as a conjuring trick. Most such rules however are only imprecise guides (‘rules of thumb’ or ‘heuristics’) and if applied strictly do not lead to completely determined tours, while those that do lead to a complete tour tend to be difficult to apply and work only under very restricted circumstances. However the resulting paths can be used as a basis for constructing a complete tour, say by Euler's method.

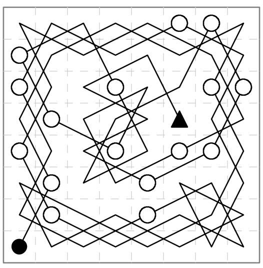
The Mani and Somesvara tours and that of Moivre follow the approximate rule: ‘Tour the border first as far as possible, and then the central area’. Collini’s method (1773) can be regarded as a more precise working of this idea. Tomlinson (1845) says that the Moivre tour follows the rule: “Play the knight to that square where he has the least power”. If we take ‘power’ to mean the maximum number of moves available to the knight on a cell then this ranks the corner cells as 2, those next to the corner as 3, and so on. This however does not generate the Mani or Moivre tours which both depart from the rule at h6 where they go on to g4 instead of back to g8.

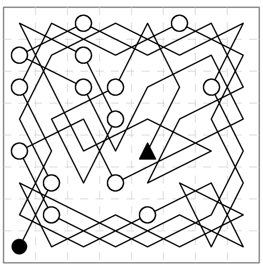
The Obtuse and Acute Rules. The paths shown below are the routes followed by ‘interchessic missiles’ programmed to make successive moves at as obtuse (including straight) or as acute an angle as possible, beginning with a1-b3, as posed in *Chessics* (#1 p.2 1976, #5 p.5, #12 p.8-9 1978). In the later article I showed that the routes followed from other cells either terminate or, if allowed to reenter cells already visited, end in a repeating ‘orbit’. The path may also encounter an ‘ambiguity’ where there is a right-angle move to left or right, in which case there is a choice of continuations.

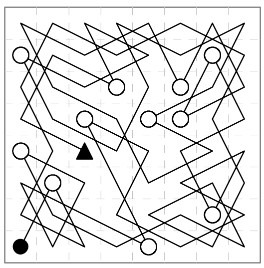

obtuse

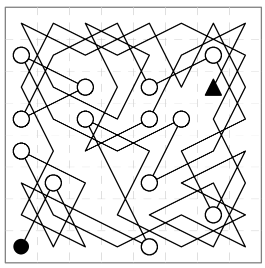

acute

The Warnsdorf Rule Augmented. In *Chessics* #22 (1985) I defined a ‘synthetic tour’ as one that is completely determined by strictly following a clearly formulated rule of construction and gave four tours beginning a1-b3, generated by combining the Warnsdorf rule (W) (See p.334) with two extra rules. Each move must be made at as obtuse (O) or as acute (A) an angle as possible to the preceding move. The rules WO and WA indicate that rule W is applied but if it breaks down on any move rule O or A is applied instead. The rules W/O and W/A are similar except that the augmented rule is applied to the choice of moves selected by W. All four rules determine a tour.


WO


W/O

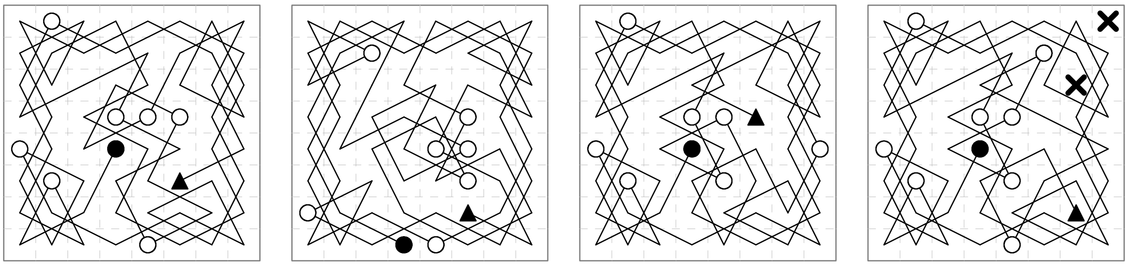

WA


W/A

The white circles indicate the points of ambiguity where the supplementary rule is applied.

KNIGHT'S TOUR NOTES

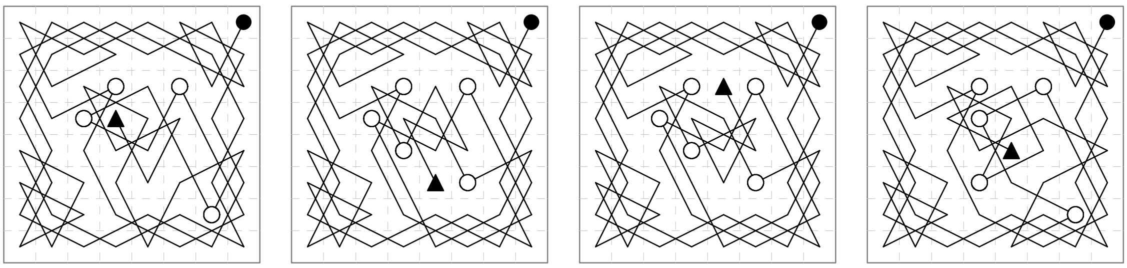
A Partial Enumeration of Warnsdorf Tours. In January 2014 I made a partial enumeration (by hand with diagrams drawn on a computer) of tours following the Warnsdorf rule on the 8×8 board. I reached the fifth ambiguity and 540 diagrams. I suspect someone must have done the full enumeration using a computer program, but have not seen any results published. Taking a couple of the more promising diagrams I followed them through to the seventh or eighth ambiguity, and show three tours with the minimum of 7 ambiguities shown by the white dots.



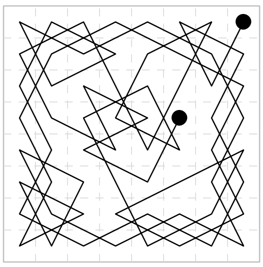
In the first the initial cell d4 could also be counted as an ambguity since the choice d4-b3 would reflect the tour in the diagonal, and the choice d4-e2 (and its reflection d4-b5) would lead to different tours. In the second tour there is no choice on the first move given the initial placement of the knight at d1. Any link to the black dot in the course of the construction is prohibited since it would result in a closed circuit covering only part of the board. The fourth diagram shown above is one of the first cases encountered where the rule fails. The choices of route available from f7 onward all lead to a dead end that leaves two cells, marked by crosses, unvisited.

The rule also involves a degree of backtracking, since you have to examine all possible choices of the next two moves to determine the next single move. This enables purists to claim that it is not a proper tour-generating rule, in which all backtracking is prohibited.

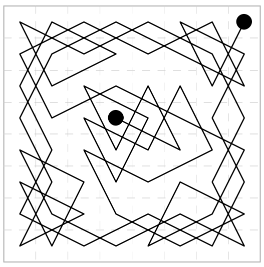
Tomlinson's Rule: "Keep as far from the centre as possible" is advocated in *Amusements in Chess* (1845). The rule is valid, it generates four tours, beginning from a corner, but also leads to numerous dead ends on the way. The white dots again show where there is a choice of moves. The centre-outwards coding works for this rule, the numbers being in sequence of distance from the centre, but note that the cells coded 5 and 6 are equidistant from the centre point.



Unfortunately the example Tomlinson gives has an error at h6, going to g4 instead of g8.



Tomlinson



Stonebridge

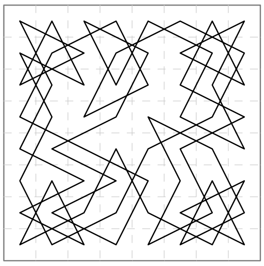
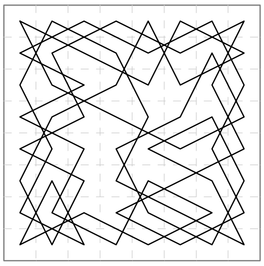
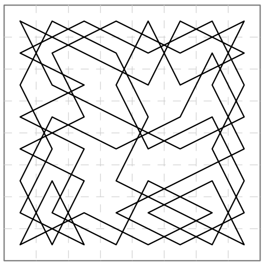
An algorithm giving similar results to this rule is proposed by Brian R. Stonebridge in *Mathematical Spectrum* (vol.25 #4 1992/3).

His diagrammed tour from a corner (reoriented above to start h8-g6) is identical to the above tours as far as the first ambiguity at d6 but then goes to f5 and h4. His algorithm decides which route to take by applying the rule to each choice. As a result I think it prefers the route to h4 since it gives access to g2, near to two edges, whereas via f5 there is no access to b2 since it has been used. He shows ten routes from the ten generically different initial cells. Two cases apply an extra condition.

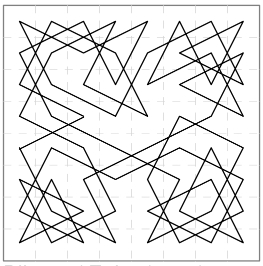
Irregularity in Tours

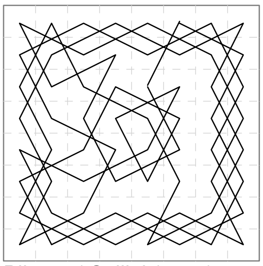
Despite the work of Euler (1759) on symmetric chessboard tours, most of the knight tours constructed since his work continued to be asymmetric. This is perhaps not so surprising since the vast majority of tours possible on the 8×8 board are asymmetric. The aim is often simply to construct a tour, or to find one between given end points, or to show other particular effects. However, the degree of symmetry to be found in many of the asymmetric tours published is still considerable. Not so many are out-and-out ‘irregular’. Here we look at a selection of the most irregular from various sources, and consider how their ‘degree’ of asymmetry or irregularity can be judged. We also make use of some of these irregular tours in the Symmetrisation section.

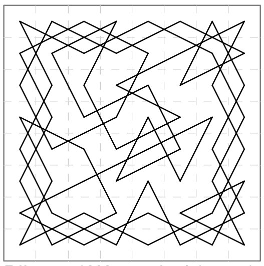
Three of five irregular tours by A. M. Legendre (1752-1833) constructed by Euler's method in his *Théorie des Nombres* (1830) are shown here.

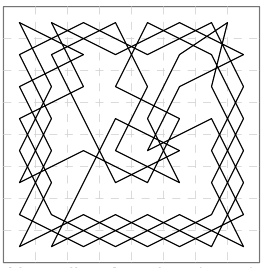




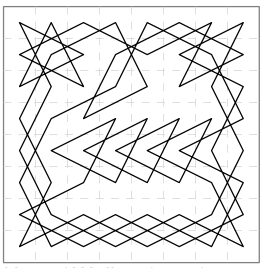
Bilguer‘s *Handbuch des Schachspiels* (1843) has three irregular tours (Einleitung p.4-5) which are described as ‘after’ Euler (a4-c5), Collini (f8-e6) and Warnsdorf (c3-e4). However the latter example fails to follow the Warnsdorf rule at h4 where it should go to f5 and at c5 where it should go to e4. I show them as examples of asymmetric irregularity. The fourth is from *Nouvelles Annales de Mathematiques* (1852) where it illustrates Euler's method.

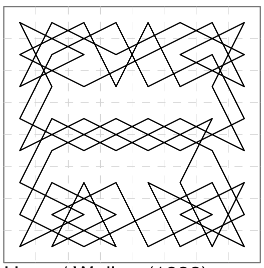

Bilguer / Euler (1843)

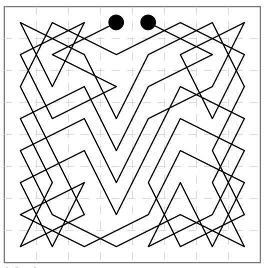

Bilguer / Collini (1843)


Bilguer / Warnsdorf (1843)


'Nouvelles Annales' (1852)


Hone / Walker (1832)


Hone / Walker (1832)

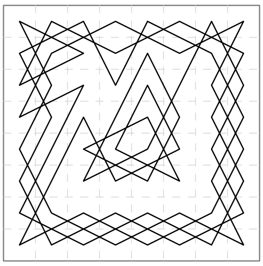

Kafer 1842

Prettier tours, but nevertheless asymmetric overall, are the above two from Hone (1832), probably provided by George Walker, and the open tour from Käfer (1842).

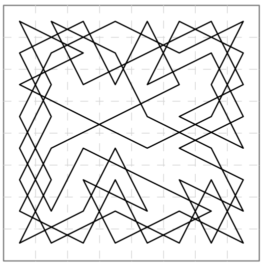
Another particularly irregular tour is that in an article by Glaszer (1841) who cites a closed tour by his late colleague Prof. Rothe constructed by Euler's method. Murray (1930) thought this “Horrible!” Contemplating these and other irregular tours, such as the Chevalier W. (1773) tour, made me wonder if it is possible to measure the degree of asymmetry or irregularity of a tour and to find one that was the most asymmetric.

In a symmetric tour all the moves that appear must obviously occur an even number of times since each has its symmetric counterpart. (Since zero is an even number, this is also true of those that don't appear!) It occurred to me that in the case of a closed tour the asymmetry might be measured by the number of the 21 generic moves that occur an odd number of times.

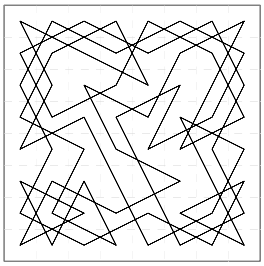
The third tour below is, on this criterion, the most asymmetric tour I could construct (it was published in my blog, *Jeepyjay Diary*, 28 September 2014).



Chevalier W 1773



Rothe in Glaszer



Jelliss 2014

In this tour the total of generic moves occurring an odd number of times is 18 (in a closed tour the number is always even), and moreover 12 of these occur only once. The only moves that occur an even number of times are the corner moves like a1-c2 (where there are eight, which occur in every closed tour) the edge-to-edge moves of the type a2-c1 (eight again), and the pair of moves a3-c2 and h3-f2 which are of the same generic type. Two moves are considered to be of the same type if one can be put in the position of the other by a rotation or reflection of the board (or in other words they have the same cell code, a3-c2 and h3-f2 being 7-4 type). The 12 move types that occur only once in this tour are b1-c3, d1-c3, f1-e3, b2-c4, d2-b3, d2-f3, d3-c5, e3-d5, a4-b6, e4-g5, b5-d6, e5-f7.

For comparison, I give the number of generic moves used, the number of these odd, and the number occurring just once, for some other tours. The Nilakant-ha (1640) tour, being symmetric, is 16-0-0. The highly symmetric Vandermonde (1771) tour is 12-4-2. Euler's (1759) first closed tour 16-8-5. The first Legendre tour is 17-10-3. The Bilguer tours are: 18-10-5, 16-8-6, 20-11-7. The Hone tours are 17-11-4, 16-6-0. The Nixon T, R, D, Figured tours (p.427) which seemed to me particularly irregular, are 20-6-3, 17-12-2, 20-10-7. The Chevalier W (1773) tour 18-12-9. The Rothe tour 18-10-4. My tour is 21-18-12. However I must admit that the Rothe tour still somehow gives a more asymmetric impression, so perhaps some better measure of irregularity is possible.

Approximate Symmetry

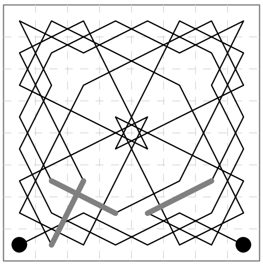
Exact symmetry on the chessboard can only be of the 180° rotational kind. On square boards without holes it is impossible to construct a knight's tour that has exact axial symmetry, in which each move in the left half of the board has a corresponding move in the right half. A formal proof of this does not seem to have been given until Jaenisch (1862). It follows that biaxial (direct quaternary) symmetry is even more impossible. Birotary (oblique quaternary) symmetry (unchanged by 90° rotation) is also impossible on the 8×8 board. The best that can be achieved is Mixed Quaternary Symmetry which we study later in this volume (p.41).

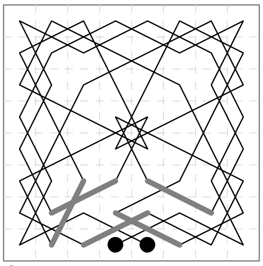
Many tours have however been constructed on the 8×8 board showing approximations to these forms of symmetry. The best have the minimum number of moves deviating from symmetry (shown in the following diagrams by grey shading). In the most effective tours these moves, or end-points of open tours, are contrived to be in a corner of the board, away from the main pattern, where they are less noticeable. Eliminating the non-matching moves and end-points to make the pattern exactly symmetric always results in the tour being converted into a pseudotour consisting of two or more superimposed circuits, or in the quaternary symmetry being reduced to normal binary symmetry. Gathered here is a collection of tours from historical sources classified according to symmetry.

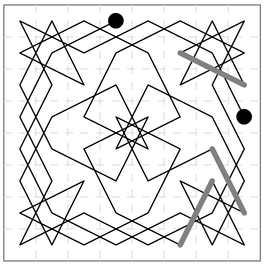
Tours of biaxial and birotary types can readily be formed from the patterns formed of symmetrically placed 16-move paths enumerated by de Hijo (1882) and diagrammed in the Pseudotour volume ⌘ 8). A closed tour can be formed if it is possible to find a rhomb with one side in each circuit. To make an open tour it is just a matter of finding a link between the two circuits, and deleting one move from each circuit at the ends of the link (4 choices). I think it is always possible to find such a link, probably several, though it may disrupt the centre pattern.

Approximate Octonary Symmetry

The following examples approach octonary symmetry. The first is from Käfer (1842). The third from Falkener (1892). These are open tours using a seven-move linkage. The second from *Szachy od A do Z* by Litmanowicz and Gizycki uses a longer linkage than the similar Käfer tour.

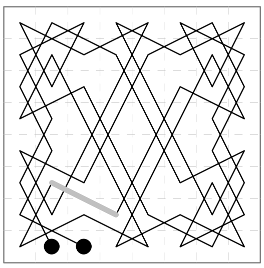
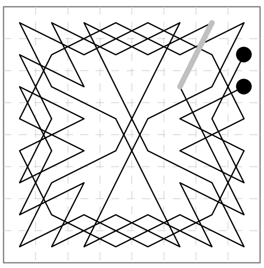
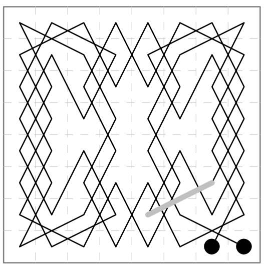
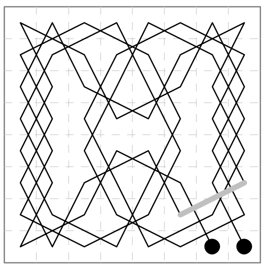

Kafer 1842


Szachy od A do Z

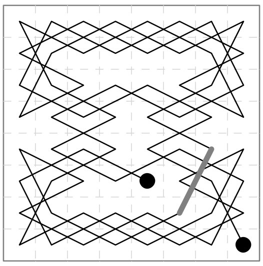

Falkener 1892

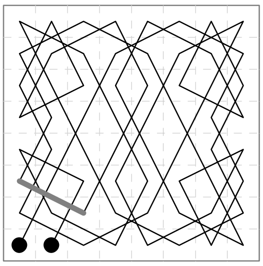
Approximate Biaxial Symmetry

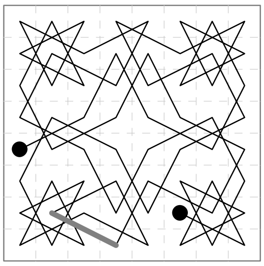
In Käfer (1842) there are 4 open tours with maximum biaxial symmetry, as shown here. In the first tour I have corrected the diagram. The two ends link to ends of the move marked in grey. Deleting this and joining up the ends results in a pseudotour with exact symmetry.

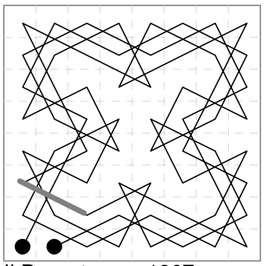





Other examples in historical sequence. Tomlinson (1845) with piece-wise symmetry differing by 16. The ‘Figure Eight’ tour by Staeker (1849). The open four-star tour by Slyvons (1856). And one with a large internal space from *Il Passatempo*, Bologna, 1867-69.

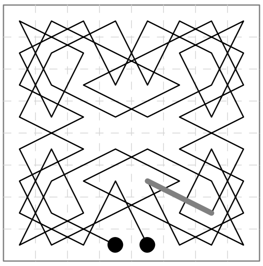
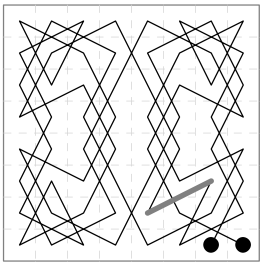
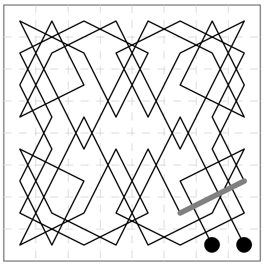
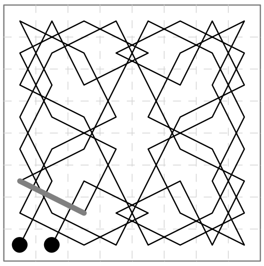

Tomlinson 1845

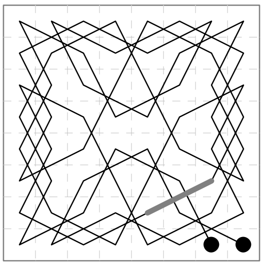
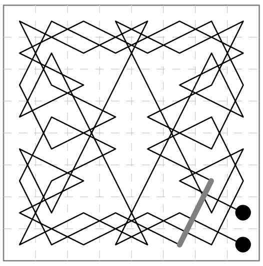
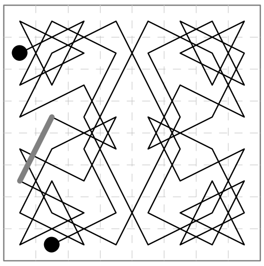
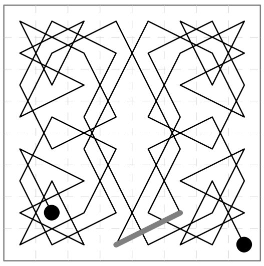

Staeker 1849


Slyvons 1856

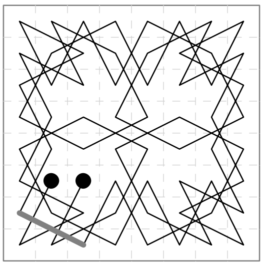
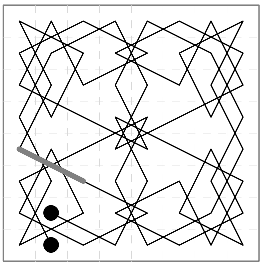
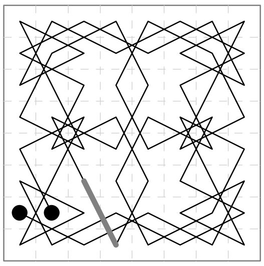
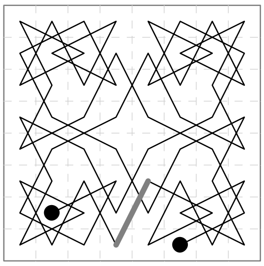

Il Passatempo 1867

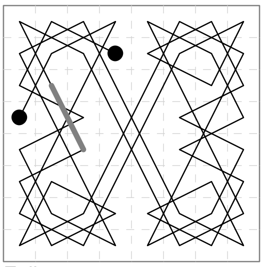
The following selection of eight open tours with max biaxial symmetry is from Adam (1867).

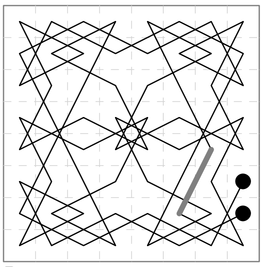





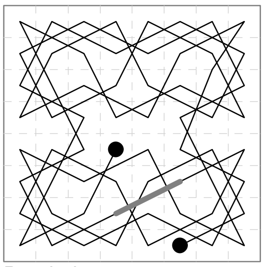





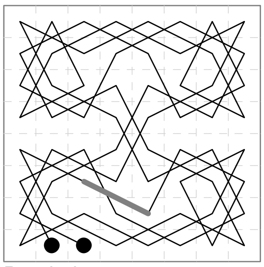
Four tours with maximal biaxial symmetry from V. Gorgias (three 1871, the fourth 1872).


Falkener 1892


Pearson 1907

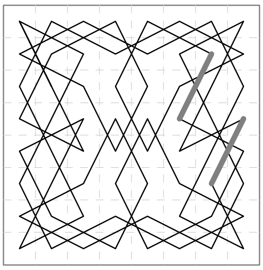

Bergholt 1917

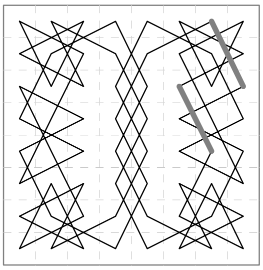

Bergholt 1917

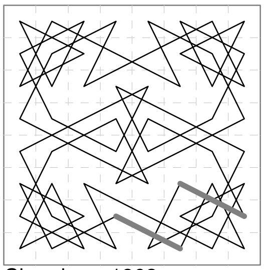
Others from Falkener (1892, #2 p.352), Pearson 1907, Bergholt 1917.

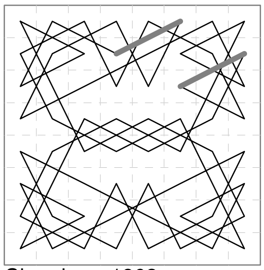
Closed tour examples from the History pages in ⌘ 6 include the Vandermonde (1771) tour, three from Laisement (1782), his 10A and two with stars in the corner, and the tour by Addison (1837).

Figure 13 from Wenzelides (1849) is shown below. Eugéne Pelletier de Chambure (1862, Plate 9) has three closed tours with approximate biaxial symmetry. The first has the maximum of 32 slants.

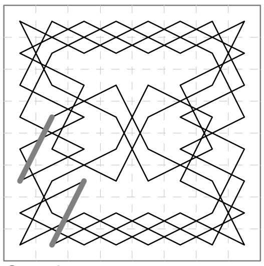

Wenzelides 1849 Fig.13

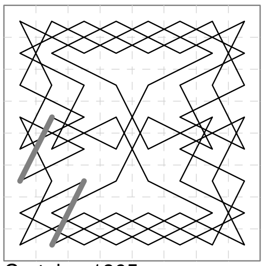

Chambure 1862

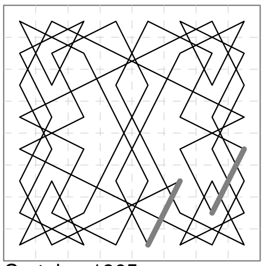

Chambure 1862

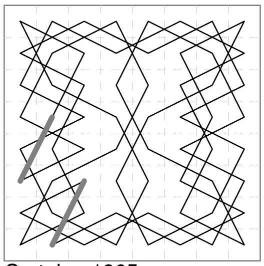

Chambure 1862

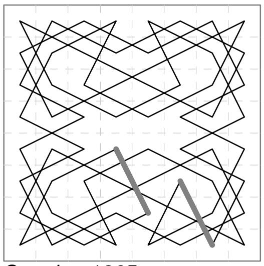
Six closed tours by Cretaine (1865) from his Plate B, after Vandermonde, and two of my own (2015) derived from the de Hijo (1882) pseudotour enumeration.

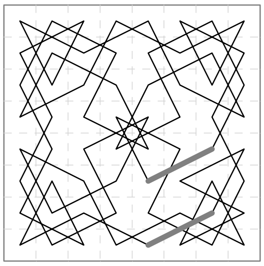

Cretaine 1865

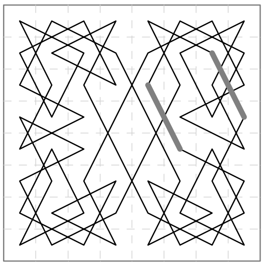

Cretaine 1865

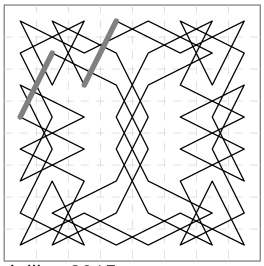

Cretaine 1865


Cretaine 1865

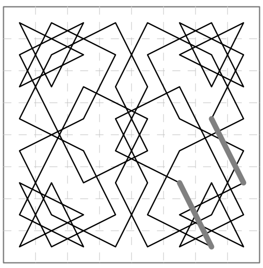

Cretaine 1865

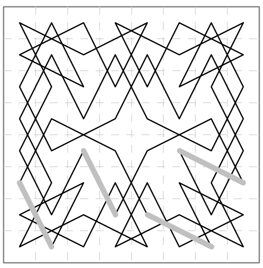

Cretaine 1865

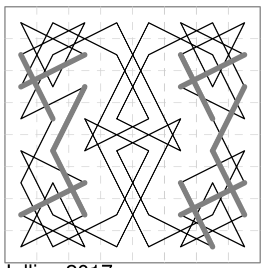

Jelliss 2015


Jelliss 2015

The following tour from the E. H. column (*Glasgow Weekly Herald* Jan 18 1873, problem III, numbered from f3 to g1) has the comment: "If 32 is connected with 1, and 64 with 33, the figure formed shows plicate symmetry"' This is a description of direct quaternary symmetry, meaning folding or pleated, that I've not seen used elsewhere.


E.H-. 1873

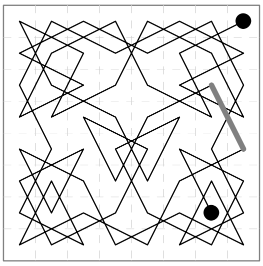
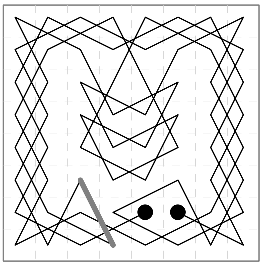
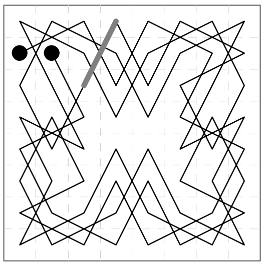
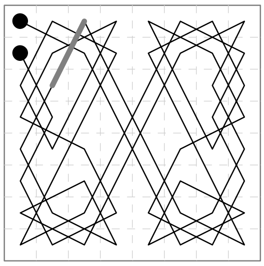

Parmentier c.1890


Jelliss 2017

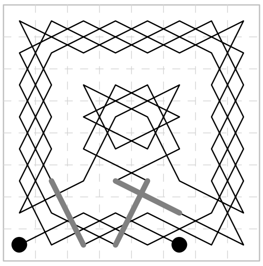
I also show a closed tour by General Parmentier (who was active in the 1890s) which is quoted by Kraitchik (1927) as 'remarkable for regularity'. The Jelliss (2017) example shows maximum 5×5 and mimimum 1×1 lozenges centred on the board, but it is not minimally asymmetric.

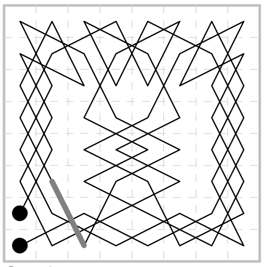
Approximate Axial Symmetry

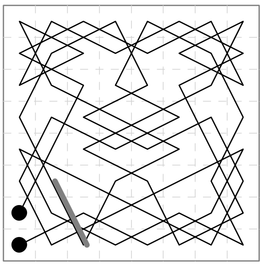
We begin with open tour examples, four from Käfer (1842). There is always a move (shown here by the thicker line) that does not have a symmetric mate. If this is replaced by a pair of moves linking its ends to the open ends of the tour the result is an axially symmetric pseudotour. Conversely, given a pseudotour of this type a tour can be constructed if a link between two of its circuits can be found.

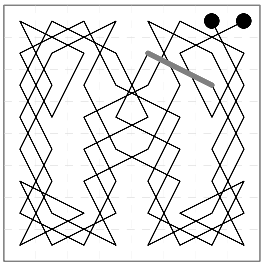





The Tomlinson (1845) example uses a 7-move linkage. Other 3-move linkage examples are the corkscrew pattern from Staeker (1849) and one with similar intersecting loxenge shapes from Charpentier (1849). The other is from Mann (1859).

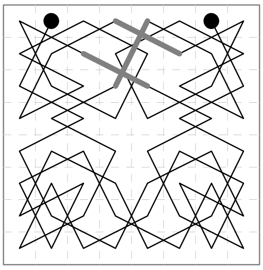

Tomlinson 1845

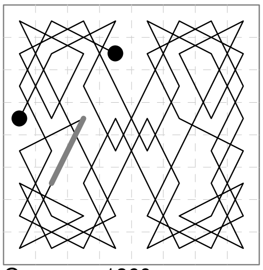

Staeker 1849

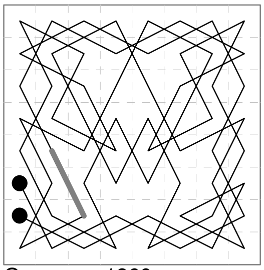

Charpentier 1849

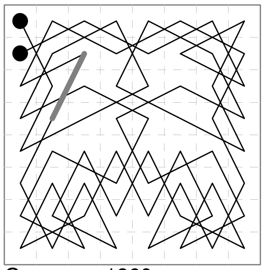

Mann 1859

A rarer type of open axially symmetric tour with the end-points symmetrically placed is #14 from a group of cuttings from a German newspaper, sent to me from Cleveland Library.


German c.1860

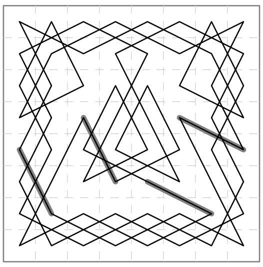

German c,1860

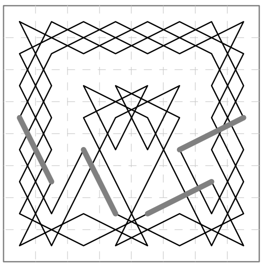

German c,1860

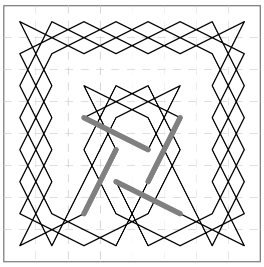

German c,1860

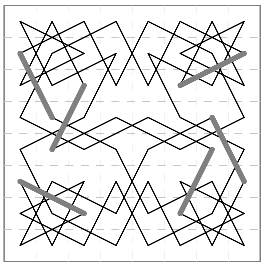
This was presented in cryptotour form (possibly from *Illustrierte Zeitung* 1852-63). It has three moves that break the symmetry. We show three others above from the German cuttings.

We now look at closed tours with maximal axial symmetry, from Wenzelides (1849), Chambure (1862), Falkener (1892) and the German cuttings.


Wenzelides 1849 Fig.5

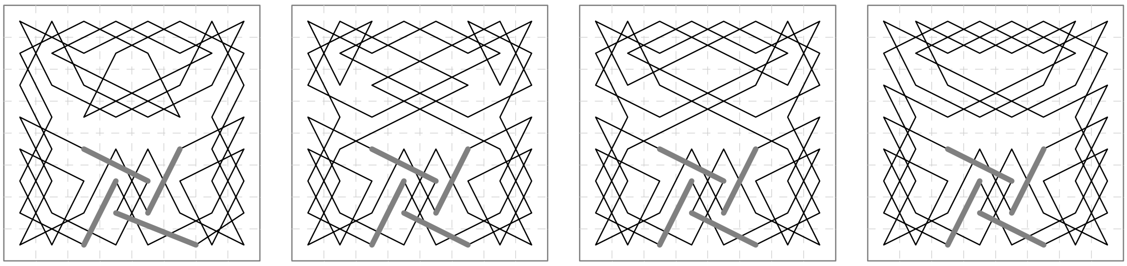

Chambure 1862


Falkener 1892

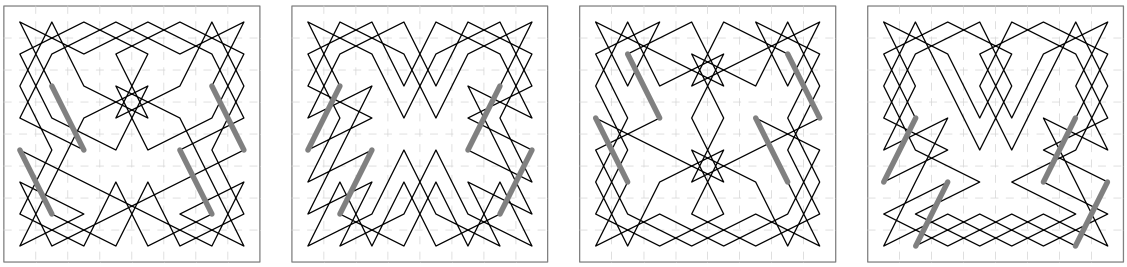

German c.1860

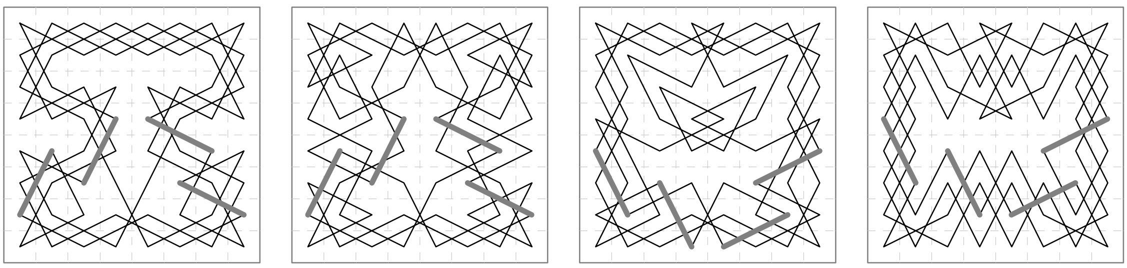
In these there are always two pairs of parallel moves that do not match. If either is replaced by the other sides of its rhomb the result is an axially symmetric pseudotour. Sometimes one pair is more obvious than the other. The earliest example in the History pages in ⌘ 6 is by Laisement (1782 4A). The second is #5 from Wenzelides shown above. The Chambure example shows a framed ‘butterfly’. The Falkener example, where the four mismatched links form a birotary pattern, can be regarded as a type of mixed symmetry. The fourth tour, the only closed example from the German cuttings described above, has six mismatched lnks.

In Cretaine (1865) There are 59 tours with maximal axial symmetry. Diagrams of 38 of these follow. They are arranged here according to the pattern of their four mismatched links. There are 4 with a cycle, formed of two overlapping diamonds, as in the Falkener tour above.

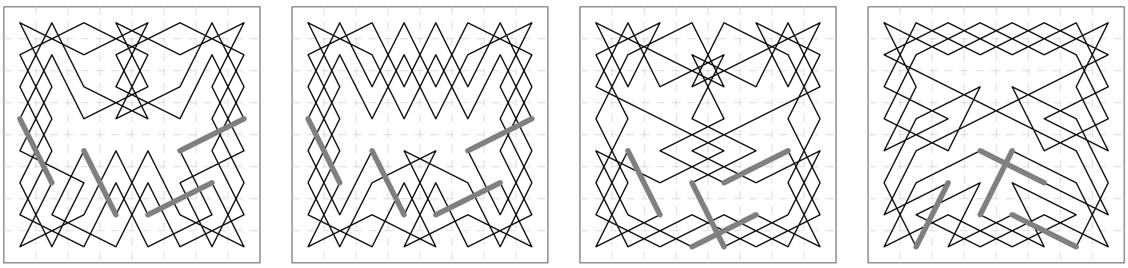


Here are eight using a pair of vertical lozenges, which seems to be the most effective type. The fourth tour, showing a nice ornamental letter W, not needing to be picked out in darker lines, was cited by Dawson (1928). There is a similar open tour design by G. Mann (1859).

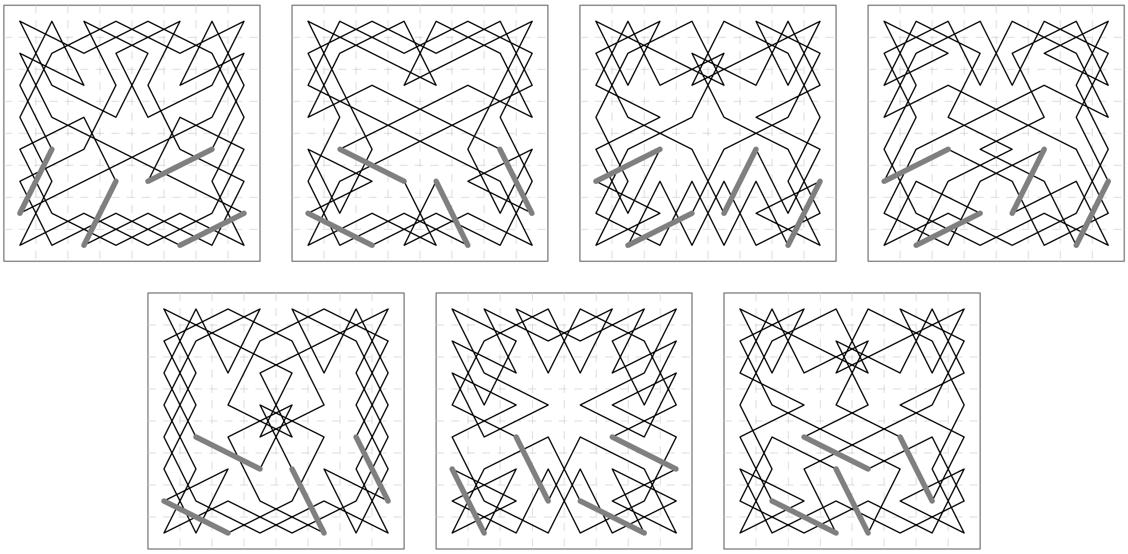




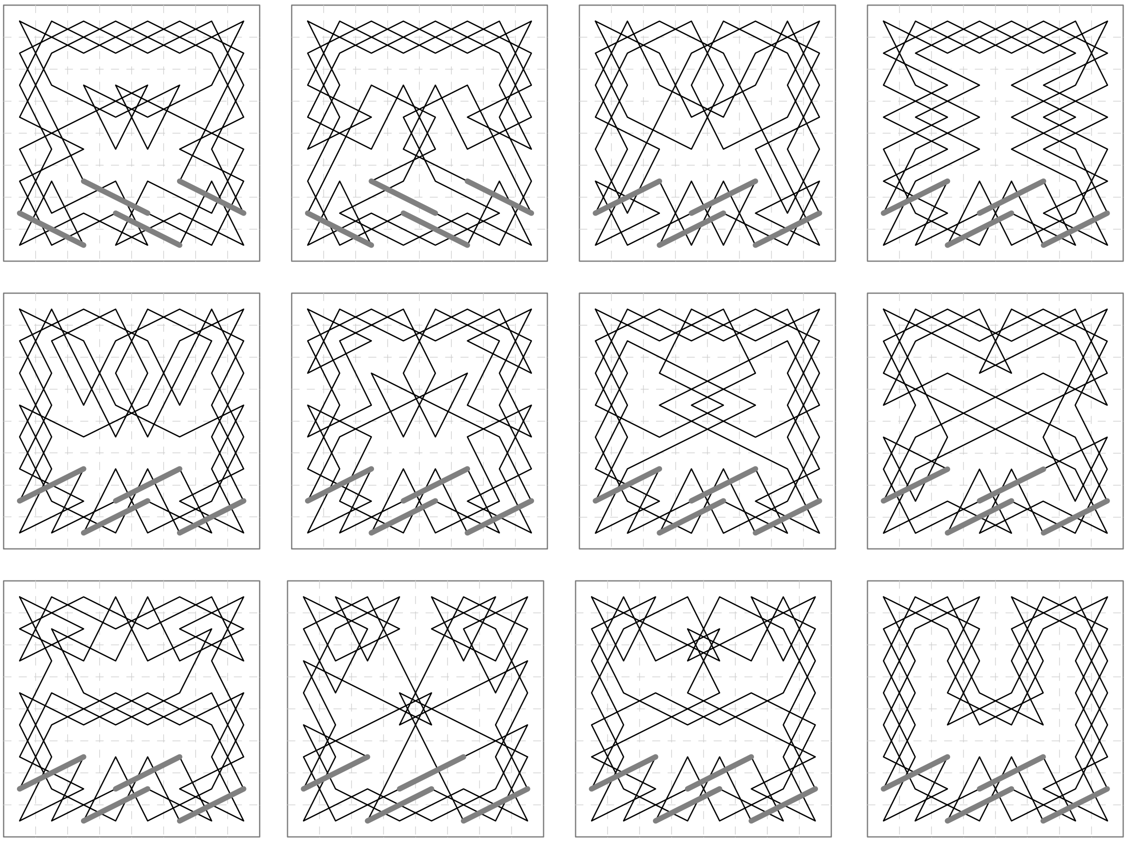
Others of two-diamond types:



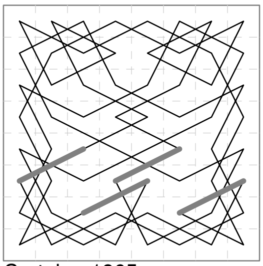
Now some from Cetaine with squares

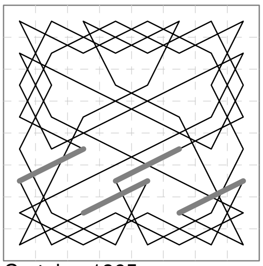


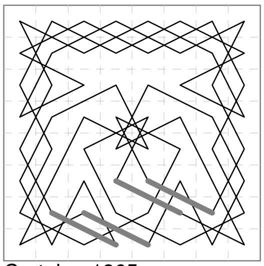
By far the most popular linkage is the use of two lozenges in the bottom three ranks. Here are 12 of this type out of 31 examples.

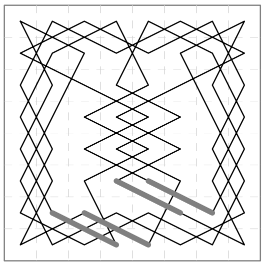


The lozenges can also be shown in different positions, we include one from Pearson (1907):

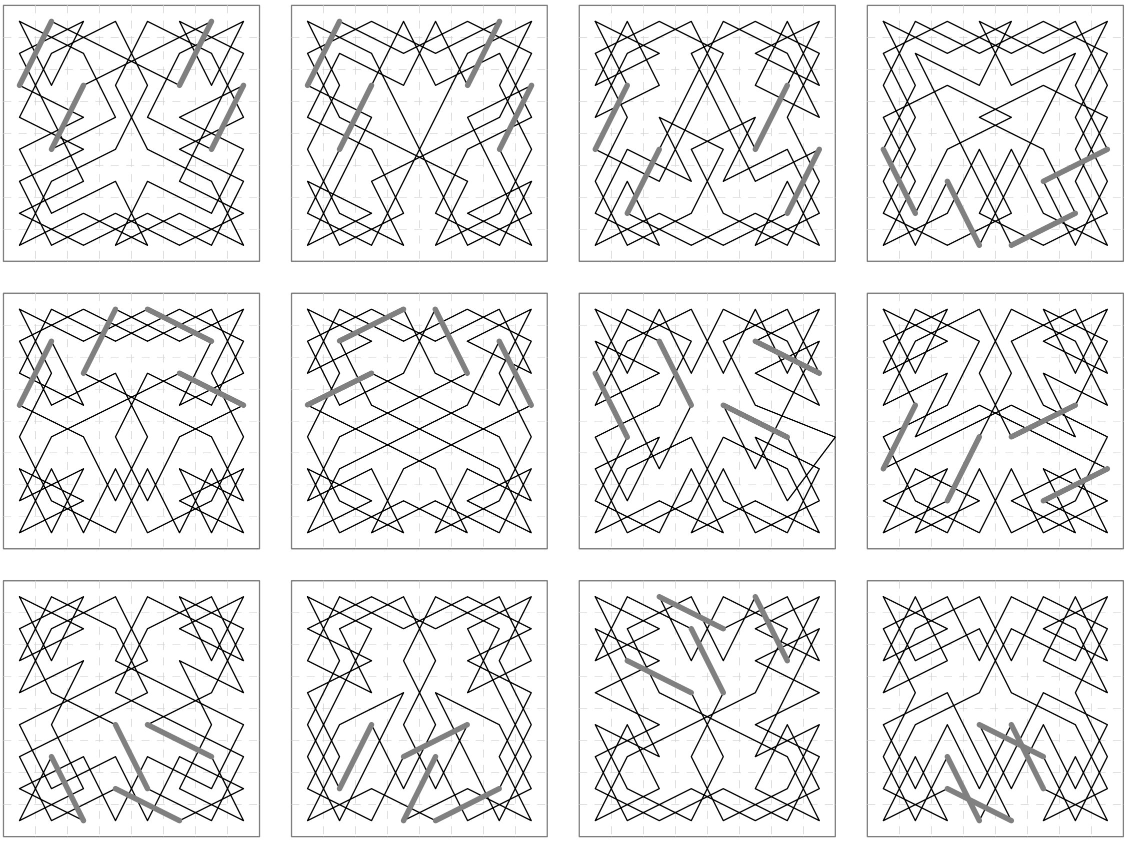

Cretaine 1865


Cretaine 1865

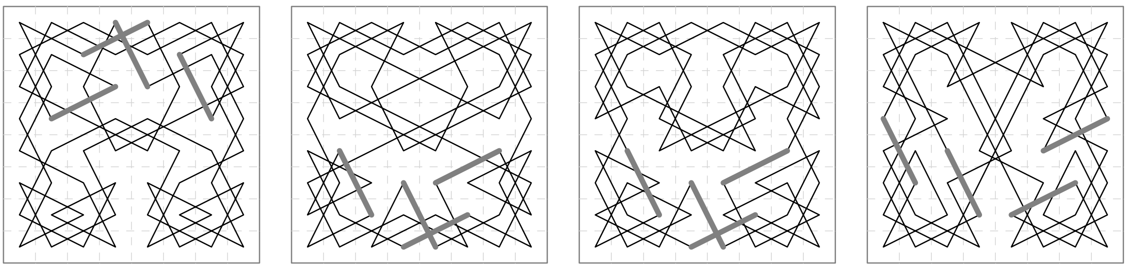

Cretaine 1865


Pearson 1907

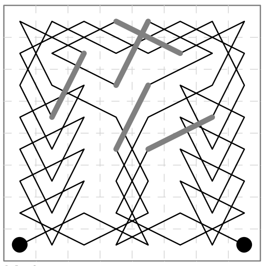
The next diagrams show 12 closed tours that solve a group of 24 unnumbered cuttings from a German newspaper, of similar date to Cretaine, sent to me from Cleveland Library.

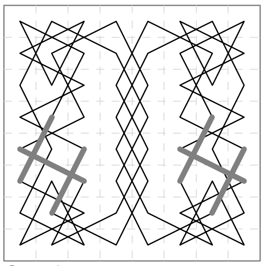


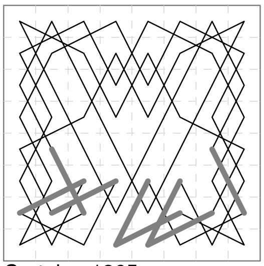
From Carle Adam, le jeune (1867) here are four closed tours.

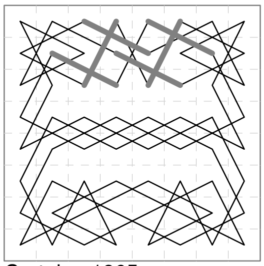


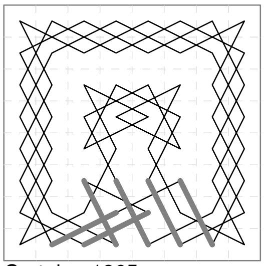
Examples with more mismatched links. Kafer (1842) open tour with five links. Cretaine (1865) and Schwenk (1991) c;losed with six links. Parmentier (1891) and Falkener (1892) open with seven links. The Parmentier (cited by Kraitchik 1927) has nine three-move lines.

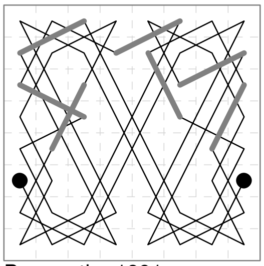

Kafer 1842

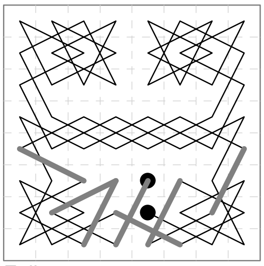

Cretaine 1865

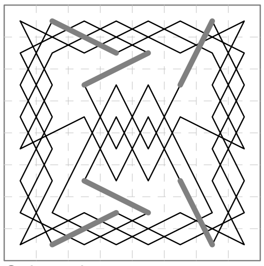

Cretaine 1865


Cretaine 1865


Cretaine 1865

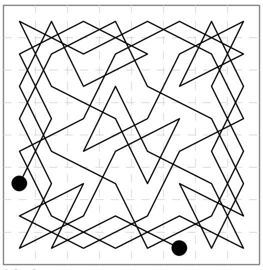

Parmentier 1891

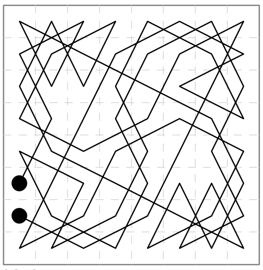

Falkener 1892

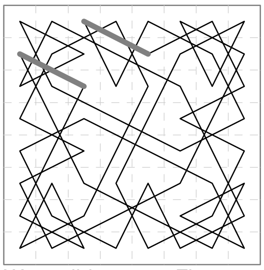

Schwenk 1991

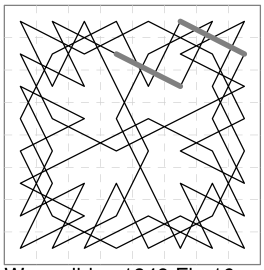
Approximate Birotary Symmetry

These two examples from Käfer (1842) are not quite of maximum birotary symmetry, but are approximately rotary with birotary components. An example to be found in the History pages in ⌘ 6 is one by Laisement (1782 6A). This is identical (apart from rotation) to #14 by Wenzelides (1849) who gives two other closed tour examples, #15 and #16 shown here.


Kafer 1842

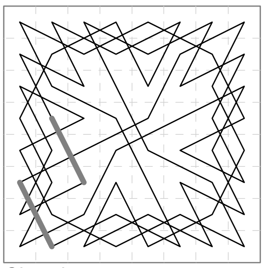

Kafer 1842

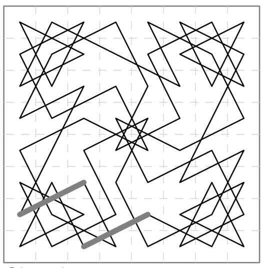

Wenzelides 1849 Fig.15

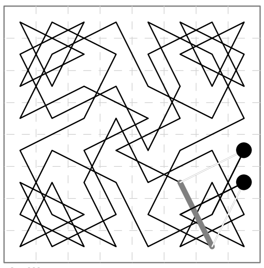

Wenzelides 1849 Fig.16

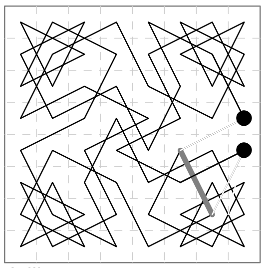
Surprisingly Cretaine 1865 has none strictly of this type.

Two from Chambure 1862 follow. As noted earlier instead of seeking a pair of parallel links, which cannot always be found, an alternative is to insert a single link, connecting the two 32-move circuits, and deleting one move from each circuit, thus forming an open tour. The other two diagrams show this method applied to Vandermonde's circuits in their birotary formation (Jelliss 2015).

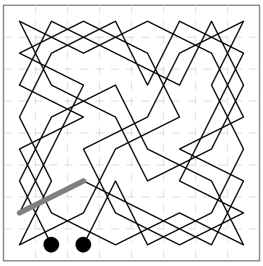

Chambure 1862

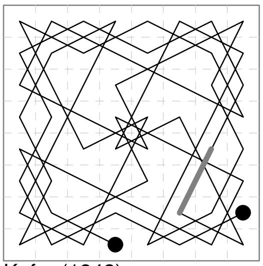

Chambure 1862

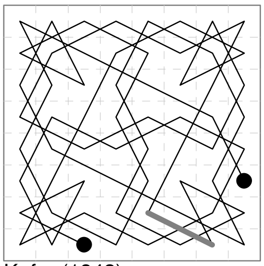

Jelliss 2015

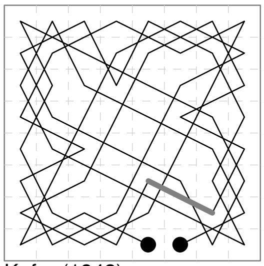

Jelliss 2015

In Käfer (1842) there are five open tours with maximal birotary symmetry, as shown here.

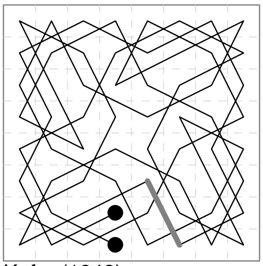

Kafer (1842)

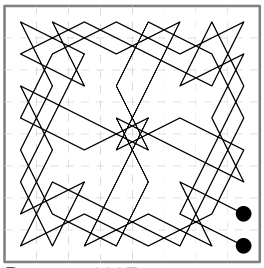

Kafer (1842)

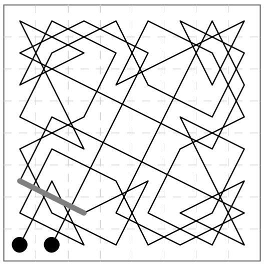

Kafer (1842)

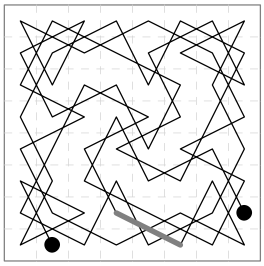

Kafer (1842)

In these the two ends link to ends of the move marked in grey. Deleting this link and joining up the ends results in a pseudotour with exact symmetry. The other examples are by Pearson (1907), Bergholt (1917) and Jelliss (2015) derived from the de Hijo (1882) pseudotour enumeration.


Kafer (1842)


Pearson 1907


Bergholt 1917


Jelliss 2015

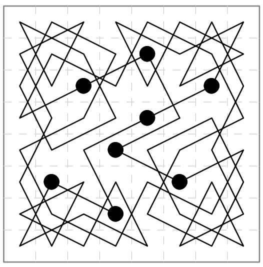
Approximate Bergholtian Symmetry

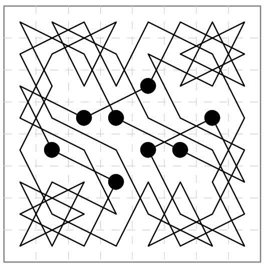
Bergholtian symmetry is centrosymmetry of the type that passes twice through the centre, and is possible only on rectangular boards that have one side odd and the other singly even (e.g. 6×7 or 5×10). See ⌘ 4 for examples. When numbered from the ends of the central cross it has the property that diametrally opposite numbers add to a constant **sum**. This is in contrast to the more common tours with Eulerian symmetry in which opposite numbers have a constant **difference**.

On other boards it is possible to construct tours which have partial Bergholtian symmetry, not all of the pairs adding to the constant. An example of this was sent to me by Prof Knuth, which inspired the following examples, which I think show the maximum amount of Bergholtian symmetry on the 8×8 board (see also a 10×10 example in ⌘ 5).

The dots which, form two rhombs, mark the ends of the symmetric parts. One rhomb is connected in the opposite way to the other and this is the only asymmetry. Diametrally opposite numbers add to 62 with the exception of the three pairs in bold and circled. If one of the rhombs is reversed this results in a tour with Eulerian symmetry. The crosslinks in the first are c6-e7-f5-d4-f3 and b3-d2-c4-e5-g6, and in the second d5-f4-h3-g5-e4 and d3-b4-a6-c5-e6.

47	52	45	26	5	24	21	28
44	3	48	53	**64**	27	6	23
51	46	1	4	25	22	29	20
2	43	54	49	**30**	**63**	12	7
55	50	**31**	**62**	13	8	19	60
42	33	40	37	58	61	16	11
39	56	35	**32**	9	14	59	18
34	41	38	57	36	17	10	15

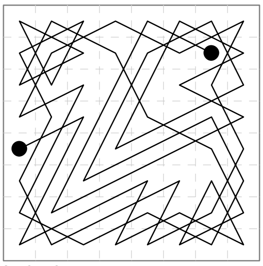


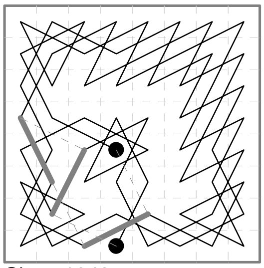


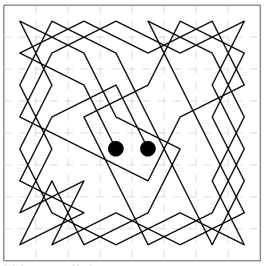
3	14	53	16	55	34	37	40
52	17	2	13	42	39	56	35
31	4	15	54	33	36	41	38
18	51	**32**	1	12	43	**62**	57
5	**30**	19	50	61	**64**	11	44
24	21	26	29	8	47	58	**63**
27	6	23	20	49	60	45	10
22	25	28	7	46	9	48	59

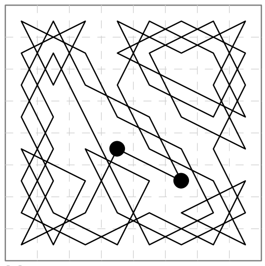
Approximate Diagonal Symmetry

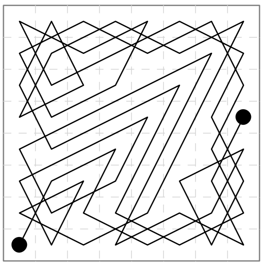
It is also possible to construct tours with a noticeable diagonal axis. However, this is a subject where little work seems to have been done, and the results are not very precise. A few examples are: (1) *Leipziger Illustrirte Zeitung* (1845). (2) Solution to a cryptotour posed in *Sissa* (1848). This tour is also #87 in Mann (1859). (3) The only open tour #2 in Wenzelides (1849) *Schachzeitung*, is similar to the Monneron Malabar tour of 1776 in having ends on the middle cells. It shows a certain amount of symmetry about the a1-h8 diagonal. (4) Tour #6 from G. Mann (1859). (5) solution to ¶196 in *Le Siecle* 15 Jun 1877 (6) solution to ¶472 by A. Béligne in *Le Siecle* 3 May 1878 (7) Collinian tour with triangular half-board of braids (Jelliss 2016). (8) Five chevrons along a1-h8 diagonal (Jelliss 2017).

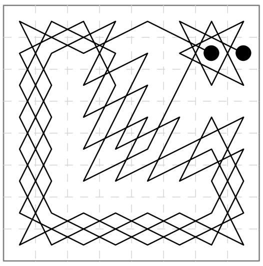

Leipzig 1845

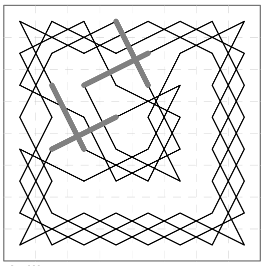

Sissa 1848

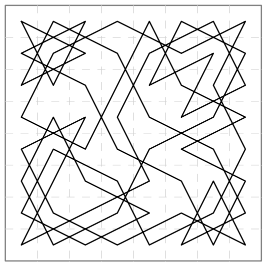

Wenzelides 1849


Mann 1859

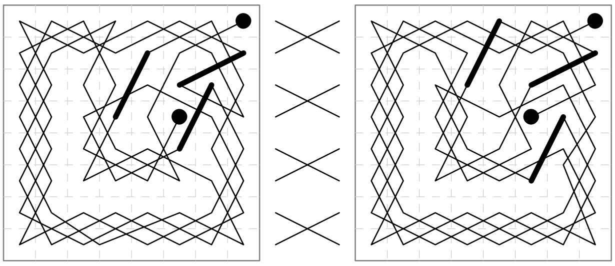

Le Siecle 1877


Beligne 1878


Jelliss 2016


Jelliss 22 Dec 2017

At one time I considered the question: Can a tour remain a tour if the pairs of ranks are transposed? That is ranks 1-2, 3-4, 5-6, 7-8. I noted that in this transformation vertical moves and horizontal straights remain knight moves, but not horizontal slants. So I considered if a tour without horizontal slants is possible, and concluded No. There must be at least one horizontal slant. It appeared that such a horizontal slant can be incorporated, but only if the ends of the tour are a zebra move apart! Here is the result I found. A pair of dual tours.



The slants are shown in bold. The horizontal slant transforms into the zebra move between the end points, and vice versa. I suppose this is just a curiosity.

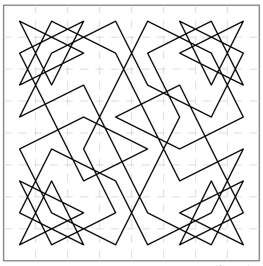
Exact Symmetry

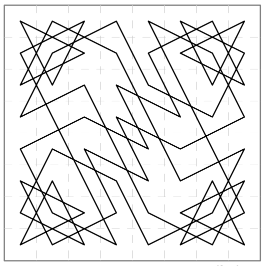
From the point of view of symmetry the 8×8 board is one of the least interesting. The only form of exact overall symmetry possible in an 8×8 knight tour is the type unaltered by a 180 degree rotation, known as binary oblique symmetry, or rotary symmetry. Furthermore it is of the Eulerian type which circles round the centre as opposed to the Bergholtian type which passes twice through the centre, which is possible on some rectangular boards. Professor Donald E. Knuth, in a letter to me on 5 Jan 1993, stated the number of symmetric closed knight tours on the 8×8 board to be **608,233**. This total has been confirmed by the work of Brendan D. McKay (1997).

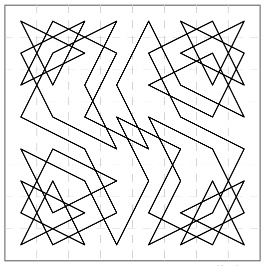
Symmetric Tours from Historical Sources

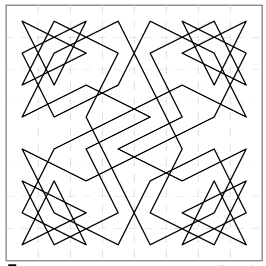
By far the earliest fully symmetric 8×8 tour is that by Bhatta Nilakantha of Sri Lanka (1640). The next to compose symmetric tours was Leonhard Euler (1757) and (1759). Strangely Euler's work in this regard was followed up only very slowly by other authors (see ⌘ 6). The emphasis at that time was on constructing tours between given end-points, regardless of pattern, or trying to show reflective rather than rotative symmetry. Five of Euler's symmetric tours are of 'Double Halfboard' type, joining a 4×8 tour to a rotated copy. The next six symmetric tours published were also of this type.

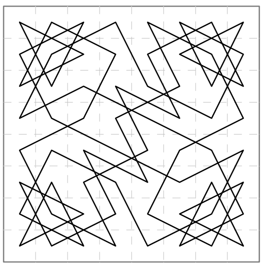
1780 Gaspard Monge. The 13 tours recently found by H. Bastian and attributed to G. Monge (1780), but not published at that time, run contrary to the above trends.

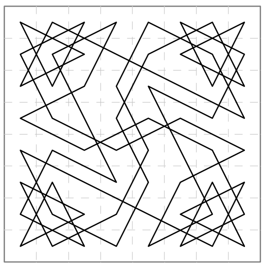

1 (io~)

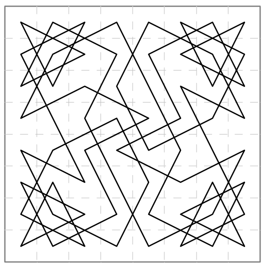

2 (ip)

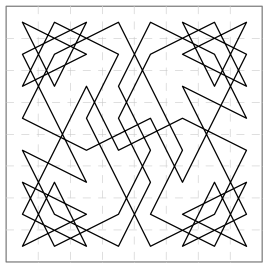

3 (lp)

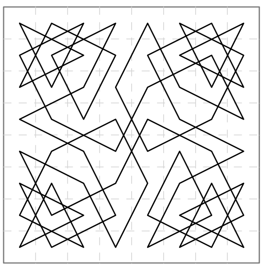

5 (io=)

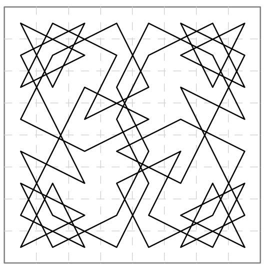

6 (in=)

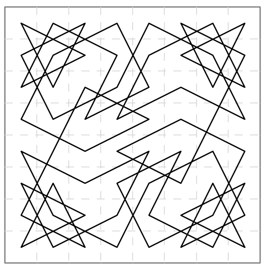

7 (jj~)

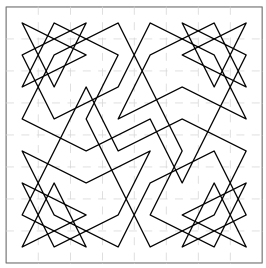

8 (ko=)


9 (jn=)


10 (kk=)

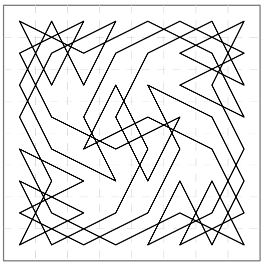

11 (jo~)


12 (lo~)

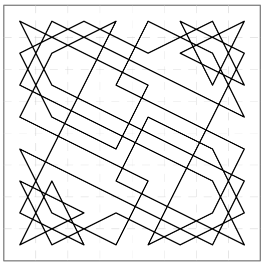

13 (ln=)

They all have the same 'meteor' corner pattern but show varied centres (in=) (io=) (io~) (ip) (jj~) (jn=) (jo~) (kk=) (ko=) (ln=) (lo~) (lp), anticipating the work of Wenzelides (1849). See the History in ⌘ 6 for Monge tour #4 which has the same centre (lp) as #3.

1842 Victor Käfer is the first tour composer since Euler to place emphasis on showing symmetry in his tours, though only two have exact symmetry. The first of these is in fact a tour of mixed quaternary symmetry with maximum oblique symmetry; the only direct symmetry being in the four moves c4-d6, c5-d3, e3-f5, e6-f4. The other includes the largest and smallest squares centred on the board that can be delineated by knight moves, a task that has been shown in many tours since. The knight moves within the square run in only two directions. Now that Käfer had shown what was possible (the Monge tours being unknown), a wider range of symmetric tours began to appear.



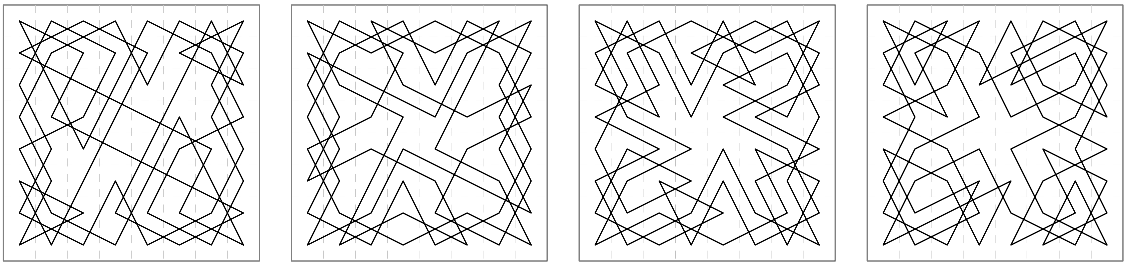
jj~



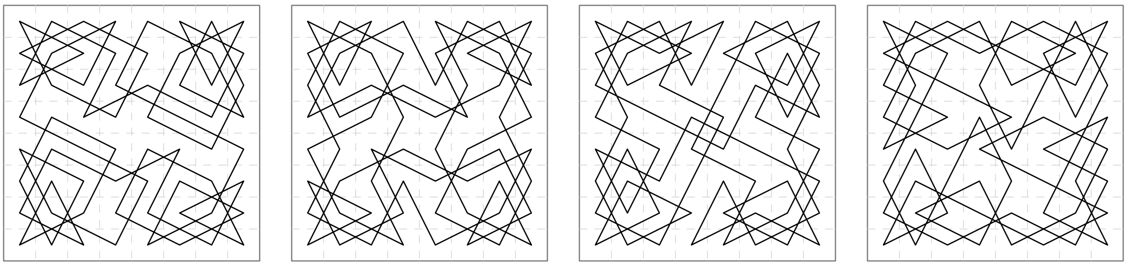
dk=

1844: Ferdinand Julius Brede (1800-1849) *Almanach für Freunde vom Schachspiel* Altona 1844. This work, like Käfer (1842), has 24 tours in diagram form at the end of the book, but printed one to a page instead of on a fold-out sheet. All the tours show 180 degree rotational symmetry, apart from #14 which is identical to Laisement (1782 #6A) but rotated 90 degrees, and shows approximate birotary symmetry. All are diagrammed here, except #14.

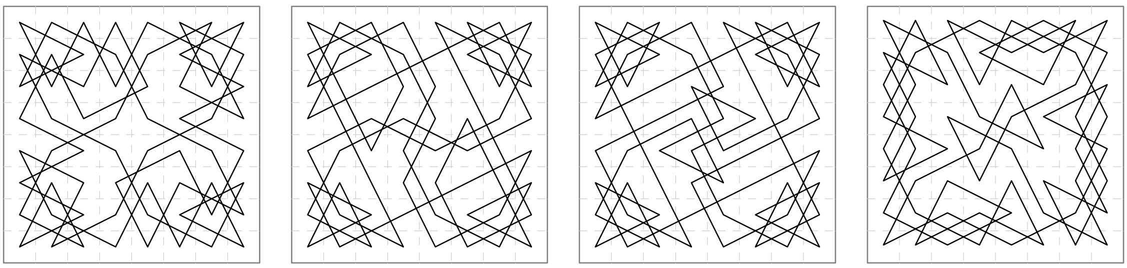
#1 (ii~), #2 (ff~), #3 (gg~), #4 (bb~).



#5 (bb~), #6 (cc~), #7 (nn~), #8 (oo~).

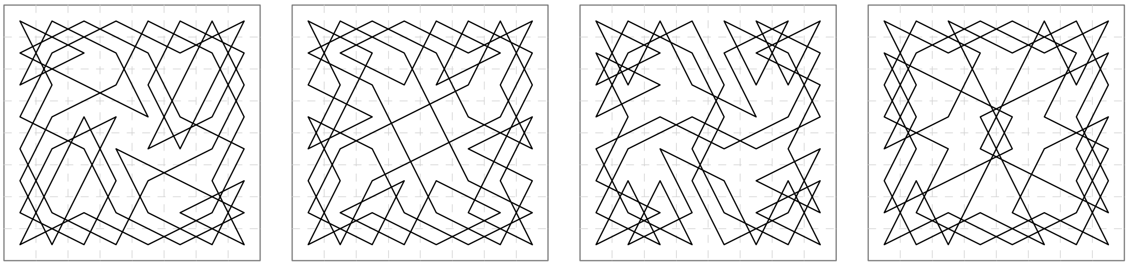


#9 (hh), #10 (jj~), #11 (kk~), #12 (ee~).

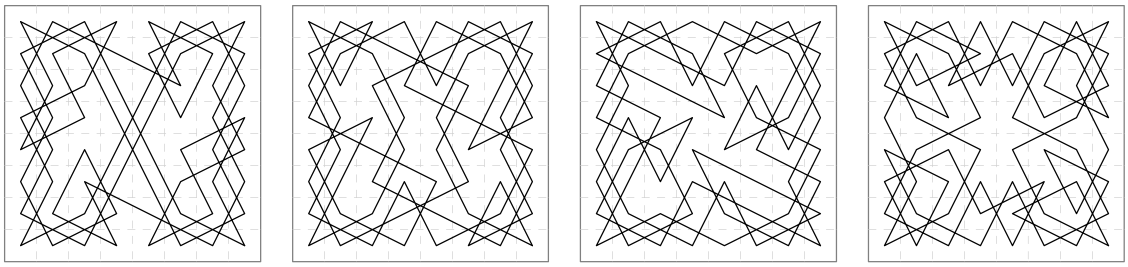


Three of the tours (#5, #6, #12) are cited by Bergholt (1918) as examples of mixed quaternary symmetry; in fact #5 shows maximum oblique quaternary symmetry. Tours #10 and #11 both have sizes 9 and 1 squares centred (as shown by Käfer). Apart from #22 and #23 they all have the four central angles alike (bb~) (cc~) (dd=) (ee~) (ff~) (gg=~) (go=) (ll~) (lp) (nn~) (oo~).

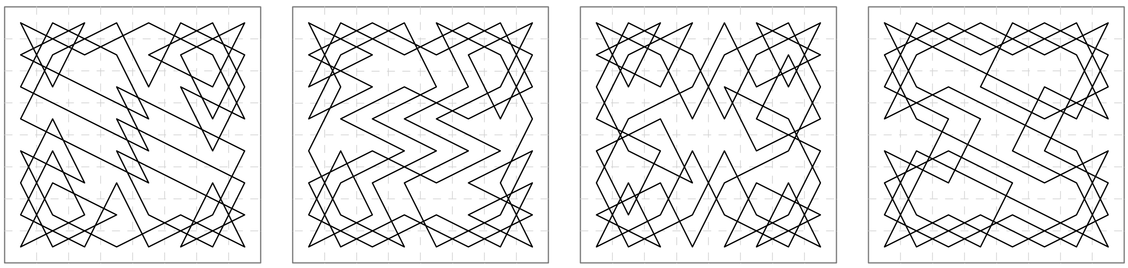
#13 (ll~), #15 (ii~), #16 (jj~) similar to Käfer. #17 (nn=)



#18 (dd=), #19 (cc=), #20 (ll~),#21 (bb=).



#22 (lp), #23 (go=), #24 (gg=). and a tour by Tomlinson (fi=)



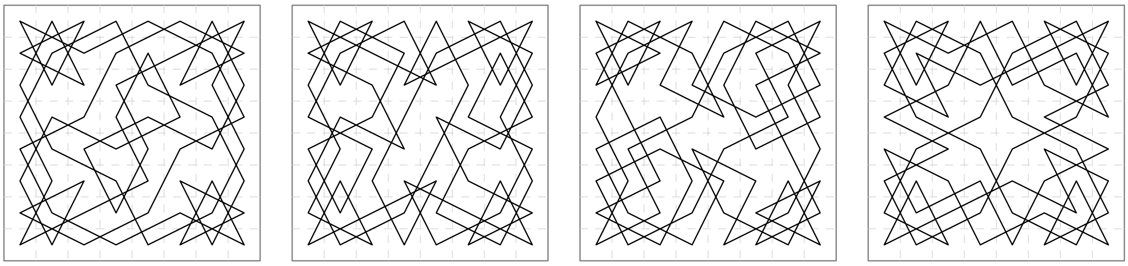
Tomlinson 1845

In #24 the incomplete hexagons such as d3-b4-a6-b8-d7-e5 were emphasised. They do not immediately strike the eye otherwise.

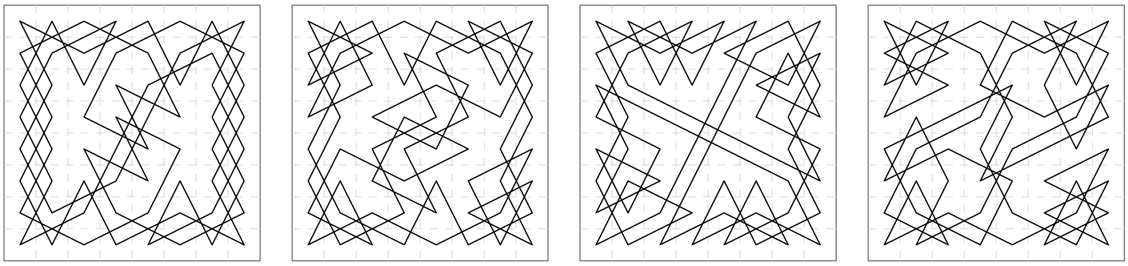
1845: Charles Tomlinson (fi=) has one symmetric tour. In a discussion of symmetry he gives the closed tour shown above (14/16 squares and diamonds, centre angles fi= type), making much of the fact that diametrally opposite cells add to 32 no matter where the numbering is started. Tours from this source are widely cited (e.g. Basterot 1853, Knight 1859).

1849: Carl Wenzelides gave tours with a wider variety of centres (cn~) (co=) (dd~) (di=) (dp) (fl=) (gh) (hj) (ii~) (il~) (in=) (jp) (kn=) (ln=) (pp). His symmetric magic tours are not included here. For ease of reference I show diagrams of these tours in the sequence they were presented. Figures not included here are either pseudotours, or indications of construction methods. Other similar tours can probably be identified by joining up the pseudotour circuits differently.

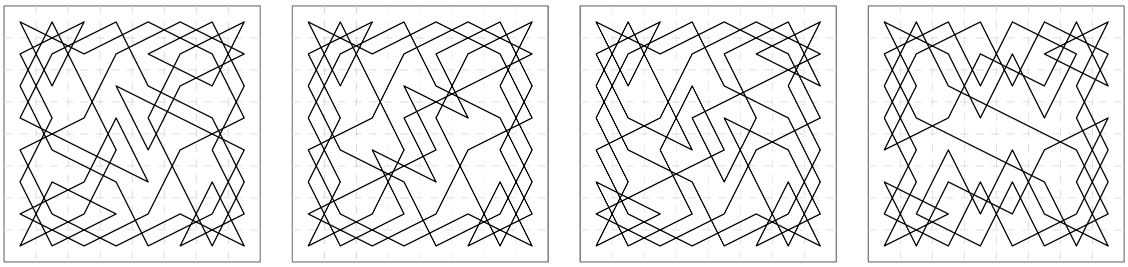
#4 (jj~), #6 (il~), #7 (hl), #8 (hh)



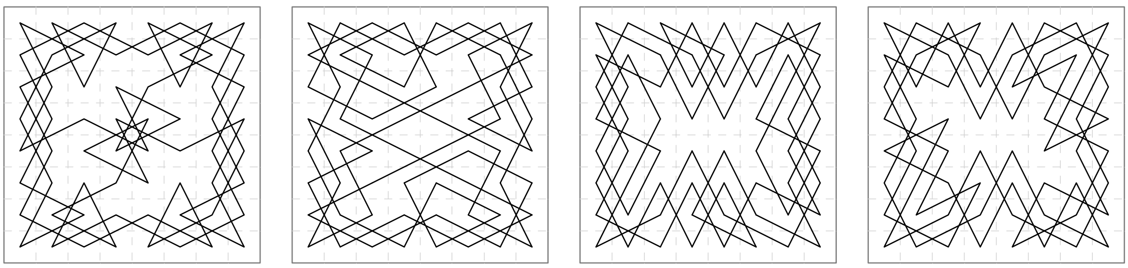
#9 (dp), #17 (kn=), #18 (dd~), #19 (hj),



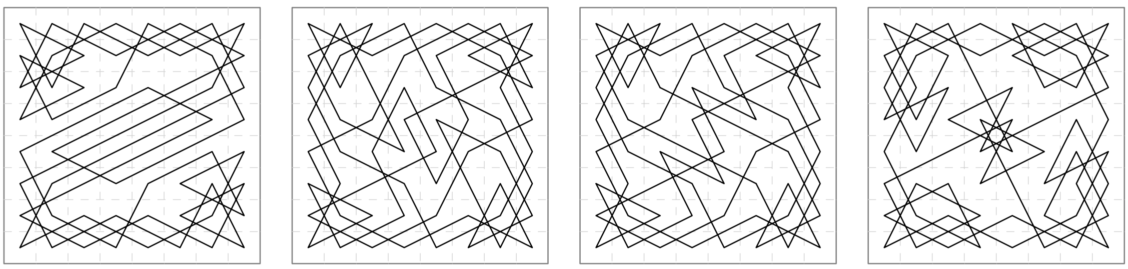
#20 (co=), #21 (cn~), #22(ln=), #26 (gi~),



#28 (pp), #29 (ii=), #30 (gg=), #32 (gg=).



#34 (di-), #35 (ln=), #36 (in=), #38 (pp),



#40 (ii~), #42 (gh), #43 (fl=), #52 (jp)

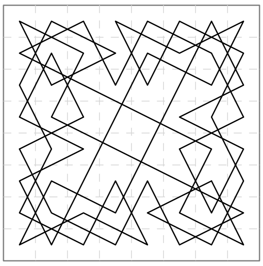
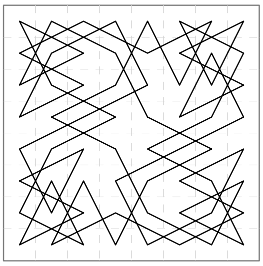
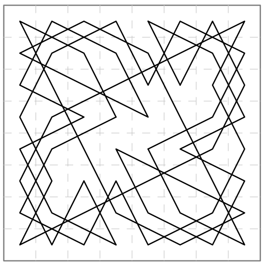
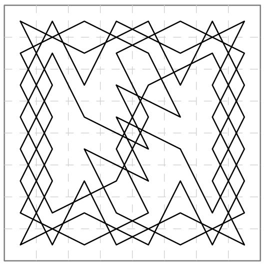
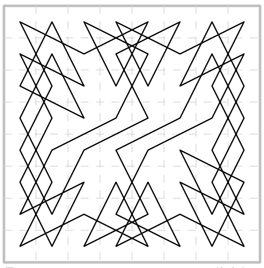
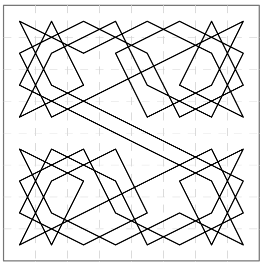
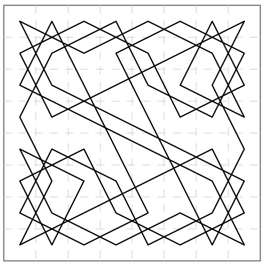
   

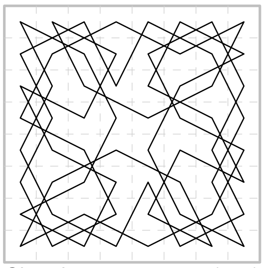
Fig.53 is the same as Fig.19. A third tour of (gg=) type (Fig.54) is the same as Brede (1844, #24). Fig.48 (kn=) is in fact a symmetric linking of the Vandermonde circuits. The final section of this article, Fig.55 onwards, is about half-board tours

This striking symmetric closed tour (hk type) is by **Arnold Pongracz** (1855) and joins tours of two irregular-shaped half-boards. Symmetric double half-board and fullboard tours by **Slyvons** (1856 #13 and #23). Symmetric tour by **Chambure** (1862 #27).


Pongracz 1855 (hk)

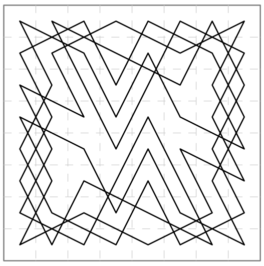
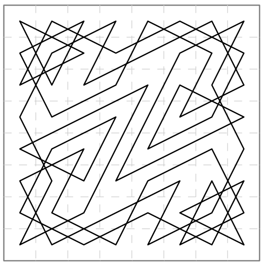
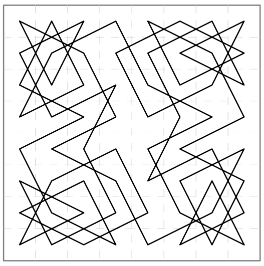
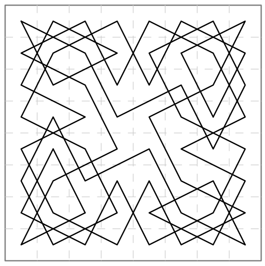

Slyvons 1856 (bd=)


Slyvons 1856 (di~)

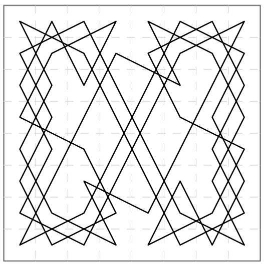
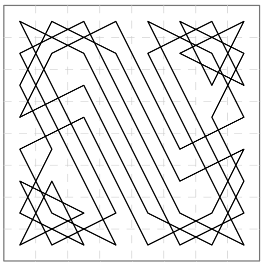
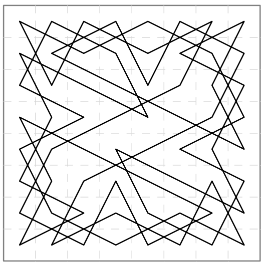
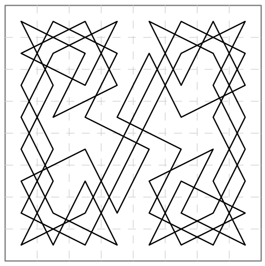

Chambure 1862 (cc~)

Here are the 36 symmetric tours by **Haldeman** (1864) with some striking designs.

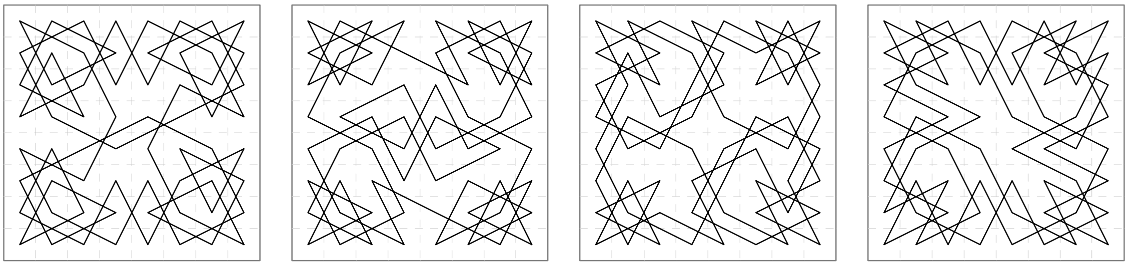
#55 (go=), #56 (dl~), #57 (bb~), #58 (ff~)

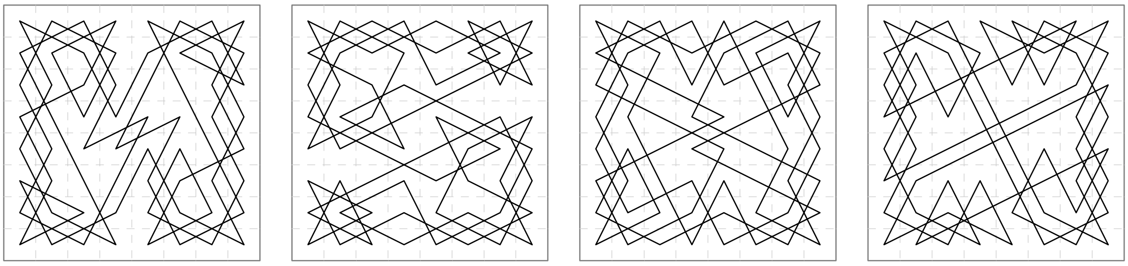
#59 (dd=), #60 (di=), #61 (il~), #62 (dk=)

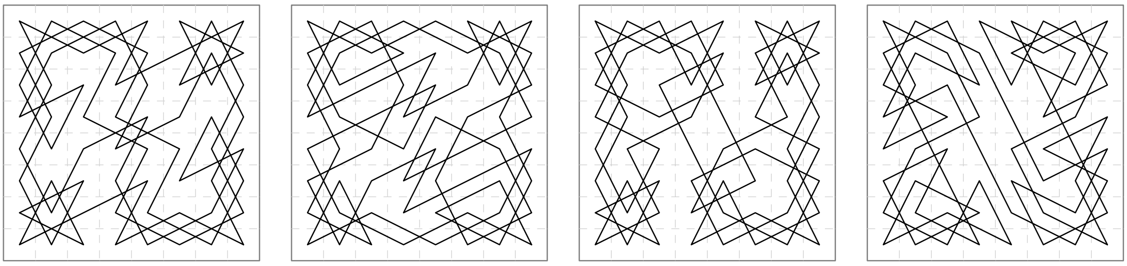
#63 (cj~), #64 (kk=), #65 (hh), #66 (gh)



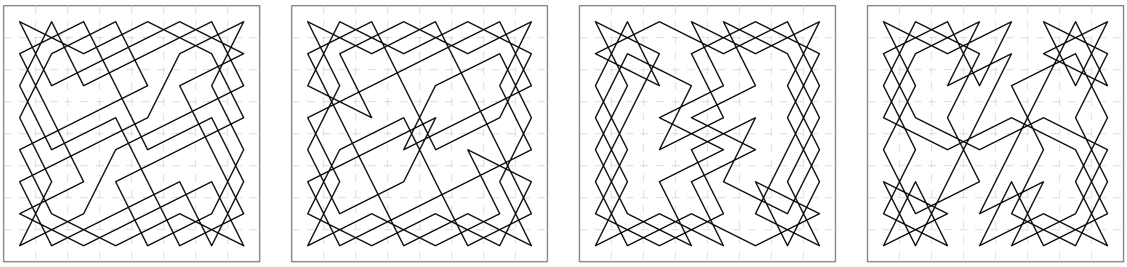
#67 (gp), #68 (il=), #69 (ko~), #70 (dd~),



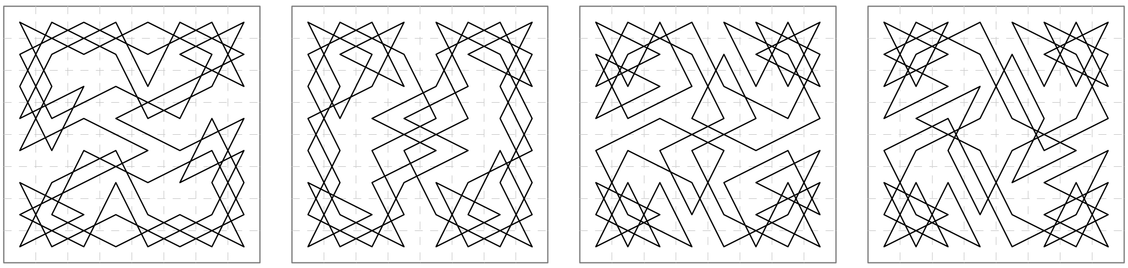
#71 (kp), #72 (kl~), #73 (hi), #74 (bi=),



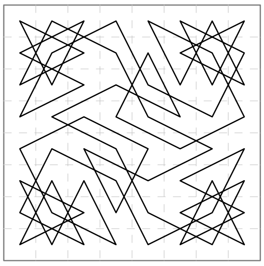
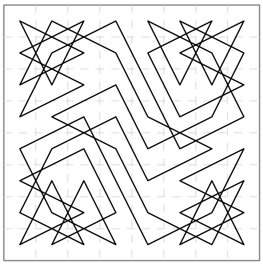
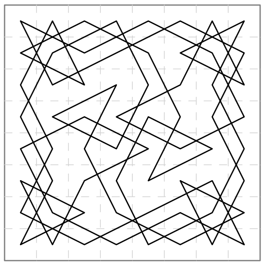
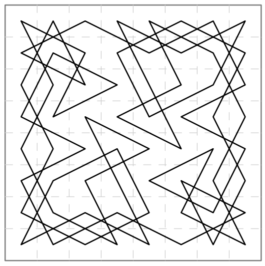
#75 (ek~), #76 (kp), #77 (ko~), #78 (ij~)



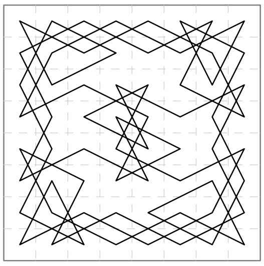
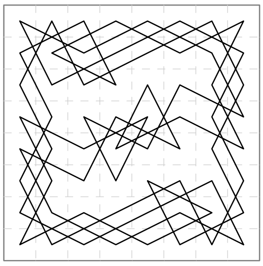
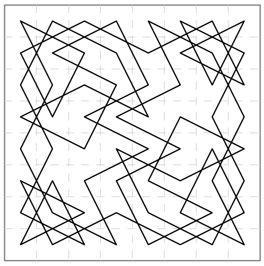
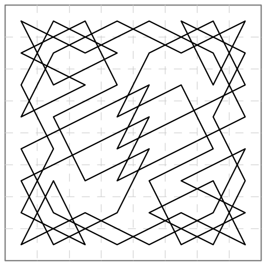
#79 (bo=), #80 (ko~), #81 (kk=), #82 (hh),



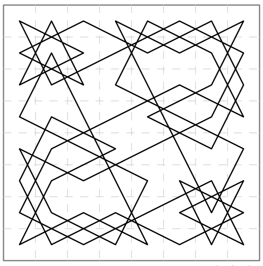
#83 (hk), #84 (hk), #85 (ko=), #86 (bo=),

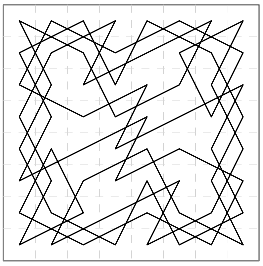





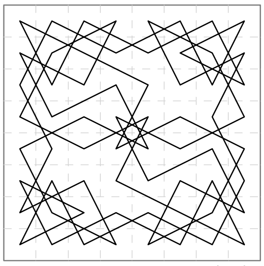
#87 (np), #88 (np), #89 (lo~), #90 (lp).

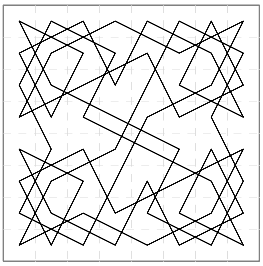
Cretaine (1865) gives 5 symmetric tours on Plate A one being a double half-board tour by Euler. His (nn=) case is the same as Brede #17. Here are the other three.

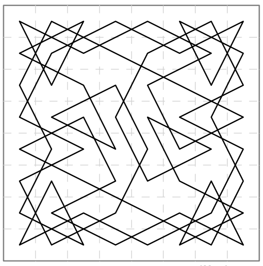

(gi=)

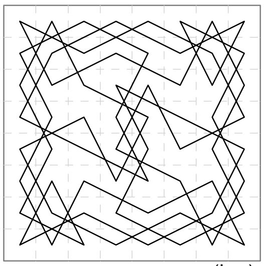

(fp)

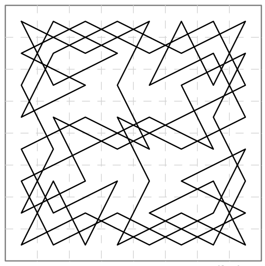

(pp)

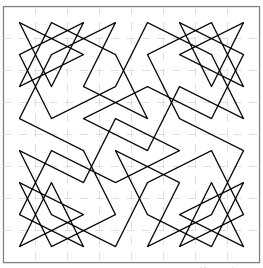
Feisthamel's column in *Le Siècle* 1876-77 has the following twelve symmetric tours:

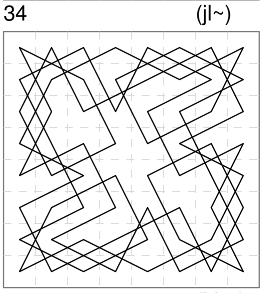

22 (de=)

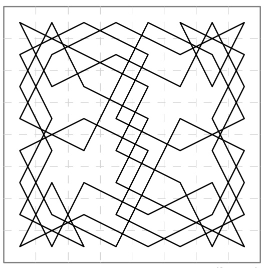

34 (jl~)

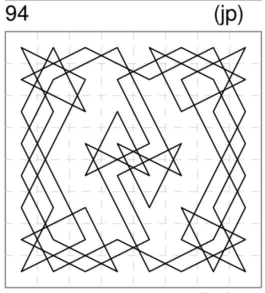

82 (jn~)


94 (jp)

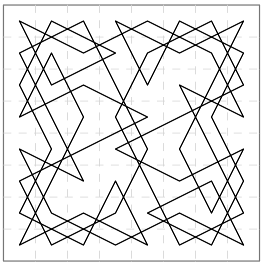

106 (ln=)

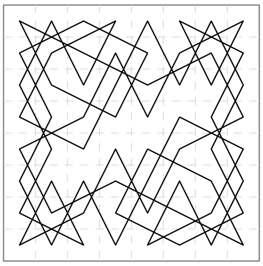

124 (bb~)

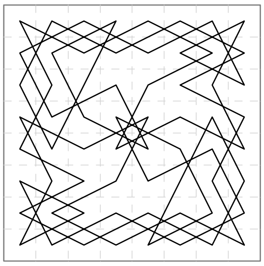

130 (kn~)

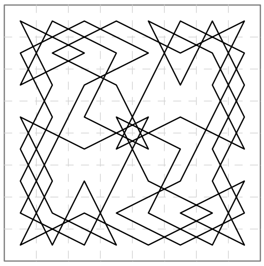

136 (kp)

¶22 (24 Nov 1876), ¶34 (8 Dec 1876) modified from a pseudotour, ¶82 (2 Feb 1877), ¶94 (16 Feb 1877), ¶106 (2 Mar 1877) modified from pseudotour, ¶124 (23 Mar 1877), ¶130 (30 Mar 1877), ¶136 (6 Apr 1877) modified from a pseudotour, shown above. And ¶148 (20 Apr 1877), ¶160 (4 May 1877) modified from pseudotour, ¶202 (22 Jun 1877), ¶238 (3 Aug 1877). The modification from the pseudotour examples is in every case the rotation of a pair of 1-1 links.

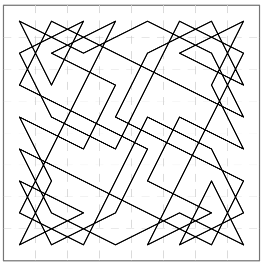

148 (jo~)

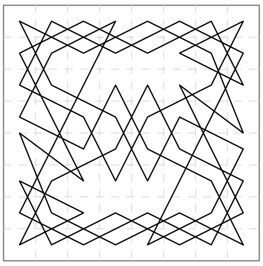

160 (bg~)

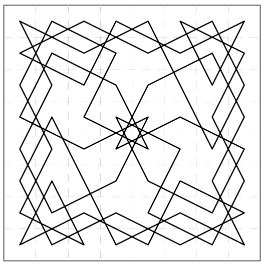

202 (pp)

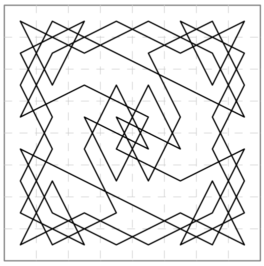

238 (pp)

The next tour solves two cryptotours ¶12 (24 Apr 1877) and ¶19 (1 May 1877) in *Le Gaulois*.in the column ‘Un Probleme Quotidien’ run by Edouard Fy and later Francois Accloque. Like a tour by Käfer (1842) it divides the maximum knight square into rectangles. The others solve ¶26 (8 May 1877) design by E. Portanguen, and ¶40 (22 May 1877) anonymous, and ¶75 (26 Jun 1877) design by Ernest Forestier.

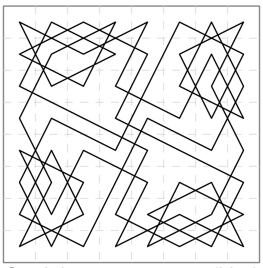

Gaulois 12 & 19 (kk~)

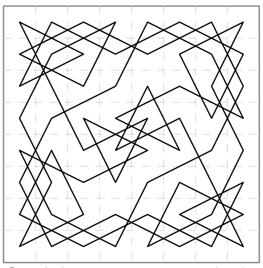

Gaulois 26 (ee=)

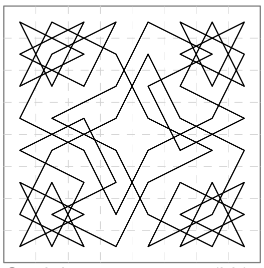

Gaulois 40 (pp)

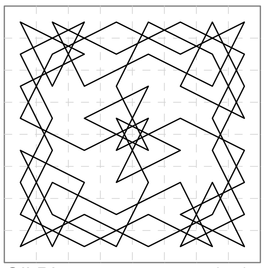

Gaulois 75 (np)

Tours that solve ¶89 (11 Jul 1877) design by M. A. F. and ¶131 (25 Aug 1877) and ¶138 (1 Sep 1877) in ‘Un Probleme Quotidien’ in *Le Gaulois* and ¶7 (25 Nov 1879) in the ‘Passe-Temps Quotidien’ column conducted by E. Framery in *Gil Blas*.


Gaulois 89 (kk~)


Gaulois 131 (op)

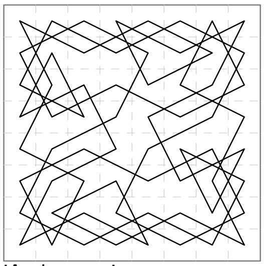

Gaulois 138 (hh)

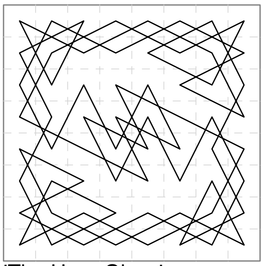

Gil Blas 7 (pp)

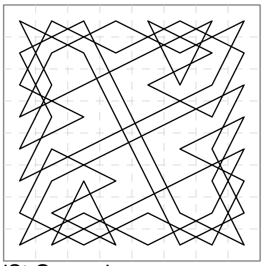
Tours from these historical collections are often diagrammed in later works without due acknowledgment of the sources, which I hope this account will go some way to correct.

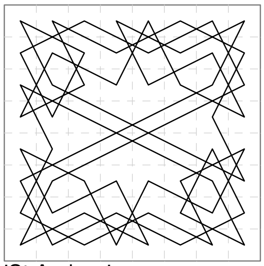
Among the magic tours there are 16 symmetric tours, found during the 1849-1888 period, namely those coded 00m and 12a to 12o in my catalogue. The central angles in them are: (bd=) 12m Wenzelides, (bg=) 00m, 12a, 12b Wenzelides, (cd=) 12o Jaenisch, (df=) 12n Jaenisch, (ef=) 12e Wenzelides, (fg=) 12c Béligne, (fn~) 12h Jolivald, 12i, 12j Bouvier, (fo~) 12g Jolivald, (gj=) 12k, 12l Francony, (go~) 12d Jolivald, (no=) 12f Jolivald.

Now some symmetric tours from Bergholt in *Queen* 1915. A handwritten footnote on the offprint in Murray's collection, states that The Hour Glass tour is a “Solution of the problem: In a diametrally symmetrical tour, to make the greatest possible number of consecutive moves within the central 4^2”. This task is also done by the 1877 Forestier tour from *Le Gaulois* above but with a different centre.

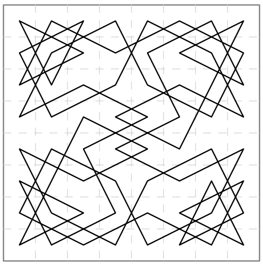

'Arabesque' (fh)

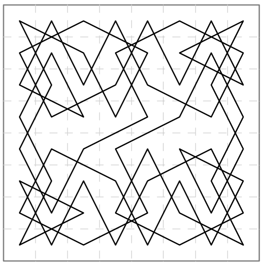

'The Hour-Glass' (op)

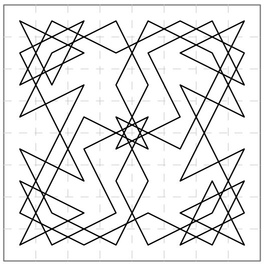

'St George' (dd~)

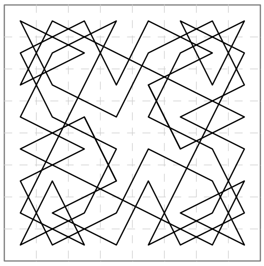

'St Andrew' (dd=)

Three by Bergholt from *Queen* 1916 and one from his 1917 memorandum


(oo=)


(gk=)


(pp)

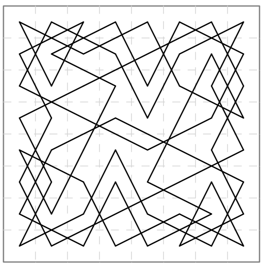

(bb~)

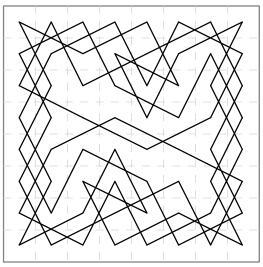
Symmetrisation

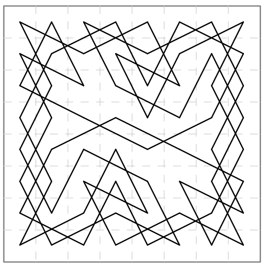
Here we look at how symmetric tours can be derived from asymmetric examples.

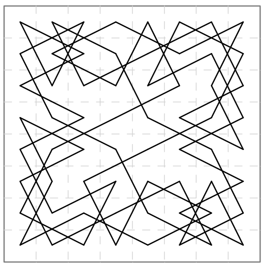
For **strict symmetrisation** two rules must be followed: 1. All pairs of moves that match diametrally must be preserved. 2. Each move in the symmetrised tour must be a move of the original tour or a diametral rotation of such a move. However it is not always possible to strictly symmetrise a tour. A looser aim is simply to try to preserve as many of the features of the original tour as possible.

Here are some examples of symmetric tours that have been derived from a few of the many irregular tours that appear in the literature, including two of those shown earlier in the Asymmetry section. These four diagrams are symmetrisations of the Rothe 1841 tour. The first is loose. The others are strict symmetrisations, all that are possible.



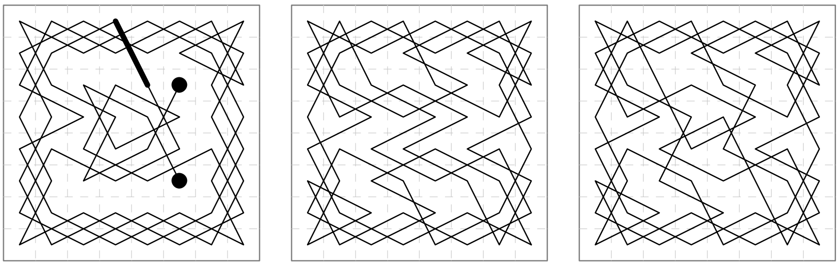




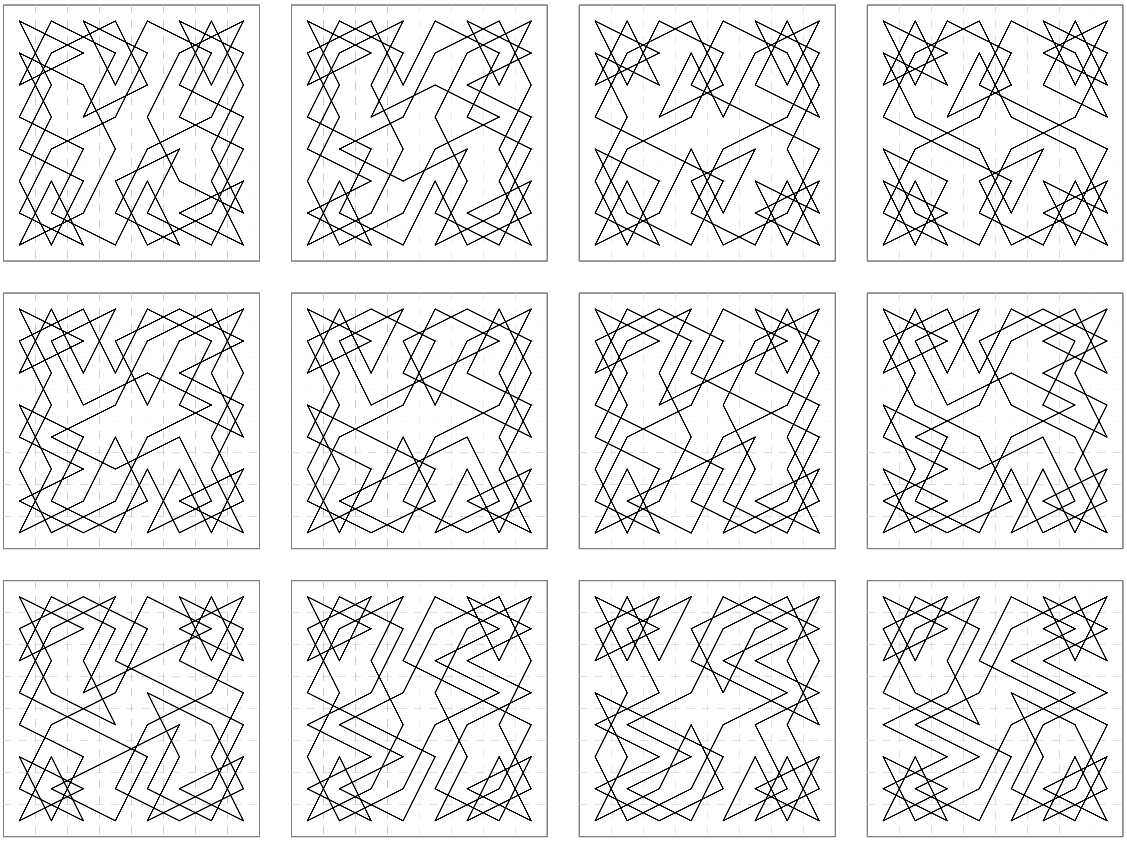


There is some mention of symmetrisation in Bergholt's manuscripts. The following two tours appear in his *Fifth Memorandum* dated 1 April 1916. The first tour is formed by the domino method, and the other is a symmetrisation of this, by Bergholt, using Euler's method.

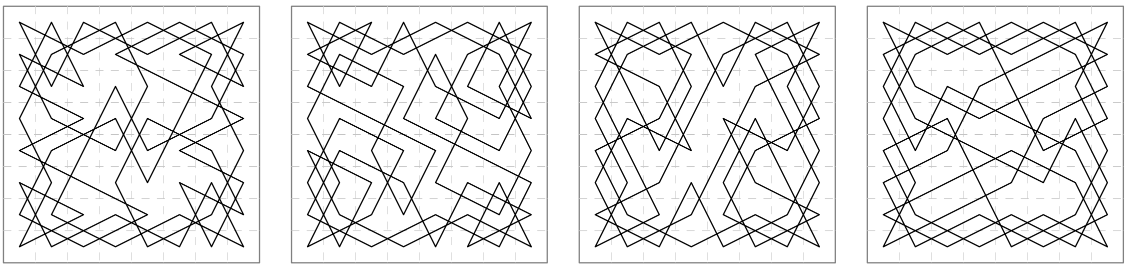
This is a case where strict symmetrisation is not possible. The third diagram shows the strictest I have found possible. The links g1-f3, d7-f6 and their rotations are not in the original tour though they do link to its end-points.



The first Legendre tour allows more freedom and can be strictly symmetrised in twelve ways:



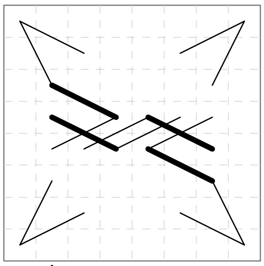
The next three tours are strict symmetrisations of the T, R, D, figured tours by Dennison Nixon that were mentioned earlier among the very irregular tours (centres kk=, bk=, il~, ci=).

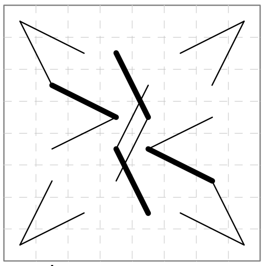


The fourth is a symmetrisation of the M. Abraham tour (⌘ 6) of Collinian type. Others derived from Collinian tours are (cl~) (fi~) (no=), (no~) in the following collection.

Constructing Symmetric Tours

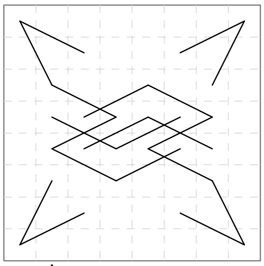
This may be a good place to explain methods that can be used for consructing symmetric tours. Let us begin by making a tour on the 8×8 chessboard with a given central pattern. Since there are 183 possibilities rather than choose one at random I will use the central angles (see ⌘ 1) corresponding to my initials gj. This gives two possible patterns:

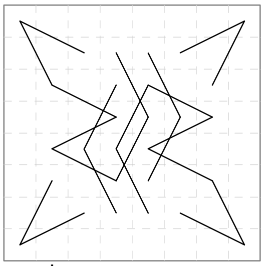

gj=


gj~

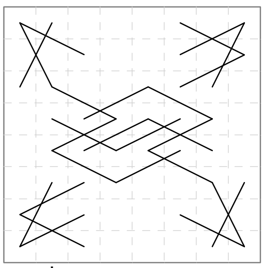
In the (gj=) version the leading arms of the angles are in the same direction, taken here to be horizontal (meaning the two-step component of the knight move is across the page). In the (gj~) version the leading arms of the g components are horizontal while those of the j components are vertical. The next step is to put in the corner moves which are forced in any closed tour: The corner moves meet the (g) moves at b6, g3 which means that the moves available at the cells a4, c8, f1, h5 are reduced from four to three, but not yet sufficient to mean the path through them is forced.

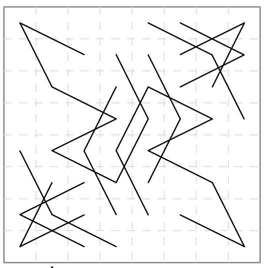
We now have some freedom to put in further moves. How might we proceed to produce an interesting pattern? In the (gj~) pattern for example we can join up the angles by b4-d3 and e6-g5, but in the (gj=) pattern no such joins are possible. There are many possibilities. One might be to put in extra moves to match the (j) angles. Let us do this in both. We thus arrive at:


gj=

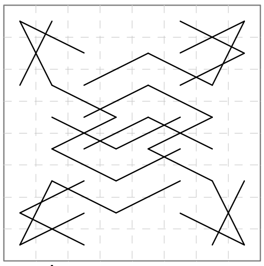

gj~

Examining the moves available at cells on the edges you may notice that the moves at a2 and h7 in both are reduced to two, so these must be put in. In the gj= pattern the move choices at a6 and h3 are reduced to three, but if we take them to be c7-a6-c5 and f4-h3-f2 this will result in a short circuit of 16 moves. This means we must take a6-b8 and h3-g1. In gj~ moves through b2, g7 are forced.

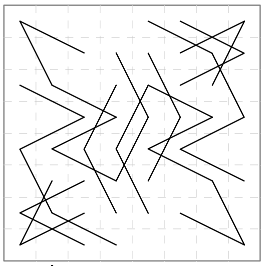

gj=


gj~

We now have some freedom again, let's put in extra j-type moves in gj= to create triple chevrons. In gj~ we can put in a pair of sharper chevrons a4-c5-a6, and its cousin.

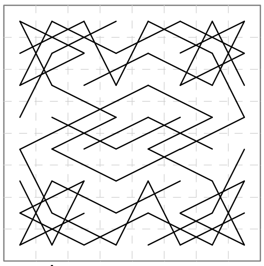


gj=

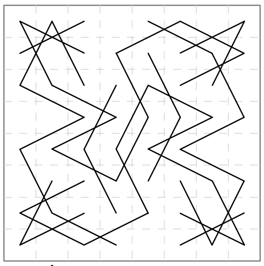


gj~

Now in gj= moves are forced at b1, g8, c8, f1, c1, f8. Further since c3, f6 are now used, moves are forced at a4, h5. Yet further single moves are forced at a6, h3. Then two at d1, e8. Now two at b7, g2. Now most loose ends appear to have two choices, but closer examination shows only one is available at d7, e2. Nothing seems to be forced in gj~ so let's try c1-e2 and its cousin. Moves are now forced at g1, b8. At f1 we cannot join to d2 and e3 since at the other side of the board, for symmetry, we would have e7-c8-d6, forming an eight-move circuit. So we must take f1-h2, c8-a7.



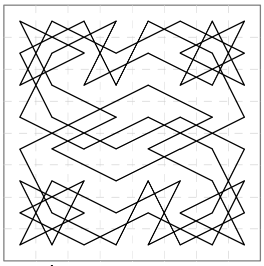
gj=



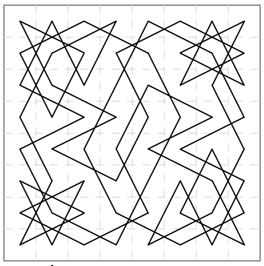
gj~

In gj= let's take a3-c2, completing a 7-move corner star. Other links are now forced. We now need to follow the path to check it is a tour and not two circuits. It is a tour.

In gj~ let us try the same a3-c2. Moves are now forced through e1, d8. Then at a5, h4, a7, h2. There are still two or three choices at all the loose ends. Let's try d1-e3 making a two-move line. Moves at d8, c7, b1 are now forced. It will be found that this is a also tour.



gj=



gj~

The reader should set up some designs for central patterns and try to construct tours employing these moves. This is the question facing the composer of a monogram or pictorial tour (see ⌘ 6). Usually, given a pattern, it is possible to complete the tour in several ways. A closed tour is preferable to an open one, and a symmetric one to an asymmetric. Try a pattern of your own design, or work through the centre angles as I have done in my complete collection below.

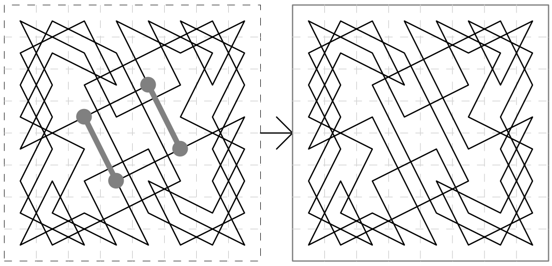
Readers with initials a, m or q to z I suggest could use the correspondence bcdefg => qrtsuv, ijkl => wxyz, no => ma.

According to Prof. Knuth there are 608233 geometrically distinct symmetric knight tours. Dividing by 183 gives an average of over 3000 tours from each angle pattern. This seems surprisingly high considering the difficulty often found in completing a tour.

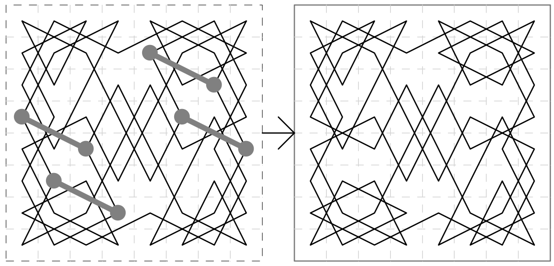
Converting Pseudotours to Tours

The procedure outlined above resulted in tours. However the final result will often turn out to be a closed pseudotour, or even open paths where the end-points have no knight-move connections. I show here 12 cases I encountered myself that are solved by finding a linkage polygon of alternate inserted and deleted moves that will convert the pseudotour into a tour.

The simplest case is a circuit of four moves. In our diagrams the inserted moves are shown by thicker grey line, the moves between their dotted ends being deleted. In the first example a pair of central 1-1 moves is rotated to change two half-tours into a single tour. In the second example two pairs of rhombs are reversed to connect two 20-move symmetrically placed circuits to a 24-move symmetric circuit to complete a tour.

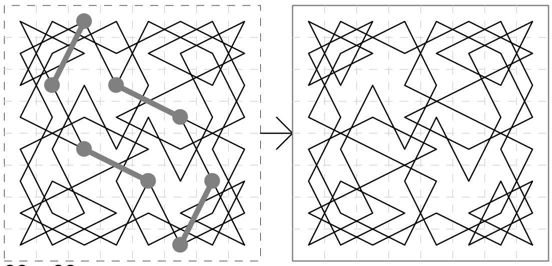


32 + 32 fi=

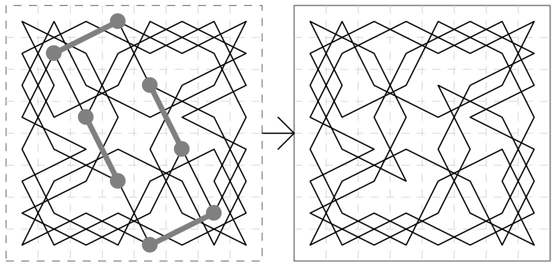


24 + 20 + 20

More often there will be a polygon of eight links, four insertions and four deletions.

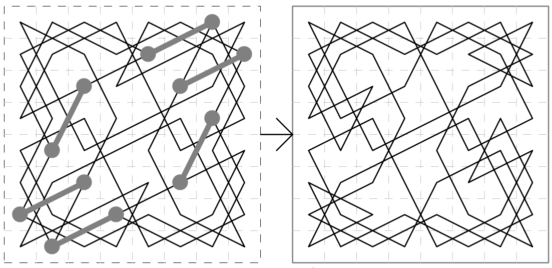


32 + 32 oo~

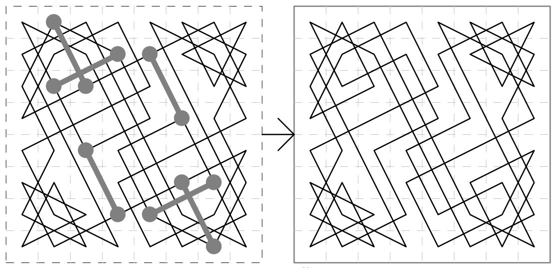


20 + 44 cc~

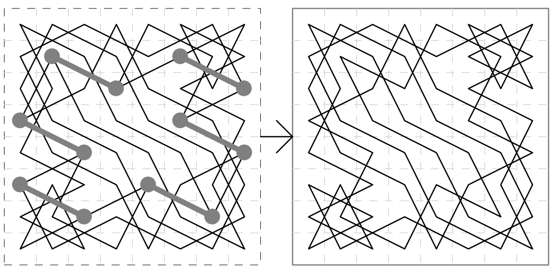
Here are some examples that use polygons of twelve moves, circling the centre to avoid disrupting the central angle pattern. The twelve-move circuit round the edges is a common fault.



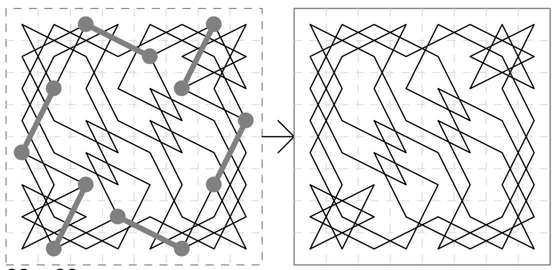
26 + 26 + 12 cd~



32 + 32 dk=

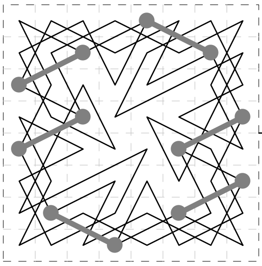


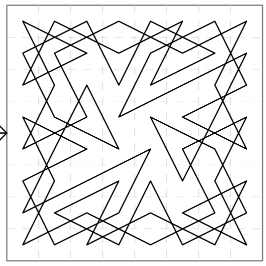
32 + 32 eh

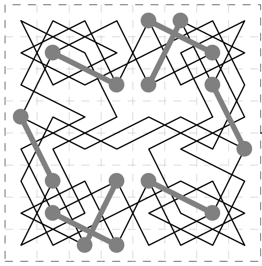


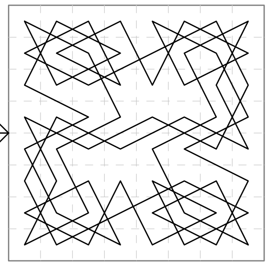
32 + 32 el~

Finally some 16-move circuits.

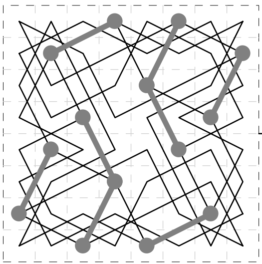

32 + 32

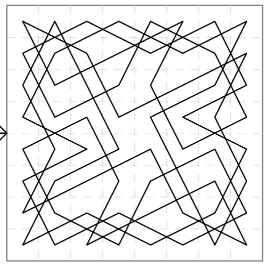

ll~

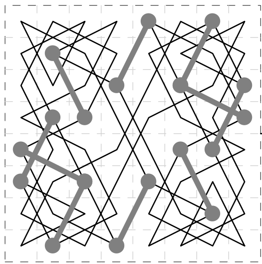

32 + 32

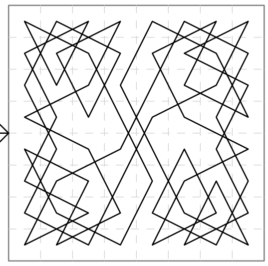

bj~

Linkage polygons can also be combined together. These cases show the two-move routes through d3 and e6 or c3 and f6 being replaced by two other moves.


8 + 8 + 10 + 10 + 14 + 14


ff~


32 + 32


de~

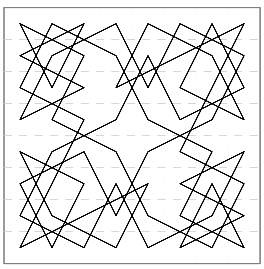
A Complete Central Angle Collection

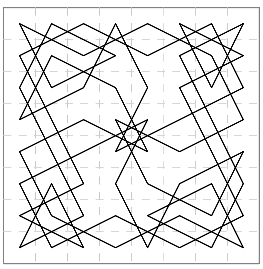
The following collection consists of symmetric tours of my own composition that show all possible central angle patterns. I show one tour for each of the 183 cases. Of course many more examples are possible of each case, because the moves between the centre and the corners can be varied considerably. Some centres allow more variation than others. Part of the construction process for 12 of these tours were shown above. Most of the tours were composed Aug-Nov 2015, but about 50 are older, some dating back to the 1970s, while 25 were constructed Jul 2019 to avoid duplicating tours elsewhere in the text. Some particular angle combinations occur frequently in tours of the Mixed Quaternary Symmetry (MQS) type (see the next section). See also in ⌘ 6 and ⌘ 8.

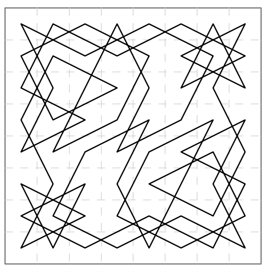
Tours with Central Angles Symmetric

We begin with the symmetric angles h and p (angles a and m are not possible).

hh, pp, hp (See the MQS tours for 26 more hh tours and 3 more pp)

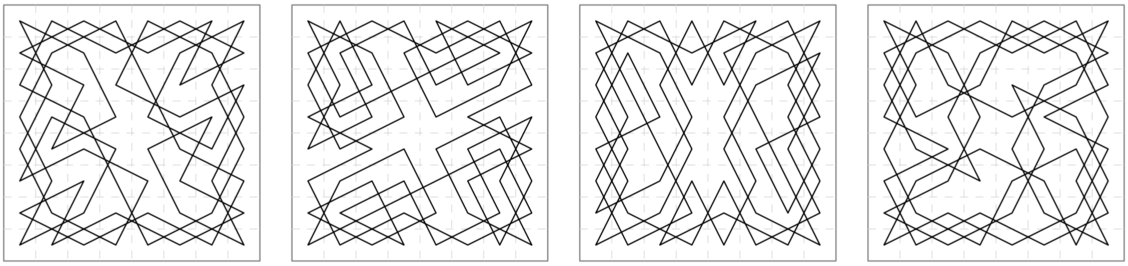




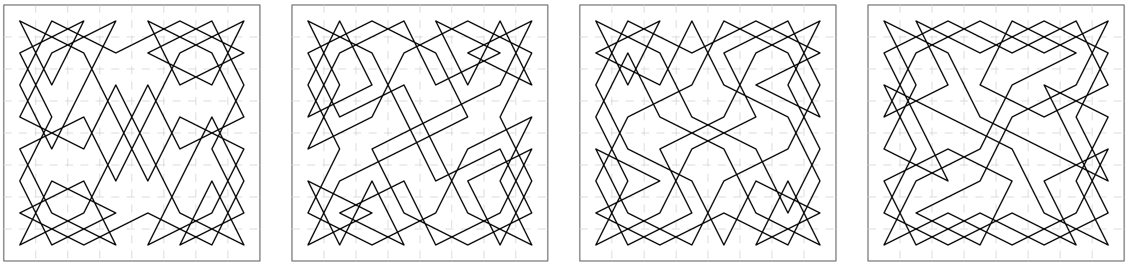


Tours with One Type of Central Asymmetric Angle

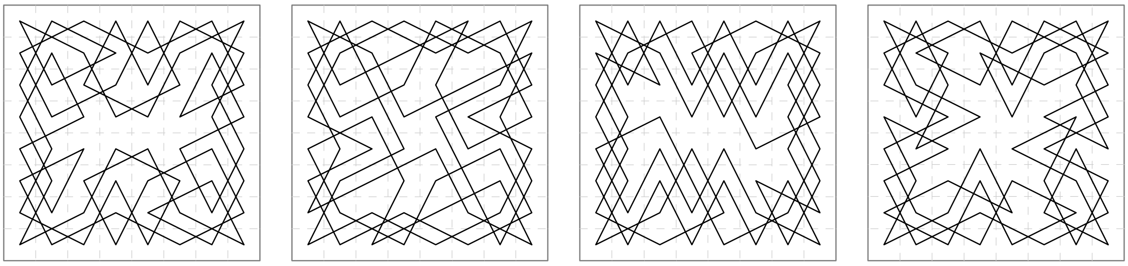
bb=, bb~, cc=, cc~ (See MQS 15 more bb and 5 more cc)



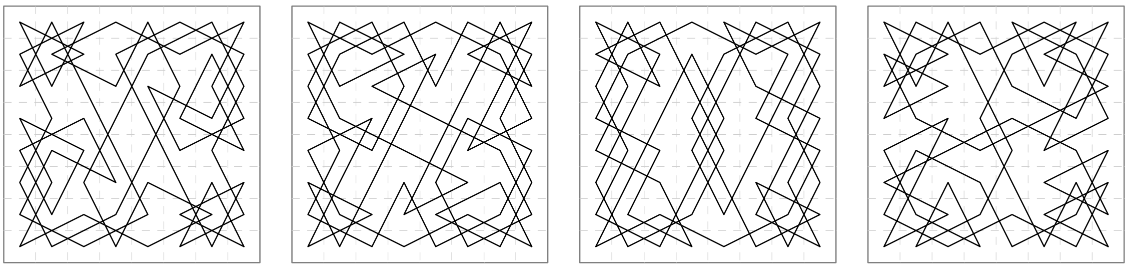
dd=, dd~, ee=, ee~ (See MQS for 5 more ee and 2 dd)



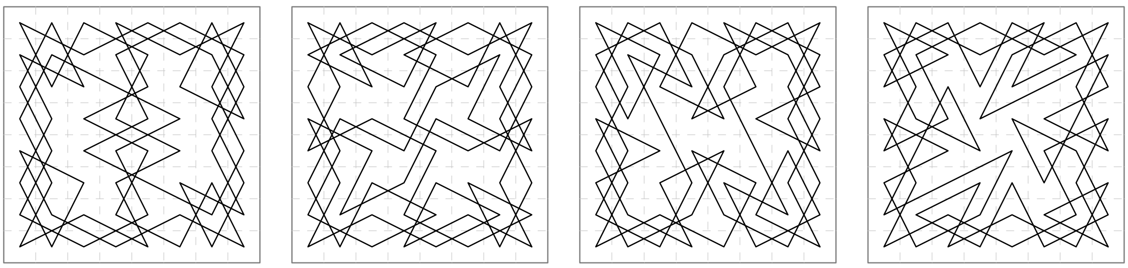
ff=, ff~, gg=, gg~ (See MQS for 5 more ff, and 20 more gg)



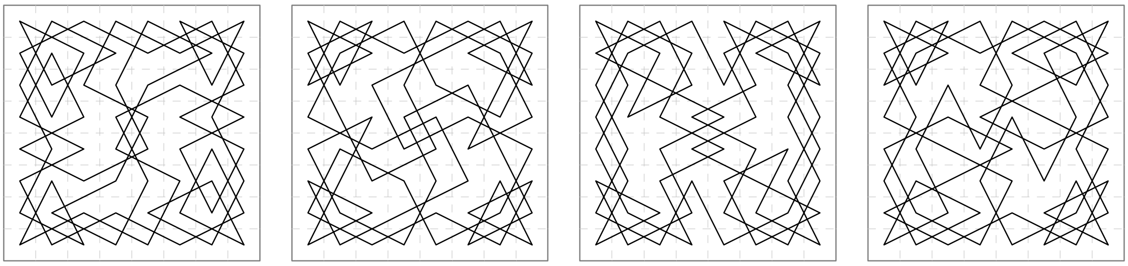
ii=, ii~, jj=, jj~ (See MQS for 15 more ii and 28 more jj)



kk=, kk~, ll=, ll~(See MQS for 26 more kk and 16 more ll)

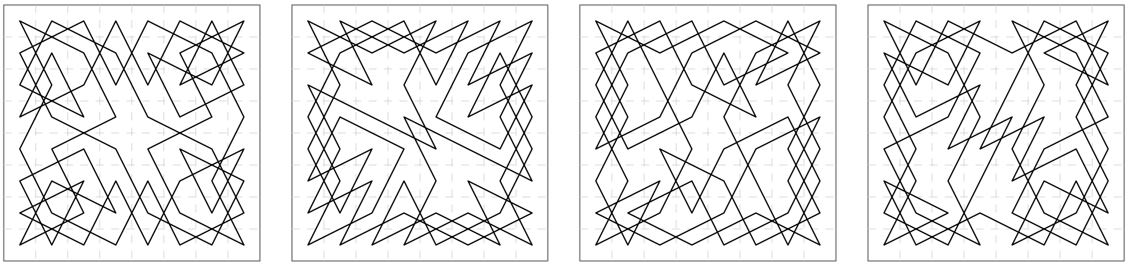


nn=, nn~, oo=, oo~(See MQS for 22 more nn and 22 more oo)

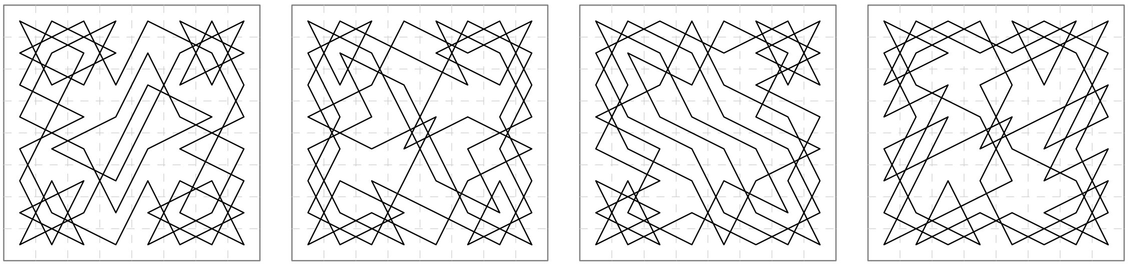


Tours with Two Types of Central Angles (One Symmetric)

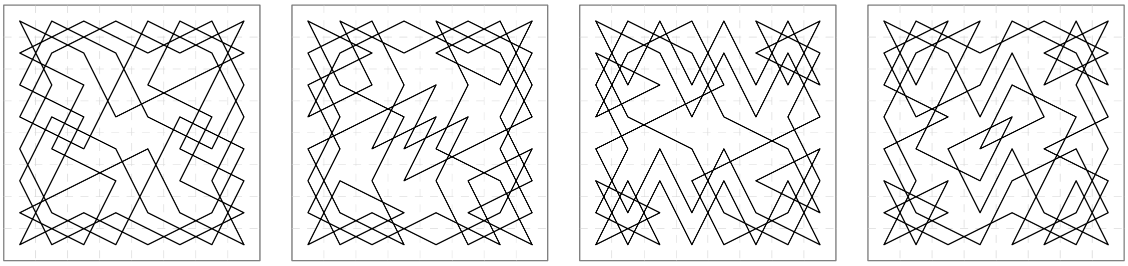
bh (parallels), bp, ch, cp



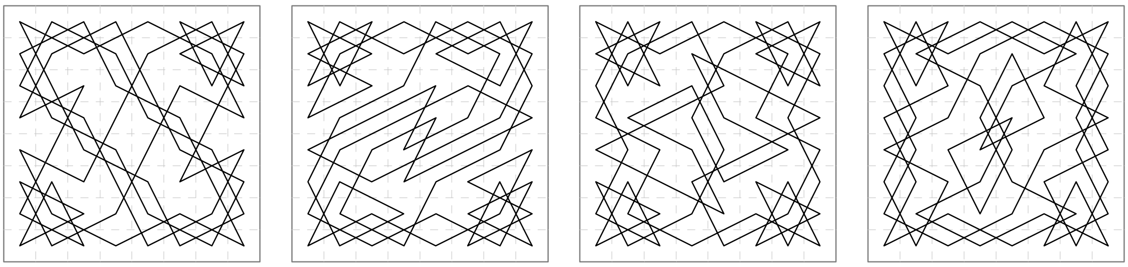
dh, dp, eh. ep



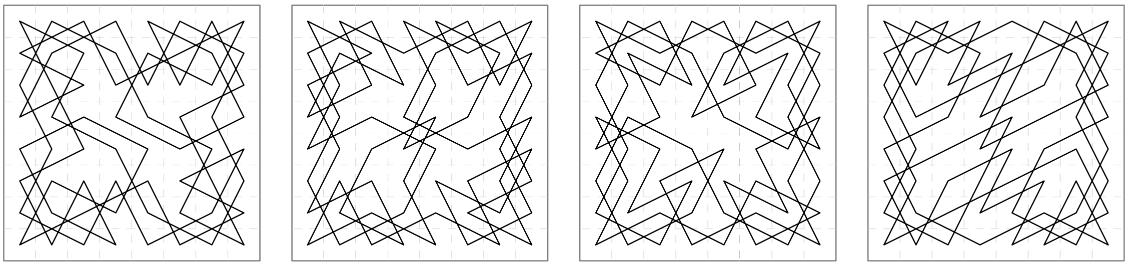
fh, fp, gh, gp,



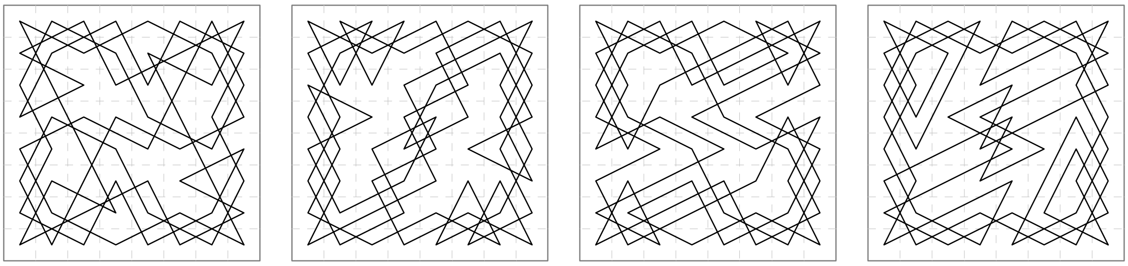
ih, ip, jh (this has a4-d2 a sequence of angles 615234), jp



kh, kp, lh, lp

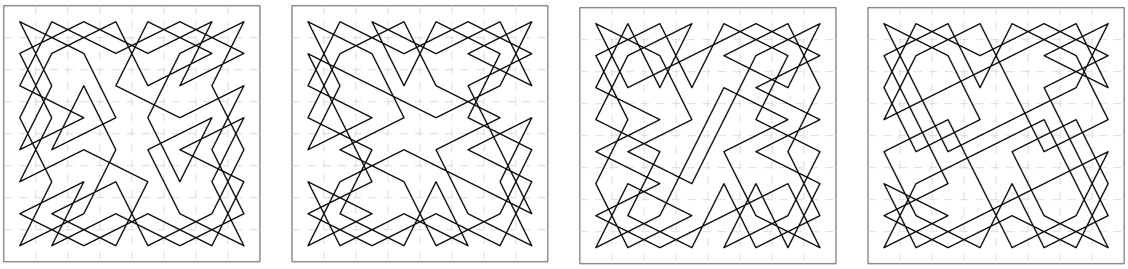


nh (shows all triangles 1 to 12 and 15), np, oh, op

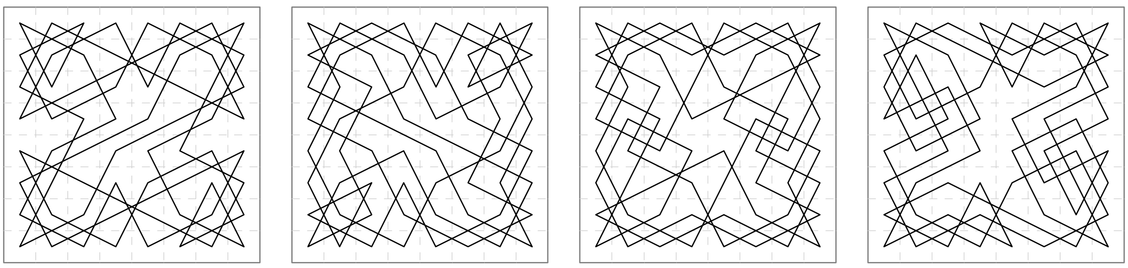


Tours with Two Types of Central Angles (Both Asymmetric)

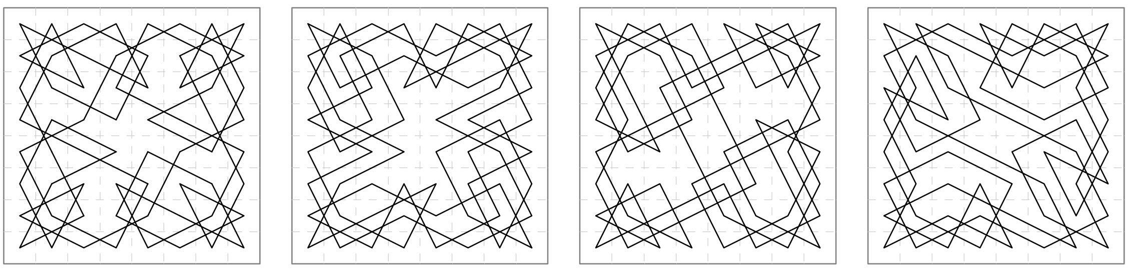
bc=, bc~, bd=, bd~



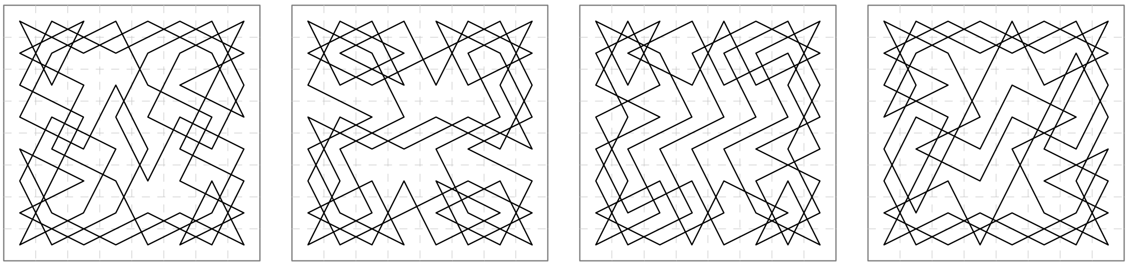
be=, be~, bf=, bf~



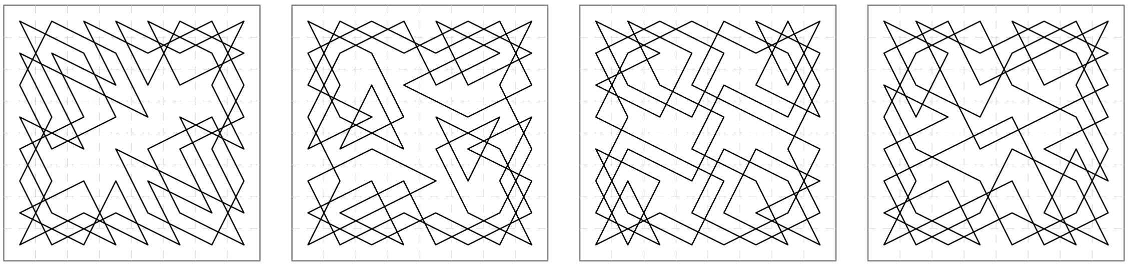
bg= (48 octonary moves), bg~, bi=, bi~ (See MQS for 29 more bg tours)



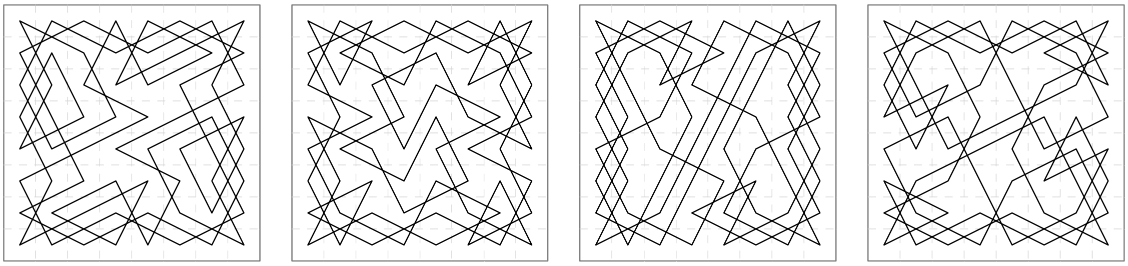
bj=, bj~, bk=, bk~



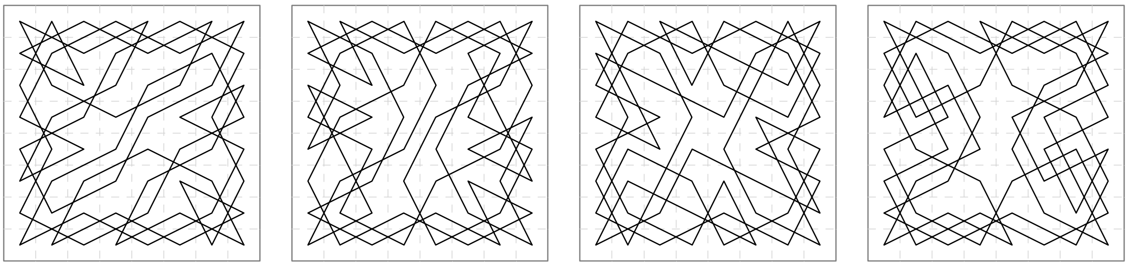
bl=, bl~, bn=, bn~



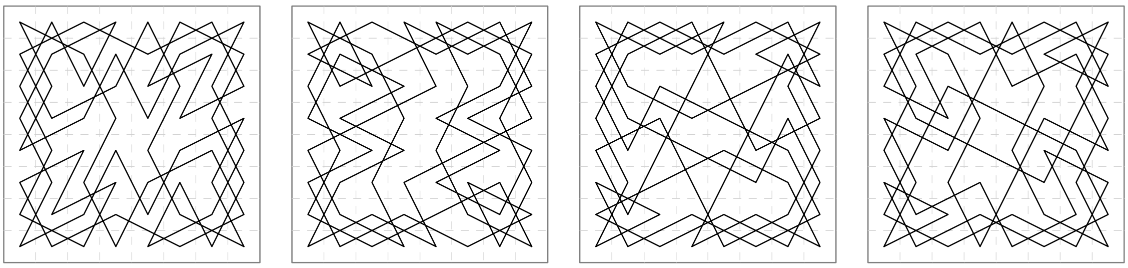
bo=, bo~, cd=, cd~



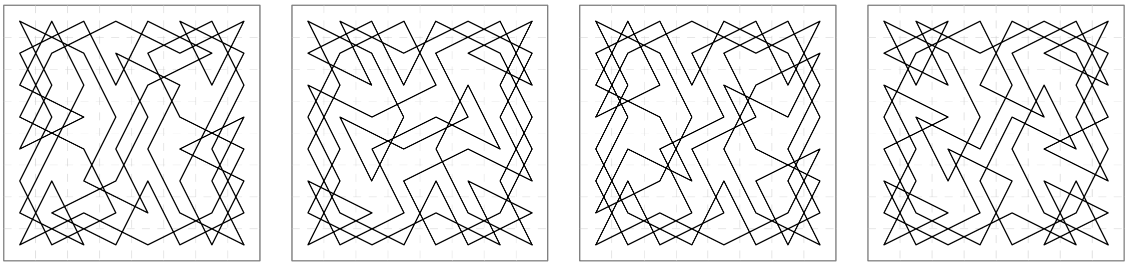
ce=, ce~, cf=, cf~ (See MQS for 10 more cf tours)



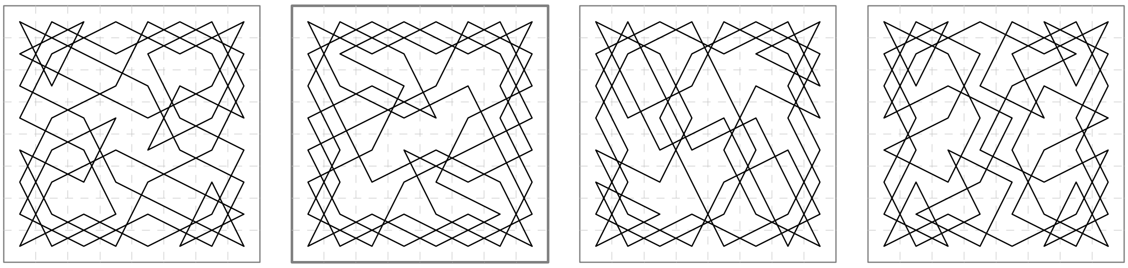
cg=, cg~, ci=, ci~



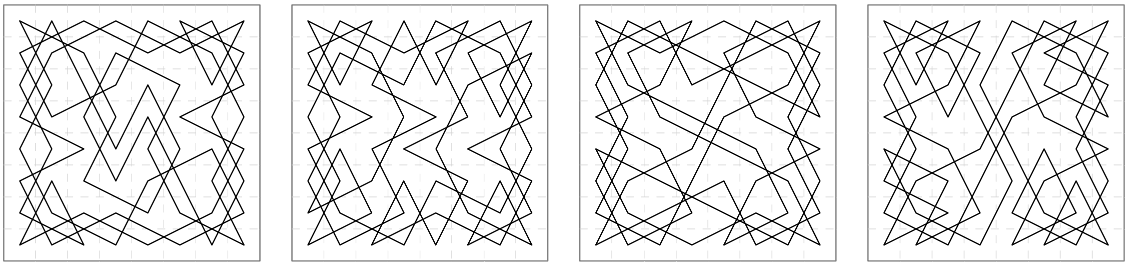
cj=, cj~, ck=, ck~



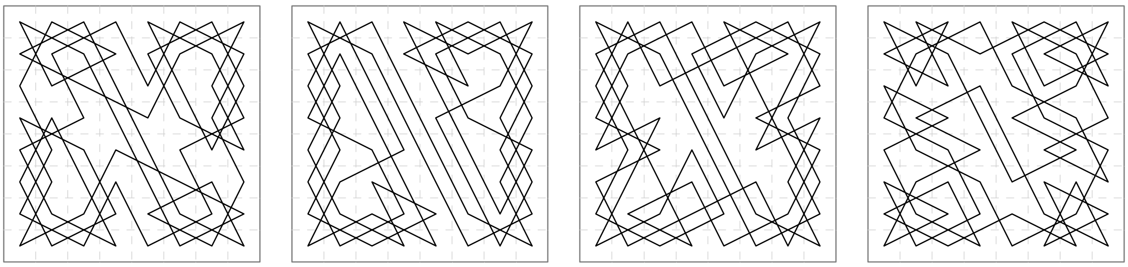
cl=, cl~, cn=, cn~



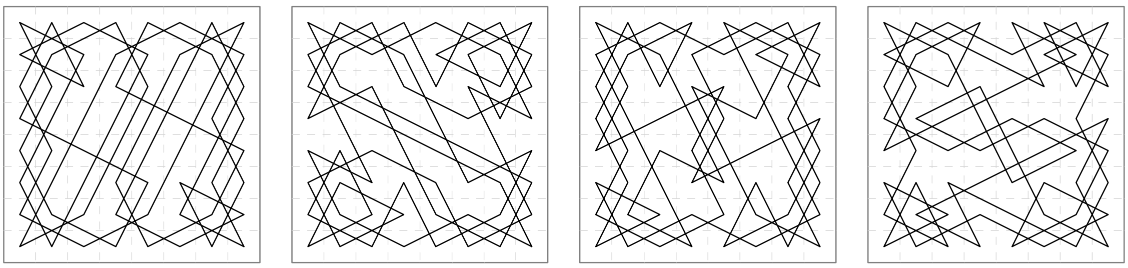
co=, co~ , de=, de~ (See MQS for 34 more de tours)



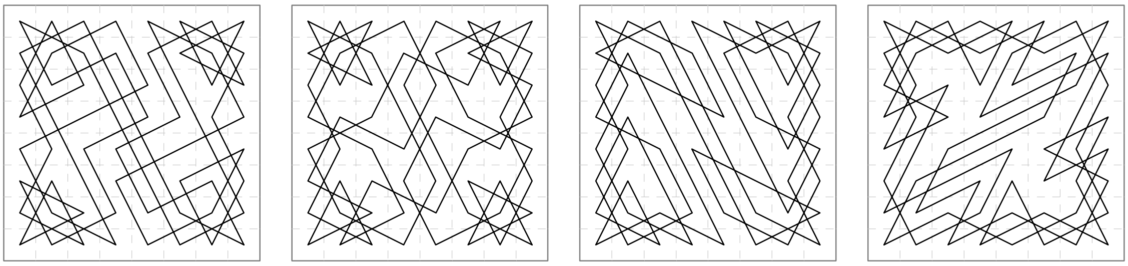
df=, df~, dg=, dg~



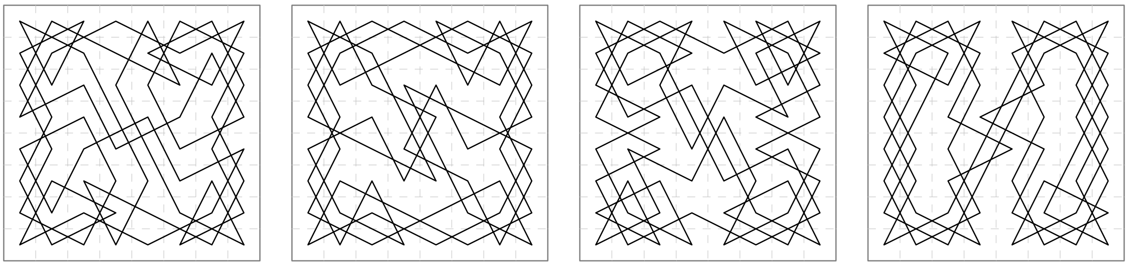
di=, di~, dj=, dj~



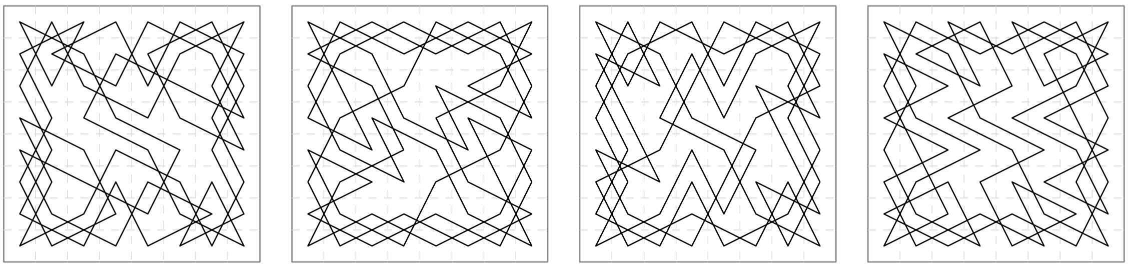
dk=, dk~, dl=, dl~



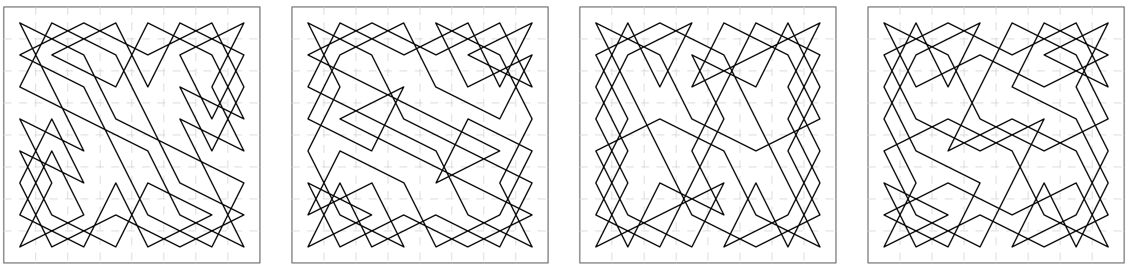
dn=, dn~, do=, do~



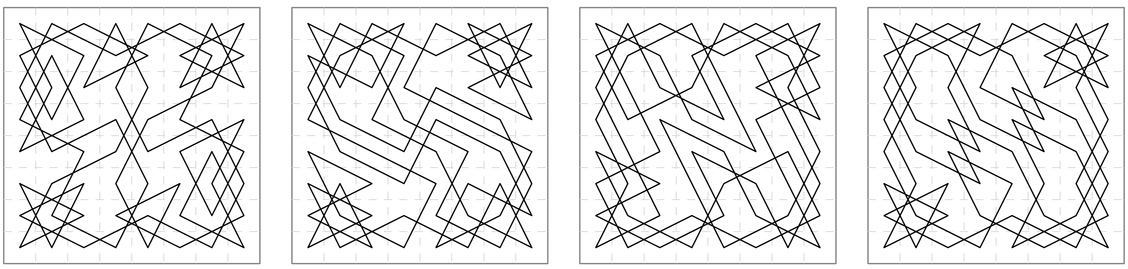
ef=, ef~, eg=, eg~



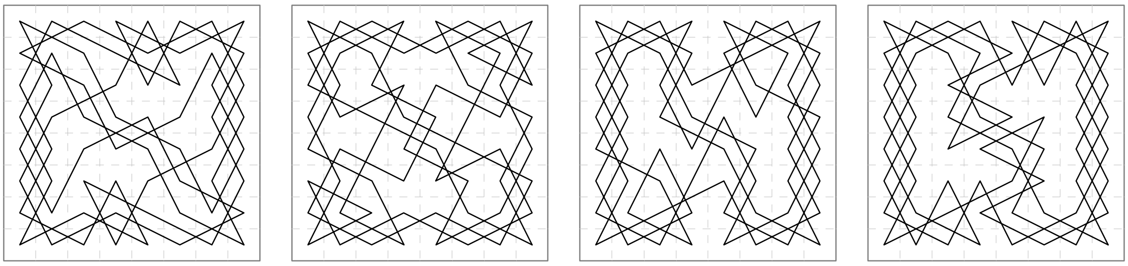
ei=, ei~, ej=, ej~



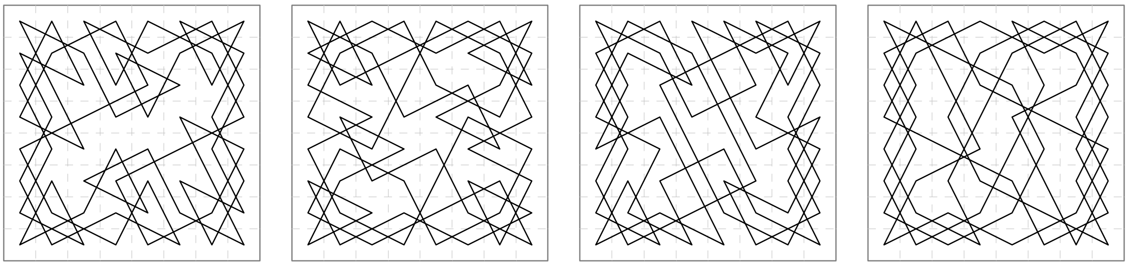
ek=, ek~, el=, el~



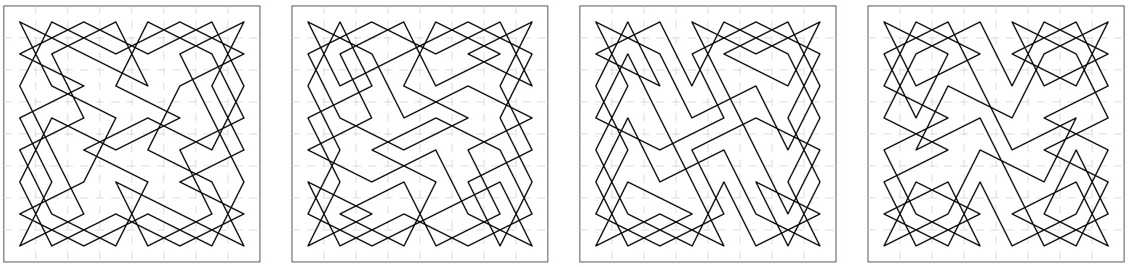
en=, en~, eo=, eo~



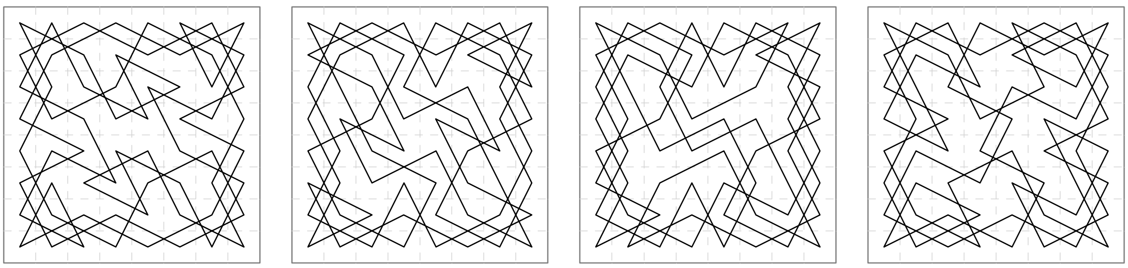
fg=, fg~, fi=, fi~



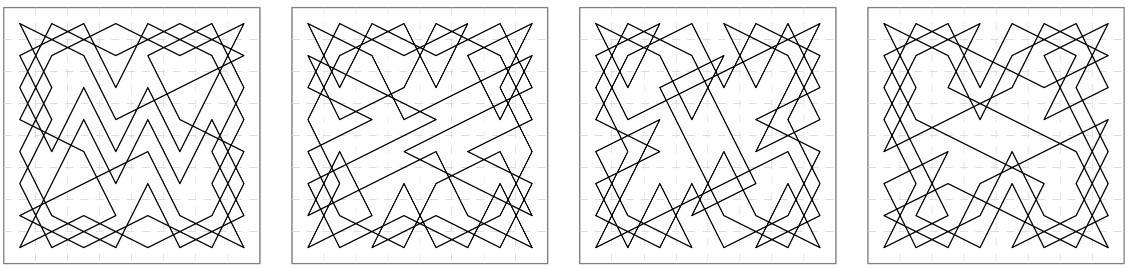
fj=, fj~, fk=, fk~



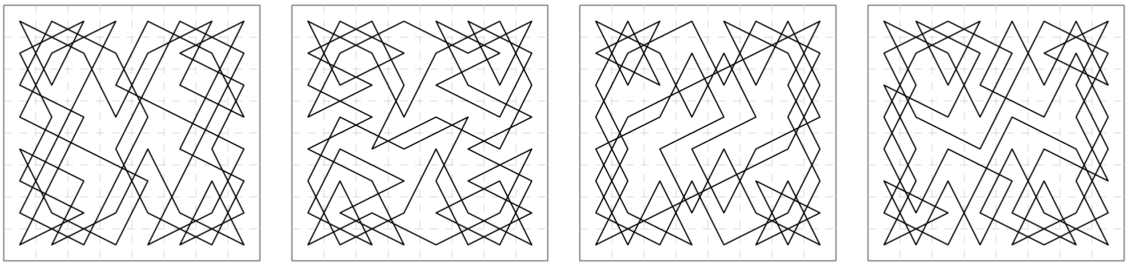
fl=, fl~, fn=, fn~



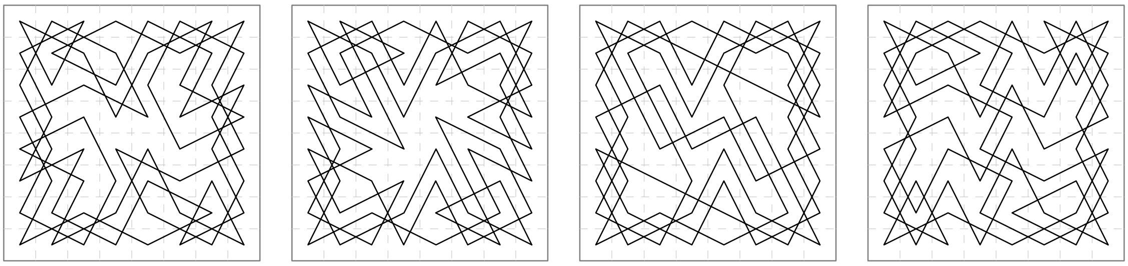
fo=, fo~, gi=, gi~



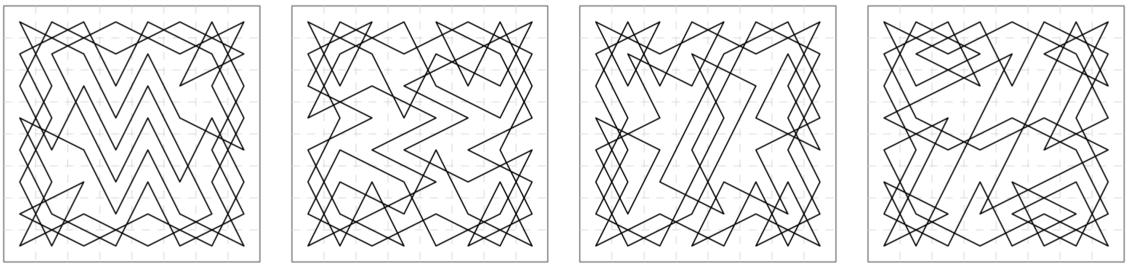
gj=, gj~, gk=, gk~



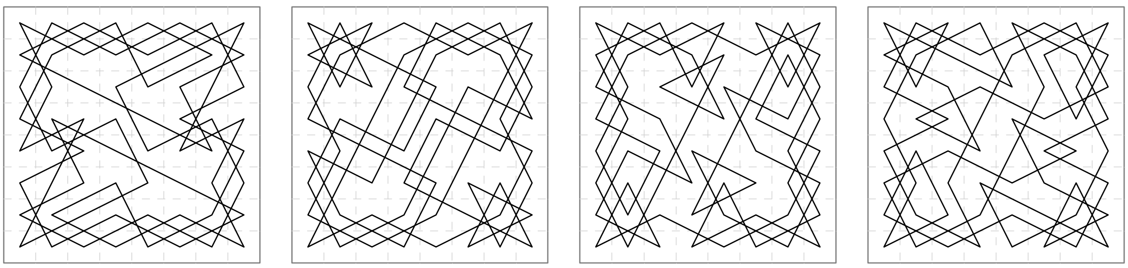
gl=, gl~, gn=, gn~



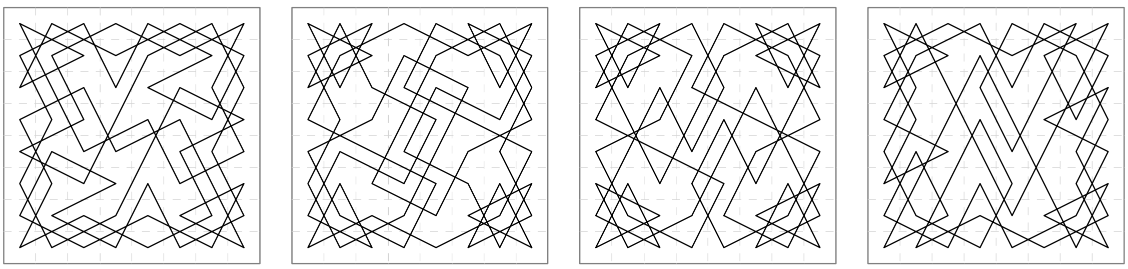
go=, go~, ij=, ij~



ik=, ik~, il=, il~ (See MQS for 14 more il tours)

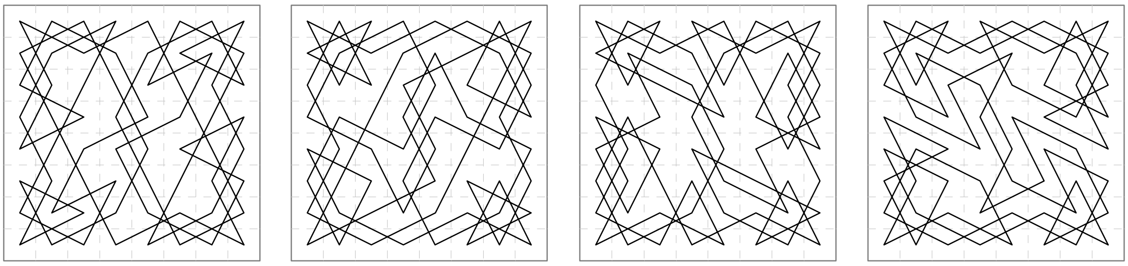


in=, in~, io=, io~

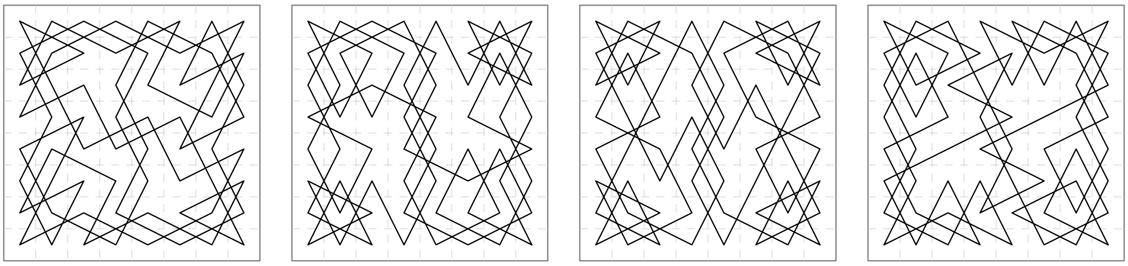


io= has 48 maximum octonary moves.

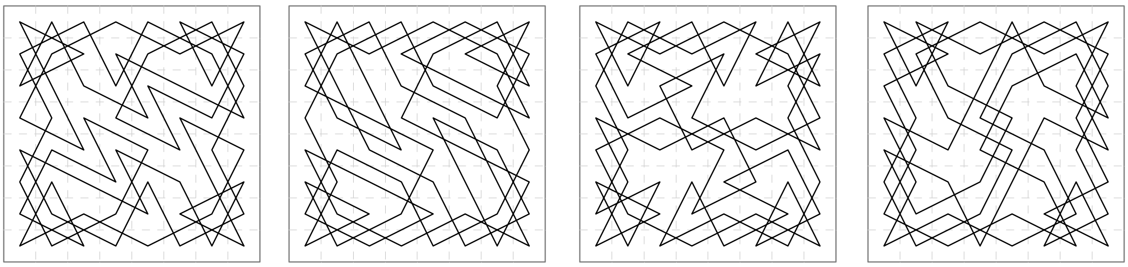
jk=, jk~, jl=, jl~(See MQS for 65 more jk tours)



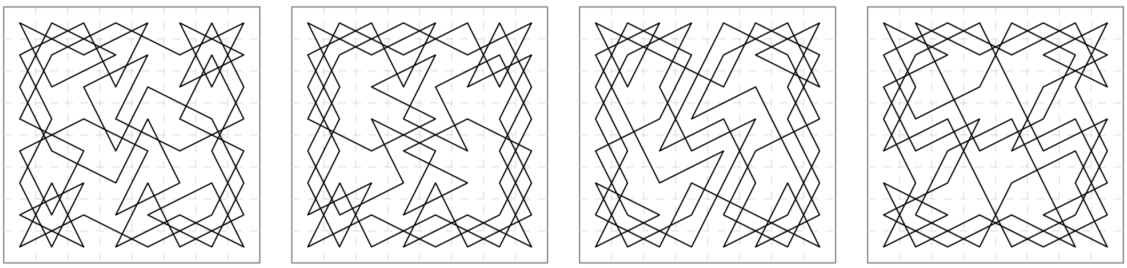
jn=, jn~, jo=, jo~



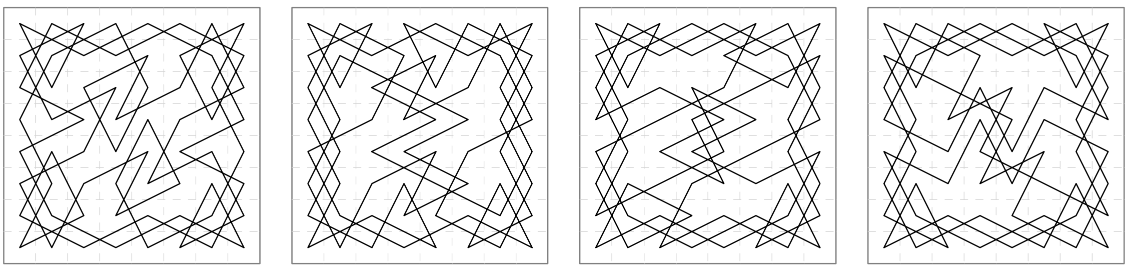
kl=, kl~, kn=, kn~



ko=, ko~, ln=, ln~



lo=, lo~, no=, no~ (See MQS for 14 more no tours)



Mixed Quaternary Symmetry

General Principles

A knight's tour is said to show **mixed quaternary symmetry** if its moves can be regarded as forming three classes, one in biaxial symmetry (direct quaternary symmetry), one in birotary symmetry (oblique quaternary symmetry), and the remainder in octonary symmetry. It follows that the tour as a whole is in rotary symmetry (also called 180° rotational symmetry or diametral symmetry) this is because both types of quaternary symmetry include this lesser form of symmetry. It also follows that the cells used by the three sets all form arrays with octonary symmetry, since they must link together.

The idea of constructing tours that show a combination of oblique and direct quaternary symmetries was introduced by Ernest Bergholt in a series of puzzles published in the *British Chess Magazine* in 1918, and elaborated in three memoirs that he sent to H. J. R. Murray that year, which are now among Murray's papers in the Bodleian Library, Oxford. These memoirs were reproduced in issues 13, 14 and 18 of *the Games and Puzzles Journal* (1996-2001). They give examples on the 8×8 and 12×12 boards. Bergholt variously described his tours as showing 'approximate', 'complete' or 'perfect' quaternary symmetry. Murray preferred 'mixed quaternary symmetry' as a more accurate description, however it should be borne in mind that the actual overall symmetry is binary oblique.

The treatment given here includes the results of Bergholt and Murray that I am aware of, but expands and clarifies their method by taking proper account of the octonary symmetry component, which they tended to regard as part of either the direct or the oblique component, depending on the focus of attention. There is probably further work on this subject among Murray's extensive papers.

On a square board of side $(2\cdot m)$ the tour has $(4\cdot m^2)$ moves, and cells, so if the biaxial class has $(4\cdot h)$ moves, the octonary class $(4\cdot j)$ and the birotary class $(4\cdot k)$, then $h + j + k = m^2$, with j even. I put the octonary number, j, in the middle since it is counted with h when assessing the direct quaternary symmetry, and with k when assessing the oblique quaternary symmetry. The tour can thus be described as of **type (h:j:k)**. Mixed quaternary symmetry is possible on any square boards of even side greater than 4, but is mainly of interest on boards of side a multiple of 4 on which a tour with quaternary symmetry is impossible, in particular the 8×8 and 12×12.

Here we cover the general principles of mixed symmetry. See the following sections on 8×8 and 12×12 boards for catalogues of examples.

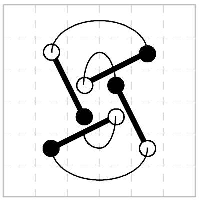
THEOREM: <u>The number of purely oblique components, k, must be odd.</u>

Proof: In terms of coordinates a knight's move is a {1,2} leap, and can be classified as 'horizontal' or 'vertical' according to the direction of the two-step part of the move. Four moves in direct symmetry are either all horizontal or all vertical, since reflection in a median does not alter this property. On the other hand, four moves in oblique symmetry consist of two vertical and two horizontal, since 90° rotation of a vertical move makes it horizontal, and vice versa. Moves in octonary symmetry can be split up into two direct or two oblique quartets. A tour with oblique symmetry on a board of side $2\cdot m$ consists of two congruent paths each of $2\cdot m^2$ moves, from corner to opposite corner. A direct quartet contributes two vertical or two horizontal moves to this half-tour. An oblique quartet however contributes one of each type. The direct and octonary moves thus always contribute a displacement of the knight by an even number of ranks or files from the initial corner. But the displacement from corner to corner is a displacement of $(2\cdot m - 1)$ ranks and files, an odd number. Thus k must be odd. QED.

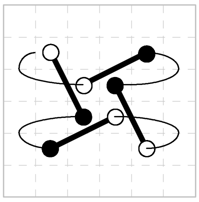
The minimum purely oblique moves is 4 that is $(k = 1)$ on boards of side $4\cdot r + 2$ (i.e. 6×6, 10×10 etc) but is 12 that is $(k = 3)$ on boards of side $4\cdot r$ (i.e. 8×8, 12×12, etc). A proof of this follows.

THEOREM: The minimum purely oblique moves cannot be $k = 1$ on boards of side $4 \cdot r$.

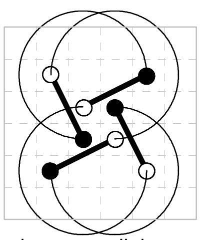
Proof: Consider the possible linkages. The four moves in oblique quaternary symmetry can only be four moves of the type 0-2, connecting cells on the diagonals in an octonary pattern, in the manner shown, or its reflection. There are three geometrically distinct ways in which these can be joined by the other $16 \cdot r^2 - 4$ moves to form a tour by paths that are in direct quaternary symmetry.



odd linkage



short even linkage

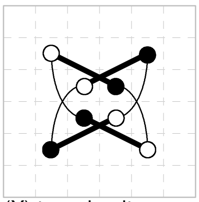


long even linkage

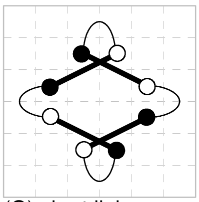
The first linkage, which can also be rotated 90 degrees, connects dark to light cells, so must use four paths of two odd lengths; but an odd length path cannot have the required symmetry, since the middle move would have to cross the median at right angles, which is impossible for a knight move. The second and third linkages connect dark to dark and light to light cells, so must use four paths of even lengths. But the lateral symmetry requires that all these be of the same length, and $(16 \cdot r^2 - 4)/4 = 4 \cdot r^2 - 1$ is an odd number. Thus $k = 1$ is impossible. QED

The minimum of purely direct moves in mixed quaternary symmetry is 0 (that is, $h = 0$) on boards of side $4 \cdot r + 2$, since these are the boards on which tours with oblique quaternary symmetry are possible. On boards of side $4 \cdot r$ the minimum is 4 (that is, $h = 1$). The four moves in direct symmetry can only be four moves of type 1-1 near the centre of the board. This also results in maximum oblique symmetry on the rest of the board, though the proportion of this that is octonary can vary. The maximum moves in (impure) oblique quaternary symmetry that can be achieved on the n×n board is therefore $n^2 - 4$, that is 60 on 8×8 and 140 on 12×12, and so on.

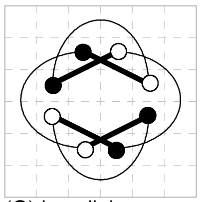
The end-points of the 4 direct moves are necessarily in octonary symmetry. The formations (X) c3-c6-d4-d5-e4-e5-f3-f6 and (O) c4-c5-d3-d6-e3-e6-f4-f5 shown below are the only octonary arrangements that admit joining up by single knight-move links in direct quaternary symmetry.



(X) two circuits



(O) short linkage



(O) long linkage

The links can be taken horizontally without loss of generality (the vertical arrangement is only a 90° rotation of the horizontal). For the remaining moves to be in oblique quaternary symmetry on a board of side $2 \cdot m$ they must form four equal paths, of $(m^2 - 1)$ moves each, and this being an odd number (on boards with m even) the end cells must be of opposite colour.

The (X) pattern can only be joined up, white to black, in oblique quaternary symmetry by joining each outer cell to the other inner cell to which it is not already linked; but this results in two circuits. The (O) pattern however allows two oblique quaternary schemes of connection, each cell joining to one of the cells in the same rank or file.

To find all the possible paths we can use the centre-outwards coding system. We require a path of $(m^2 - 1)$ moves from 1 to 1 using the diagonal indices 0, 2, 5, 9, ... once each and the off-diagonal indices twice each. The path 1...1 thus takes the form 1...d...d...d...d...1 where the ‘d’s denote diagonal cells, and there are gaps for the other numbers. These numbers can be distributed in the gaps in various ways:. The possible sequences that fill these gaps can be tabulated, classified by number of off-diagonal cells used. These result in formulae for the tours.

==

Mixed Symmetry on the 8×8 Board

A knight tour is said to show **mixed quaternary symmetry** if its moves can be regarded as forming three classes, one in biaxial symmetry, one in birotary symmetry, and the remainder in octonary symmetry. The tour can be described as of **type (h:j:k)** where h is the number of quartets of moves in the biaxial section, j (an even number) the number of quartets in octonary symmetry, and k the number of quartets in birotary formation. On the 8×8 board, 64 moves and cells, h + j + k = 16.

As shown above k, must be odd. The number of octonary quartets, j, can only take the values 4, 6, 8, 10, and since k must be odd, so is h and their possible values are 1, 3, 5, 7, 9, 11. The catalogue of tours that follows is divided into (a) those with h = 1 minimum purely direct 4 giving 48 tours (b) those with k = 3 minimum purely oblique 12, since k = 1 is impossible on the 8×8, and (c) intermediate cases with h > 1 and k > 3.

Mixed Quaternary Tours with h = 1

On boards of side 4·r like the 8×8 the minimum moves in biaxial symmetry is 4 (that is, h = 1). The four moves in direct symmetry can only be four moves of type 1-1 near the centre of the board. The maximum moves in (impure) oblique quaternary symmetry is therefore 60. The earliest tour known of this type is one by J. Brede (1844). There is also an example in E. H. (*Glasgow Weekly Herald* 1873). My study of tours of this type found 48 solutions (*The Games and Puzzles Journal* 2001 #18, p.332).

To find all the possible paths we can use the centre-outwards coding system. We require a path of 15 moves from 1 to 1 using the diagonal indices 0, 2, 5, 9 once each and the off-diagonal indices twice each. The path 1...1 thus takes the form 1...d...d...d...d...1 where the ‘d’s denote diagonal cells, and there are gaps for the other cell codes. These numbers can be distributed in the gaps in various ways: The possible sequences that fill these gaps can be tabulated, classified by number of off-diagonal cells used. These result in formulae for the tours.

On the 8×8 board we require a 15-move path from 1 to 1, and there are 5 gaps for the other 10 numbers. These numbers can be distributed in the gaps in seven ways: (44110), (43210), (43111), (42211), (33220), (33211), (32221). The possible sequences that fill these gaps are the following and their reversals, classified by number of off-diagonal cells used (to save space we omit hyphens):

zero: 01, 02, 15,
one: 031, 032, 049, 132, 149, 162, 165, 265,
two: 0349, 0371, 0382, 1349, 1382, 1649, 1732, 1749, 1782, 2349, 2649, 5649,
three: 03749, 03782, 03871, 13749, 13782, 17349, 17382, 17832, 23749, 28349, 28749,
four: 038749, 138749, 178349, 238749, 283749, 287349.

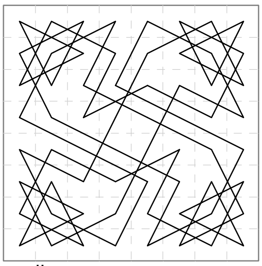
The formulae for the tours are presented above the diagrams. There are 10 formulae that generate 2 tours. The tours can be subclassified according h:j:k type. The possible values are: (1:6:9); (1:8:7); (1:10:5). These classes contain 16 cases each. The maximum of purely oblique moves is 36 (k = 9). Cases (1:4:11) and (1:12:3) are impossible. The central angles in these tours are: 6 bb~, 8 gg~, 14 hh, 4 ii~, 8 jj~, 4 kk~, 4 ll~.

The 16 tours of type (1:10:5)

There are 10 with oblique centre, 6 with octonary centre (hh). One of these is from the original puzzles by Ernest Bergholt in the *British Chess Magazine* 1918, in which he presents two open tours in approximate direct and oblique quaternary symmetry, and the solution combines features of both.

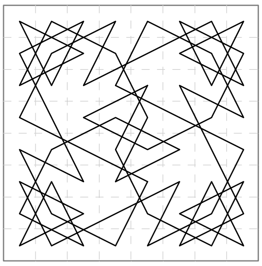
The octonary components are 3-8-7-4-9, 5-6 (4 times); 4-9, 3-8-7-1, 5-6 (4 times); 4-9, 3-7-8-2, 5-6 (2 times); and six cases of 1 each with 4-9,7-8-2, 5-6, 0-3; 8-7-4-9, 5-6, 0-3; 4-9, 3-8-7, 2-6, 0-3; 7-4-9, 3-8, 5-6, 0-3; 4-9, 3-8, 1-7, 5-6, 0-3; 4-9, 1-7-8, 5-6, 0-3.

1387494783026561



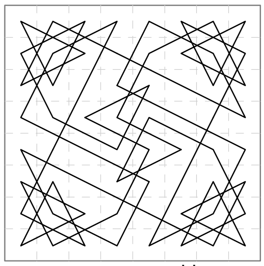
ii~

1038749478326561



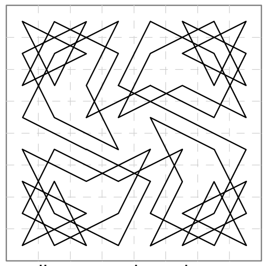
jj~

1038749478326561



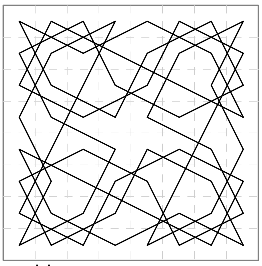
kk~

1387494783026561



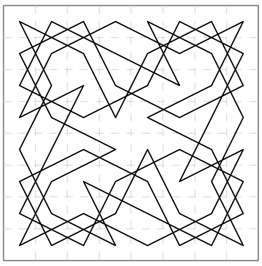
ll~ quarterboard

1783265649403871



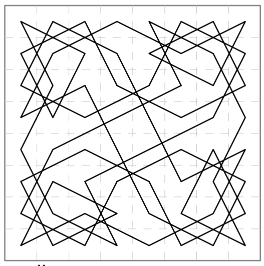
bb~

1783265649403871



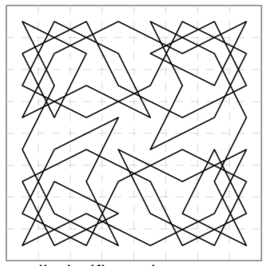
gg~

1783026564943871



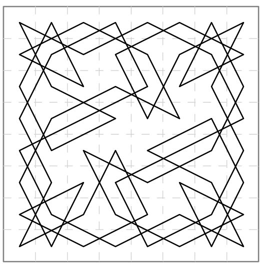
ii~

1783026564943871



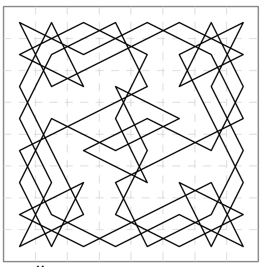
ll~ halfboard

1656494037828731



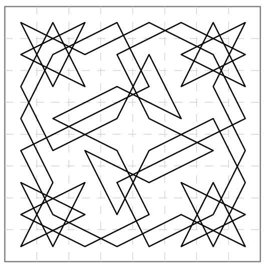
gg~

1656494378287301



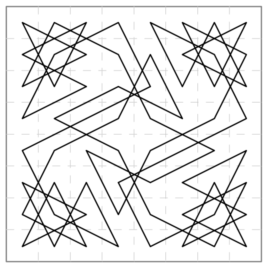
jj~

1656494782873031



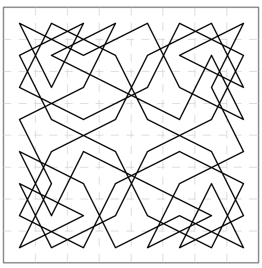
hh

1656287494783031



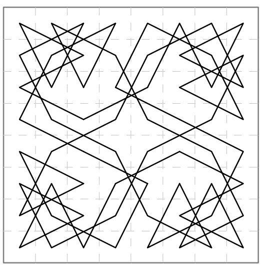
hh halfboard

1783038749462651



hh

1749478303826561



hh

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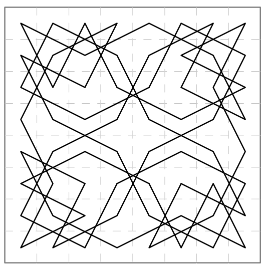
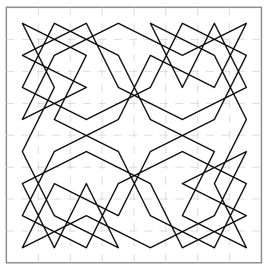


hh Bergholt 1918

1783034946562871



hh

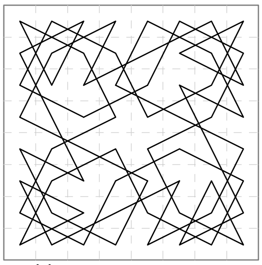
The tours marked ‘half-board’ can be divided into two by a curved line through the centre that crosses only two links. The ‘quarterboard’ tour can be divided into four parts in this way.

The 16 tours of type (1:8:7)

There are 10 with oblique centre, 6 with octonary centre (hh).

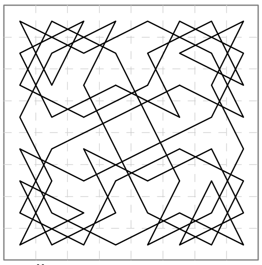
The octonary components are 4-9, 3-8-7, 5-6 (4 times); 4-9, 7-8, 5-6, 0-3 (4 times) and four cases with 2 tours: 4-9, 3-8, 1-7, 5-6; 7-4-9, 3-8, 5-6; 4-9 2-8, 5-6, 3-7; 6-4-9, 7-8, 0-3.

1783049478326561



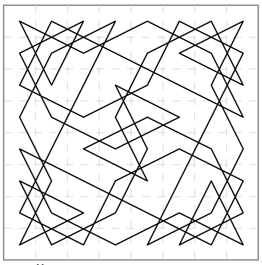
bb~

1783026564947831



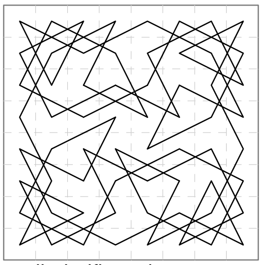
ii~

1783265649478301



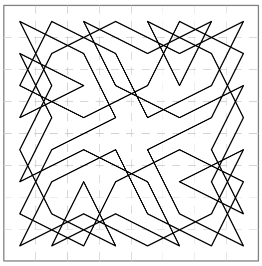
jj~

1783026564947831



ll~ halfboard

1783049465628371



bb~

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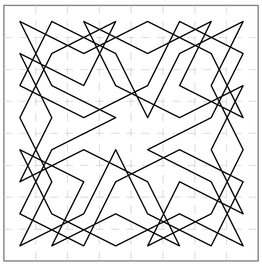
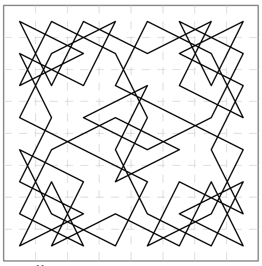


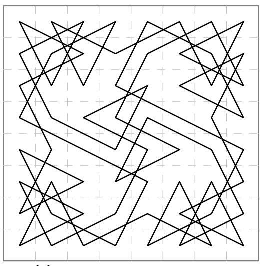
gg~

1038749473826561



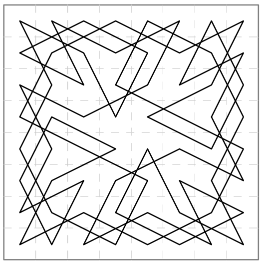
jj~

1038749473826561



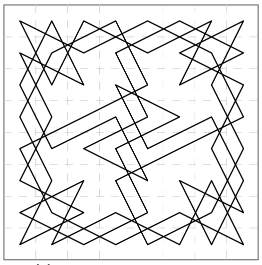
kk~

1738287304946561



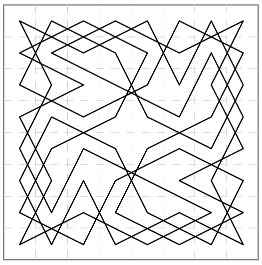
gg~

1656494738287301



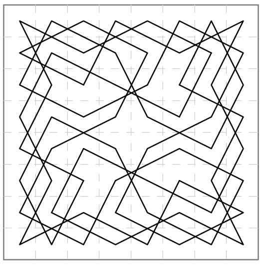
kk~

1783037826494651



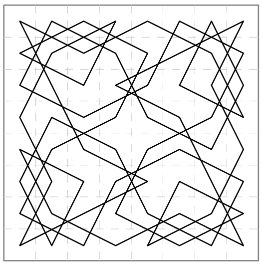
hh

1783037826494651



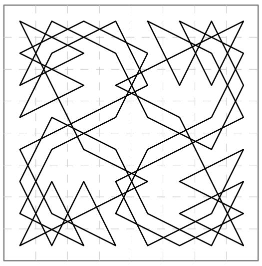
hh

1783037826564941



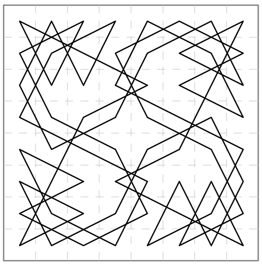
hh

1656287303874941



hh

1494783037826561



hh

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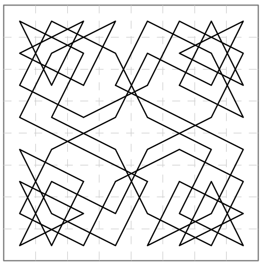
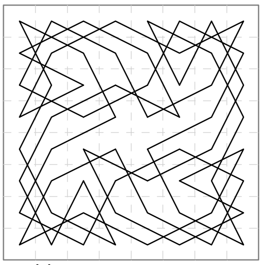


hh

The 16 tours of type (1:6:9)

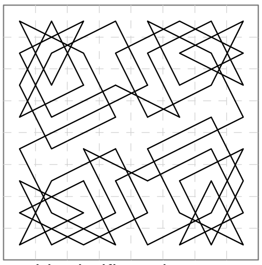
There are 14 with oblique centre, 2 with octonary centre. (hh).
The octonary components are 4-9, 7-8, 5-6 (12 times); 4-9, 3-8, 5-6 (2); 4-9, 7-8, 0-3 (2).

178265649403871



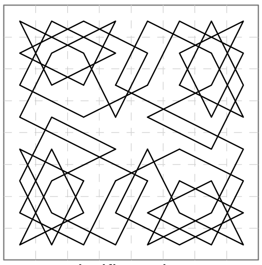
bb~

187494037826561



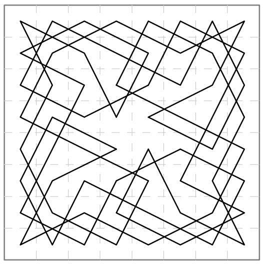
bb~ halfboard

1783494037826561



gg~ halfboard

1783287304946561



gg~

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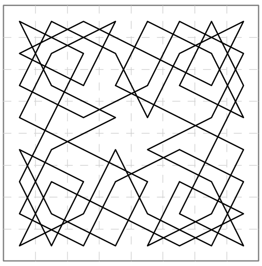

gg~

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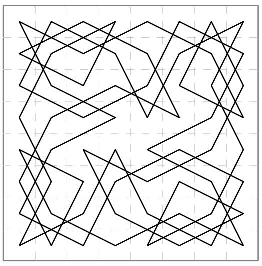
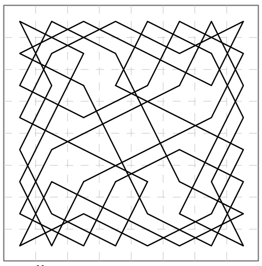


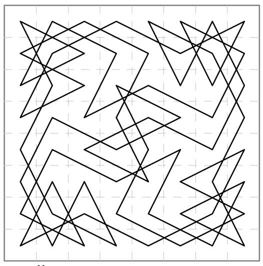
gg~

1783028734946561



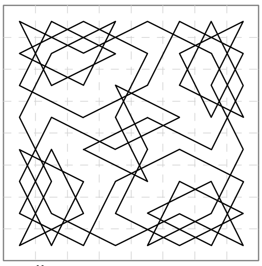
ii~

1387494656287301



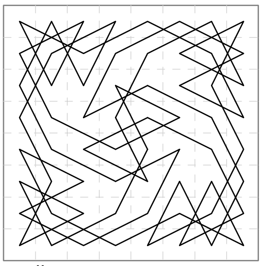
jj~

1783494656287301



jj~ Jelliss Feb 1970

1378265649478301



jj~

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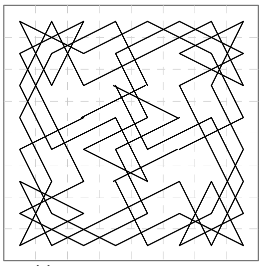

kk~

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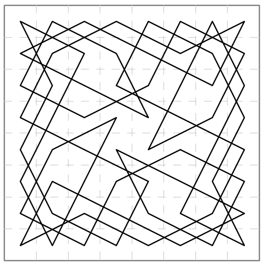
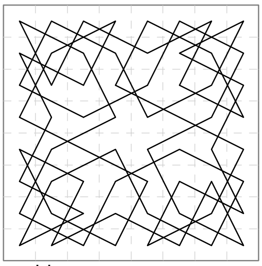


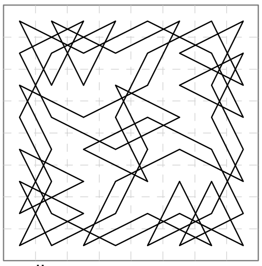
ll~

1783049473826561



bb~

1738265649478301



jj~

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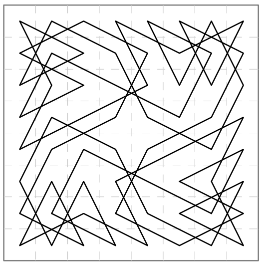
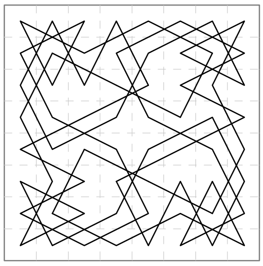


hh

1628730387494651



hh

One the jj~ cases I constructed myself as long ago as February 1970.

Mixed Quaternary Tours with k = 3

We have shown earlier that the minimum purely oblique moves in mixed quaternary symmetry is 12 (k = 3) on boards of side 4·r. The cells used by the twelve oblique moves must form an octonary pattern. If the twelve moves are all separate single moves they will use 24 cells, but this will reduce to 20 if there are four singles and four pairs, or to 16 if they join to form four three-move paths.

In an octonary pattern an off-diagonal (c) cell will contribute 8 and a diagonal (d) cell 4, so the patterns can be classified by the numbers of c and d used. Thus in general we have 10 cases:

single (24) can be (3·8) or (2·8 + 2·4) or (1·8 + 4·4) or (6·4); four cases.
double (20) can be (2·8 + 1·4) or (1·8 + 3·4) or (5·4); three cases.
triple (16) can be (2·8) or (1·8 + 2·4) or (4·4); three cases.

However the only triple moves possible using two off-diagonal cells are 6-1-1-6, 4-1-1-4, 3-1-1-3, and four 1-1 moves in oblique symmetry cannot be used since they form a circuit. This eliminates the (2·8) case. Further, since the corner cells are part of the octonary component, these 9 cases reduce to 5 on the 8×8 where only three diagonal cells can be used, and to 8 on the 12×12 where only five diagonal cells are available, but all 9 are feasible on 16×16 and larger boards.

Tours with k = 3 on the 8×8 board are those with the minimum purely oblique moves, and there are five cases: triple 16 = (1·8 + 2·4), double 20 = (2·8 + 1·4) or (1·8 + 3·4), single 24 = (3·8) or (2·8 + 2·4). The triple and double cases were extensively studied by Bergholt in 1918 (published in the *Games and Puzzles Journal* #18, 2001) though his results were not exhaustive.

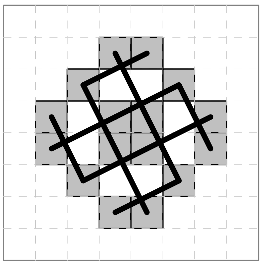
As with the h = 1 tours there are several cases depending on the number of octonary moves: (3:10:3), (5:8:3), (7:6:3), (9:4:3). Bergholt and Murray treated these classes as a single (13:3) type.

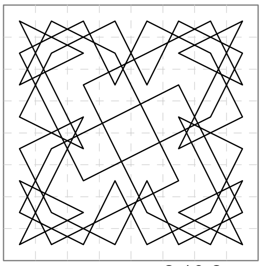
Triple move (1·8 + 2·4) class

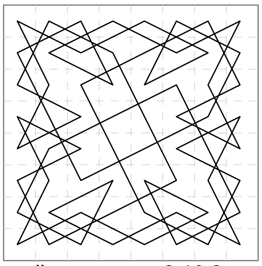
There is one type, using the (023) cells.

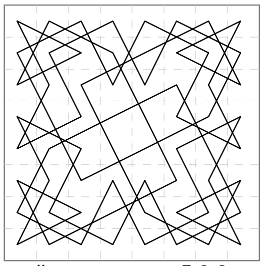
Type (023): There are two different ways of connecting these cells. Bergholt classified them as two different types. They can be distinguished by the i or l angles formed at the centre cells. Each configuration gives three tours. The second and third diagrams are given by Ernest Bergholt in *BCM* 1918 p.262 as solutions to his Problem 5.

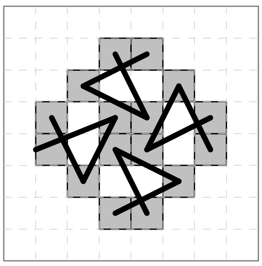
There are 4 of (3:10:3) and 2 of (5:8:3).

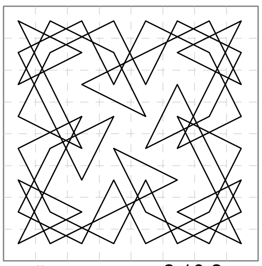

Type (023)i

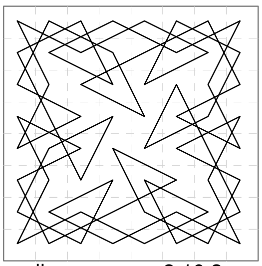

ii~ 3:10:3

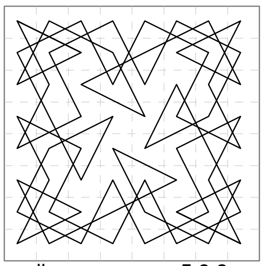

ii~ 3:10:3


ii~ 5:8:3


Type (023)l


ll~ 3:10:3

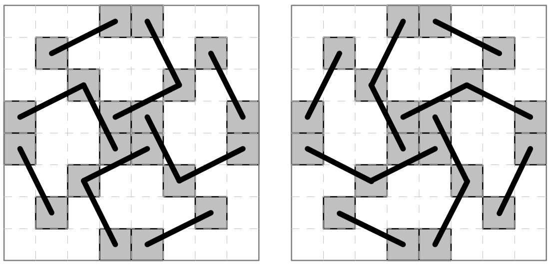

ll~ 3:10:3

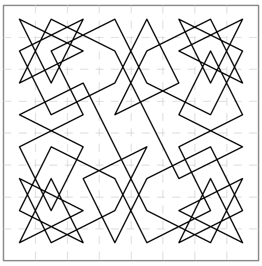

ll~ 5:8:3

Double move (1·8 + 3·4) class

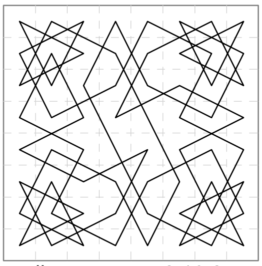
Here I find two types (0256) and (0125), both with dual connection patterns, like the (023) case.

Type (0256): 16 tours: 1 (3-10-3), 3 (5:8:3), 4 (7:6:3) in each form. The dual tours are placed alongside each other. The other moves of the tours, apart from in the diamonds like a4b2d1c3 are identical. This is less obvious than in the (023) and (0125) cases since the links are more widely distributed and not all in the centre of the picture.





il= 3:10:3



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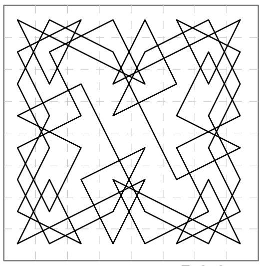

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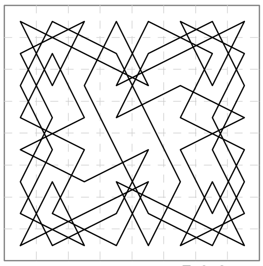

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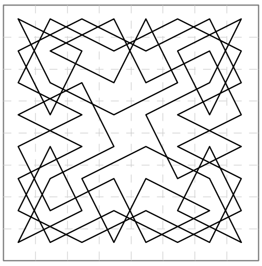
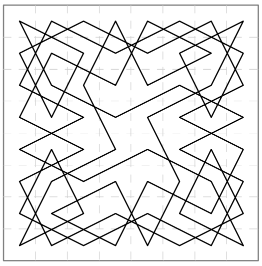


cf= 5:8:3



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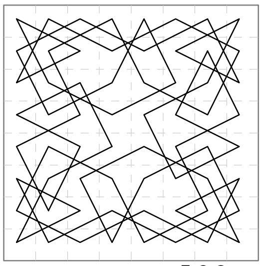
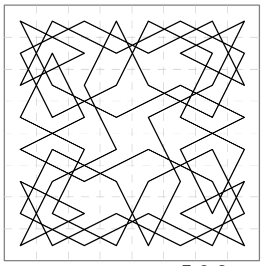


cf= 5:8:3



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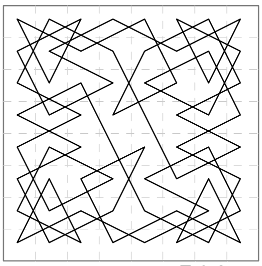
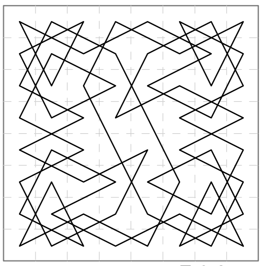


il= 7:6:3



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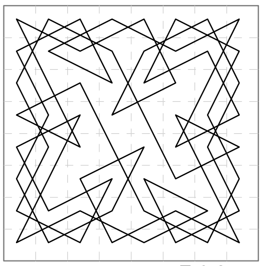

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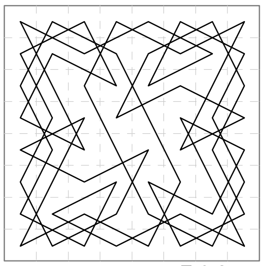

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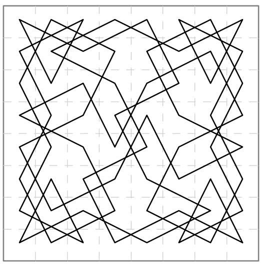
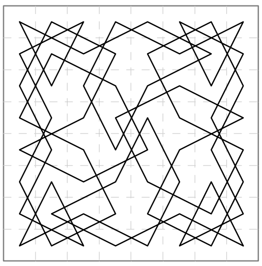
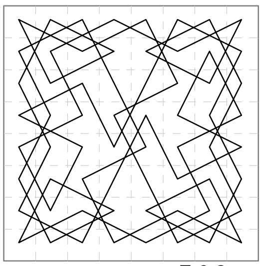


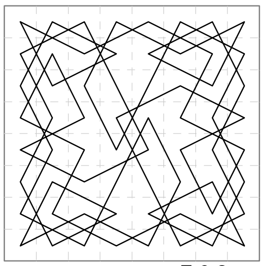
no~ 7:6:3



no~ 7:6:3

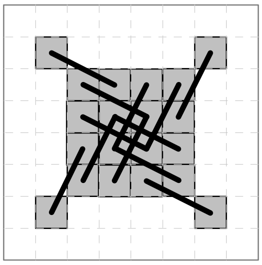


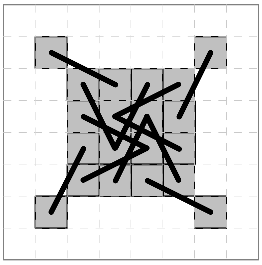
no~ 7:6:3



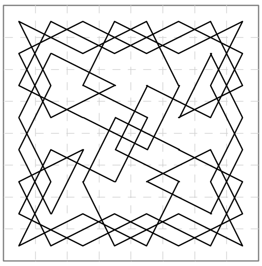
no~ 7:6:3

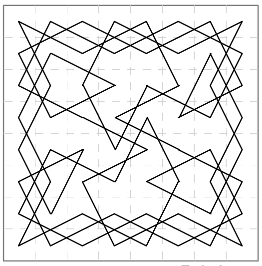
Type (0125). Two linkages, central angles n and o, each leading to 10 tours. As for the previous cases the tours occur in pairs each centre using the same border moves. Oriented so b2 angle is acute.


Type (0125) n


Type (0125) o

In each there are eight of (5:8:3) type:


5:8:3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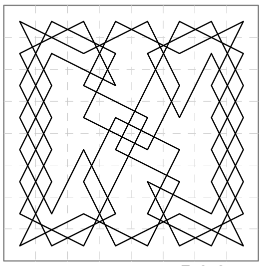
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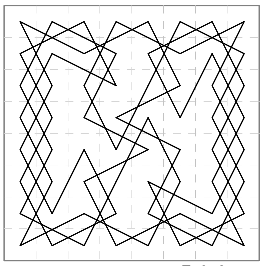
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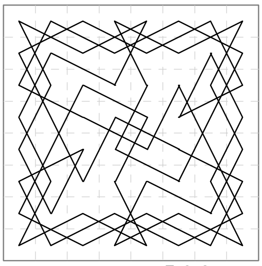
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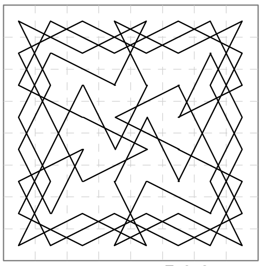
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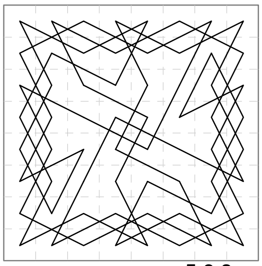
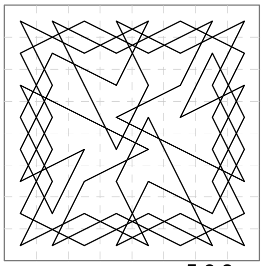

5:8:3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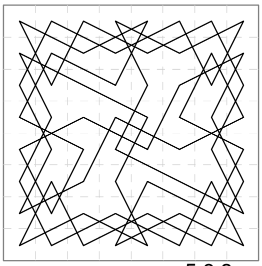
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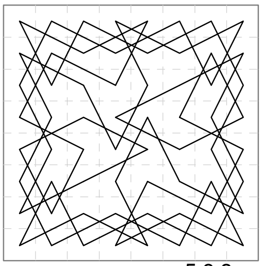
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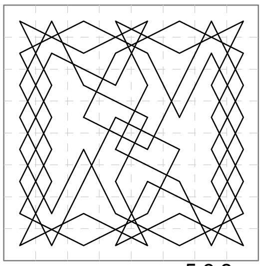
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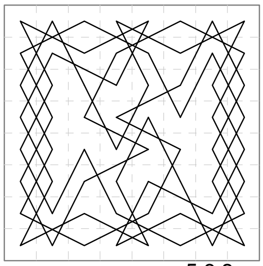
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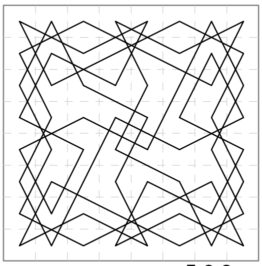
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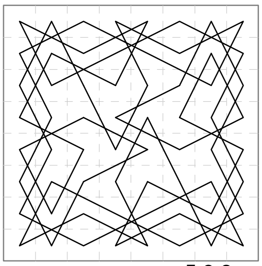
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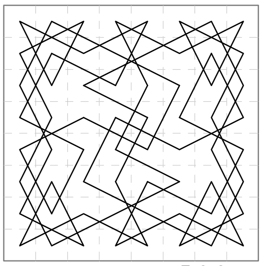
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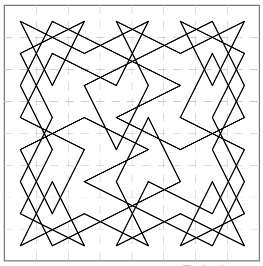
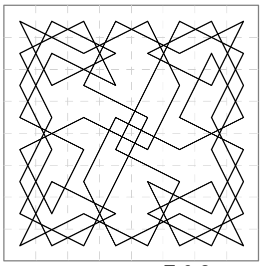
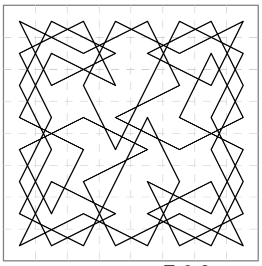
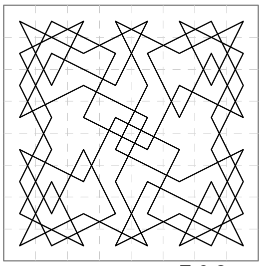

5:8:3

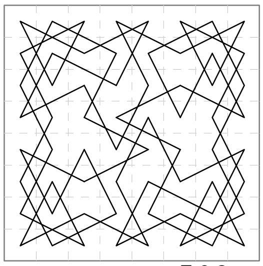
continued

and two of (7:6:3) type:


7:6:3


7:6:3

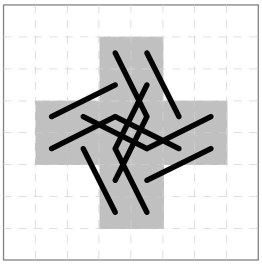

7:6:3

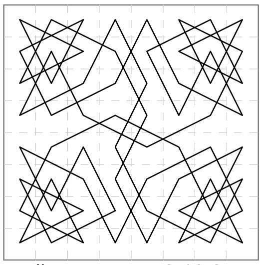

7:6:3

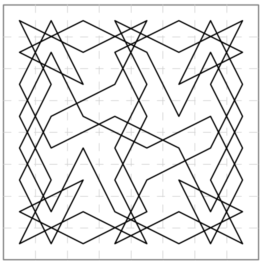
Double move (2·8 + 1·4) class

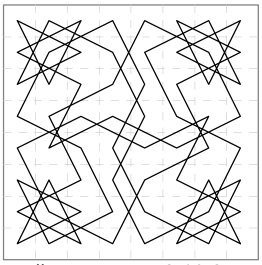
All the tours in this section were identified by Bergholt, except for the three of type (014). He labelled them by Roman numerals. There are five possible types: (013 = IIIb) 9 tours, (014 = IIIc) 3 tours, (034 = IIIa) 5 tours, (156 = I) 49 tours, (238 = II) 6 tours, total 72:

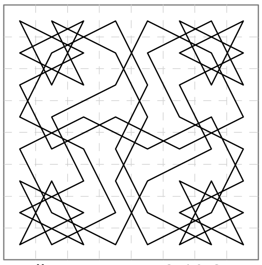
Type (013): 9 tours. 5 (3:10:3) and 4 (7:6:3)

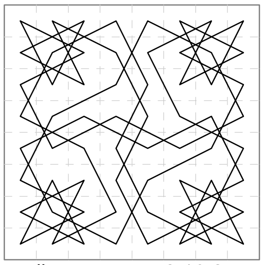

Type (013) 9 tours

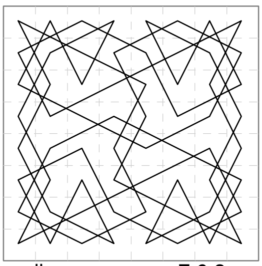

jj~ 3:10:3

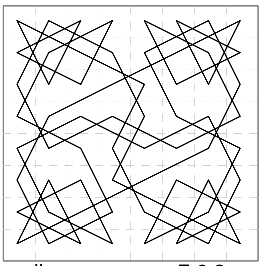

jj~ 3:10:3

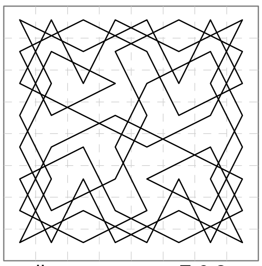

jj~ 3:10:3

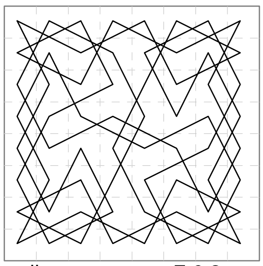

jj~ 3:10:3


jj~ 3:10:3

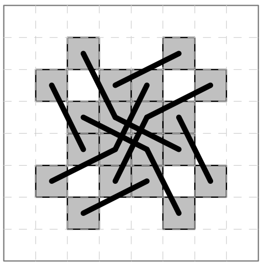

jj~ 7:6:3

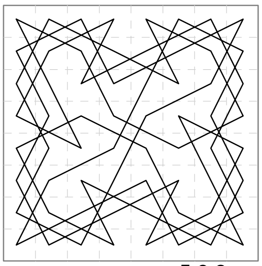

jj~ 7:6:3

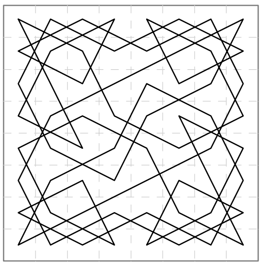

jj~ 7:6:3

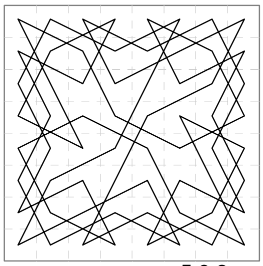

jj~ 7:6:3

Type (014): 3 tours. all (5:8:3):

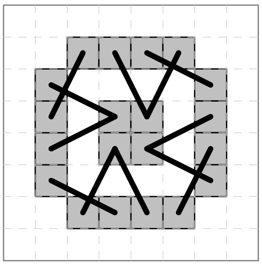

Type (014) 3 tours

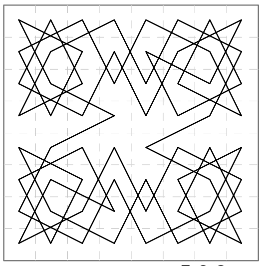

ee~ 5:8:3

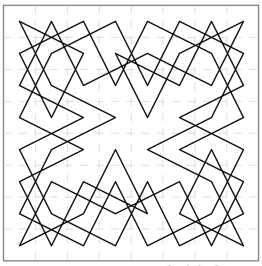

ee~ 5:8:3

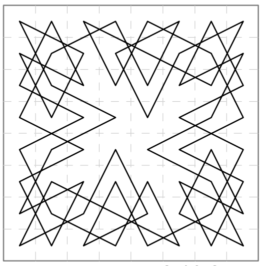

ee~ 5:8:3

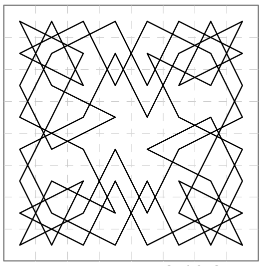
Type (034): 5 tours. 4 (3:10:3) and 1 (5:8:3)

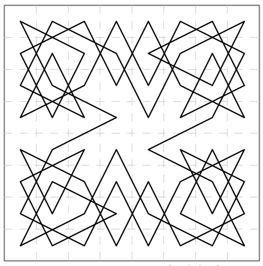

Type (034) 5 tours


gg~ 5:8:3

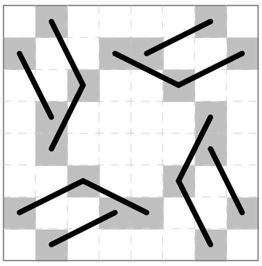

gg~ 3:10:3

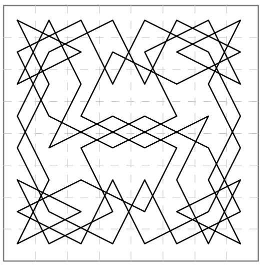

gg~ 3:10:3

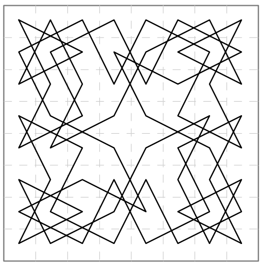

gg~ 3:10:3

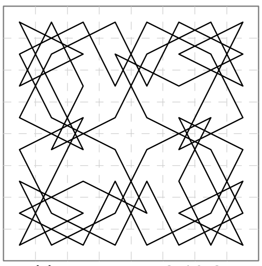

gg~ 3:10:3

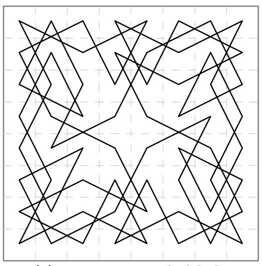
Type (238): 6 tours. 4 (3:10:3) and 1 (5:8:3) and 1 (7:6:3)

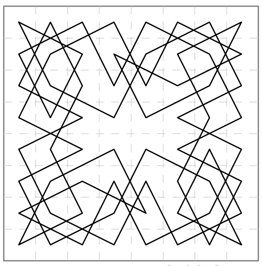

Type (238) 6 tours

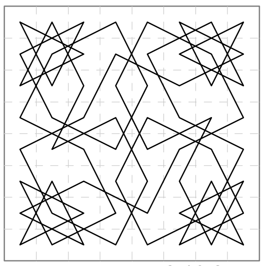

jj= 7:6:3


hh 5:8:3

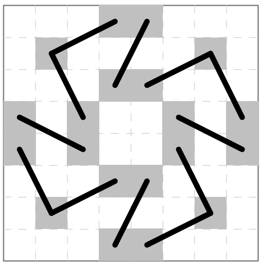

hh 3:10:3


hh 3:10:3

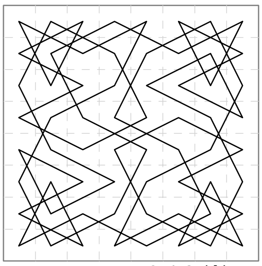

bb= 3:10:3


kk= 3:10:3

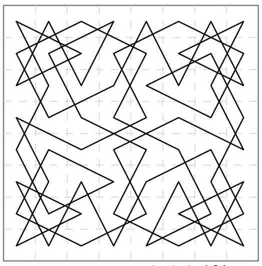
Type (156): 49 tours.



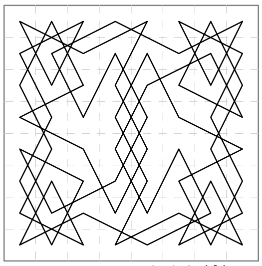
(156) 3 (9:4:3) and 9 (3:10:3)



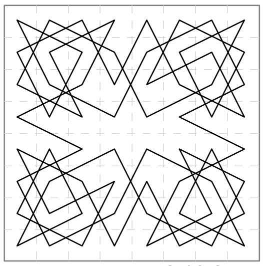
kk= 9:4:3 (A)



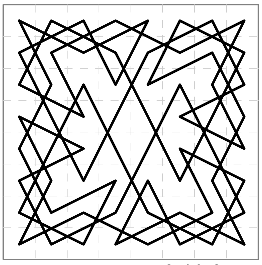
kk= 9:4:3 (A)



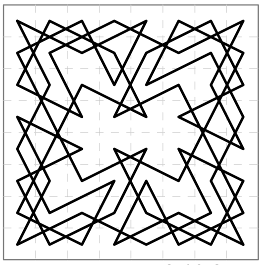
jj= 9:4:3 (A)



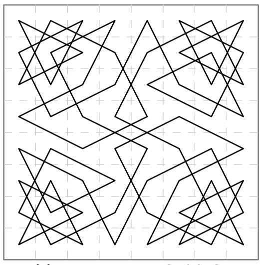
bb= 3:10:3



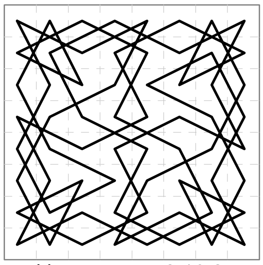
ii= 3:10:3



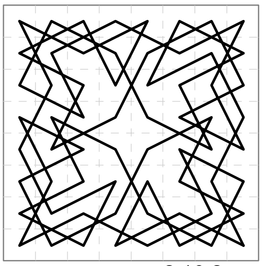
ll= 3:10:3



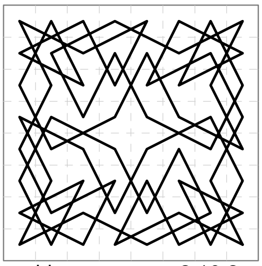
kk= 3:10:3



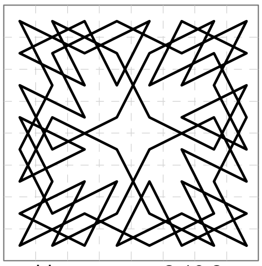
kk= 3:10:3



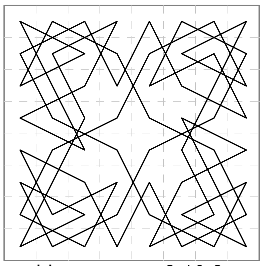
hh 3:10:3



hh 3:10:3

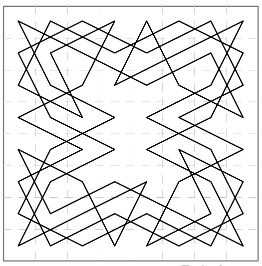


hh 3:10:3

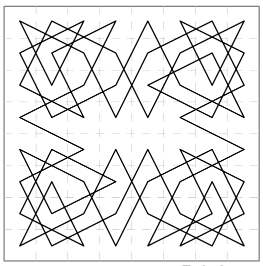


hh 3:10:3

(156) 14 (5:8:3) two with gg= centre



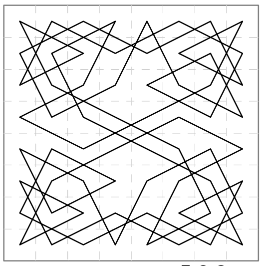
gg= 5:8:3

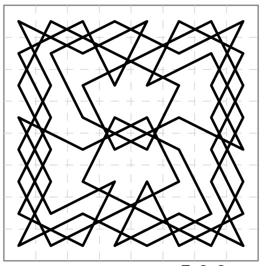


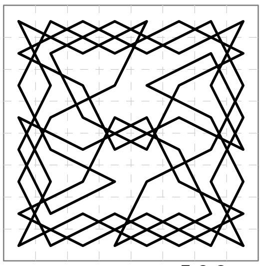
gg= 5:8:3

continued

(156) 14 (5:8:3) twelve with other centres


dd= 5:8:3


nn= 5:8:3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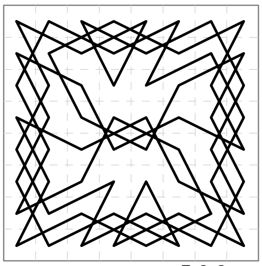
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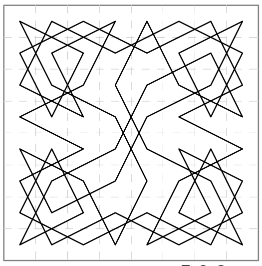
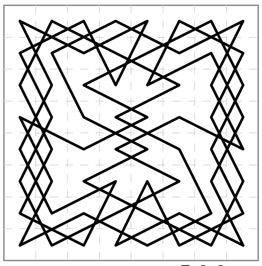
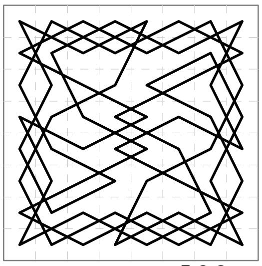
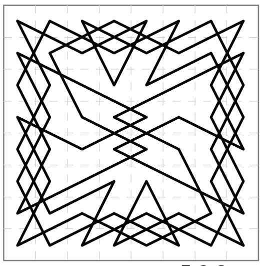
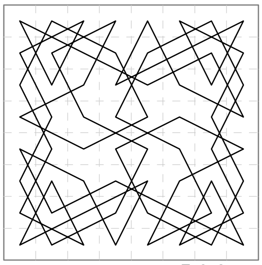
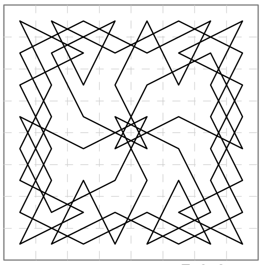

ee= 5:8:3

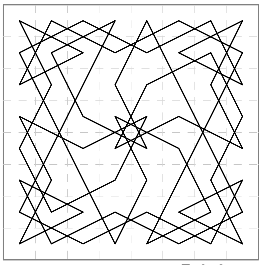

oo= 5:8:3

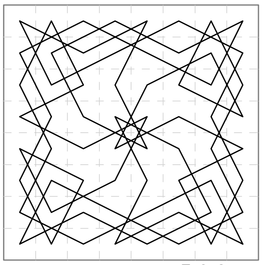

oo= 5:8:3


oo= 5:8:3

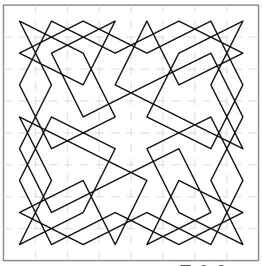

kk= 5:8:3

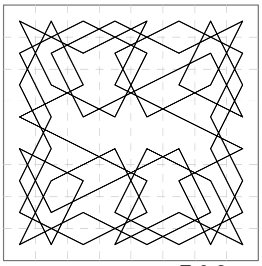

pp 5:8:3

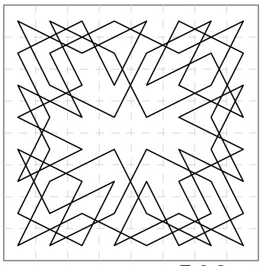

pp 5:8:3

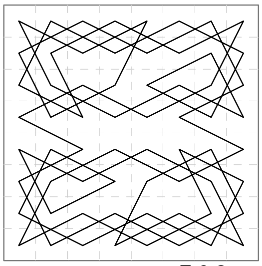

pp 5:8:3

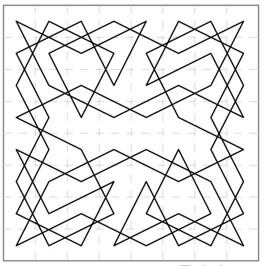
(156) 23 (7:6:3) eight with bb=, cc=, gg= centres

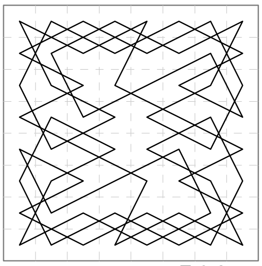

bb= 7:6:3

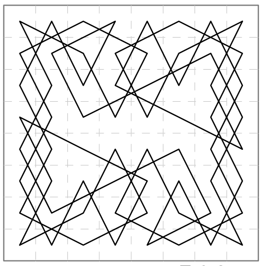

bb= 7:6:3

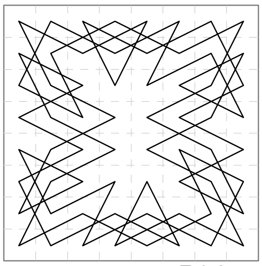

bb= 7:6:3


cc= 7:6:3


cc= 7:6:3

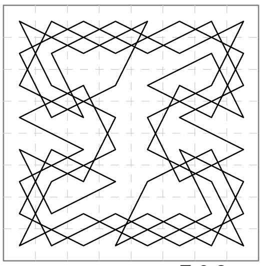

gg= 7:6:3

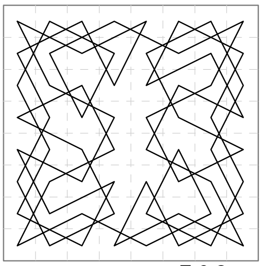

gg= 7:6:3

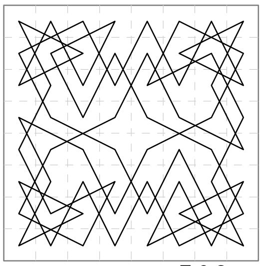

gg= 7:6:3

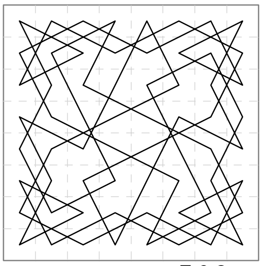
continued

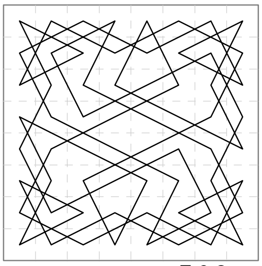
(156) 23 (7:6:3) the other 15 with f, h, i, j, k, l centres


ff= 7:6:3


ff= 7:6:3


hh 7:6:3


ii= 7:6:3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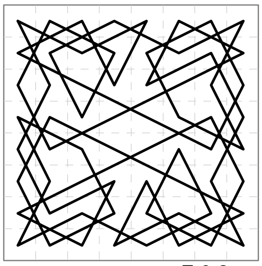
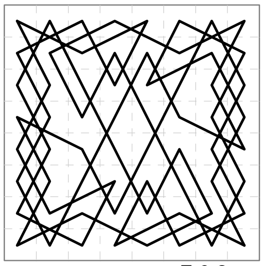
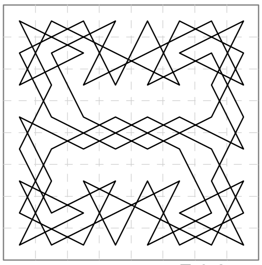
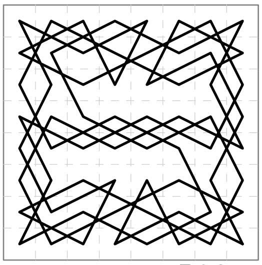
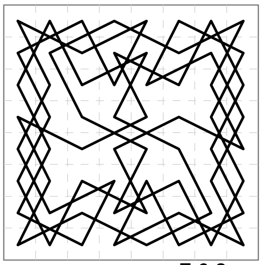
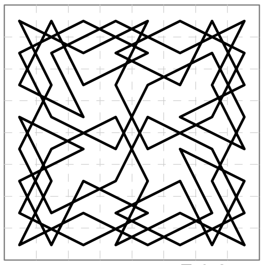

ii= 7:6:3

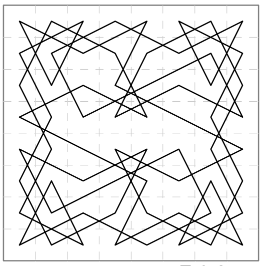

ii= 7:6:3

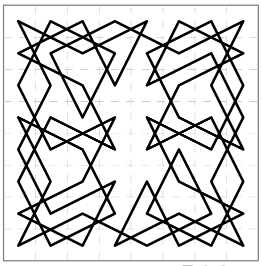

jj= 7:6:3

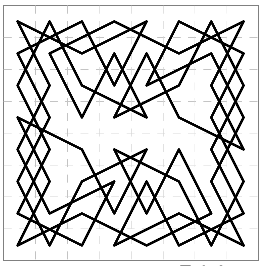

jj= 7:6:3

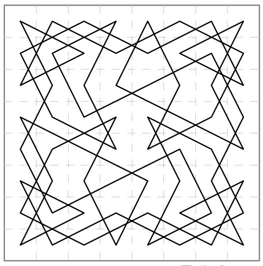

kk= 7:6:3


kk= 7:6:3


ll= 7:6:3

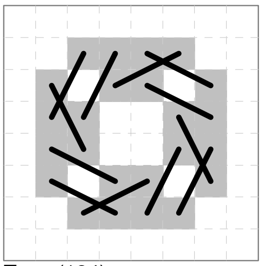

ll= 7:6:3

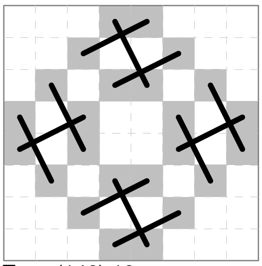

ll= 7:6:3

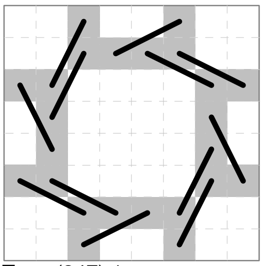

ll= 7:6:3

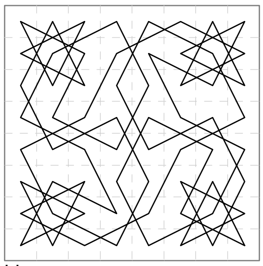
Single moves (3·8) class

In this class I find three choices of off-diagonal cells that will connect in the required manner, namely (134), (146), (347) as shown in the diagrams below. Of these (134) produces no tours on the 8×8, though it can be used on larger boards. (347) gives a single tour, and (146) gives 19 tours.


Type (134) no tour

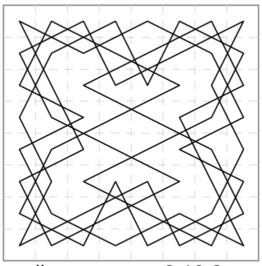

Type (146) 19 tours

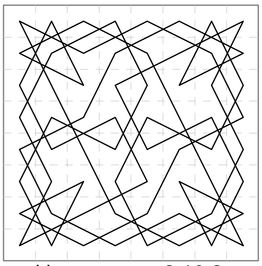

Type (347) 1 tour

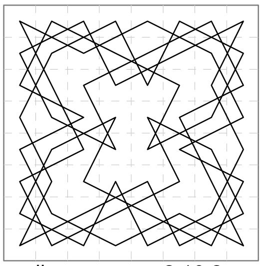

kk= Bergholt 3:10:3

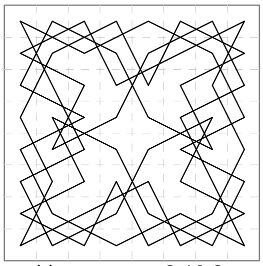
Type (347): (3:10:3) unique case shown above. This unique (347) solution (reflected) appears as a solution to Problem 4 by Ernest Bergholt in *BCM* 1918 p.195.

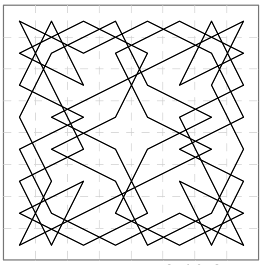
Type (146): 6 (3:10:3) and 6 (5:8:3) ...

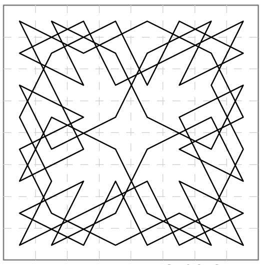

ii= 3:10:3

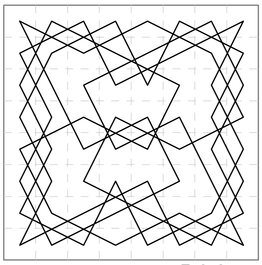

kk= 3:10:3

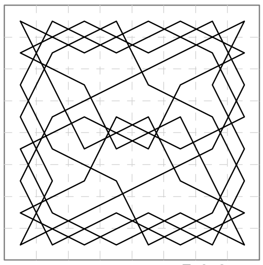

ll= 3:10:3

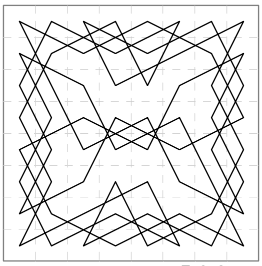

hh 3:10:3

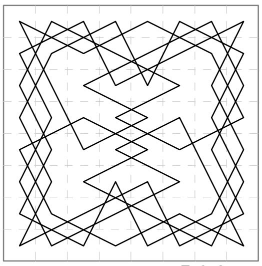

hh 3:10:3

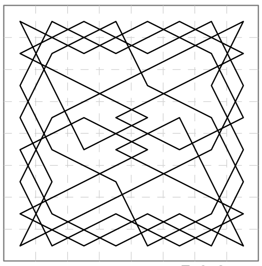

hh 3:10:3

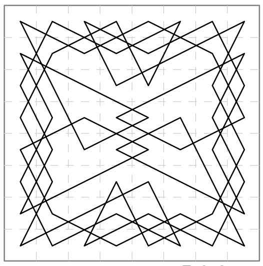

nn= 5:8:3


nn= 5:8:3

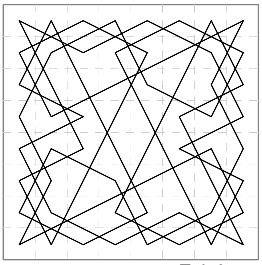

nn= 5:8:3

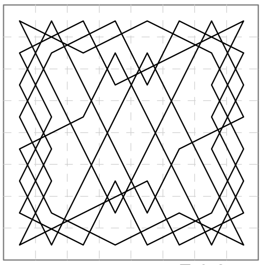

oo= 5:8:3

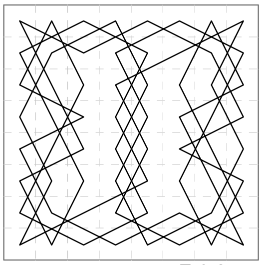

oo= 5:8:3

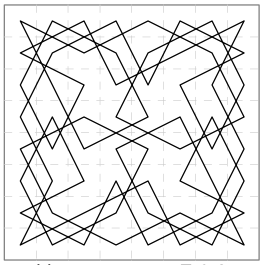

oo= 5:8:3

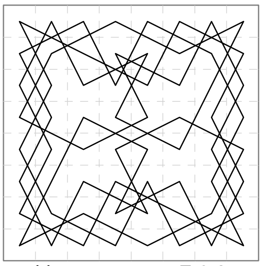
and 7 (7:6:3). The second kk= tour is given as a solution to Problem 4 by Bergholt in BCM 1918 p.195. The second ii= tour and the jj= tour are in Murray (1942).

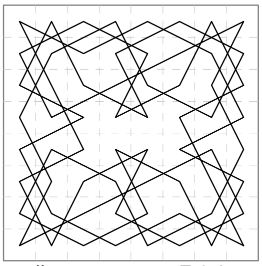

ii= 7:6:3

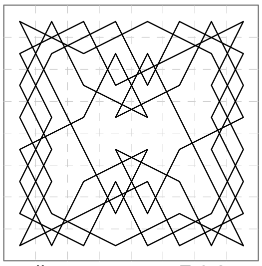

ii= Murray 7:6:3


jj= Murray 7:6:3


kk= 7:6:3


kk= Bergholt 7:6:3

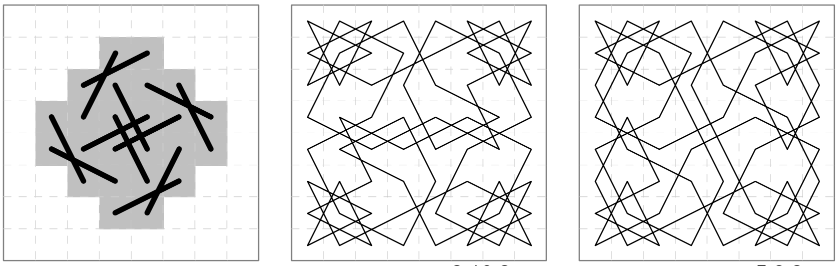

ll= 7:6:3


ll= 7:6:3

Single moves (2·8 + 2·4) class

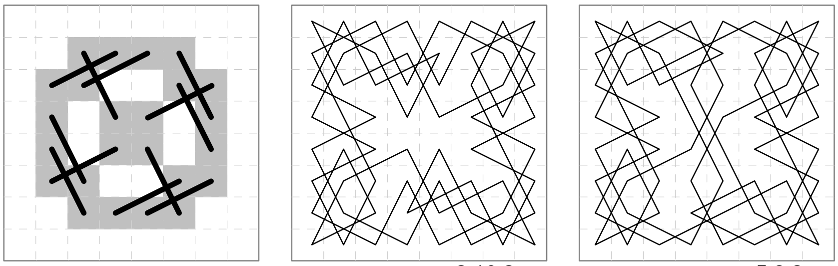
This has 12 separate oblique moves and uses two diagonal and two off-diagonal cells. The nine combinations I have found are: (0123) 2 tours, (0234) 2 tours , (0145) 3 tours, (0135) 12 tours, (0456) 20 tours, (0156) 28 tours, (0238) 31 tours , (0126) 33 tours, (1256) 22 tours, total 153 tours.

Type (0123) two tours 3:10:3 and 5:8:3:



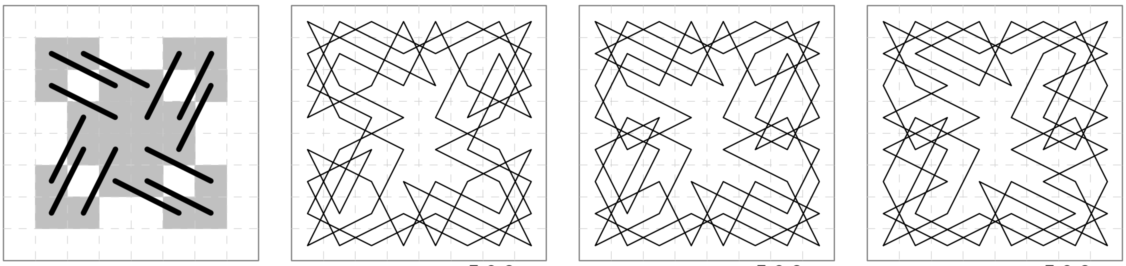
jk= 3:10:3 de= 5:8:3

Type (0234) two tours 3:10:3 and 5:8:3:



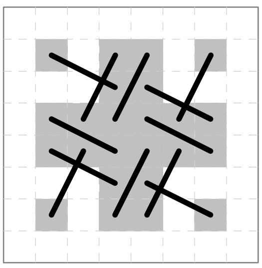
bg~ 3:10:3 de~ 5:8:3

Type (0145) three tours 5:8:3:



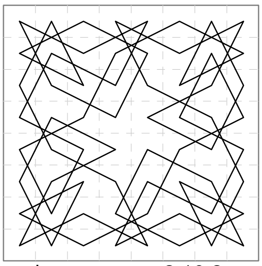
bg~ 5:8:3 bg~ 5:8:3 bg~ 5:8:3

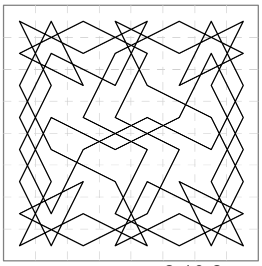
Type (0135) twelve tours. 5 (3:10:3) and 6 (7:6:3) and 1 (9:4:3).

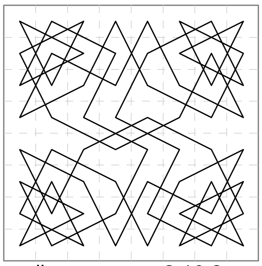


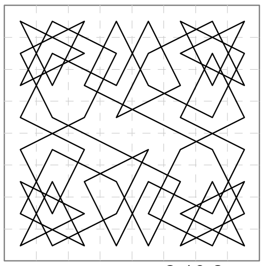
tour diagrams follow

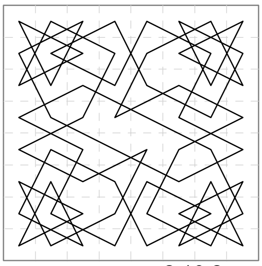
(0135) twelve tours. 5 (3:10:3) and 6 (7:6:3) and 1 (9:4:3).

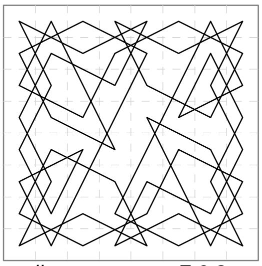

bg= 3:10:3

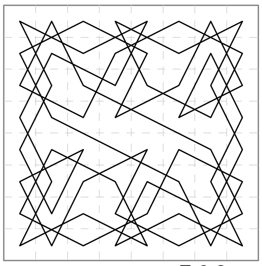

jk~ 3:10:3

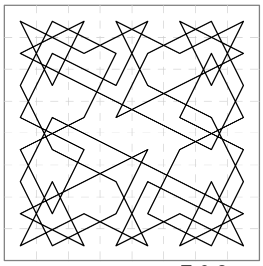

jk~ 3:10:3


il~ 3:10:3


il~ 3:10:3


il~ 7:6:3


il~ 7:6:3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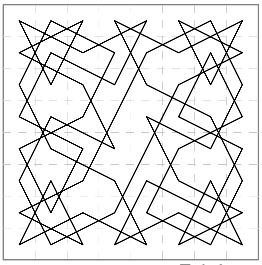
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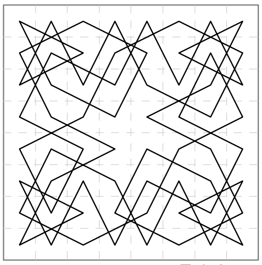
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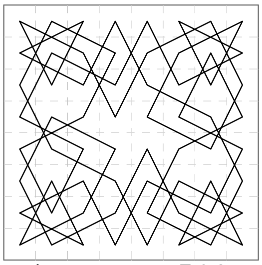
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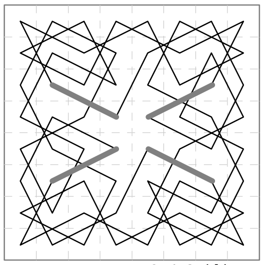
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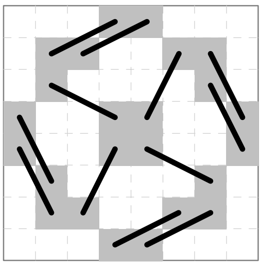
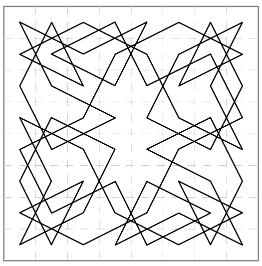
Type (0456) 20 tours



(0456) 1 (3:10:3) and 7 (5:8:3) and 9 (7:6:3) and 3 (9:4:3)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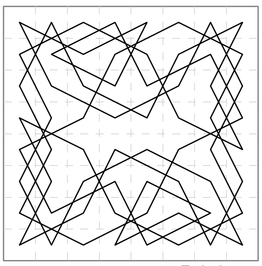
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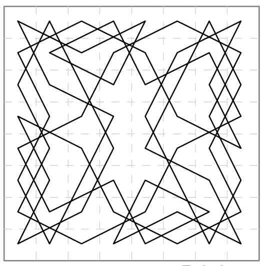
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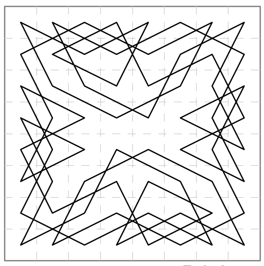
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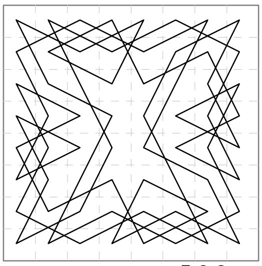

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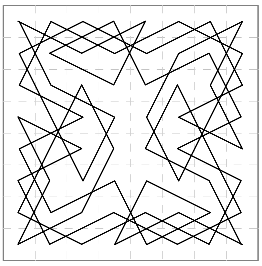

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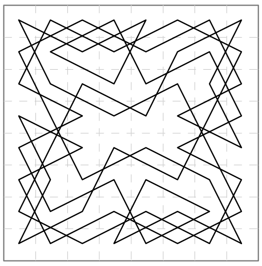

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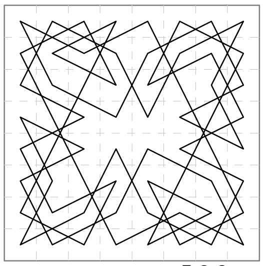

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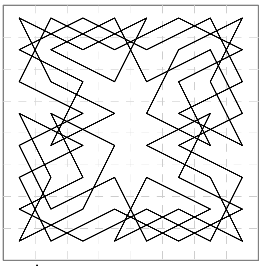

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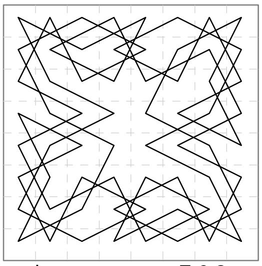

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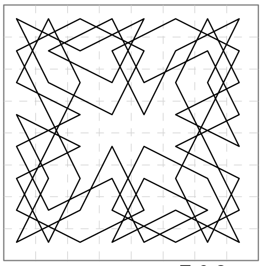

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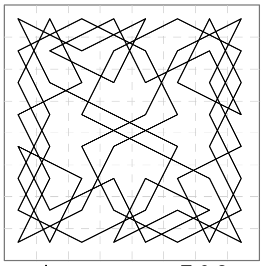

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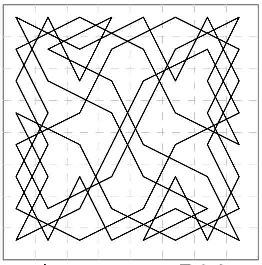

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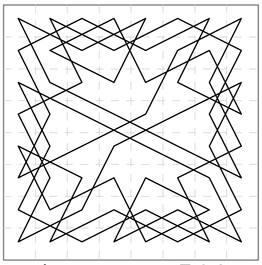

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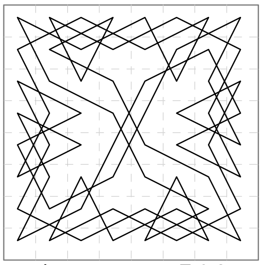

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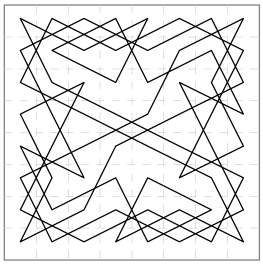

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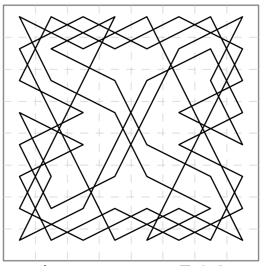

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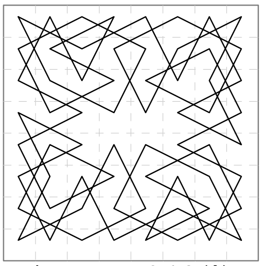

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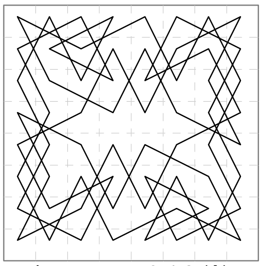

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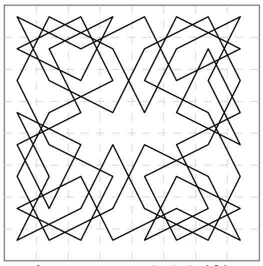
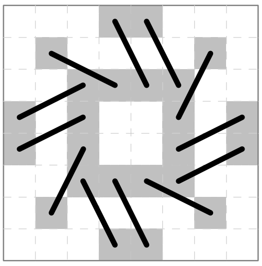
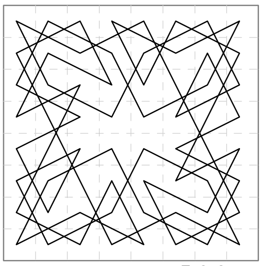


bg~ 9:4:3 (A)

Type (1256) 22 tours. 5 (5:8:3) and 10 (7:6:3) and 7 (9:4:3)





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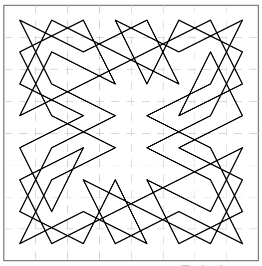

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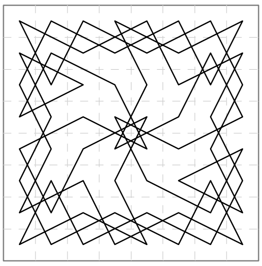
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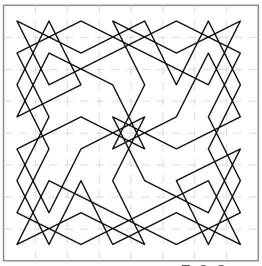
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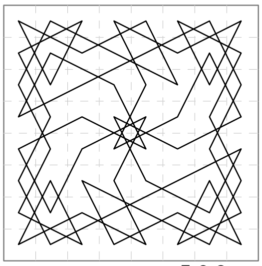
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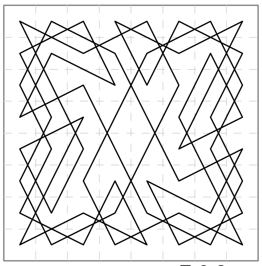
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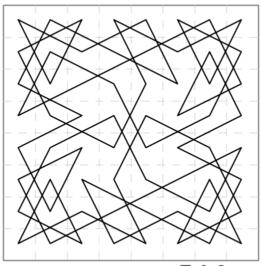
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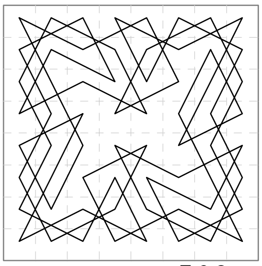
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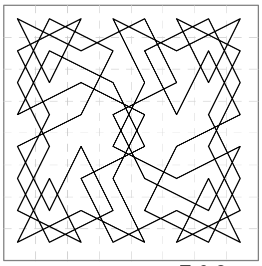
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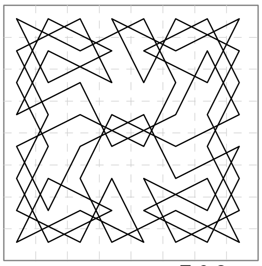
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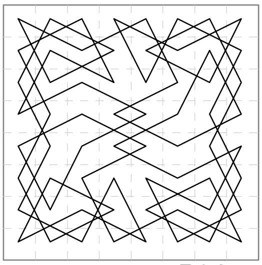
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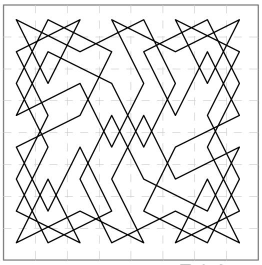
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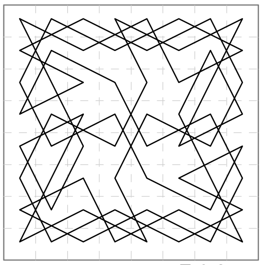
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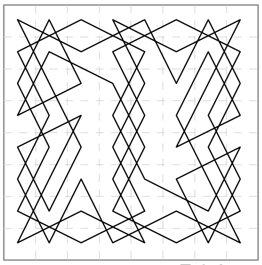
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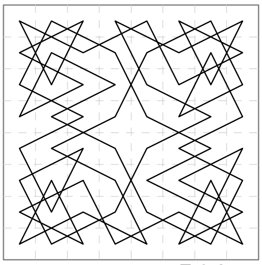
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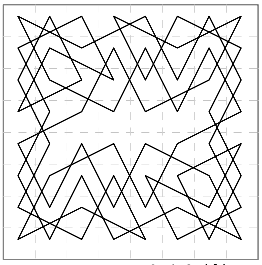
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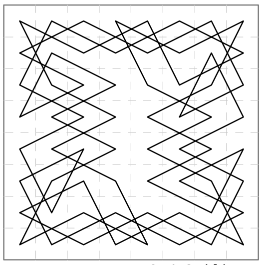
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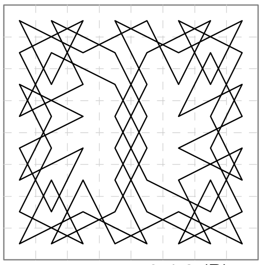
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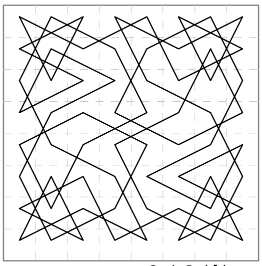
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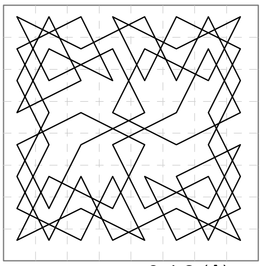
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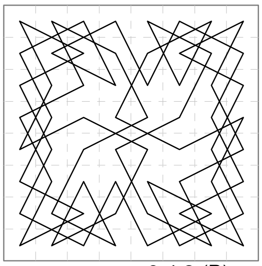
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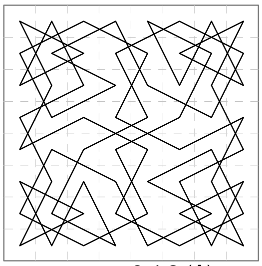
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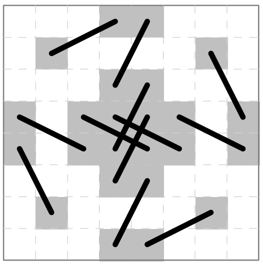
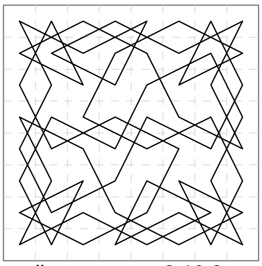
Type (0156) 28 tours



(0156) 2 (3:10:3) and 9 (5:8:3) and 13 (7:6:3) and 4 (9:4:3)



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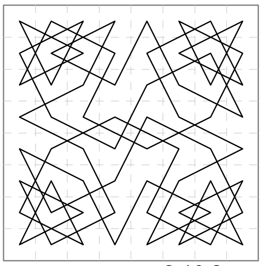

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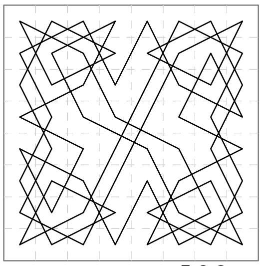
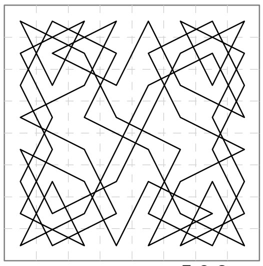


de= 5:8:3



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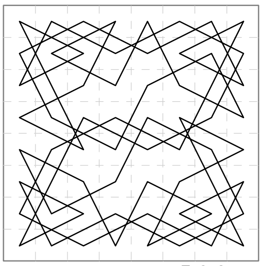

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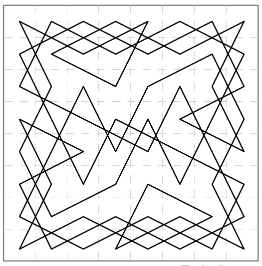

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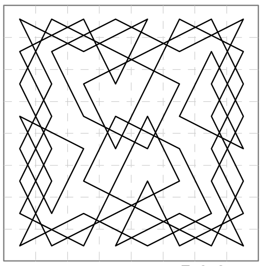

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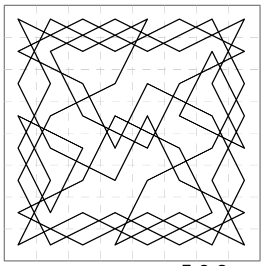

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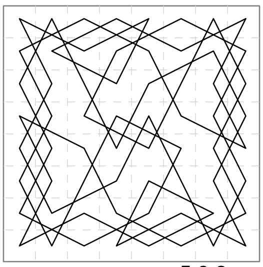

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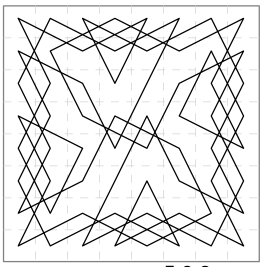

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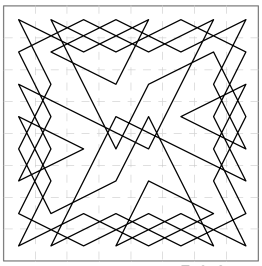

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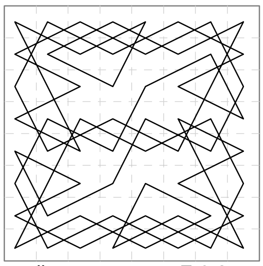

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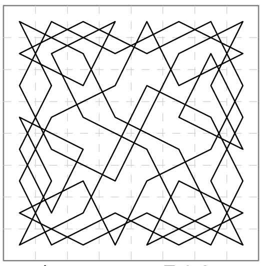

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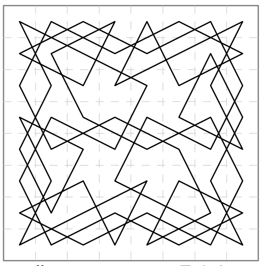

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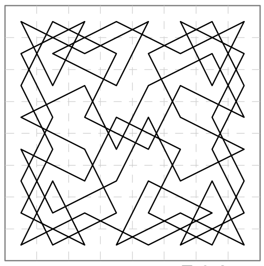

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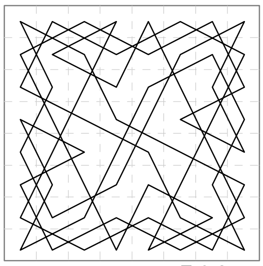

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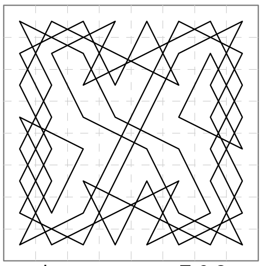
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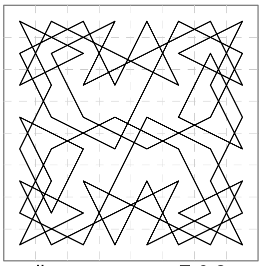
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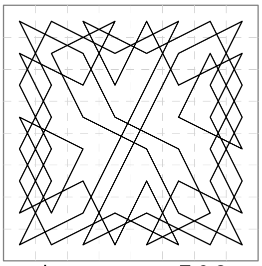
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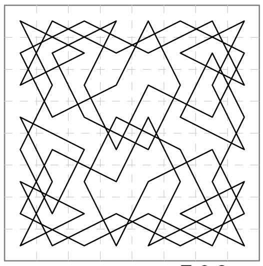
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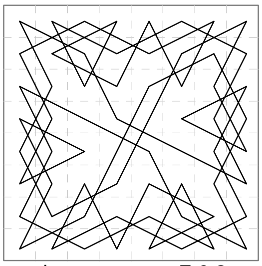
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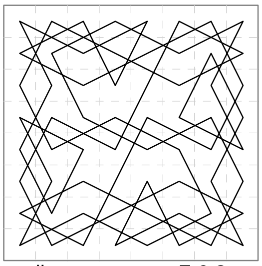
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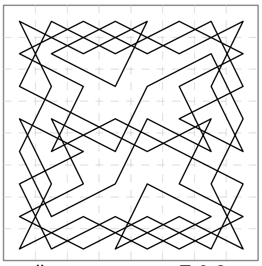
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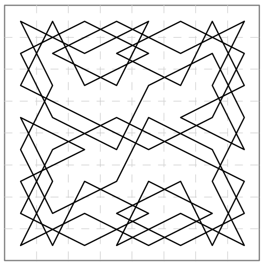
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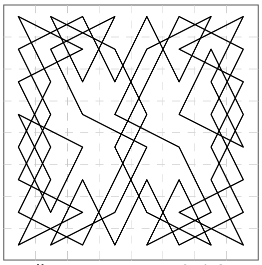
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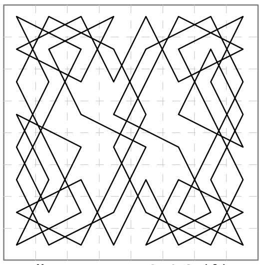
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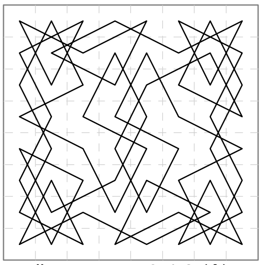
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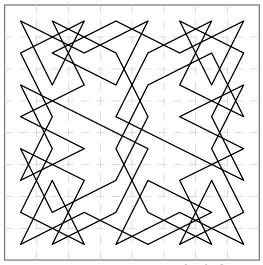
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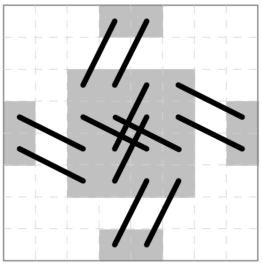
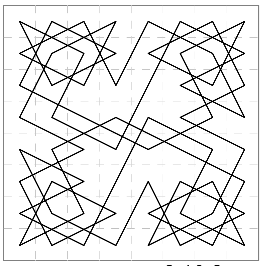
Type (0126) 33 tours. 4 (3:10:3) and 13 (5:8:3)

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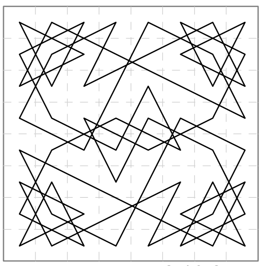
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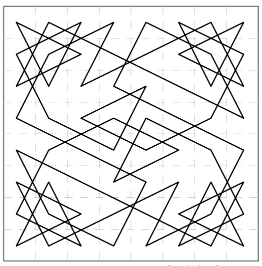
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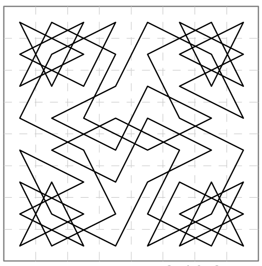
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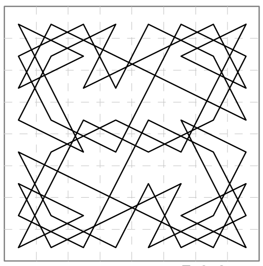
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continued

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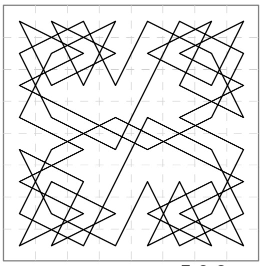
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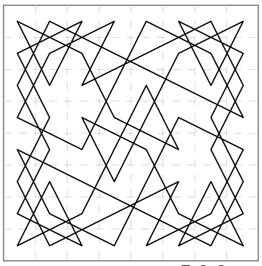
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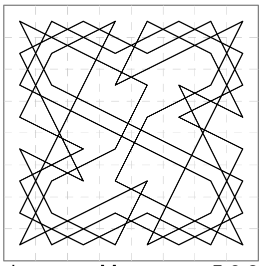
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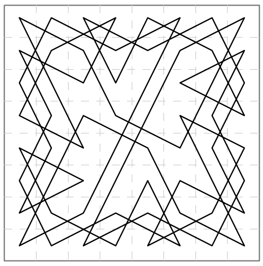
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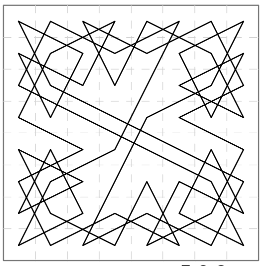
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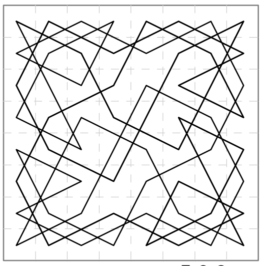
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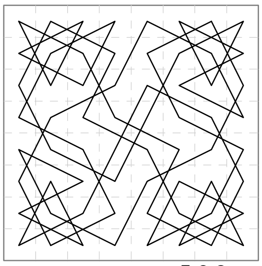
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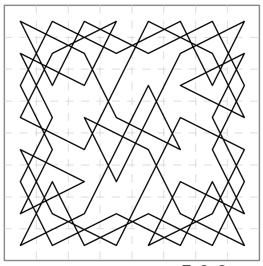
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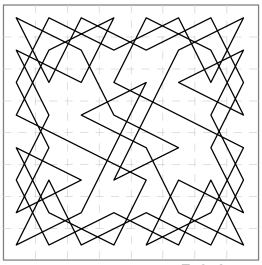
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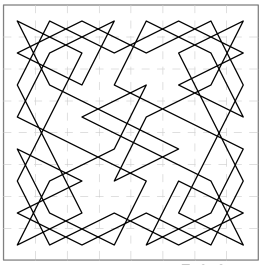
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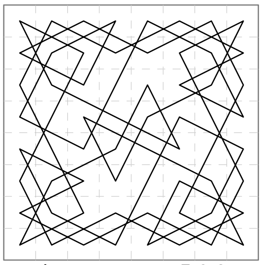
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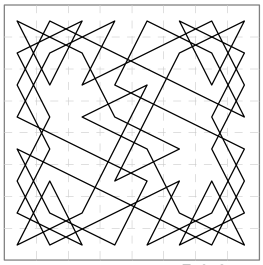
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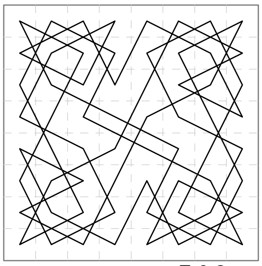
(0126) continued 14 (7:6:3) and 2 (9:4:3)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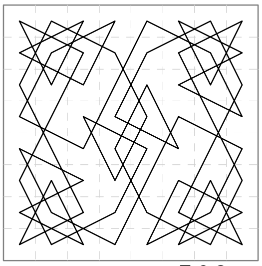
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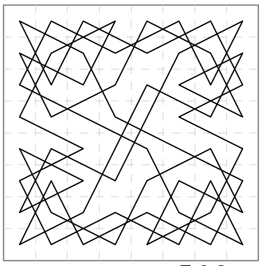
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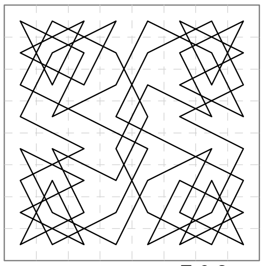
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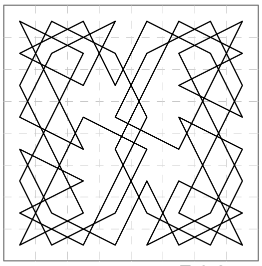
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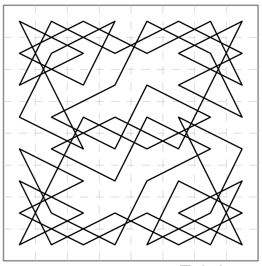
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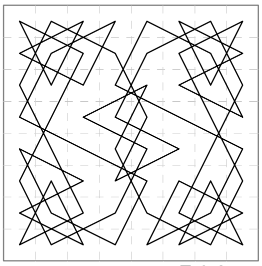
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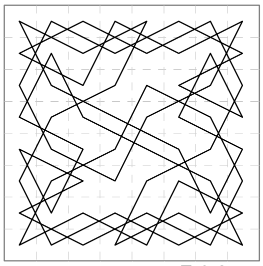
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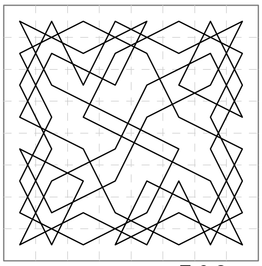
(0126) continued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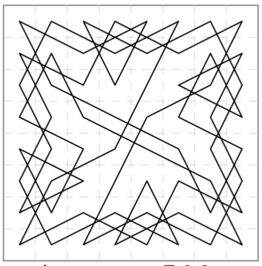
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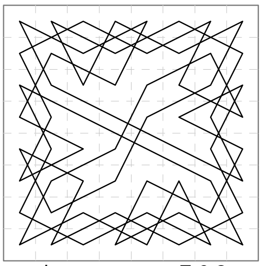
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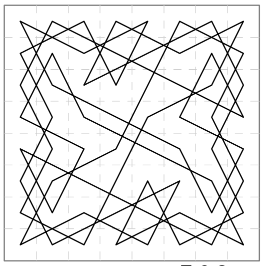
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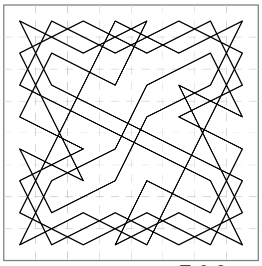
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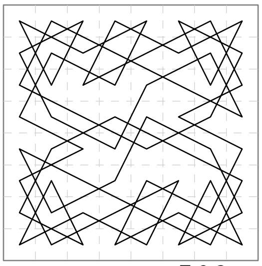
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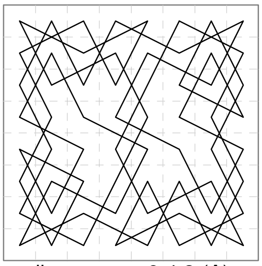
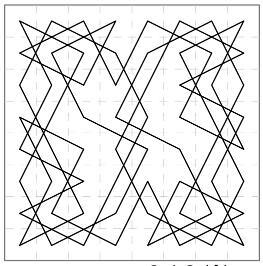

jk= 9:4:3 (A)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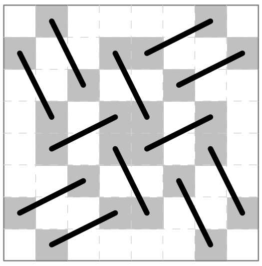
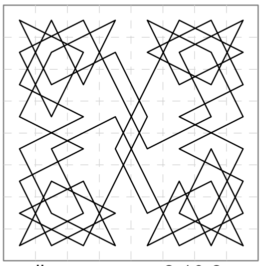
Type (0238) 31 tours. 4 (3:10:3) and 2 (5:8:3) and 19 (7:6:3) and 6 (9:4:3)

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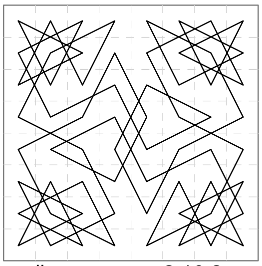
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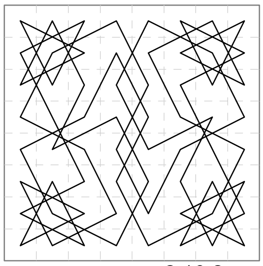
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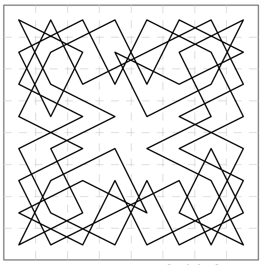
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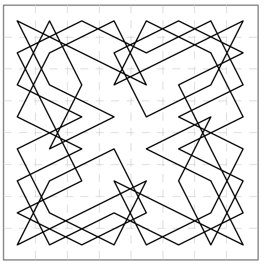
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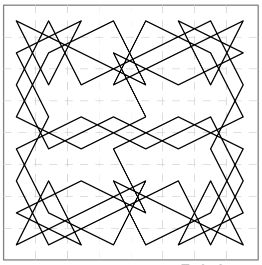
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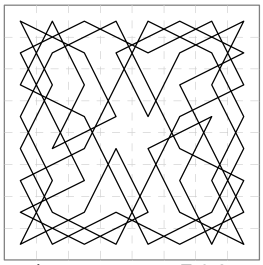
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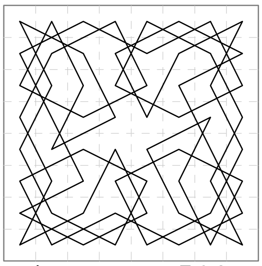
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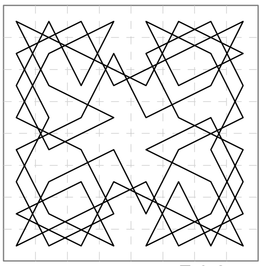
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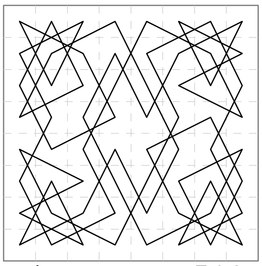
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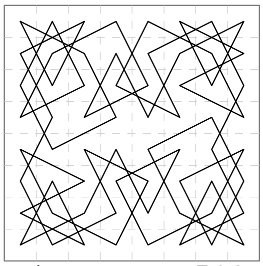
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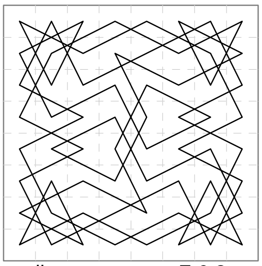
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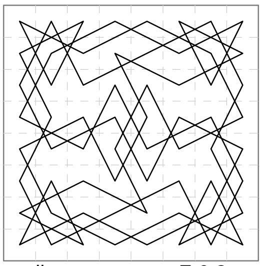
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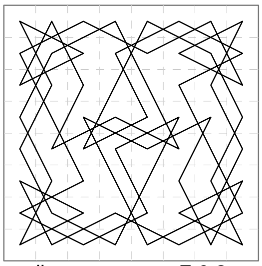
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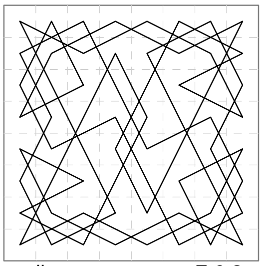
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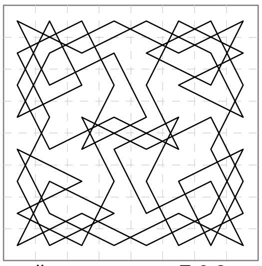
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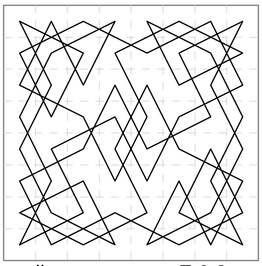
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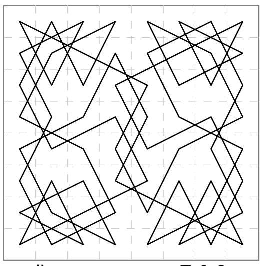
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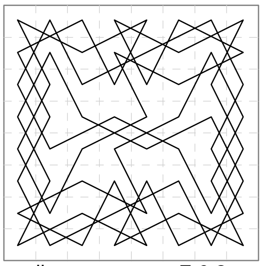
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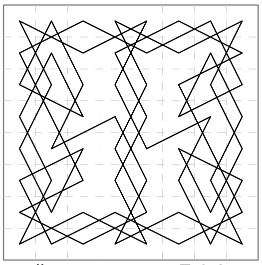
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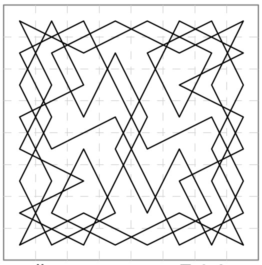
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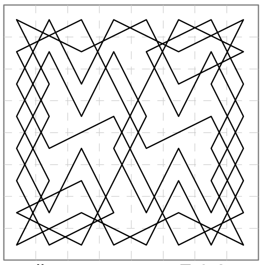
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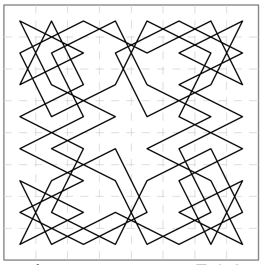
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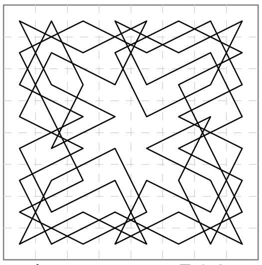
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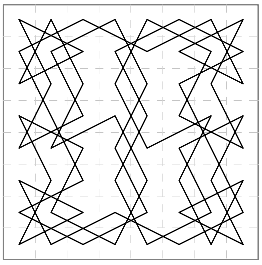
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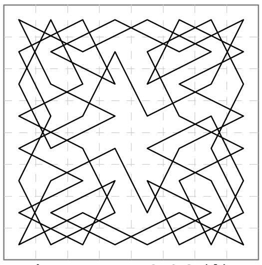
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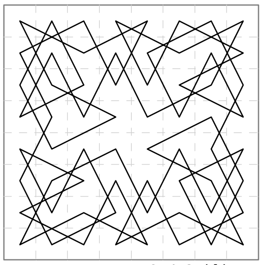
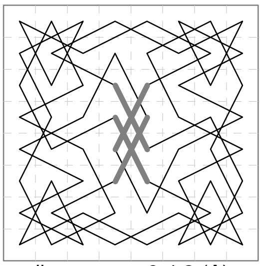
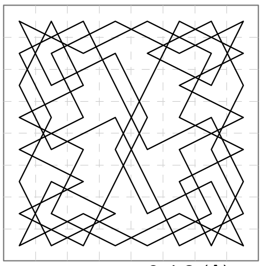
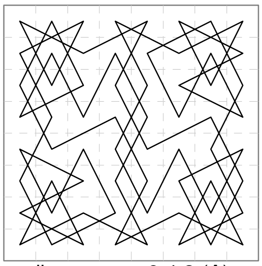
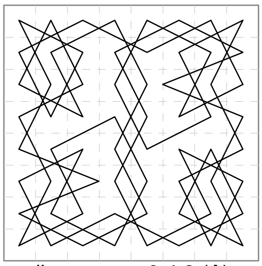

bg= 9:4:3 (A)


jk~ 9:4:3 (A)


jk~ 9:4:3 (A)


jk~ 9:4:3 (A)


jk~ 9:4:3 (A)

Mixed Quaternary Tours with h > 1 and k > 3

Above we have enumerated the cases with minimum direct (h=1) and minimum oblique (k=3). The feasible intermediate h:j:k values are: 3:8:5, 3:6:7, 3:4:9, 5:6:5, 5:4:7, 7:4:5. I have a few examples of all of these, diagrammed below.

H. J. R. Murray continued research into mixed quaternary symmetry after Bergholt's death in 1925 and gives the tabulated figures below in his 1942 ms which add to 580. He says that each tour is composed of a number of chains alternately in direct and oblique symmetry. However it is not clear how he deals with sections that show octonary symmetry, which may occur between quaternary sections and can apply to either. These figures have not been independently checked.

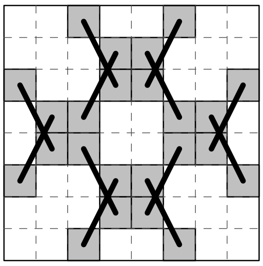
chains:	8	16	24	32	40	total
tours:	145	168	185	67	15	580

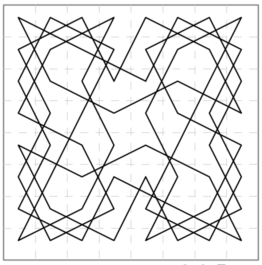
Among Murray's papers there are very many diagrams of tours and it is probable that a closer examination of them will find that he did much further work on this subject.

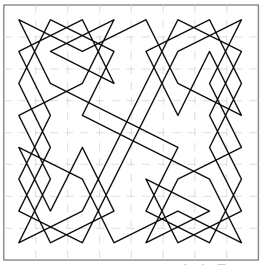
Most of the tours shown here were found by my own systematic searches made in similar fashion to those in the h=1 and k=3 sections.

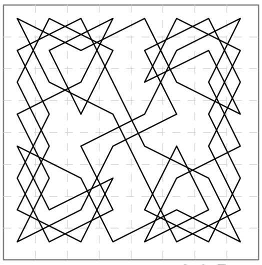
Tours with h = 3

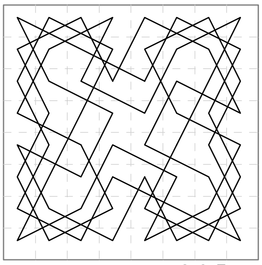
Type [137] A set of 14 tours of this type were constructed by Bergholt, all having the same pure direct quaternary component of 12 moves connecting these off-diagonal cells. The cells are joined round the corners by the moves 3-8-7 in every case. These 16 moves being in octonary symmetry. The corner moves themselves add a further 8 in octonary symmetry, and there may be another set of 8. So the tours are of types (3:6:7) or (3:8:5). Here are the 10 tours of (3:8:5) type.

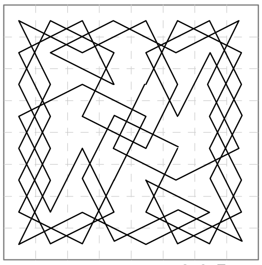

[137]

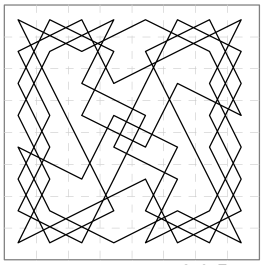

cc~ 3:8:5

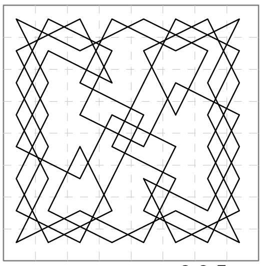

dd~ 3:8:5

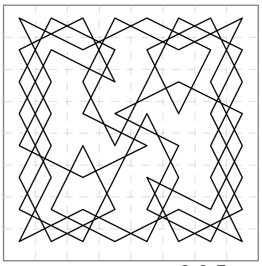

ee~ 3:8:5

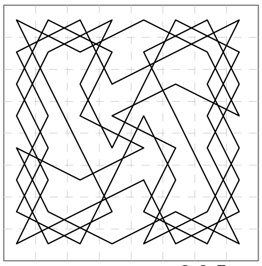

ff~ 3:8:5

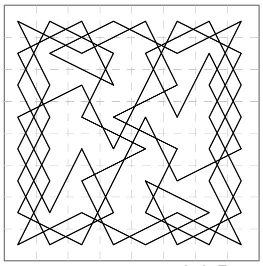

nn~ 3:8:5


nn~ 3:8:5

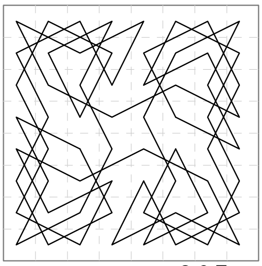

nn~ 3:8:5

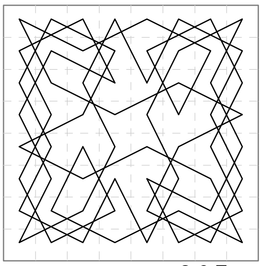

oo~ 3:8:5

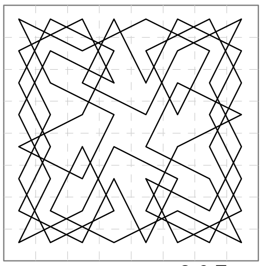

oo~ 3:8:5

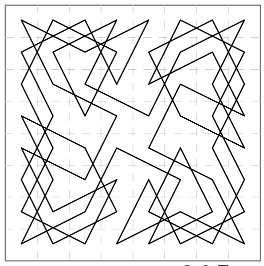

oo~ 3:8:5

Here are the 4 tours of (3:6:7) type by Bergholt from the [137] array.

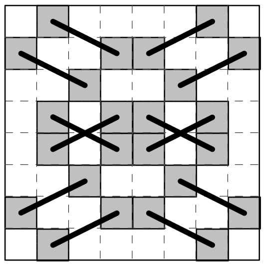

cc~ 3:6:7

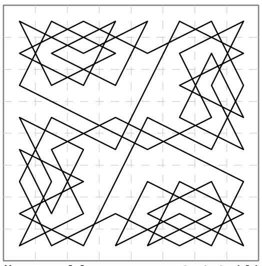

cc~ 3:6:7

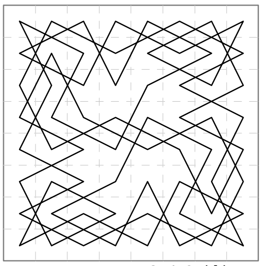

ff~ 3:6:7

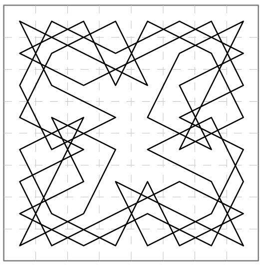

ff~ 3:6:7

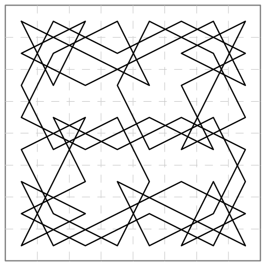
Type [0238] This is another pattern I looked at with h=3 using two diagonal and two off-diagonal cells. It generates 16 tours. Two are of (3:4:9) type which I think are the only tours of this type possible. The first was among examples given by Murray. Also two (3:8:5) type and eight (3:6:7).

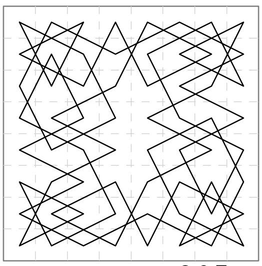

[0238]

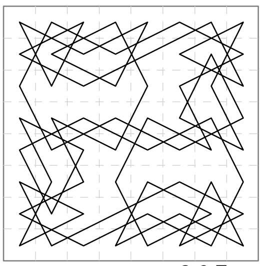

jk= Murray 3:4:9 (A)

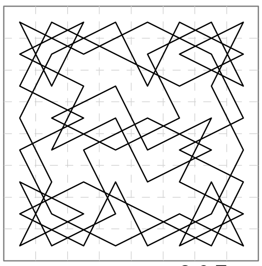

jk= 3:4:9 (A)

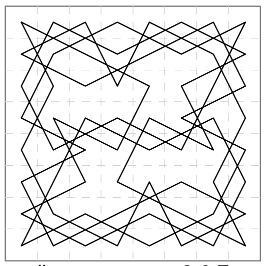

bg~ 3:8:5

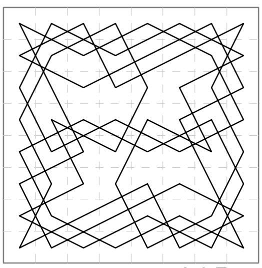

jk= 3:8:5

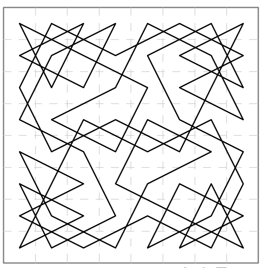

bg~ 3:6:7

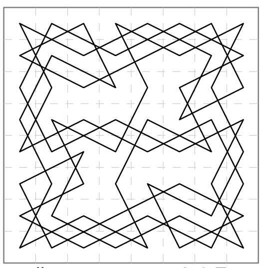

jk= 3:6:7

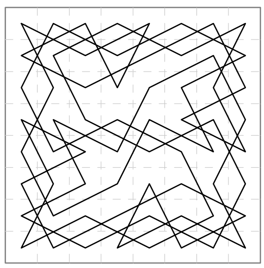

jk= 3:6:7


jk= 3:6:7


jk= 3:6:7


jk= 3:6:7

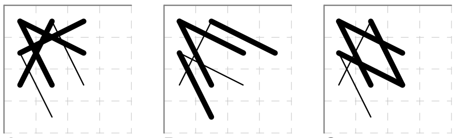

jk= 3:6:7


jk= 3:6:7

There are also four of (3:10:3) type. For these see the k=3 section above, two under (0126) and one each in (0123) and (0234).

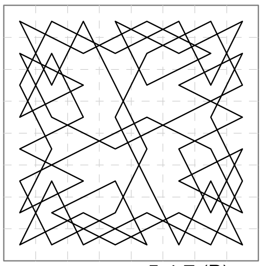
Tours with j = 4

The minimum number of octonary moves in a tour of mixed quaternary symmetry on a square board of any size is 16 (i.e. j = 4), having four moves in each quadrant, consisting of the 2 moves through the corner cell and 2 moves incident with the next-to-corner cells. There are three minimal formations on these cells. Any other routes through the next-to-corner cells will result in six octonary moves. However it turns out that case C (using moves 8-2) is impossible, at least on the 8×8 board, as other octonary moves are forced. There are numerous examples of A (8-7) and B (8-3).

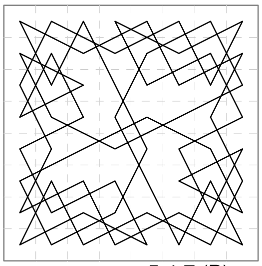


A B C

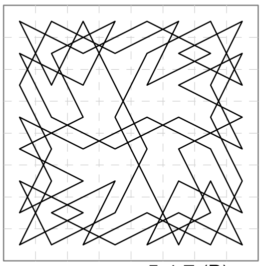
An enumeration I attempted (in 2009) of tours with j = 4 in formation B yielded 19 tours.



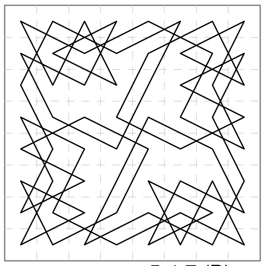
jj~ 5:4:7 (B)



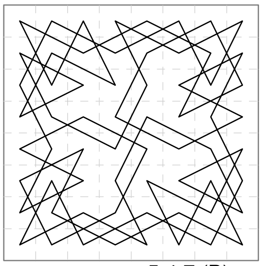
jj~ 5:4:7 (B)



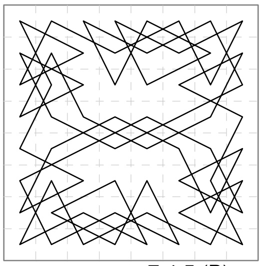
jj~ 5:4:7 (B)



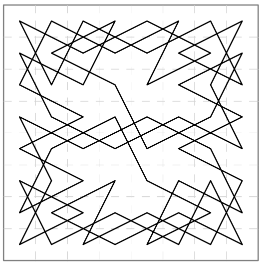
kk~ 5:4:7 (B)



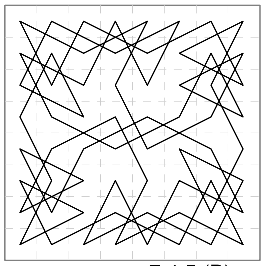
kk~ 5:4:7 (B)



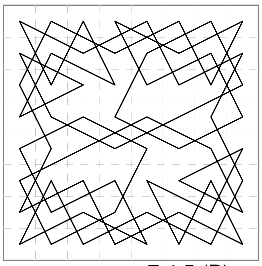
jj= 7:4:5 (B)



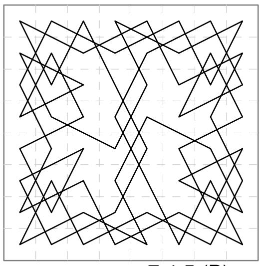
jk= 7:4:5 (B)



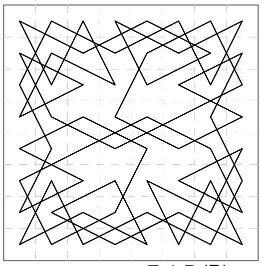
jk= 7:4:5 (B)



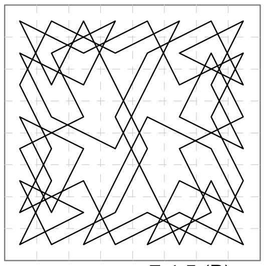
jk~ 7:4:5 (B)



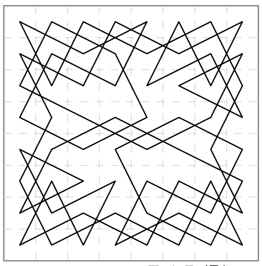
jk~ 7:4:5 (B)



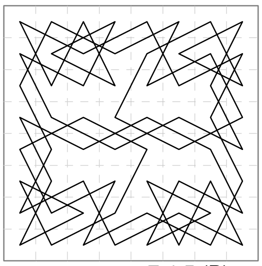
jk~ 7:4:5 (B)



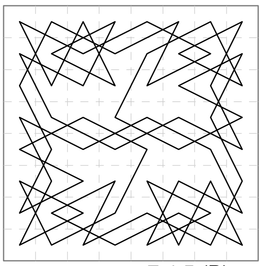
jk~ 7:4:5 (B)



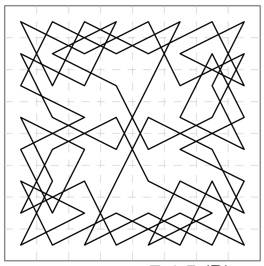
jk~ 7:4:5 (B)



jk~ 7:4:5 (B)



jk~ 7:4:5 (B)



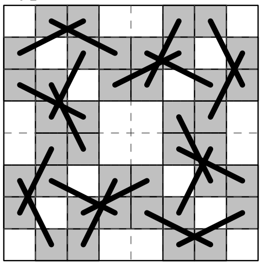
kk= 7:4:5 (B)

Three were of (9:4:3) type; these and one other are included in the k=3 section above. The 16 others of (5:4:7) and (7:4:5) type are shown here.

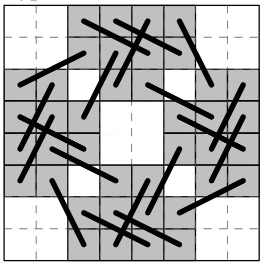
Tours with k = 5

Following on from the above k=3 examples, it is natural to consider whether there are tours with 5 sets of moves in pure oblique quaternary symmetry. If all 20 moves are separate they will use 40 cells forming an octonary pattern. If all these cells are off-diagonal we have the 40 = 5·8 case. I find that there are two types in this class. The only pattern using the next to corner cell, 8, is Type (13478), but this does not generate any tours on the 8×8 board, though it can be used on the 12×12 board. The only other case is Type (13467): This generates four tours. The three of (5:6:5) type are shown below. For the (1:10:5) case see in the h=1 (minimum direct) section above.

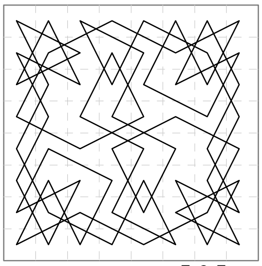
Type (13478) no tours



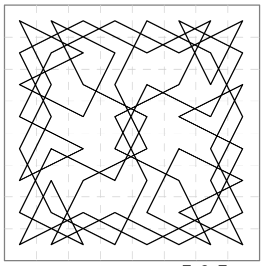
Type (13467) 4 tours



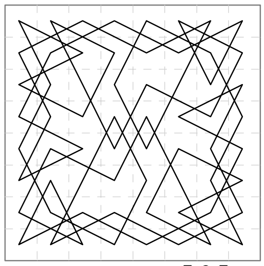
Three of the tours from the (13467) array.



kk= 5:6:5

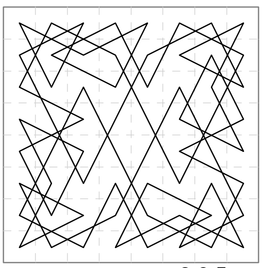


nn= 5:6:5

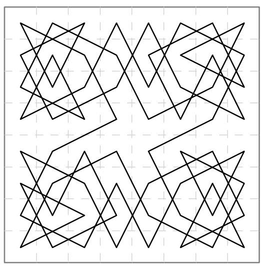


oo= 5:6:5

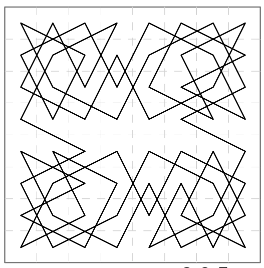
(3:8:5) Case. These three are given by Murray as examples of tours not included in the special types studied by Bergholt. Two are linked half-board tours of squares and diamonds.



ii= 3:8:5



bb~ 3:8:5



bb= 3:8:5

Here we conclude this catalogue of tours with Mixed Quaternary Symmetry. It should be borne in mind that the enumerations have been done manually and not by computer, and it is possible that cases have been missed. There is certainly more work needed on the later sections.

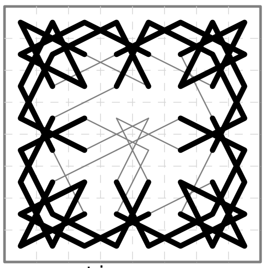
Summary of the Catalogue

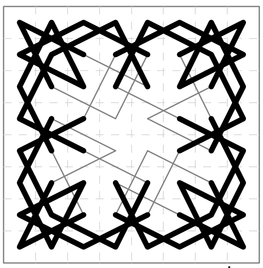
Class h = 1 minimum purely biaxial (48 tours) = 16 (1:10:5) + 16 (1:8:7) + 16 (1:6:9).
Class k = 3 minimum purely birotary (287 tours) =
53 (3:10:3) + 90 (5:8:3) + 118 (7:6:3) + 26 (9:4:3)
Triple move (1·8 + 2·4)
Type (023) dual 6 tours = 4 (3:10:3) + 2 (5:8:3).
Double move (1·8 + 3·4)
Type (0125) dual 20 tours = 16 (5:8:3) + 4 (7:6:3)
Type (0256) dual 16 tours = 2 (3-10-3) + 6 (5:8:3) + 8 (7:6:3).
Double move (2·8 + 1·4)
Type (013) 9 tours = 5 (3:10:3) + 4 (7:6:3)
Type (014) 3 tours = 3 (5:8:3):
Type (034) 5 tours = 4 (3:10:3) + 1 (5:8:3)
Type (156) 49 tours = 9 (3:10:3) + 14 (5:8:3) + 23 (7:6:3) + 3 (9:4:3)
Type (238) 6 tours = 4 (3:10:3) + 1 (5:8:3) + 1 (7:6:3)
Single moves (3·8)
Type (134) 0 tours (but usable on larger boards)
Type (146) 19 tours = 6 (3:10:3) + 6 (5:8:3) + 7 (7:6:3)
Type (347) 1 tour = 1 (3:10:3)
Single moves (2·8 + 2·4)
Type (0123) 2 tours = 1 (3:10:3) + 1 (5:8:3)
Type (0234) 2 tours = 1 (3:10:3) + 1 (5:8:3)
Type (0145) 3 tours = 3 (5:8:3)
Type (0135) 12 tours = 5 (3:10:3) + 6 (7:6:3) + 1 (9:4:3)
Type (0456) 20 tours = 1 (3:10:3) + 7 (5:8:3) + 9 (7:6:3) + 3 (9:4:3)
Type (0156) 28 tours = 2 (3:10:3) + 9 (5:8:3) + 13 (7:6:3) + 4 (9:4:3)
Type (0238) 31 tours = 4 (3:10:3) + 2 (5:8:3) + 19 (7:6:3) + 6 (9:4:3)
Type (0126) 33 tours = 4 (3:10:3) + 13 (5:8:3) + 14 (7:6:3) + 2 (9:4:3)
Type (1256) 22 tours = 5 (5:8:3) + 10 (7:6:3) + 7 (9:4:3)

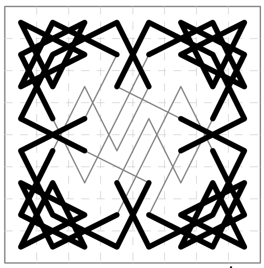
The (9:4:3) tours can be divided into two classes according to whether the second octonary move after the corner move 9-4 is either A (8-7) or B (8-3), none of C (8-2) type being possible on the 8×8 board. The 26 consist of 4 of type B and 22 of type A.

Maximum Octonary Symmetry

The minimum number of octonary moves in a closed tour is 8, since the corner moves cannot be dispensed with. The maximum octonary moves in a mixed tour depends of course on the size of the board, for the 6×6 board it is 24 (j = 6) and for the 8×8 board it is 40 (j = 10). The maximum number of moves achieved in octonary symmetry in a closed tour is 48 moves. The first example here (Jelliss 1990) is asymmetric and was formed by simple linking from an octonary pseudotour. But the same can also be achieved in tours with rotative symmetry, as in the other two examples (Jelliss 1990 and 2009). The 48 moves are marked by the thicker lines.


asymmetric

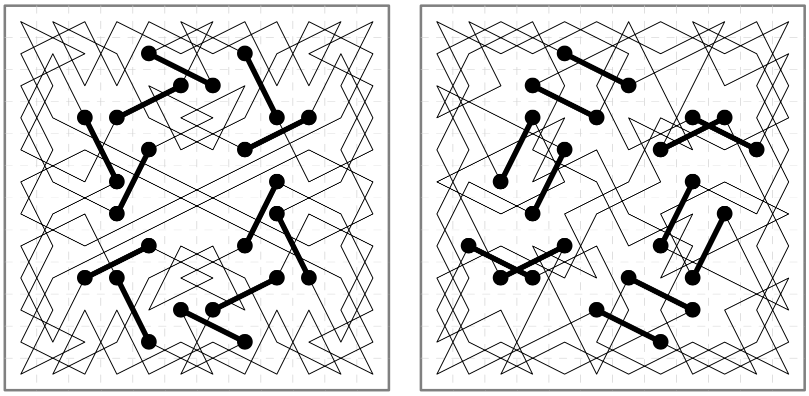

symmetric bg=


symmetric io=

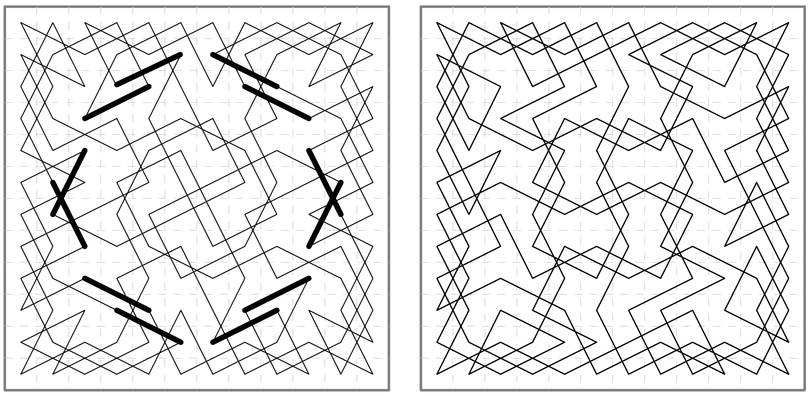
Mixed Symmetry on the 12×12 Board

Apart from magic tours the only other systematic work on the 12×12 board that I know of is Ernest Bergholt's work on constructing tours showing approximations to quaternary symmetry. The 14 tours by Ernest Bergholt that follow appeared in his three *Memoirs* (numbers 7, 8 and 9) sent to H. J. R. Murray in 1918. In the first two tours by Bergholt only 12 moves deviate from quaternary symmetry, having only binary oblique symmetry. The rest of the moves are in direct quaternary in the first diagram and in oblique quaternary in the second diagram. The first shows the maximum direct quaternary symmetry that can be achieved.

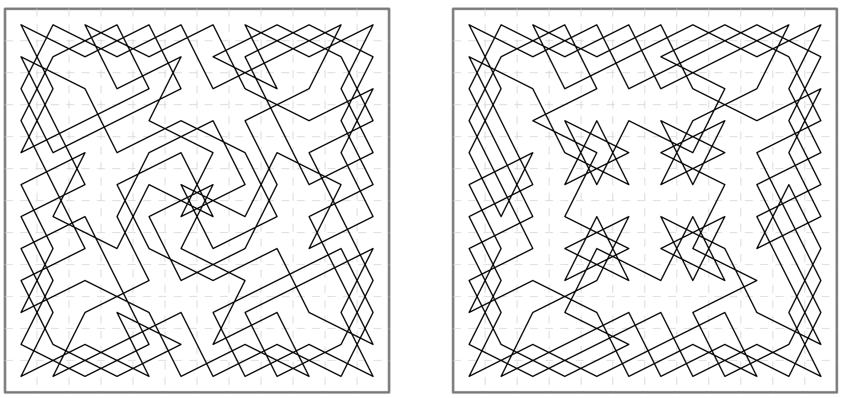
Tours 1 and 2.



Tours 3 and 4

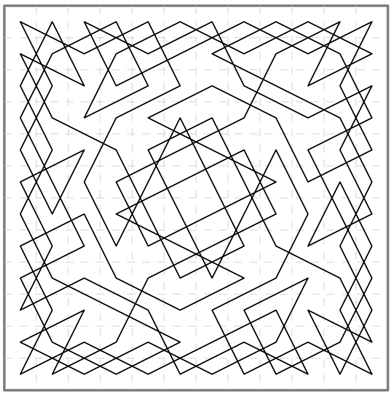


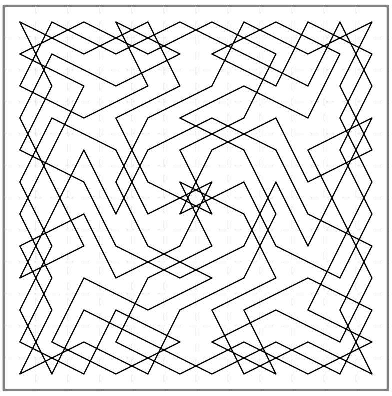
Tours 5 and 6.



The first six tours by Bergholt show oblique quaternary symmetry with twelve moves in direct quaternary symmetry. The twelve anomalous moves occur in the same positions in all the tours and are connected in the same way in all the tours, except the last. The other six examples show direct quaternary symmetry predominant. Two different arrangements of the 12 anomalous moves are used. In each case they occur in triplets that can be regarded as alternate sides of a 6-move circuit.

Tours 7 'Camouflage' and
8 (unnamed).



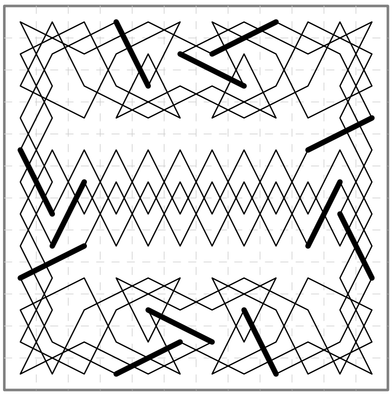


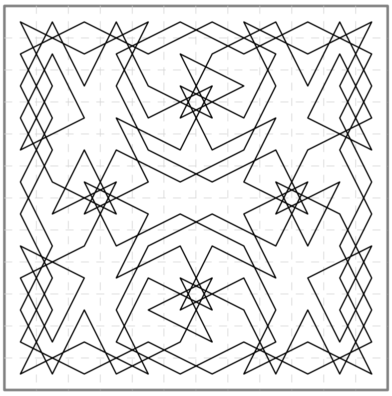
Bergholt explained how he arrived at the idea of mixed (quaternary) symmetry which is illustrated in these tour diagrams. By 'nodes' he means the cells used by the 12 anomalous moves:

“Having advanced so far it suggested itself to me that, by suitably choosing the nodes it would be possible so to distribute the six pairs of interrupting moves that (in the case of oblique symmetry) the twelve lines should form three groups of four, each group being itself in direct quaternary symmetry. This being done, the whole tour would be completely quaternary (mainly oblique, but partially direct); and an apparently impossible problem would be solved.”

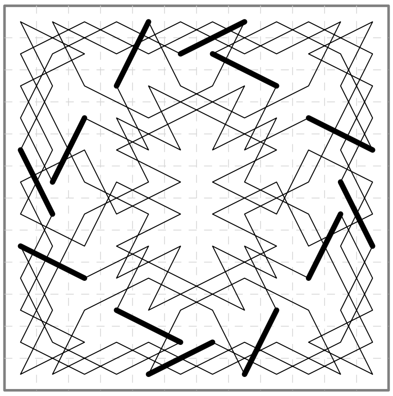
His aim was also to place the anomalous moves away from the centre so that they do not detract from the central pattern and so that the deviation from true quaternary symmetry is difficult to detect.

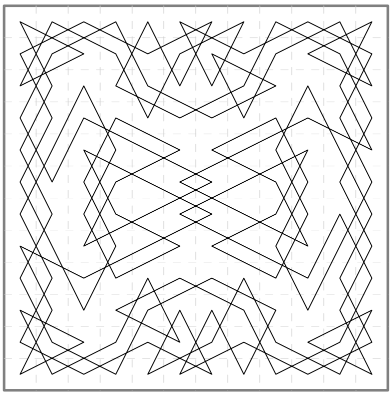
Tours 9 the 'Balcony' and
10 the 'Constellation'





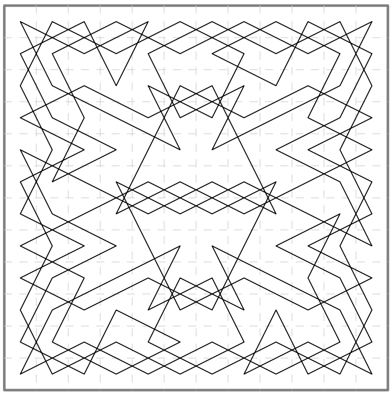
Tours 11 and 12.

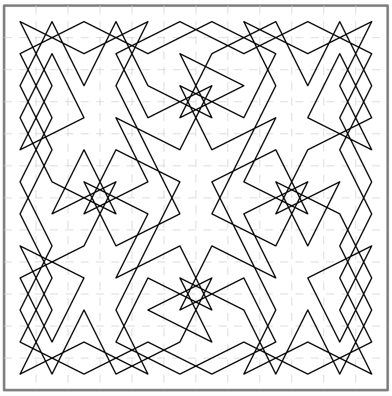




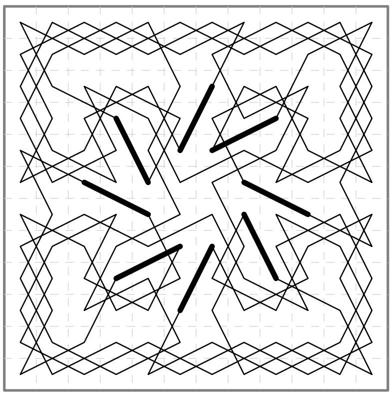
Tours 13 (diagram reflected left to right from that given in the MS, so that the 12 interpolated moves are in the same orientation as in tours 11 and 12) and 14 (a version of the "Constellation" with same connections but alternative centre).

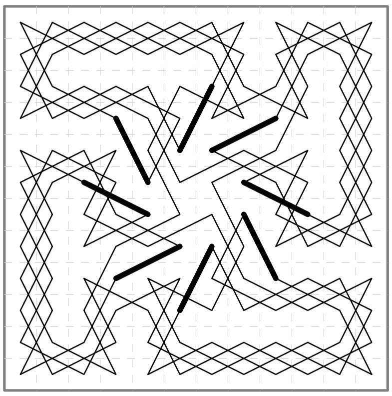
Tours 13 and 14.



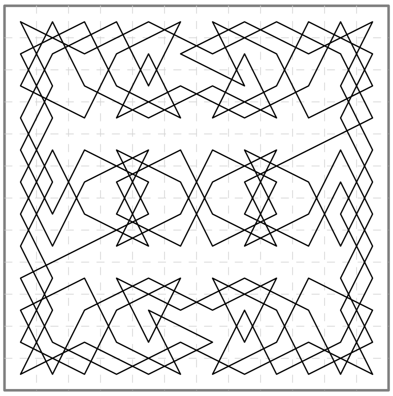


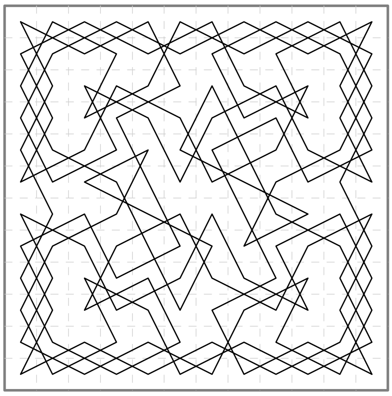
We continue with four examples by H. J. R. Murray (1942), constructed by the method of slants. The first two use the same eight slants in a "waterwheel" formation.





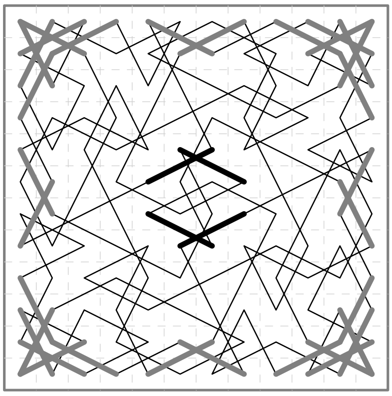
Two more by Murray:

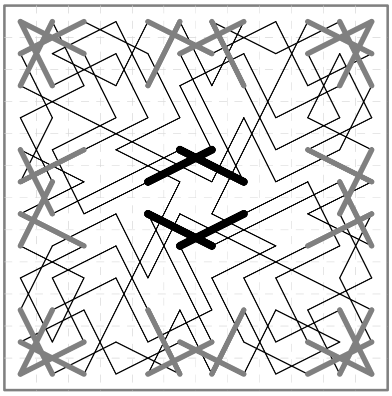




The maximum oblique quaternary symmetry that can be achieved is to have just four moves in direct quaternary symmetry, as in these two examples (Jelliss 2003). Since there are 48 of this type on the 8×8 board the number possible on the 12×12 must be considerable. The octonary components are shown by a thicker grey line.

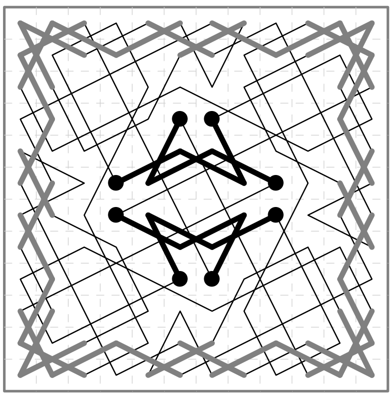
Tours of h:j:k type 1:8:27.

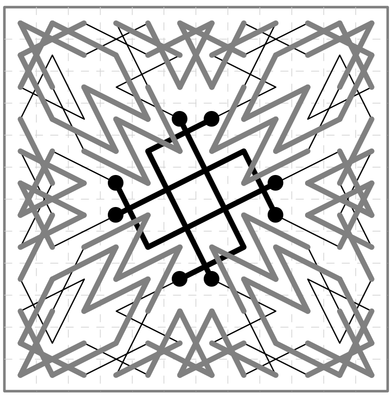




Here are 8 more tours of my own construction (Jelliss Jul 2003) in mixed quaternary symmetry. The number of cells covered by the central pattern gradually increases. The inner and outer areas are of opposite symmetry type. The bold dots mark where reflective symmetry changes to rotative symmetry or vice versa. In these frst two the octonary moves are all in the outer area.

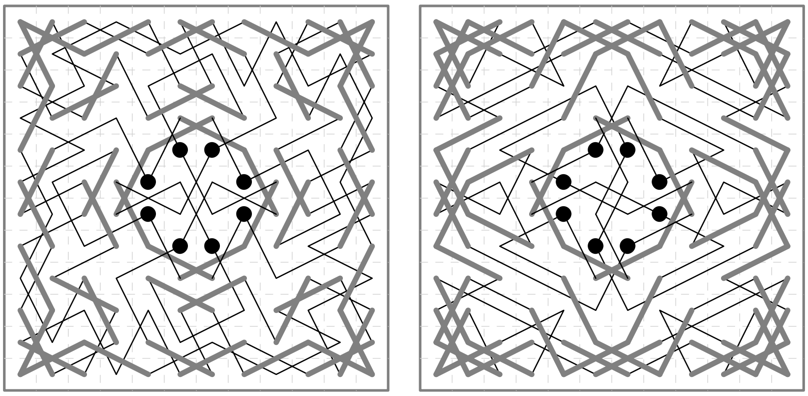
Cases 3:10:23 and 11:22:3





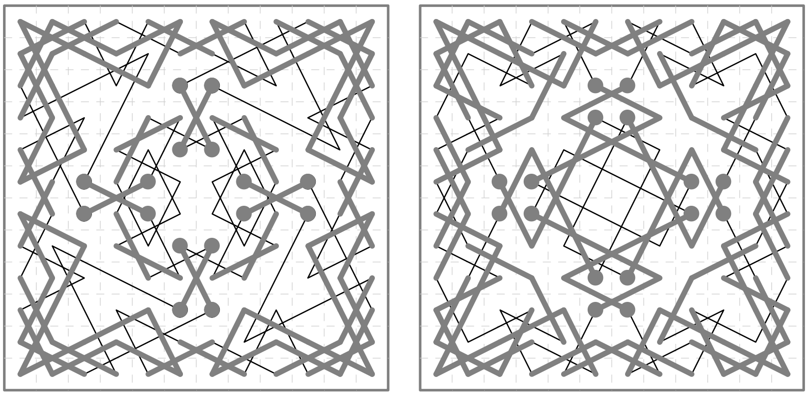
The next two indicate that octonary components can occur in both sections.

Cases: 3:2+12:19 and 15:16+2:3

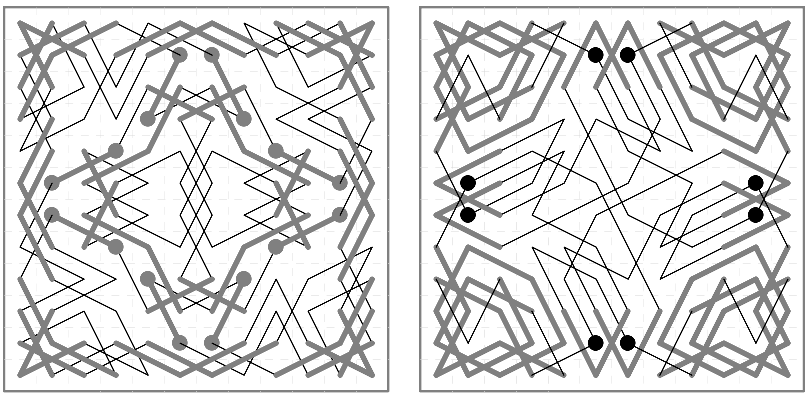


In the next three there is one or more octonary move between the inner and outer areas, and these can be taken as belonging to the direct or oblique sections, or as a transition between them.

Case 5:4+16:11 (or 5:2+18:11) and 9:18+6:3 (or 9:24:3)

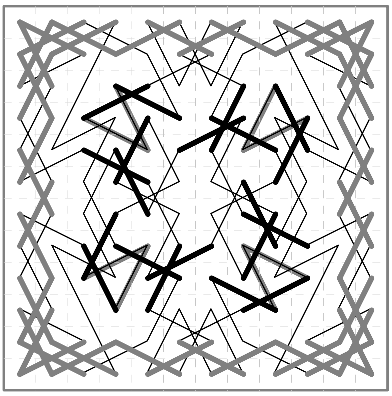


Cases 7:6+10:13 (or 7:4+12:13) and 5:16+4:11.



In the final example above the oblique and direct symmetry areas meet.

The next example (Jelliss 2009) shows that the formation (13478) of five separate moves in oblique quaternary symmetry, which does not generate any tours on the 8×8 board, can be used in a tour on the 12×12. It's h:j:k values are 17:14:5.



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Other Evenly Even Tours

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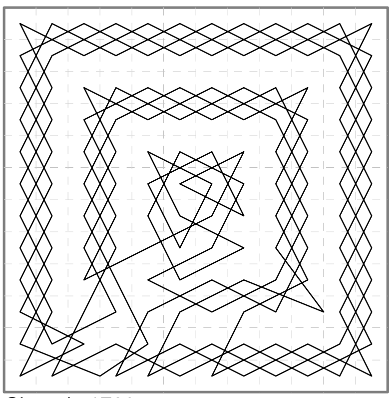
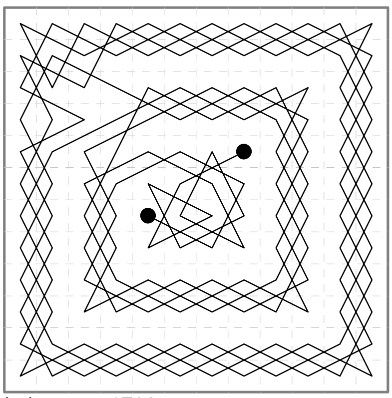
Some 12×12 Tours

By far the earliest tours known on this board are these asymmetric braid examples. A closed tour by Chapais (1780 Fig,32) and an open tour by Denis Ballière de Laisement (1782, Fig 28, Plate V). The central 4×4 in each case consists of paths following the squares and diamonds.

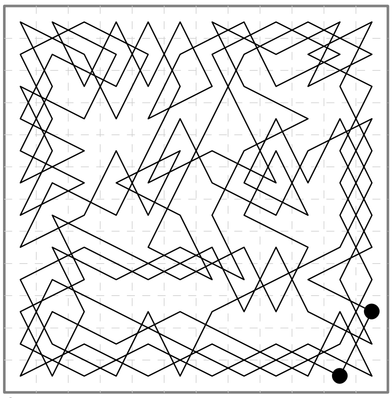


Chapais 1780

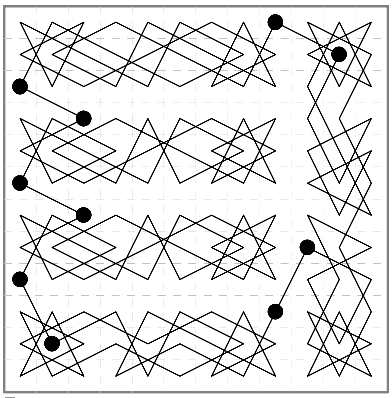


Laisement 1782

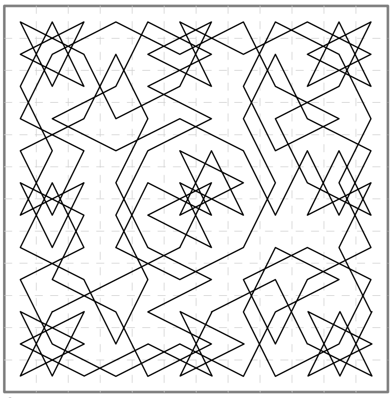
There are 11 knight tours of the 12×12 board attributed to the Rajah of Mysore, Krishnaraj Wadiar, who died in 1868. These were collected in the Harikrishna work of 1871, and reproduced by S. R. Iyer in *Indian Chess* 1982. Here we show eight of these tours (for the Magic Tour see ⌘ 9, and for the Figured Tours see ⌘ 11).



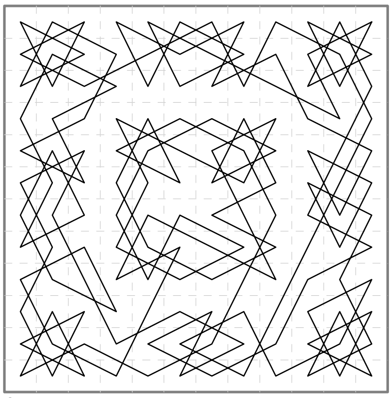
6



7



8



9

¶6, numbered from k1, has the diagonals adding to 870 (the even diagonal consists mainly of multiples of twelve: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 100, 110 and the odd diagonal is: 19, 43, 67, 69, 85, 87, 137, 73, 53, 79, 129, 29).

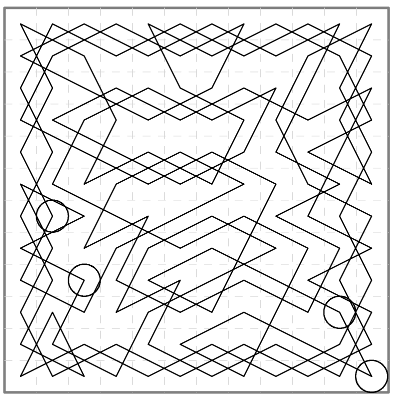
¶7, numbered from c8, is a compartmental tour.

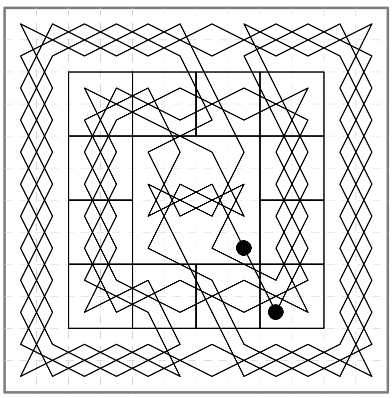
¶8, numbered from e7 to d5, combines an (Euler) central cross tour with stars in the corners.

¶9, numbered g2 to h4, surrounds an asymmetric 6×6 closed tour with a frame 3-ranks wide.

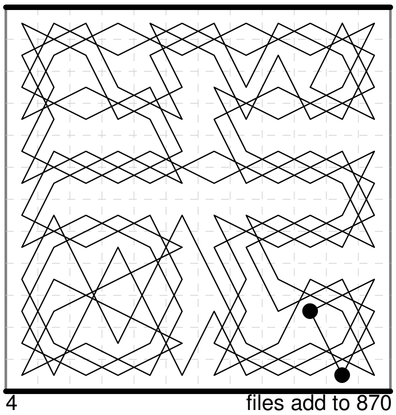
¶11, numbered from the marked corner, has the moves 1 to 43 and 144 to 102 forming congruent paths: the end-points are circled.

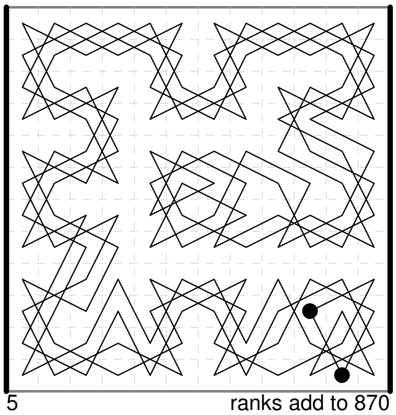
¶34, numbered from i3 to h5, is partially magic. All the 40 areas 2×2 in the outer border add to 282, as do the 12 marked areas 2×2 in the inner border, while the central 4×4, which is the double Beverley quad, adds to 354 in each rank and file.





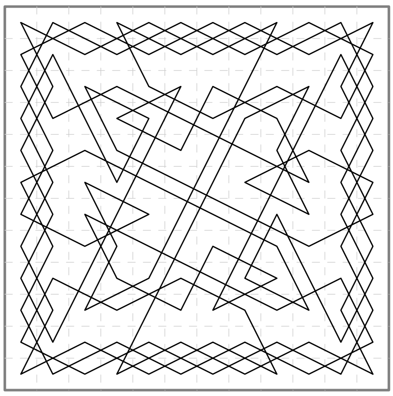
The Rajah also gave two semi-magic tours

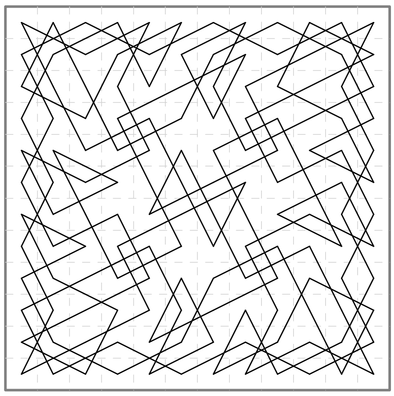

4 files add to 870


5 ranks add to 870

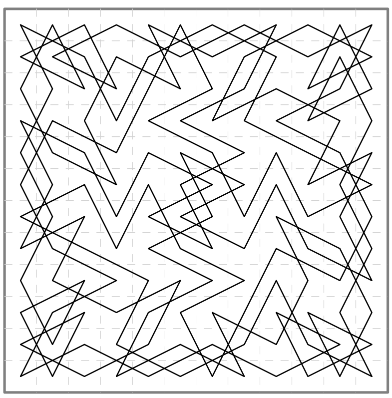
The only exact symmetry possible on evenly even boards is the 180 degree rotary type. Apart from the mixed symmetry examples above I have only two examples. They also show some partial quaternary symmetry, either birotary or biaxial.

Of about the same date as the work by the Rajah of Mysore are the symmetric tours on the 12×12 board in *Kaeidoscope Echiquieen* by Carle Adam (1867). This has 102 figures, but I have only seen the first three tours. The first is a closed symmetric tour (when numbered, opposite cells differ by 72).





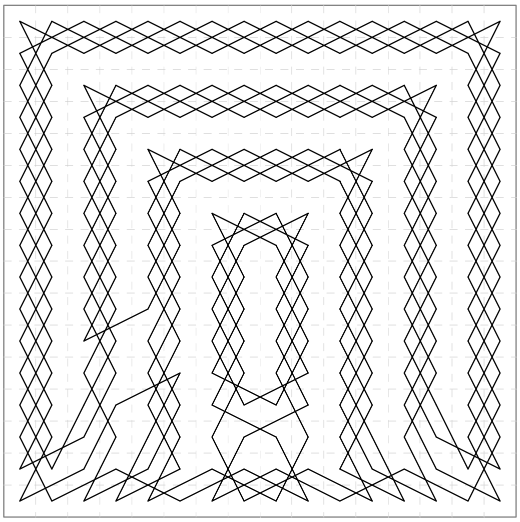
The second 12×12 tour above is from a design of crosses and reels on the back cover of *Linaludo* by Archibald Sharp (1925) which can be extended to any board whose sides are a multiple of four. See the 16×16 section in ⌘ 8 for a pseudotour using this pattern.

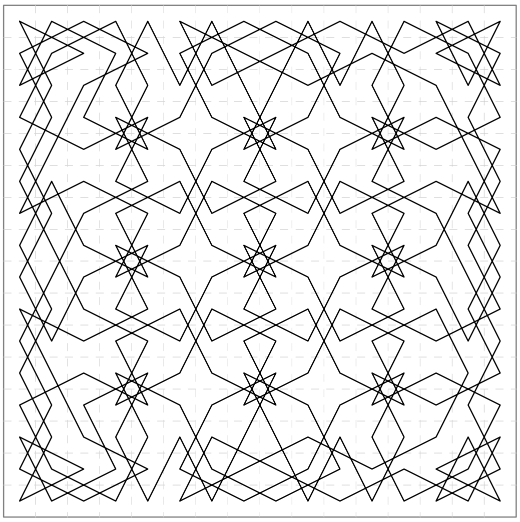


The final symmetric tour shown here is the one used by ‘Wolfram’ in the The Listener Crossword puzzle 3353 (*The Times* 1996). This tour is in 90° rotational symmetry, apart from the eight moves of {1,1} and {0,2} type in the central 4×4 area, which have 180 degree rotation.

Some 16×16 Tours

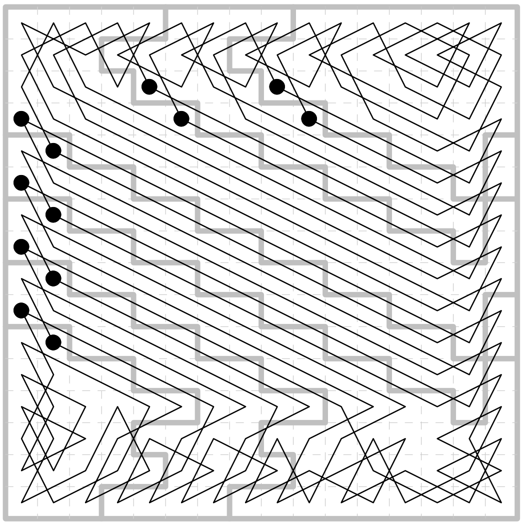
The earliest examples on this board are from Laisement (1782). His Fig 29 Plate V extends the braids of his C-shaped 8×8 tour to cover the 16×16 board, while Fig 30 uses it in the four quarters of the 16×16, arranged in direct symmetry. On Plate VI he gives two more 16×16 tours, reproduced here. Fig 31 is headed ‘Les Cambranles’ which I think means ‘Frames’ or ‘Mantelpiece’.





His Fig 32, which is almost biaxial, is headed ‘Les Croix de Malthe’ (Maltese Crosses) and has nine repeated patterns of eight moves that I call octangles. Laisement indicates in the text that the design can be extended indefinitely, specifically mentioning one of 14400 cells (120×120).

The next tour below is by Pierre Dehornoy (2003) and shows how to construct a tour with most moves in one direction on a square board of side 4n (n > 3). The tour is composed of 4n – 9 pieces of 7 types (the 16×16 case uses one of each type). The four types of corner pieces are each used once (extended where necessary), the three types of piece that fit between these are repeated (with lengthening straight pieces) an appropriate number of times. See also his 20×32 example (p.252).



Other tours of large evenly even boards can be found in the sections on Mixed Quaternary Tours (vol.7), Pseudotours (vol.8), Magic tours (vol.9) and Leaper Tours (vol.10), Figured tours (vol.11)