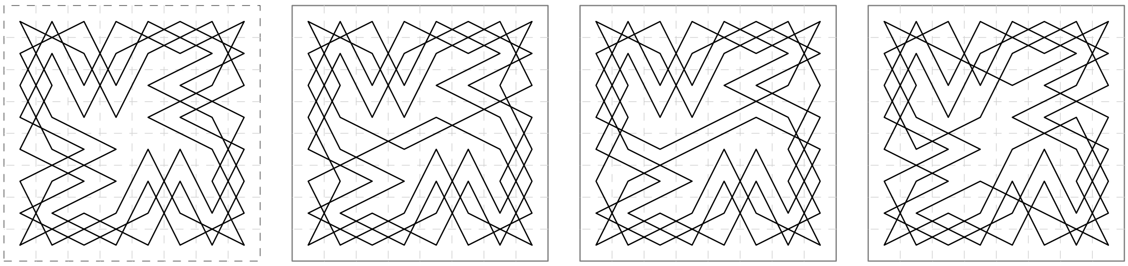
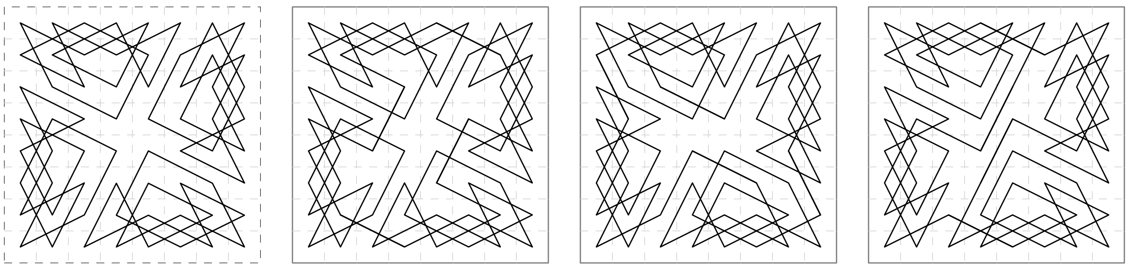
⌘ 8

Octonary & Quaternary Pseudotours



by G. P. Jelliss



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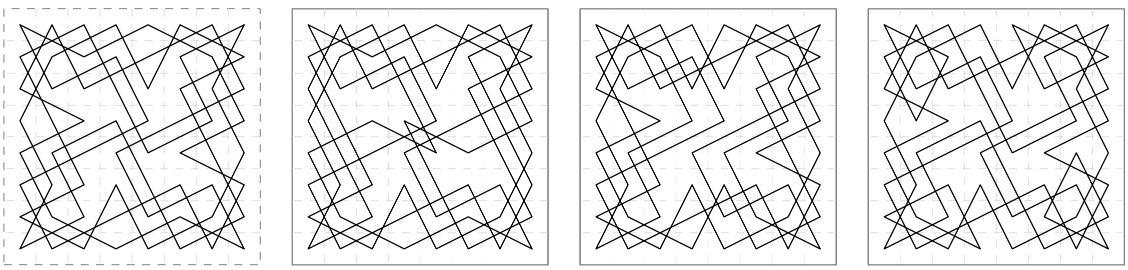

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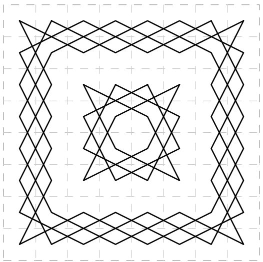
Title Page Illustrations:
Examples of Quaternary Pseudotours with Binary Symmetric Tours derived from them by simple linking.

Contents

Octonary Pseudotours

The full set of octonary pseudotours on the 8×8 board has only 33 members. Diagrams of all the cases are shown below. They include the Annular (Collini) pattern and the Squares and Diamonds pattern, shown first and last in the list below. The remaining 31 cases do not seem to have been studied systematically until I investigated them in *Chessics* (#22 p.72 1985 and #25 p.106 1986). They can be classified in various ways. The codes are listed here with shortest circuit codes first, hyphens omitted for concision, and 1-1 abbreviated to 1.

There are fifteen that, like the Collini pseudotour, have all their circuits co-centric with the board. The Collini is of type **4×12 + 4×4**. Two have the formula **2×28 + 2×4**, with central diamonds.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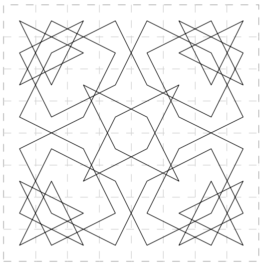
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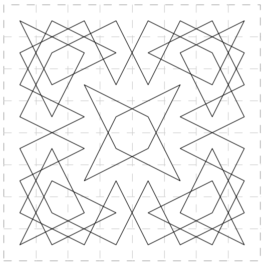
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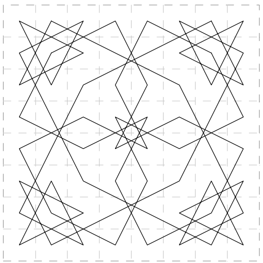
The other twelve octonary pseudotours of co-centric type have circuits of length 20. There are eleven **2×20 + 2×12**.and one **2×20 + 6×4**. Six have the pp centre and six have the hh centre.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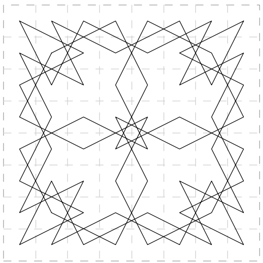
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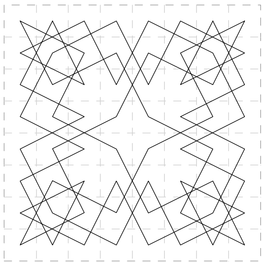
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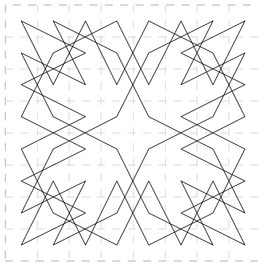
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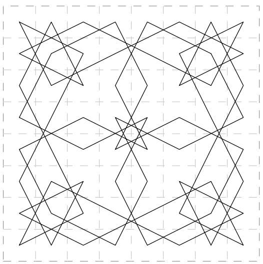
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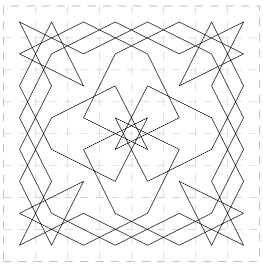
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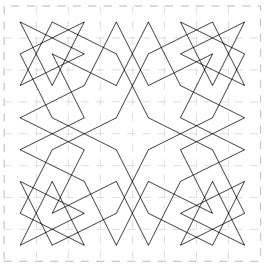
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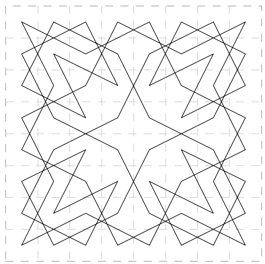
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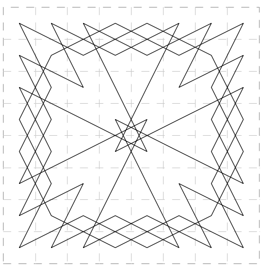
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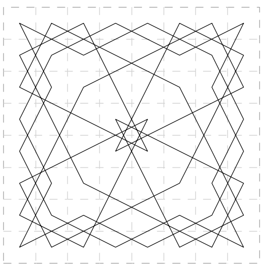
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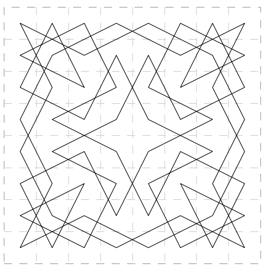
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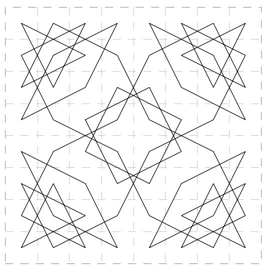
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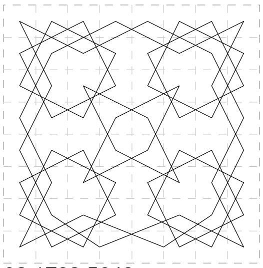
One of these diagrams appears in Laisement (1782) as Fig.33 on Plate VI. This is the earliest diagram of a pseudotour that I know of.

There are 8 octonary pseudotours that have some circuits co-centric with the board, and others off-centre. Two of type **2×12 + 10×4**. Three of type **2×12 + 4×8 + 2×4**. Three of type **4×12 + 4×4**.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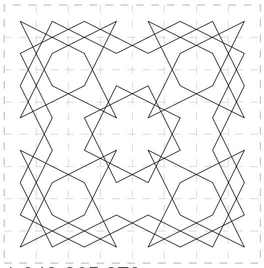
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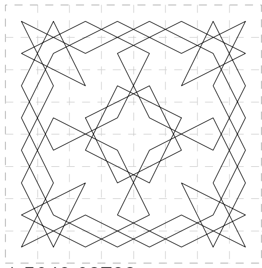
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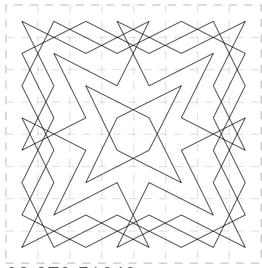
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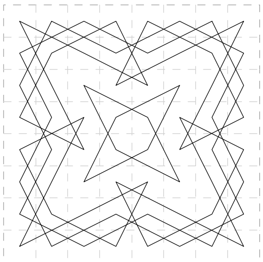
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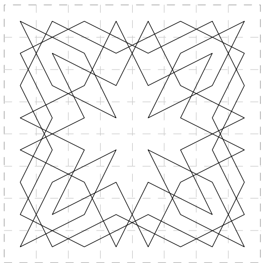
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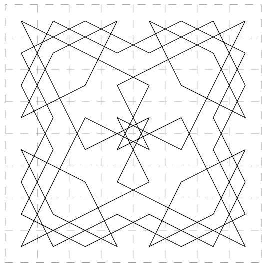
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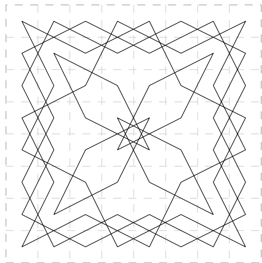
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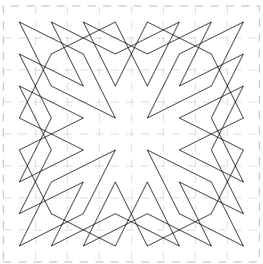
There are ten octonary pseudotours that have no circuits co-centric with the board. Six of these are of type **4×12 + 4×4**.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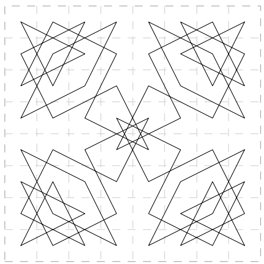
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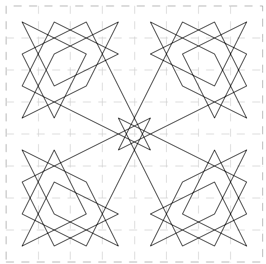
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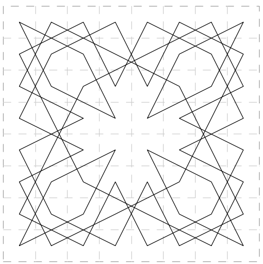
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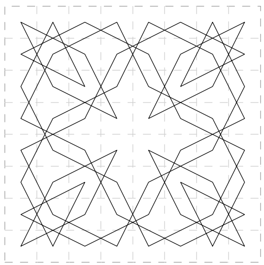
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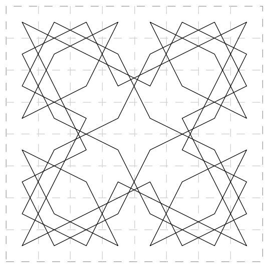
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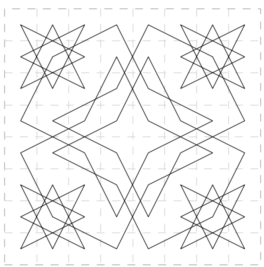
Three cases **8×8** and one of type **16×4**, namely the squares and diamonds pattern.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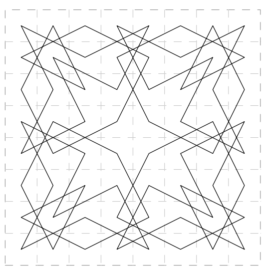
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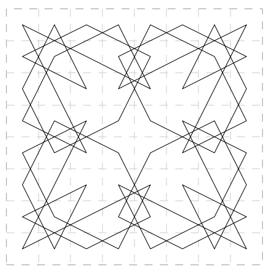
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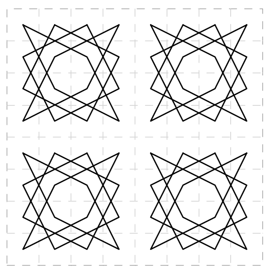
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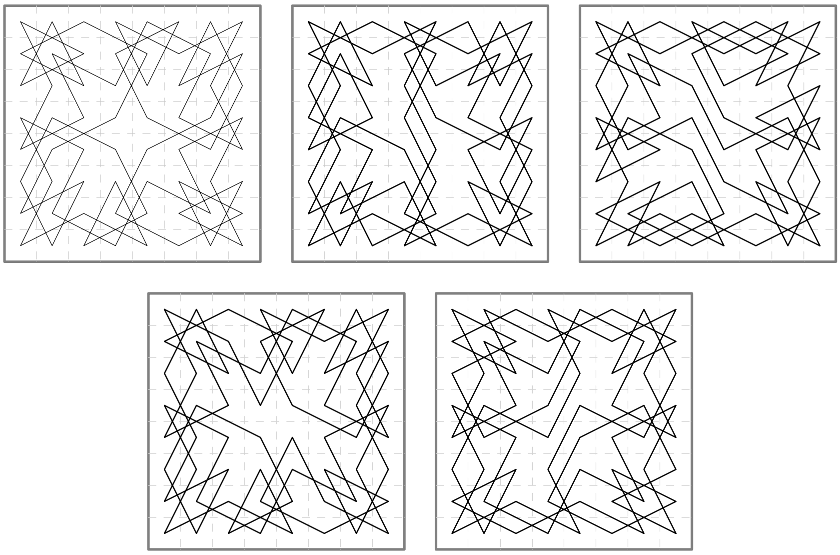
We show tours derived from these in the following sections.

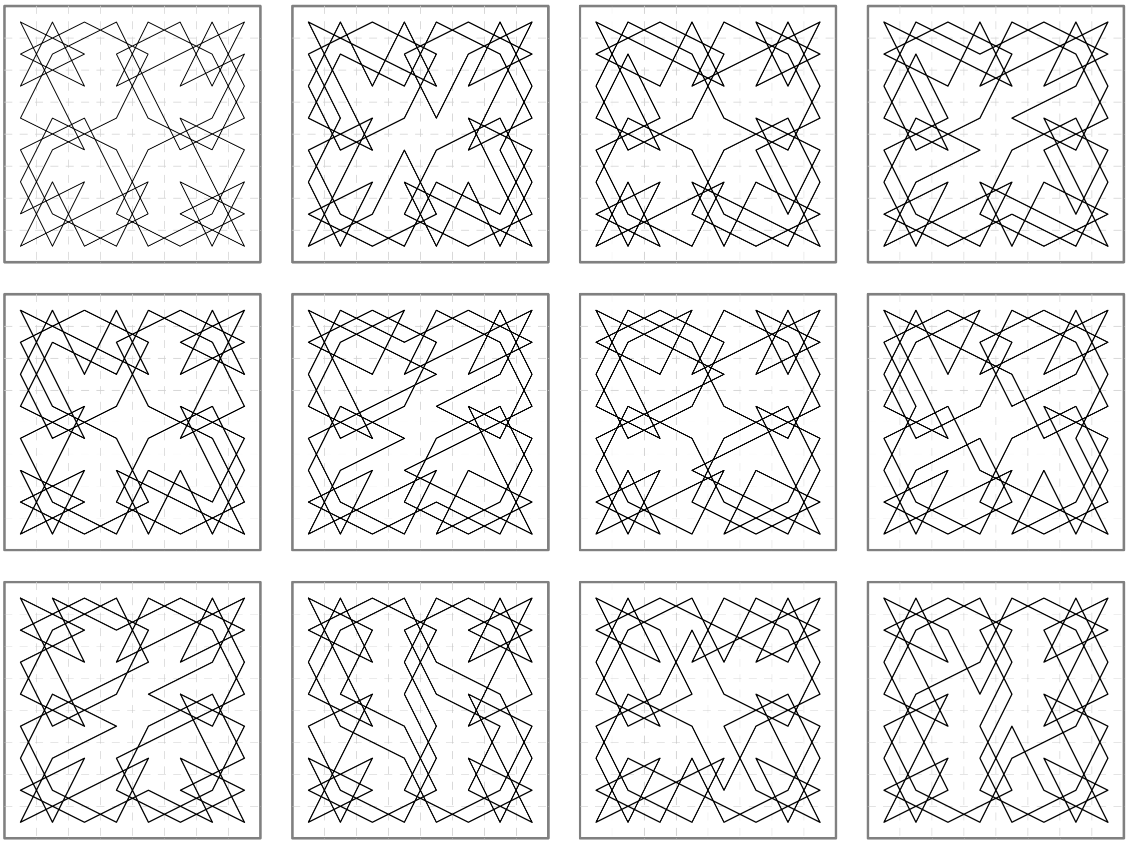
Tours from Other Octonary Pseudotours

All three of the pseudotours of 8 circuits of 8 will form symmetric tours by simple linking.

First the **8×8** 'glider' case gives five tours (centres hh, hj, hk, gh, hk):



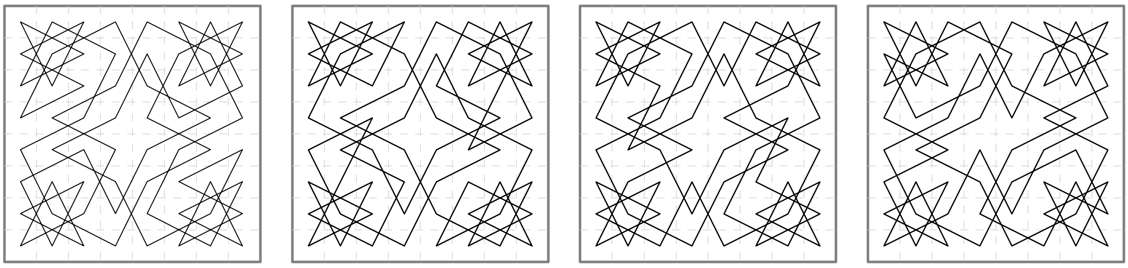
Second the **8×8** 'crab' case gives 12 tours (centres hh, gh, hh, gh, hh, gh, hh, bh, gh, hj, bh, gj=):



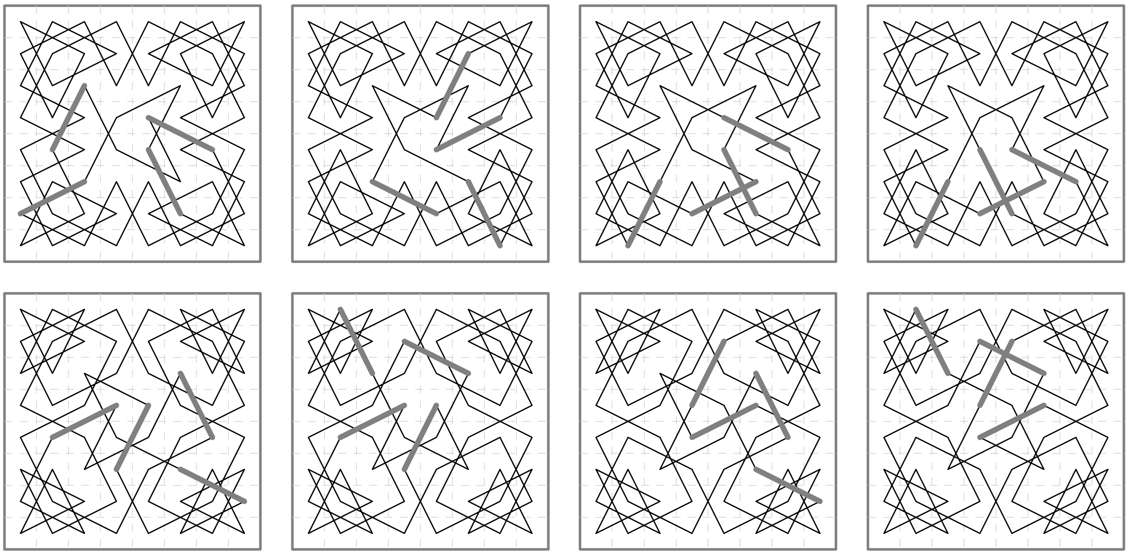
The seventh includes three successive two-move lines, twice.

In counting the above tours it is necessary to check carefully that one tour is not a reflection or rotation of another. Duplicates can easily be missed.

Lastly the **8×8** ‘stars and rockets’ case gives four tours (centres bh, hh, gh, gh):

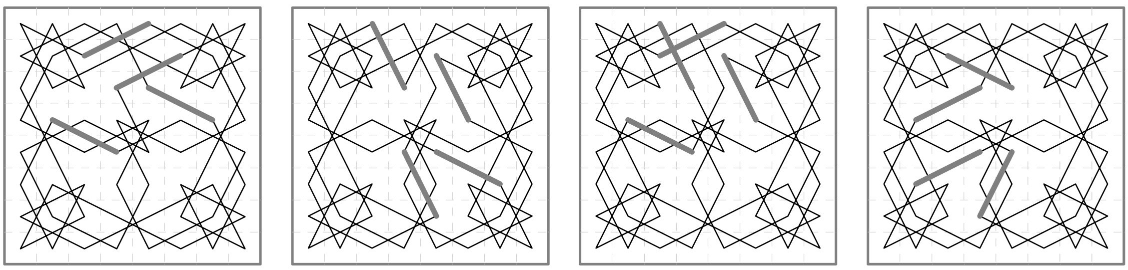


There are fifteen octonary pseudotours that, like the Collini pseudotour, have all their circuits co-centric with the board. The Collini is of type **4×12 + 4×4**. Two have the formula **2×28 + 2×4**, with central diamonds. These will by simple linking only form asymmetric tours.



Above are diagrams of the four asymmetric tours formed by simple linking from each of the pseudotours containing 28-move circuits. The first in each case has a linkage polygon without self-intersection. They all have 40 moves in octonary symmetry. This is because the 4 deleted moves are of 3 different types, a pair of the 0-2 type and two others: (8–3)×8 = 40.

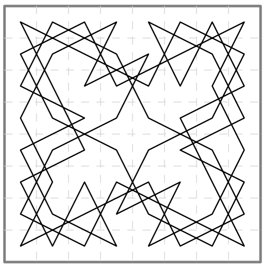
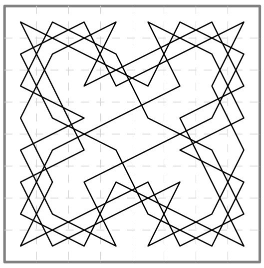
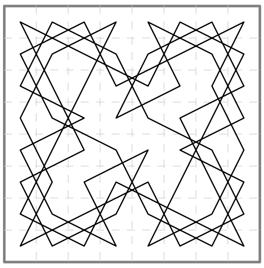
The only other case that I have considered for simple linking is the third above of pp type. This yields four asymmetric tours (Jelliss 1990), and again the first linkage polygon is nonintersecting.

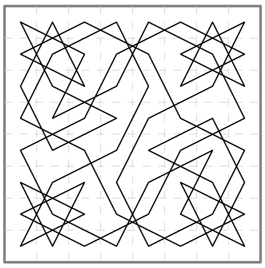
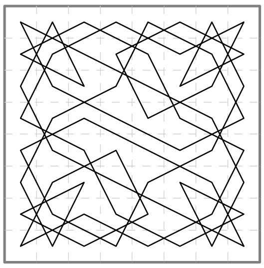
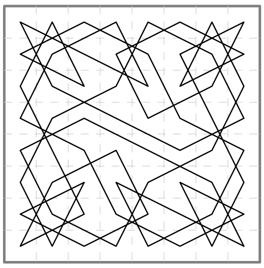


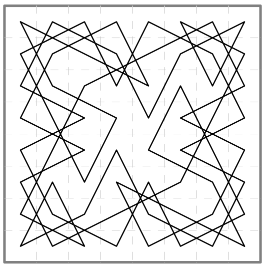
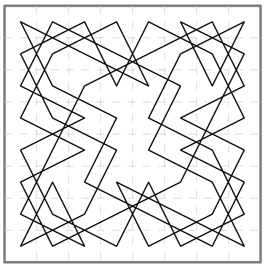
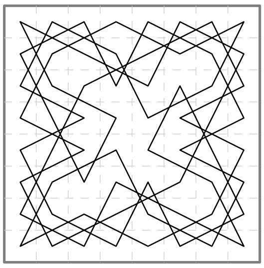
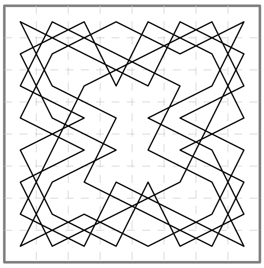
48 octonary

In the third the linkage is axially symmetric. The first two have 40 sets of 8 in octonary symmetry, the third has 32, and the fourth has 48. The theoretical upper limit for the maximum moves in octonary symmetry is 56, since having all 64 moves in octonary symmetry implies a pseudotour. The maximum of 48 moves in octonary symmetry is achieved here because the deleted moves are of only two types (coded 0-1 and 3-4). This, and examples with rotary symmetry, are shown at the end of the Mixed Symmetry study.

Three of the octonary pseudotours of the **4×12 + 4×4** type, will produce symmetric tours by simple linking. There are three tours from one, three from another and four from a third. One tour in each of the first two batches includes two three-move lines.

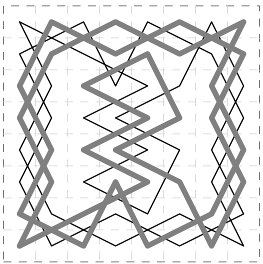




The second in the last batch includes two 1×3 non-intersected rectangles, and all have two successive two-move lines, twice (centres hh, hi, hl; eg=, bd=, bd=; fg~, bf~, bf=, fg=).

Thus among the 33 octonary pseudotours only 7 by simple linking produce symmetric tours. These 7 cases include the Squares and Diamonds which we have treated in Vol.6 where we try to enumerate the symmetric tours. The other 6 produce only 31 symmetric tours.

Addendum: This odd item is from ‘Monsieur Lamouroux’, presumably a pen-name, in *Nouvelle Regence* 1860. ‘Probleme de deux Cavaliers parcourant les 64 cases de l’echiquier en 32 coup chacun’. A pseudotour with horizontal axis. It is presented as a series of cell coordinates in descriptive notation, arranged in a lozenge shape around two knight symbols. As the white knight moves between White’s QB4 and K3 say, the black knght moves between Black’s QB4 and K3.



Quaternary Pseudotours

Method of Classification

A square pattern is said to have **quaternary symmetry** if it is unaltered by four of the transformations that preserve the square. In other words by rotations and reflections of the square it can be shown in only two distinct forms. Quaternary symmetry is of two types, **direct** where it is changed by 90 degree rotation, and **oblique** where it is changed by reflection. The catalogue at the end of this section lists all the ways of arranging 16-move knight paths to form pseudotours with quaternary symmetry on the 8×8 board.

This catalogue was first compiled by the Abbé Philippe Jolivald under the alias of ‘Paul de Hijo’ (1882). For the cases of oblique symmetry he found 140 arrangements of four 16-move circuits and 150 arrangements of two 32-move circuits. For the cases of direct quaternary symmetry he found 368 arrangements of four 16-move circuits and 378 arrangements of two 32-move circuits. An error in his book misprints the 368 total as 301 and this figure was copied by later writers, but his count is otherwise correct. The correct totals were confirmed by Thomas W. Marlow in a computer study reported in *Chessics* (#24 1985 p.92). Our catalogue shows diagrams of all these quaternary pseudotours, for the first time in complete pictorial form.

The following tables of the numbers of paths between particular squares are adapted from Tom Marlow's results. It is assumed that the path starts in a corner (9)-(4). The next move is then to any of the cells numbered (0), (1), (3), (6), (7) and the path continues until the 15th cell reached is also one of the cells numbered (0), (1), (3), (6), (7) that is a knight's move from the appropriate (4) that connects either to the initial corner (for closed 16-move circuits), or the opposite corner (for open paths which combine to form 32-move circuits). An ‘appropriate’ (4) is one related to the initial (4) by reflection in a diagonal.

Direct closed (Dc)

	(0)	(1)	(3)	(6)	(7)
(0)	0	18	16	65	36
(1)		2	7	20	24
(3)			2	41	12
(6)				43	63
(7)					19

total 368

Direct open (Do)

	(0)	(1)	(3)	(6)	(7)
(0)	0	16	24	70	32
(1)		2	4	21	23
(3)			4	38	12
(6)				43	70
(7)					19

total 378

Oblique closed (Oc)

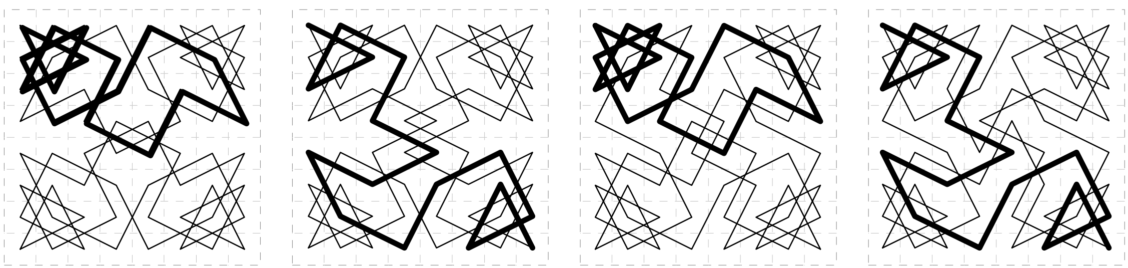
	(0)	(1)	(3)	(6)	(7)
(0)	0	3	7	12	13
(1)		0	3	15	6
(3)			3	15	8
(6)				16	25
(7)					14

total 140

Oblique open (Oo)

	(0)	(1)	(3)	(6)	(7)
(0)	0	3	13	15	17
(1)		1	2	12	9
(3)			6	13	4
(6)				13	27
(7)					15

total 150



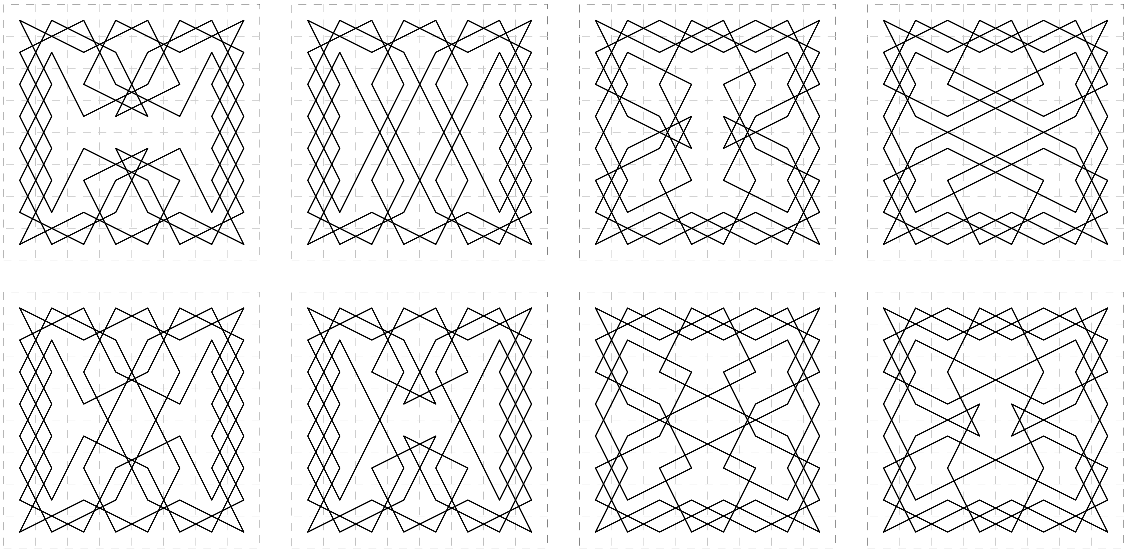
As an example to illustrate the four cases we use the circuits used by A-T. Vandermonde in his 1771 article that began the mathematical study of this subject. In terms of our cell coding the four 16-move corner-to-corner paths, both closed and open, are represented by the same sequence 9478310-2656138749. The path is closed or open according to which (2) the knight moves to following the (0). There is also an apparent choice of (6) following the (2), but the wrong choice here leads the knight into the final corner from the wrong direction.

Catalogue of Quaternary Pseudotours

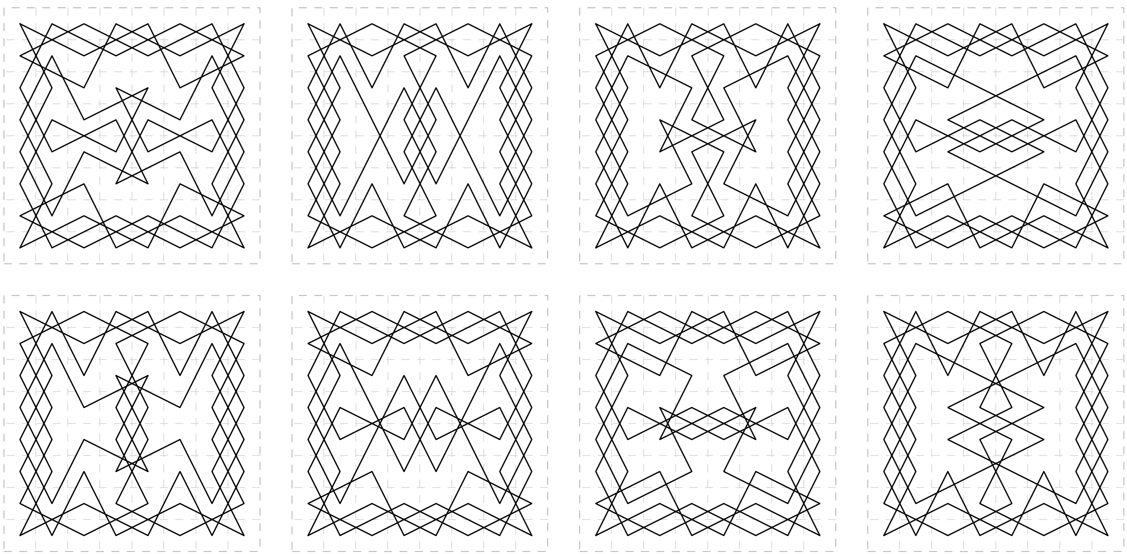
8-fold: There are 2 formulae that give 8 pseudotours, all in direct symmetry, 4 in which the 16-move path is closed and 4 in which it is open (4Dc, 4Do). Recall that each formula should be preceded by 94 and followed by 49, indicating the corner moves. The eight cases can be distinguished if required by means of the underlining convention (the code number for a diagonal cell, 0, 2, 5, is underlined if the move does not pass through the diagonal; a 1-1 move is underlined if it does not cross the diagonal through the initial corner): e.g. the first formula has the eight forms:

$6\underline{20}3873871\underline{15}6$, $62038738711\underline{5}6$, $62\underline{0}387387\underline{115}6$, $6\underline{2}03873871156$,
$620387387\underline{115}6$, $6\underline{20}38738711\underline{5}6$, $6\underline{2}0387387\underline{11}56$, $62\underline{0}3873871156$.

6203873871156



6287387301156

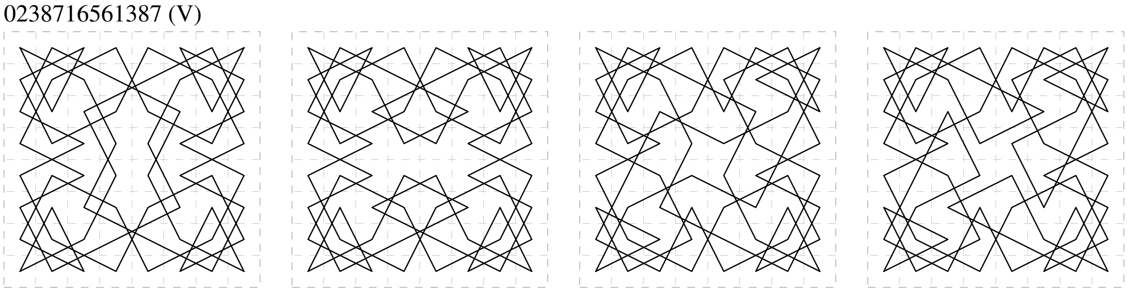


4-fold:. There are 62 formulae giving 4 pseudotours. They form four classes 4a, 4b, 4c, 4d.

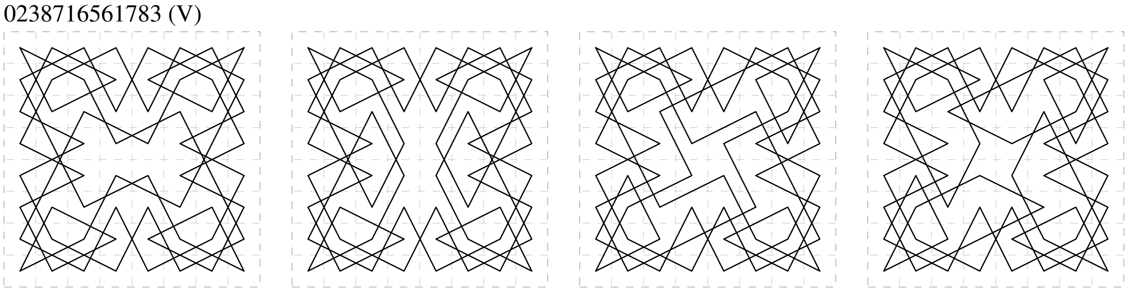
4a: 20 formulae giving one of each type (1Dc, 1Do, 1Oc, 1Oo). These include the 11 of Vandermonde type, marked (V).

4a: formulae 1-5:

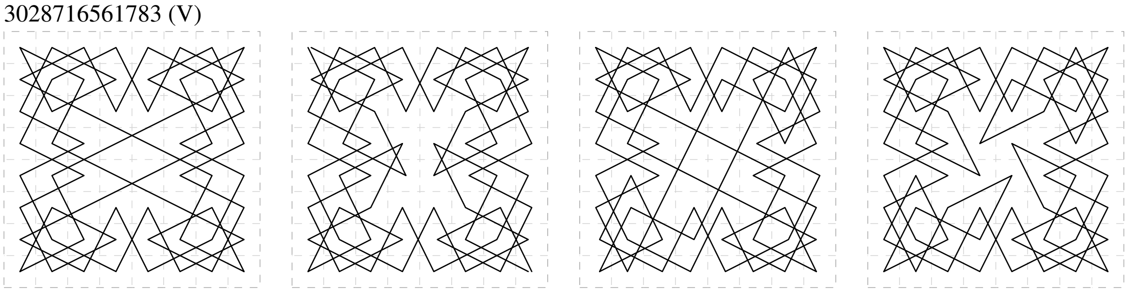
0238716561387 (V)



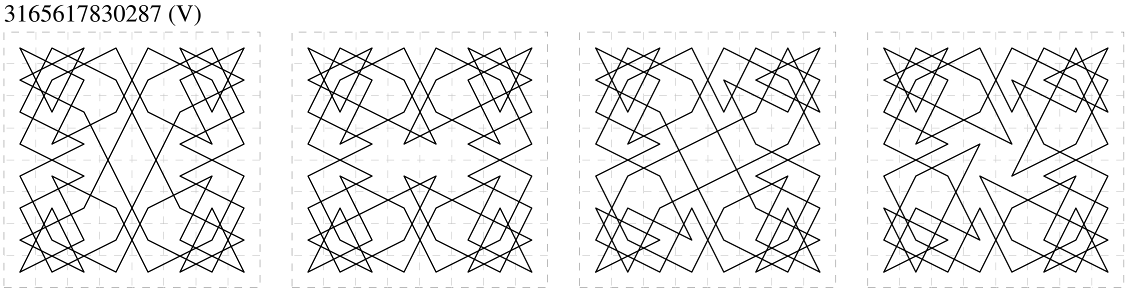
0238716561783 (V)



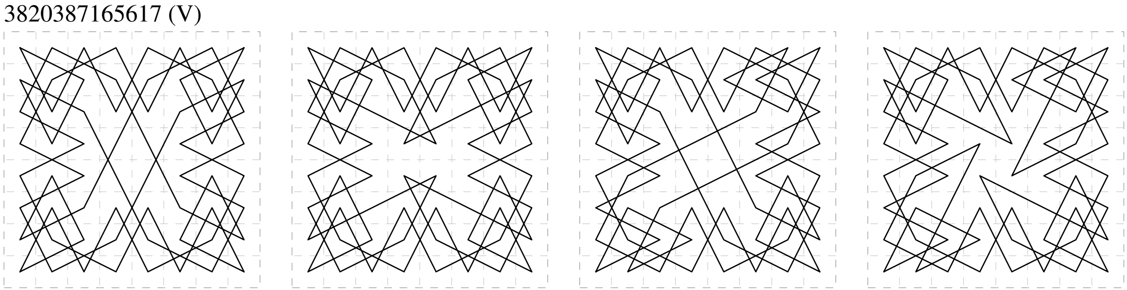
3028716561783 (V)



3165617830287 (V)

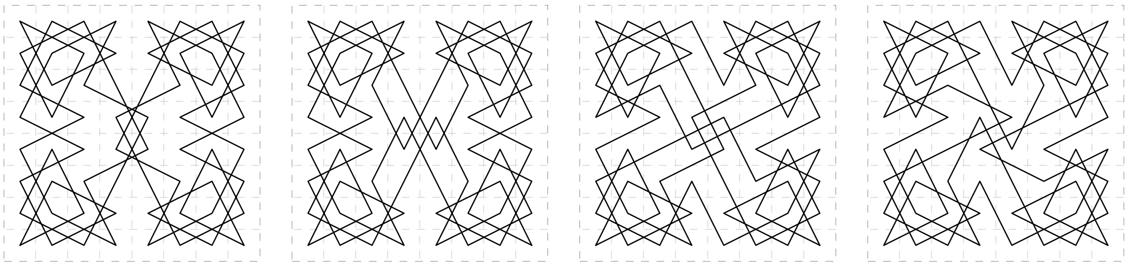


3820387165617 (V)

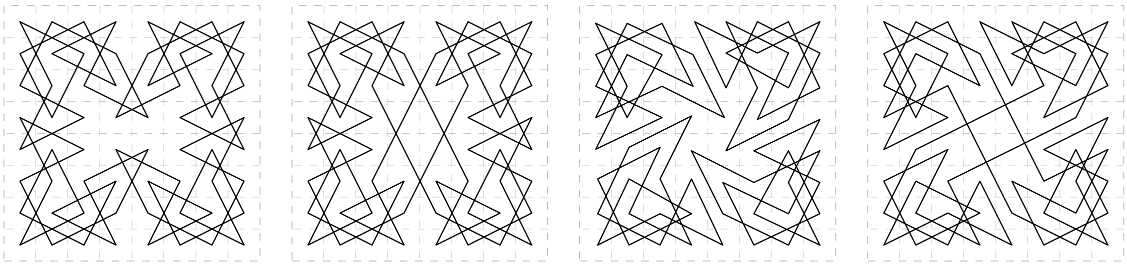


4a formulae 6-10

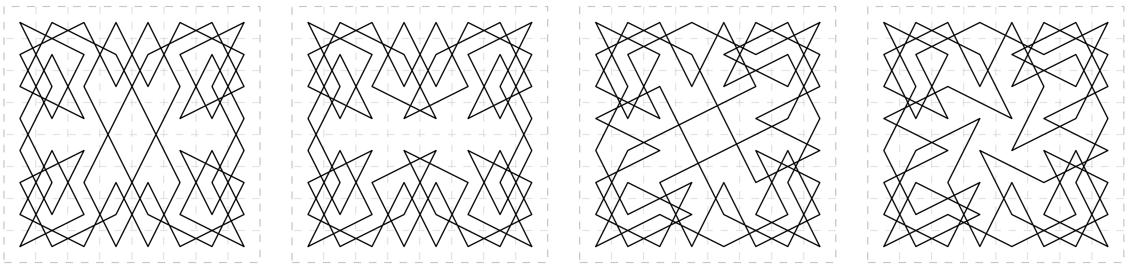
3871026561783 (V)



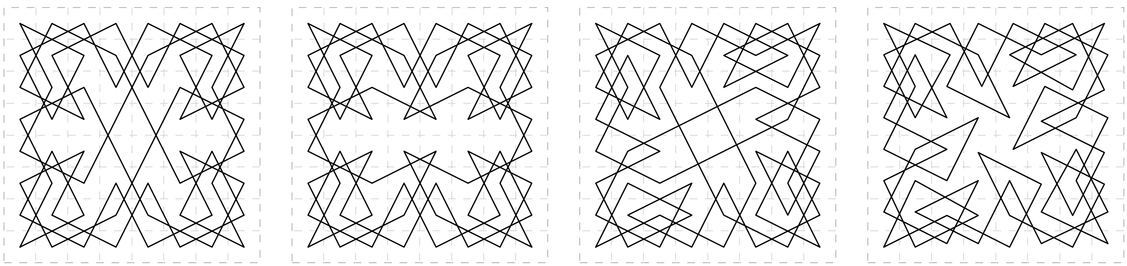
3871562038716



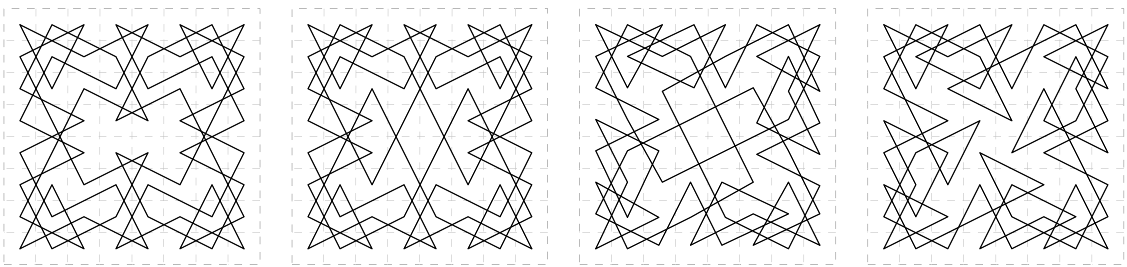
3871620387156



3871651783026

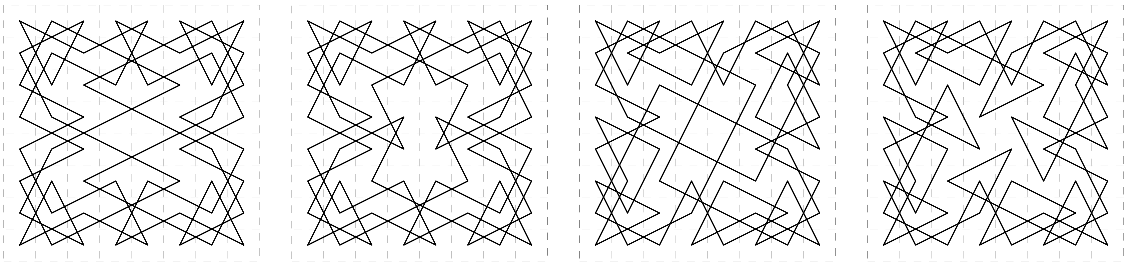


6156178302387



4a formulae 11-15

6156178320387



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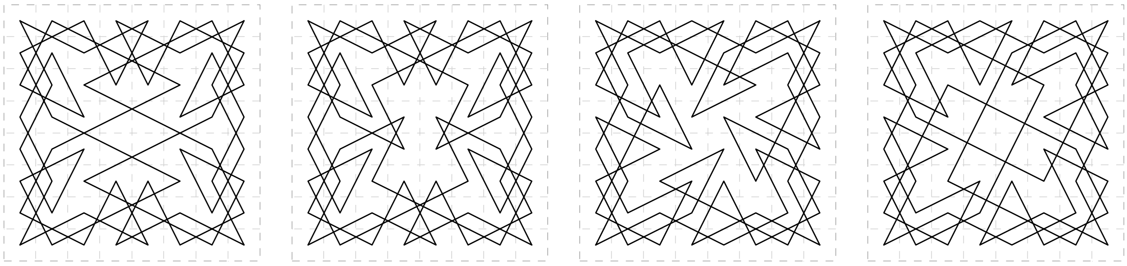
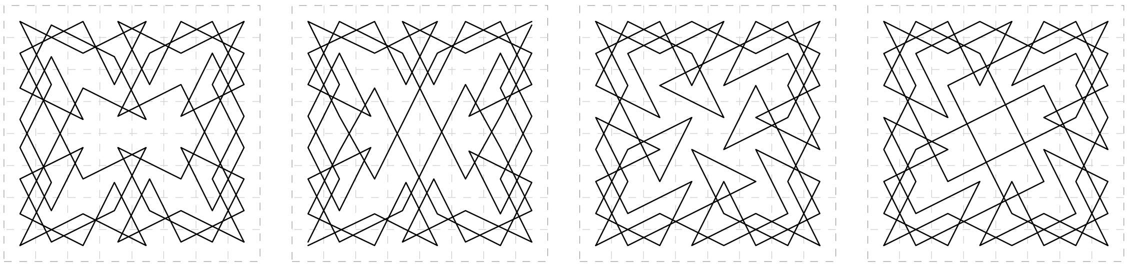
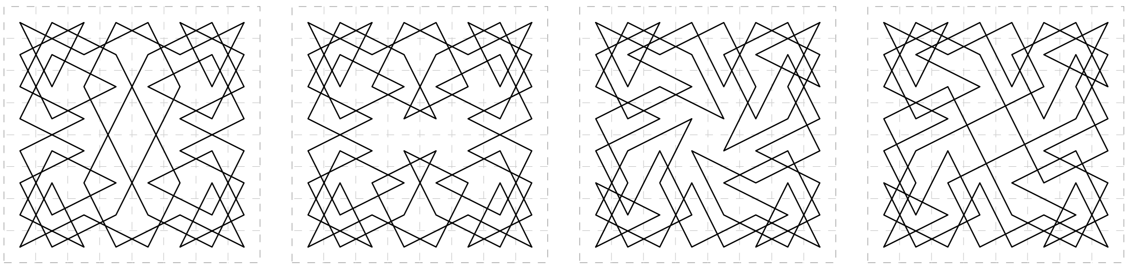


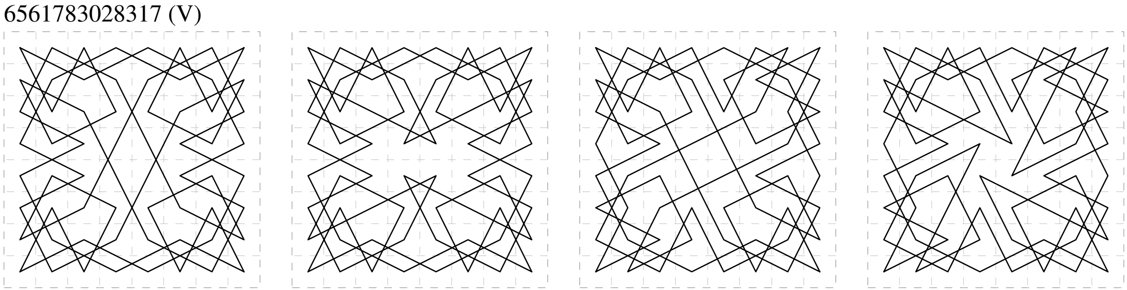
6178320387156



6203871651387

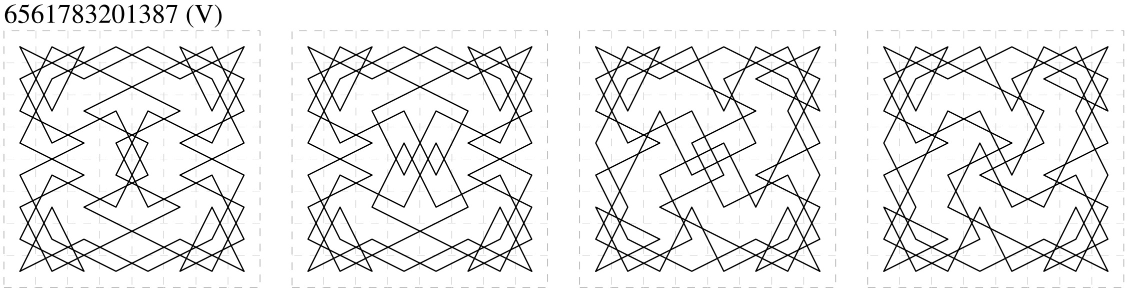


6561783028317 (V)

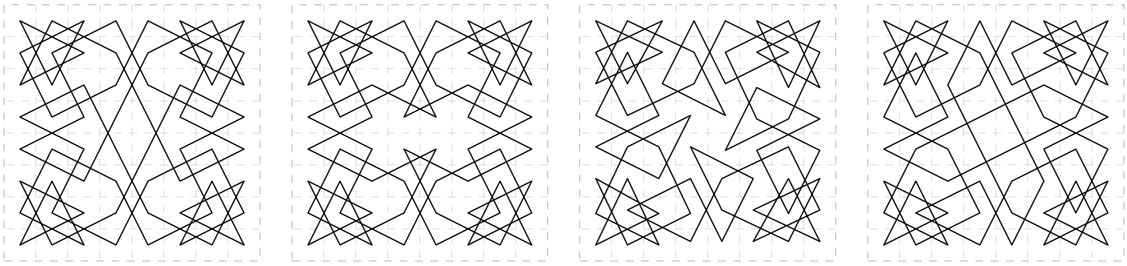


4a formulae 16-20

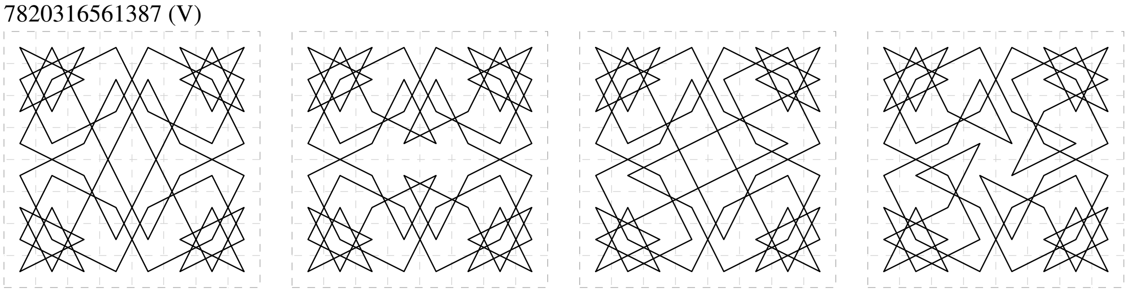
6561783201387 (V)



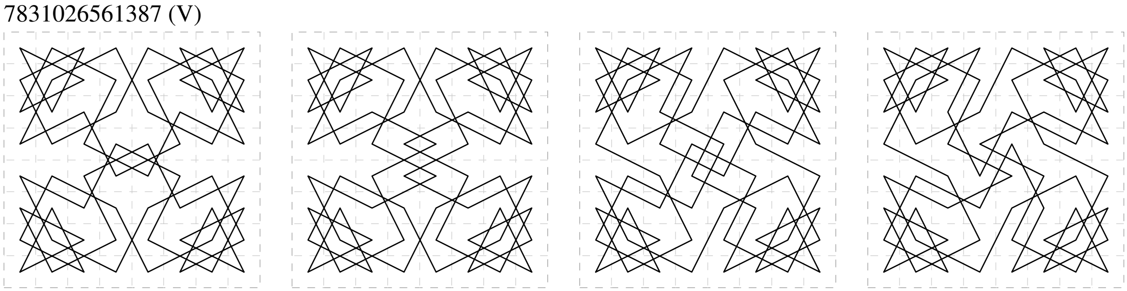
7830261561387



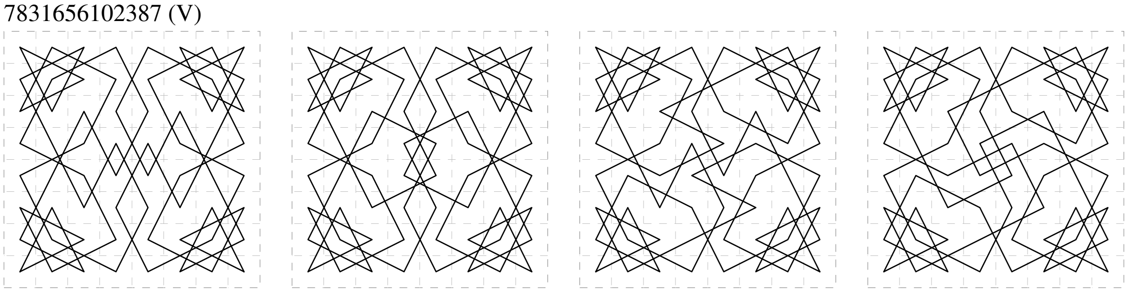
7820316561387 (V)



7831026561387 (V)



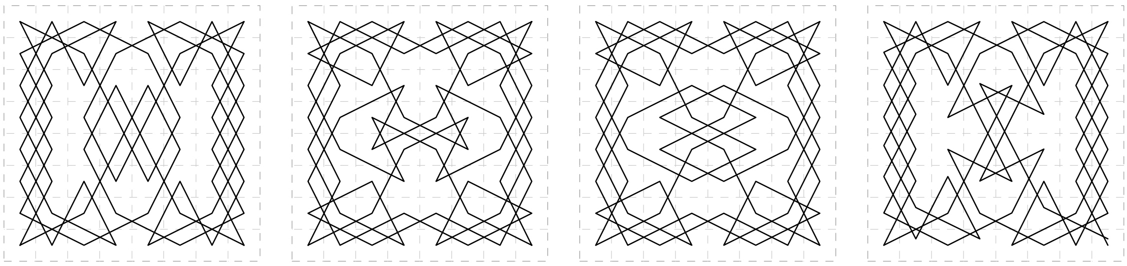
7831656102387 (V)



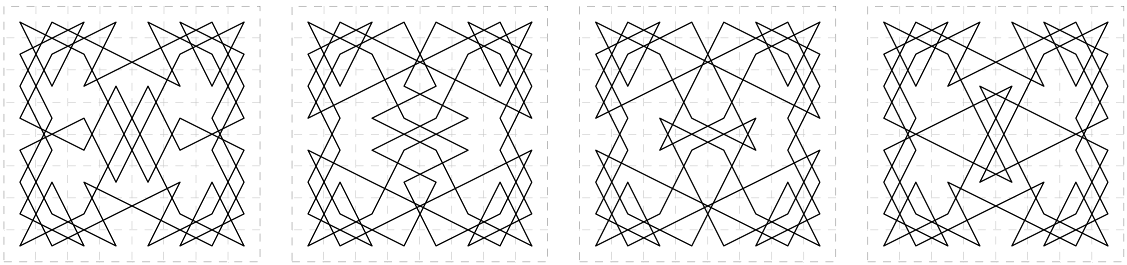
4b: 39 formulae giving pseudotours in direct symmetry, 2 closed paths and 2 open paths (2Dc, 2Do). Of these, 37 include move 1-1 which allows only direct symmetry. The other two contain 13873871 or its reverse. When all formulae are put in numerical order, the first six are 4b type.

4b formulae 1-5:

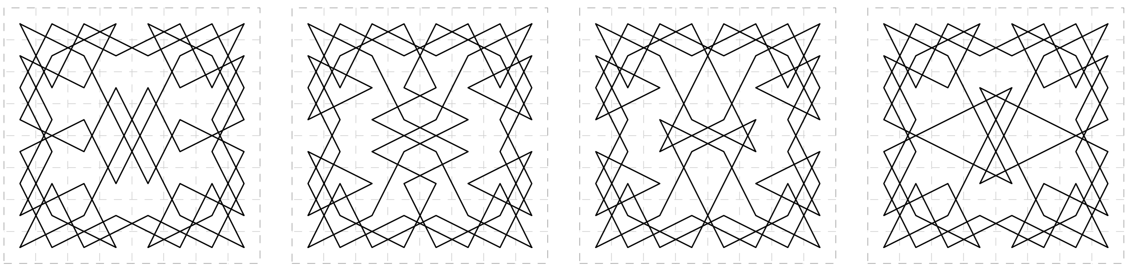
0113783782656



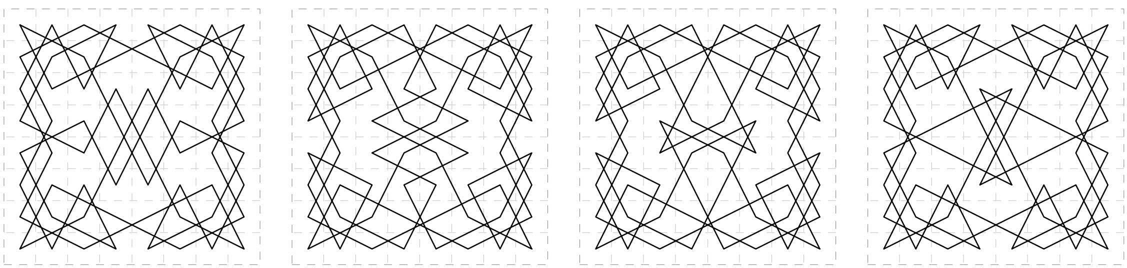
0116562387387



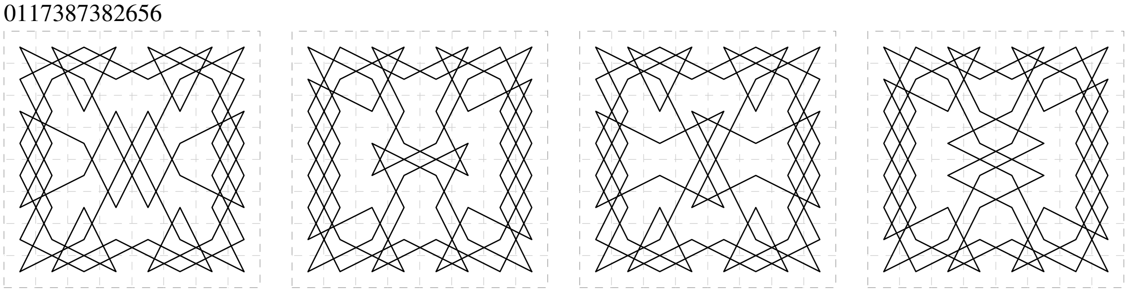
0116562837837



0116562873873

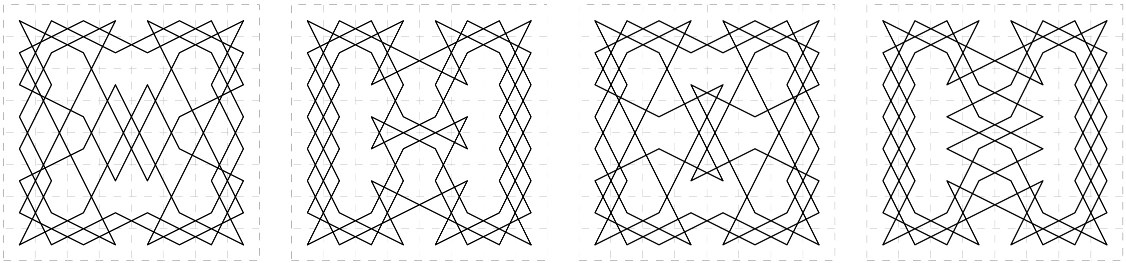


0117387382656

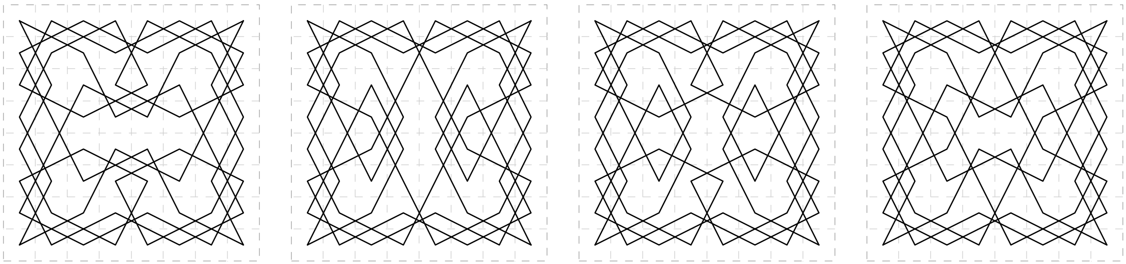


4b formulae 6-10.

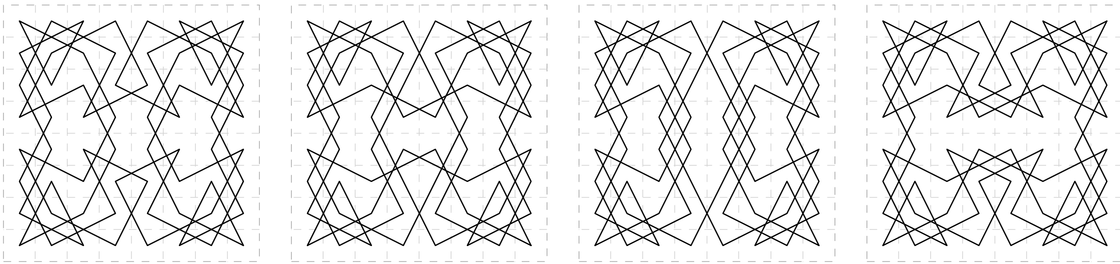
0117837832656



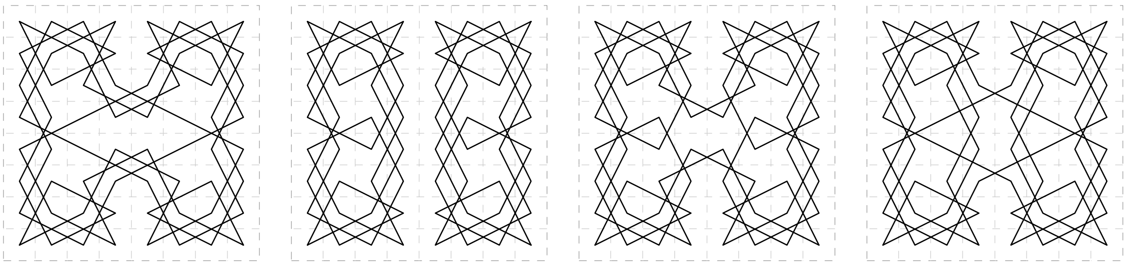
0238738711656



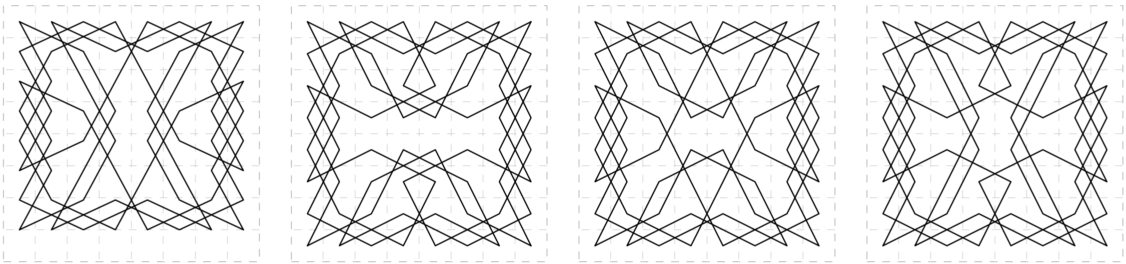
0265611387387



0265611783783

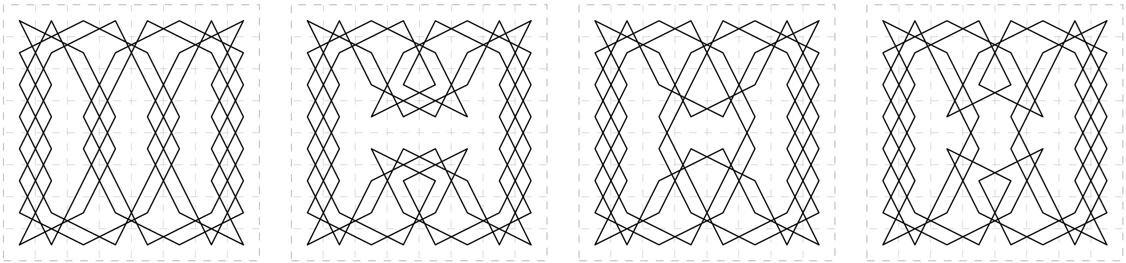


0283783711656

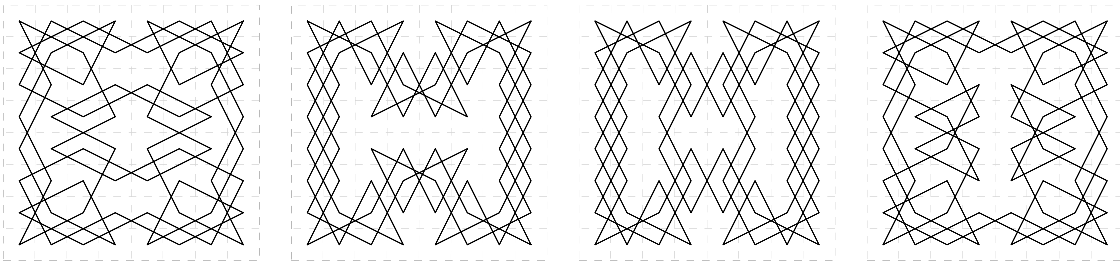


4b formulae 11-15.

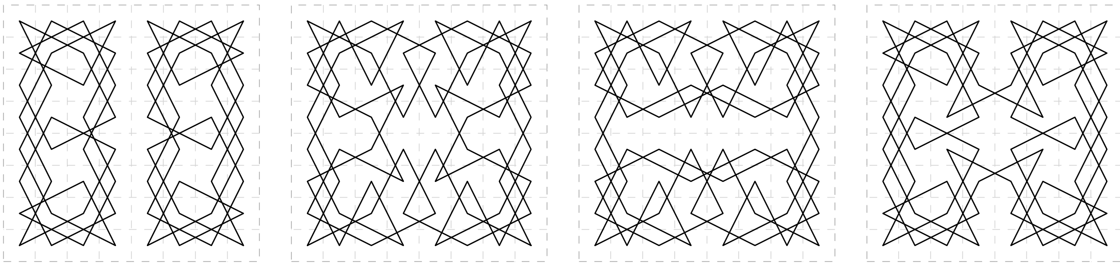
0287387311656



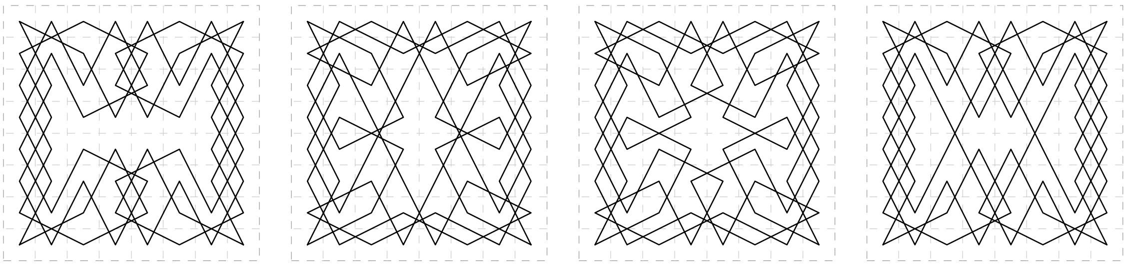
0311783782656 (Tomlinson)



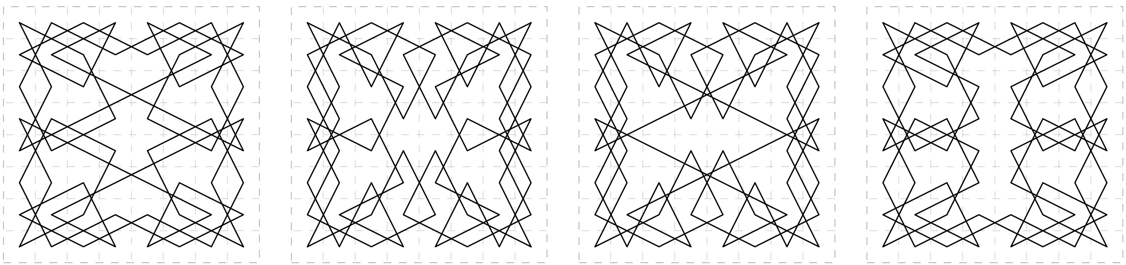
0378311782656



0378378261156

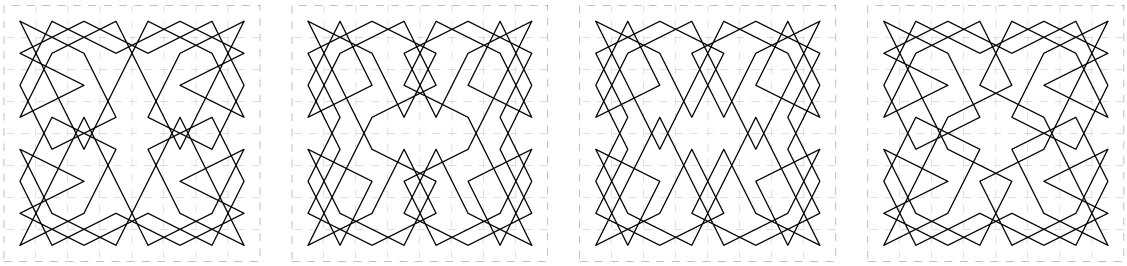


0378378265116

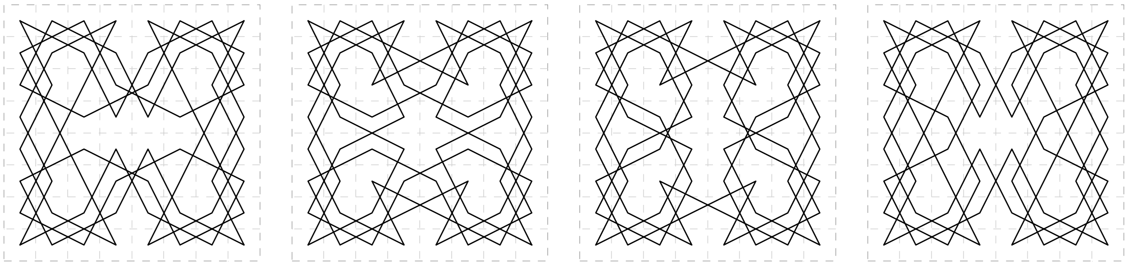


4b formulae 16-20.

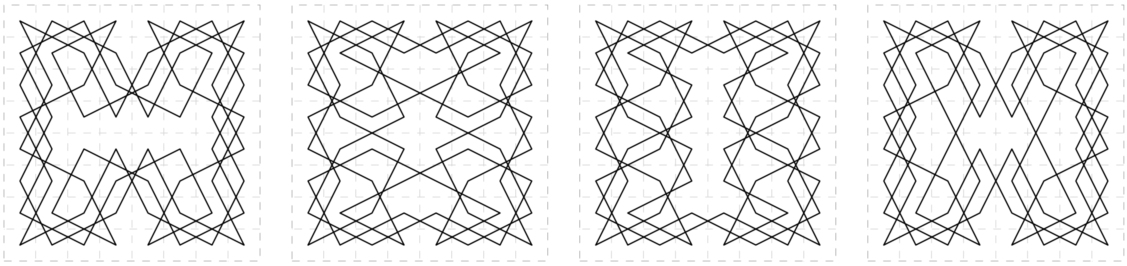
0378378265611



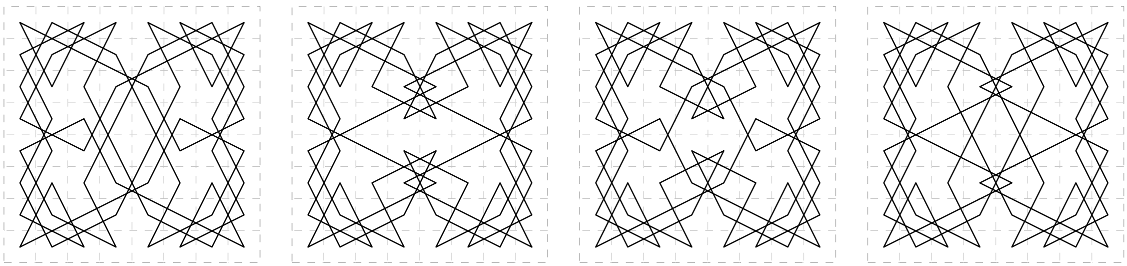
0387117832656



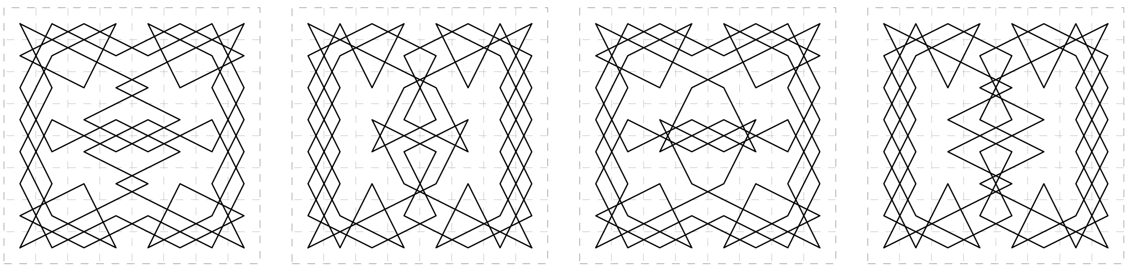
0387387115626



1165620387387

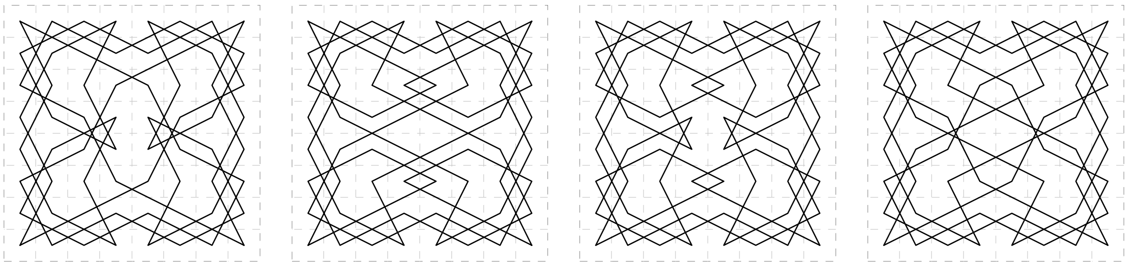


1103783782656

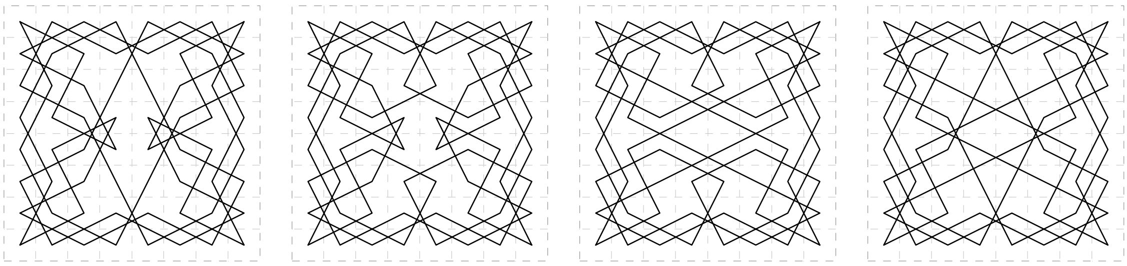


4b formulae 21-25.

1178378302656



3028738711656



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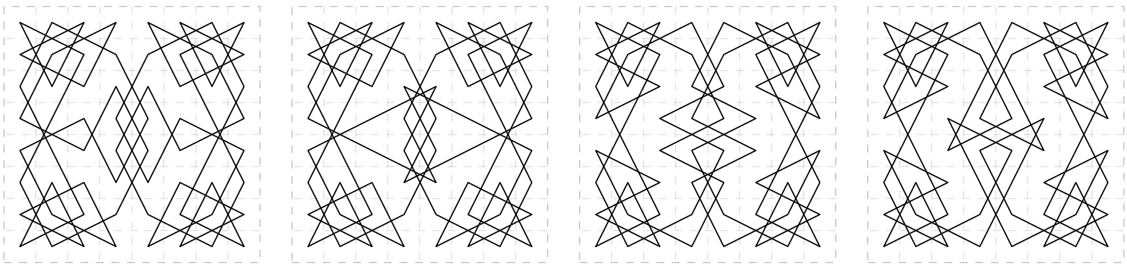
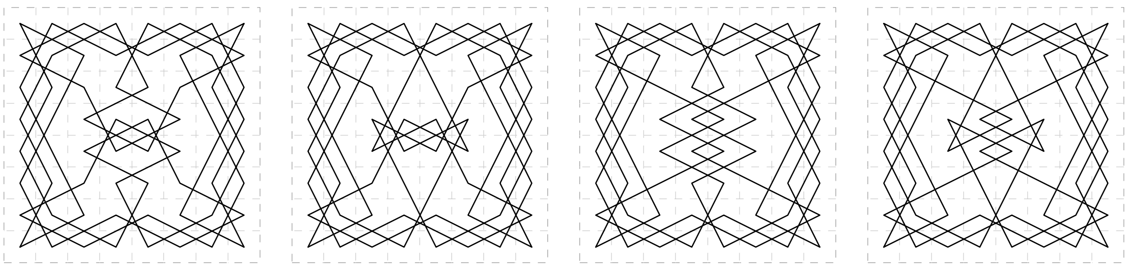
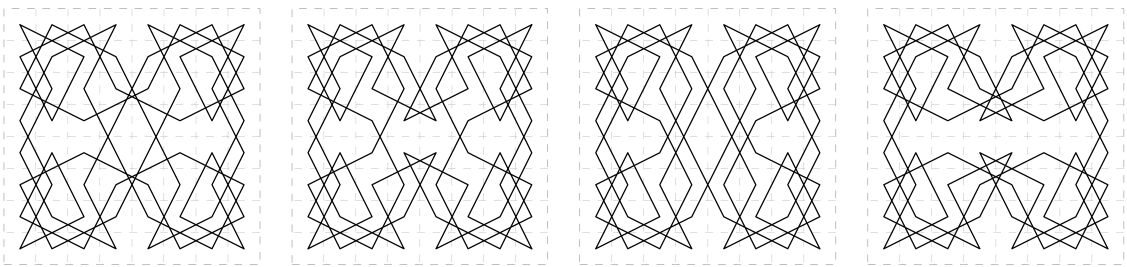


3783782011656

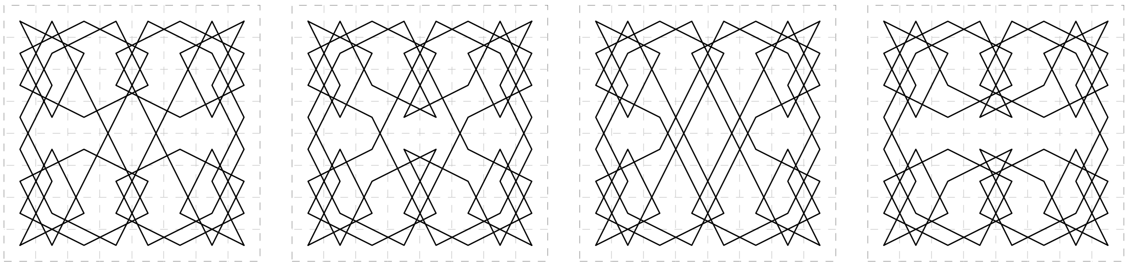


3871178302656

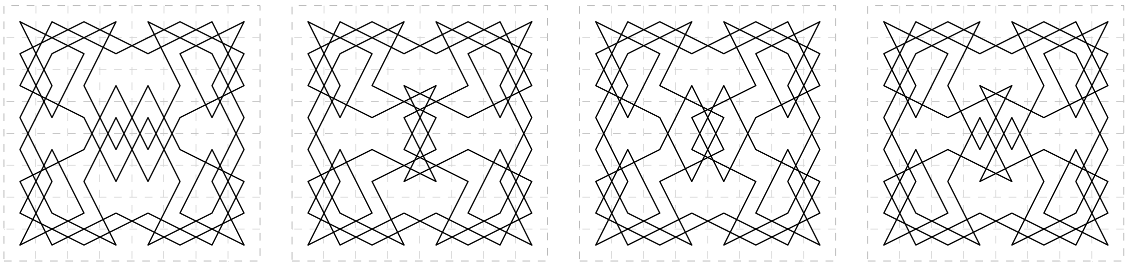


4b formulae 26-30.

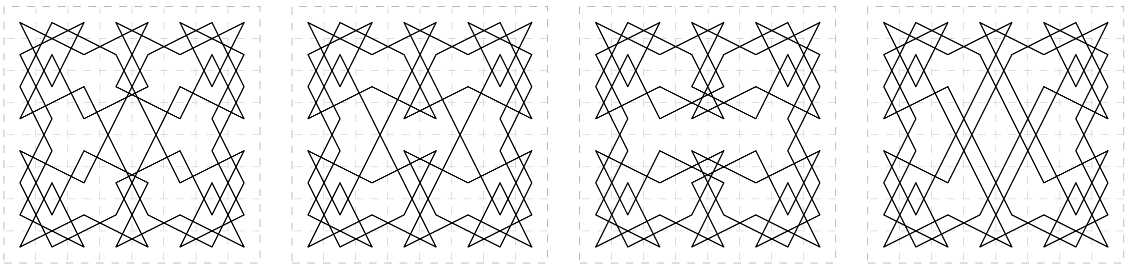
3873028711656



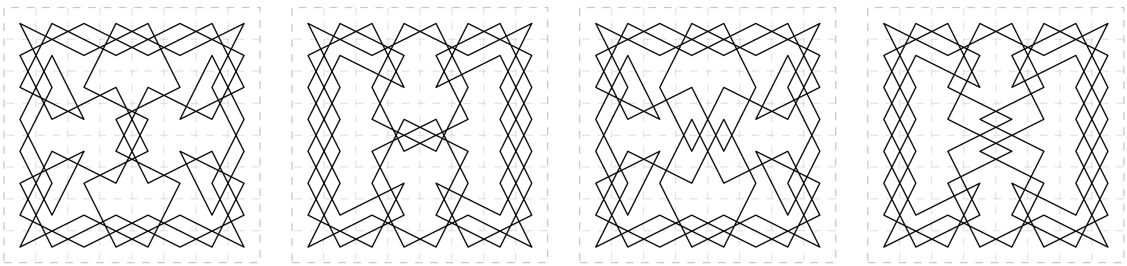
3873871102656



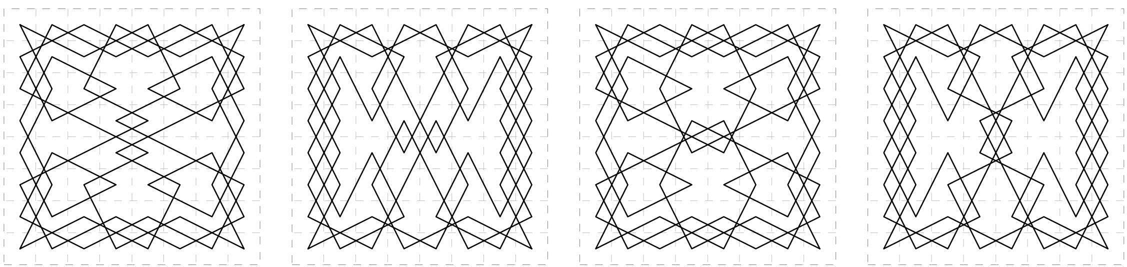
6115620387387



6201387387156

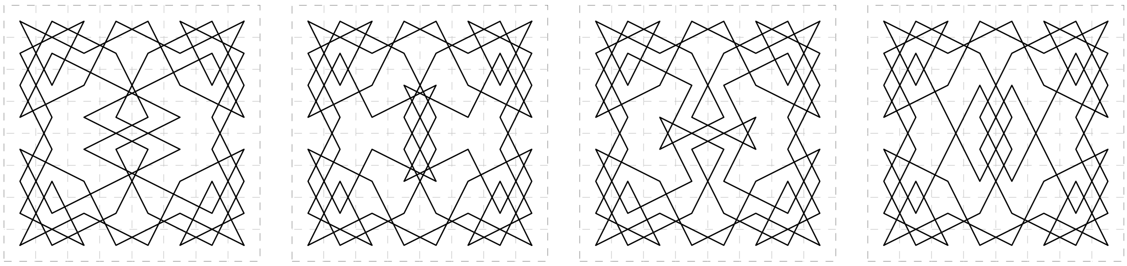


6201783783156

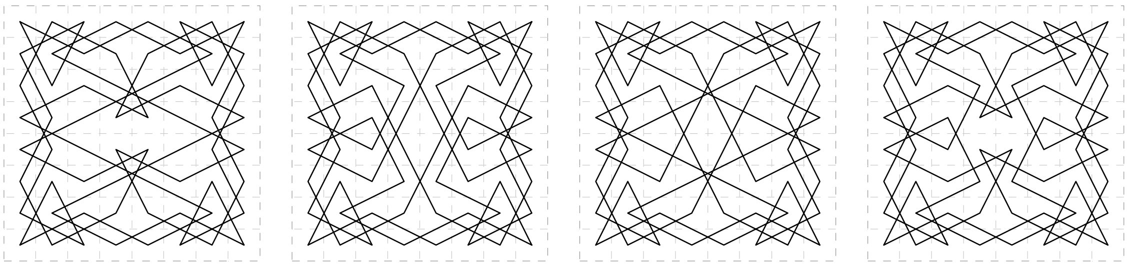


4b formulae 31-35.

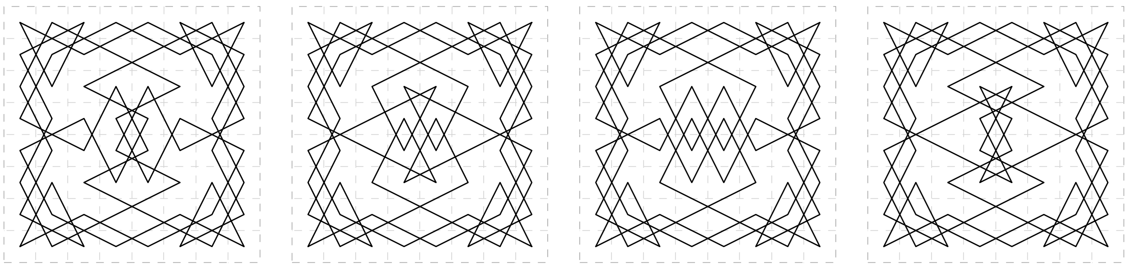
6265110387387



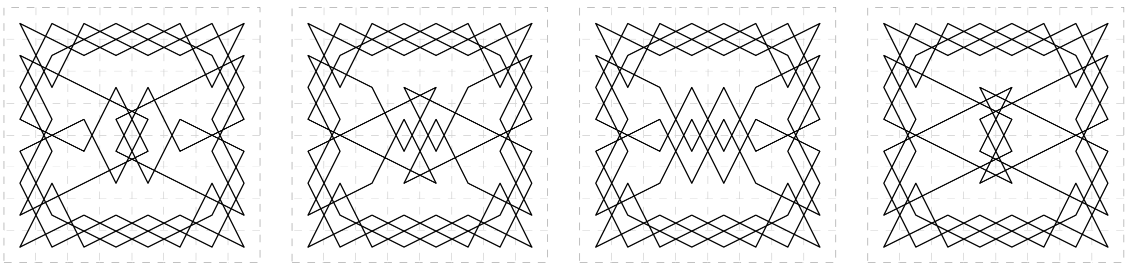
6511620387387



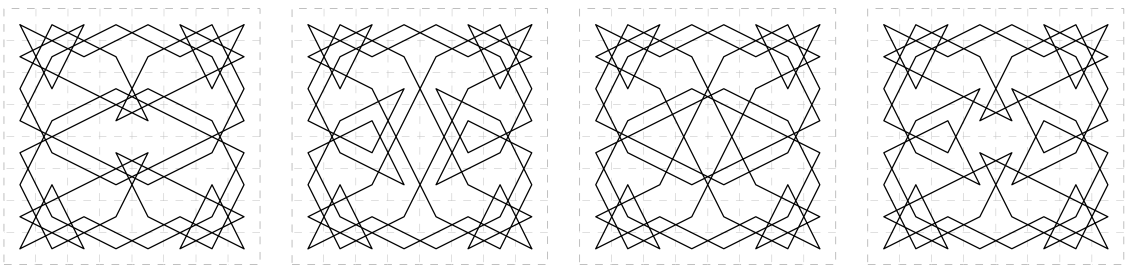
6561102387387



6561102837837

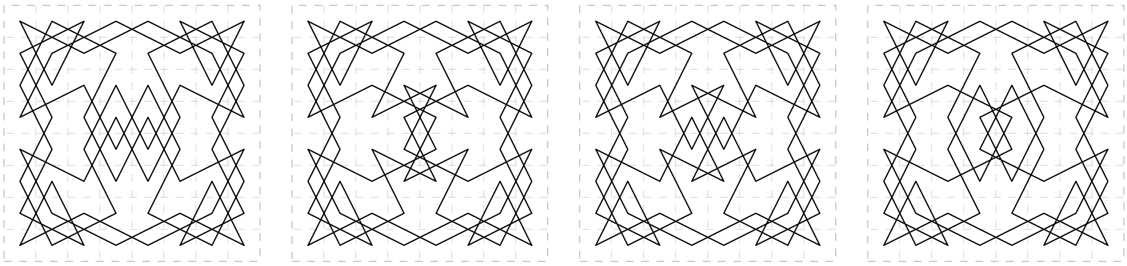


6561137830287

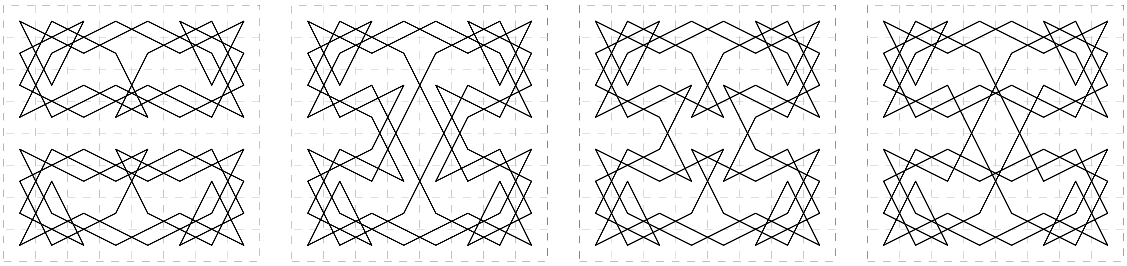


4b formulae 36-39.

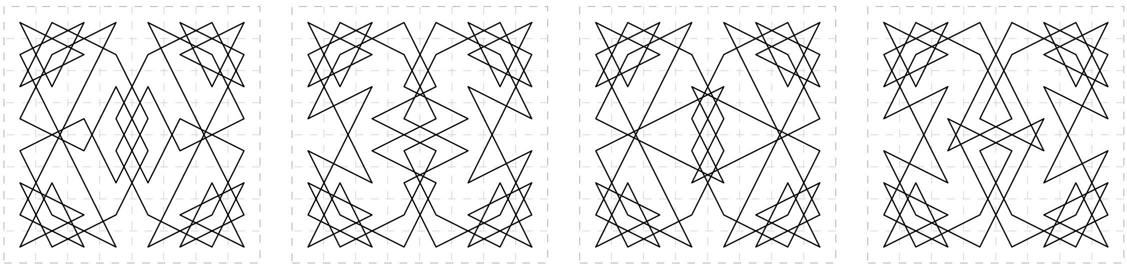
6562011387387



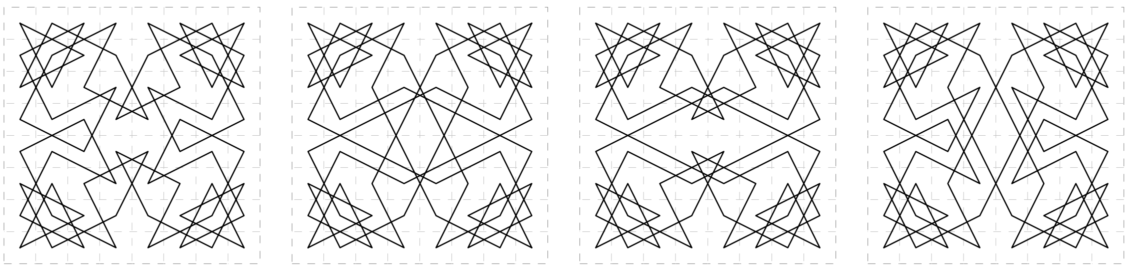
6562038711387



7830116562387

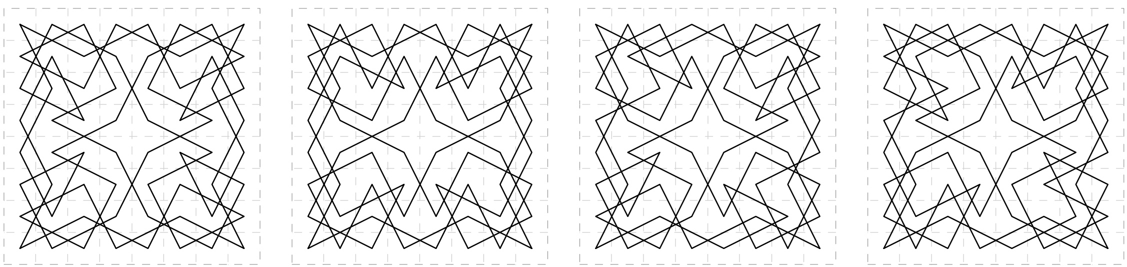


7830265611387



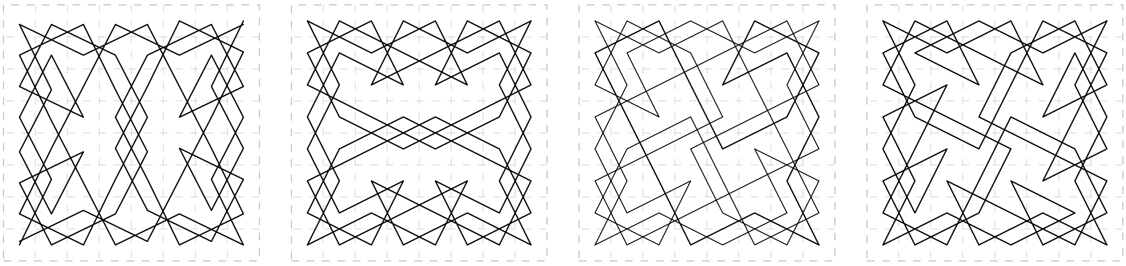
4c: 1 formula (giving 2Dc, 2Oc).

6287130387156

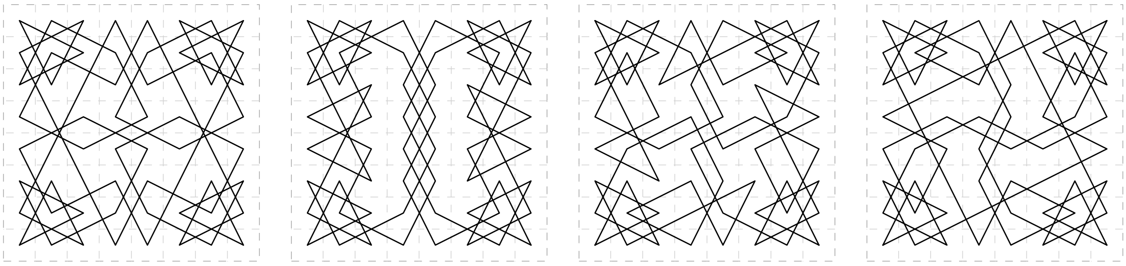


4d: 2 formulae (giving 2Do, 2Oc).

6238710387156



7830165162387

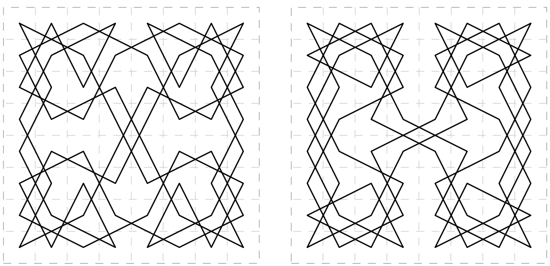


2-fold: There are 235 formulae giving two pseudotours. They form ten classes. Three classes give direct symmetry (2a, 2b, 2c). Three give oblique symmetry (2d, 2e, 2f). Four give direct and oblique (2g, 2h, 2i, 2j).

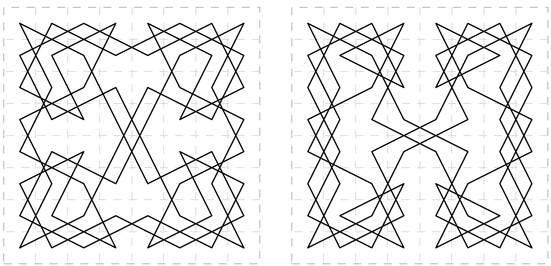
2a: 111 (1Dc, 1Do).

2a formulas 1-4

0138731782656



0138738715626



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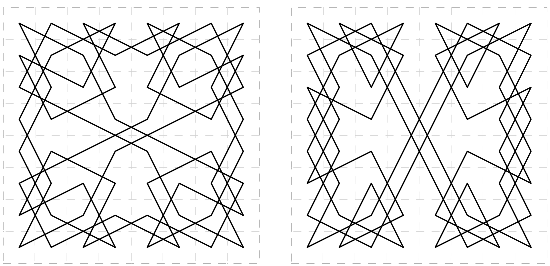

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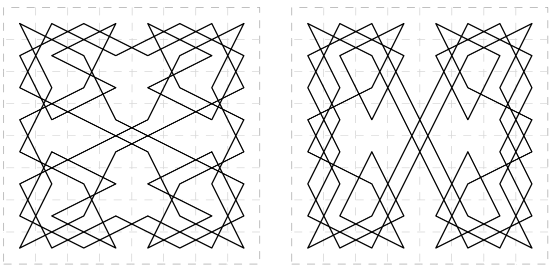

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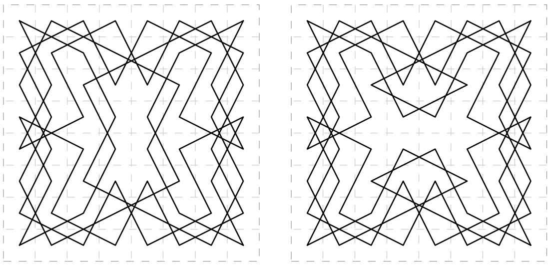

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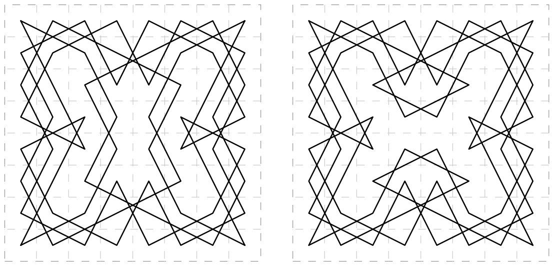

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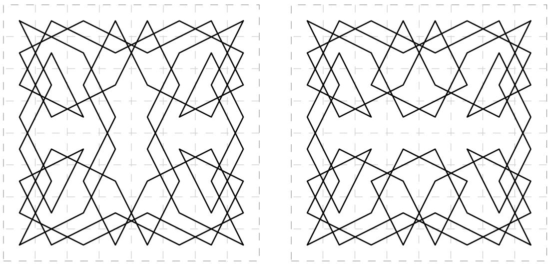

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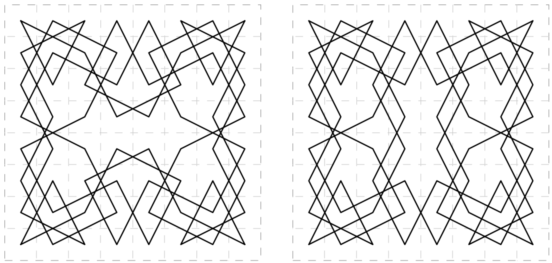

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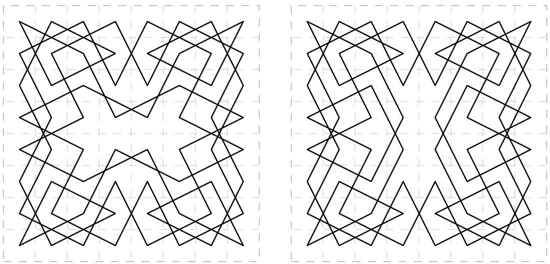

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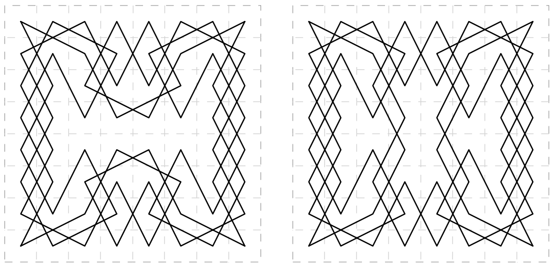

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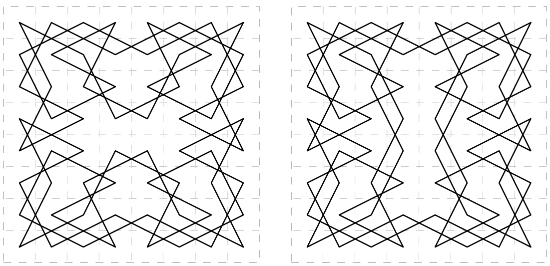

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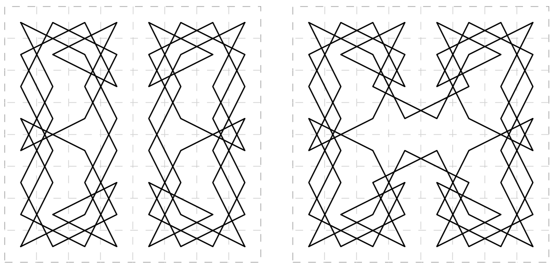

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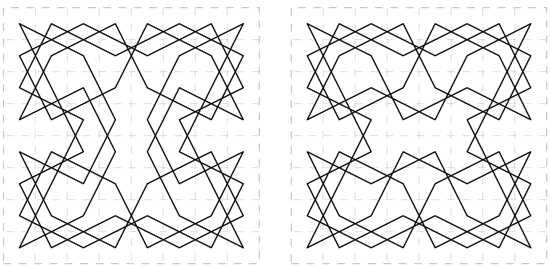

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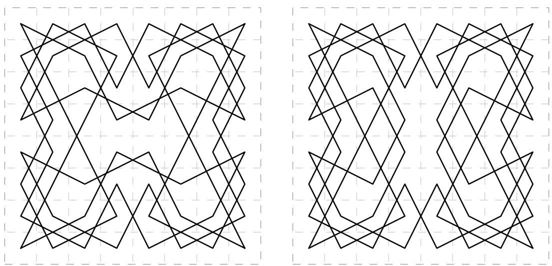

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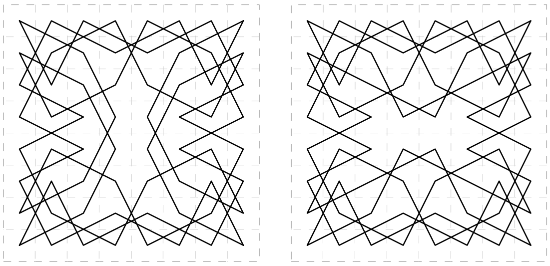

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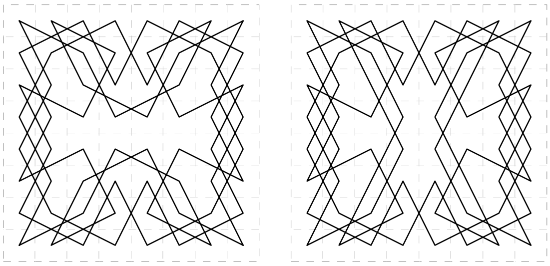

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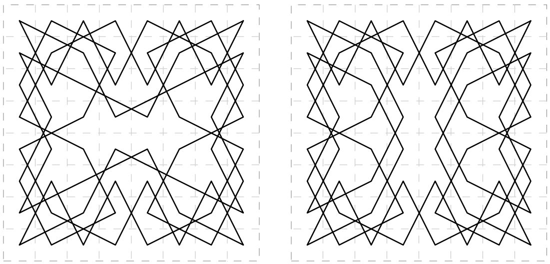

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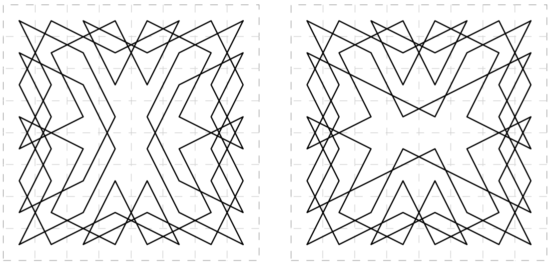

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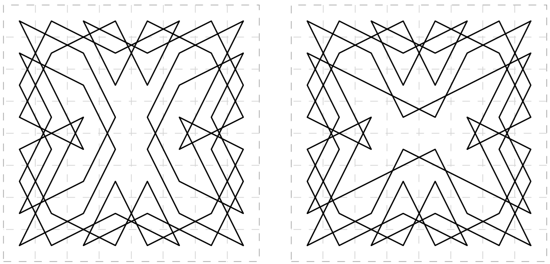

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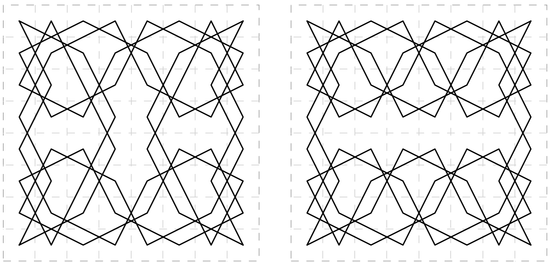

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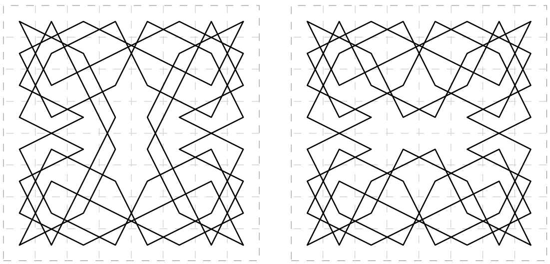

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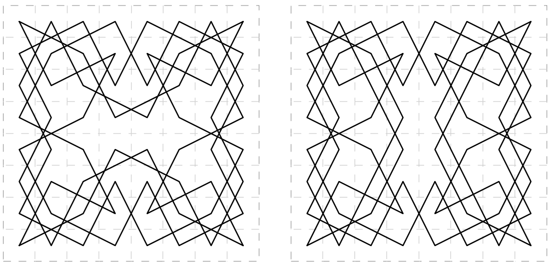

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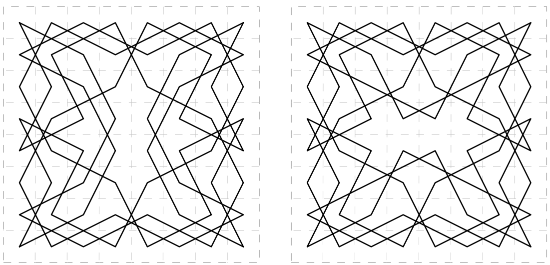

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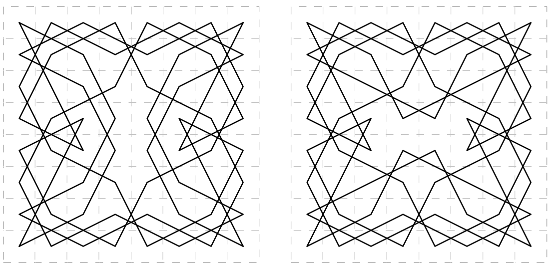

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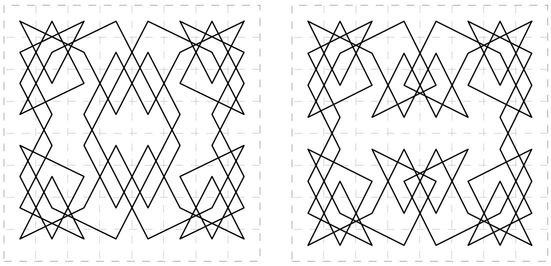

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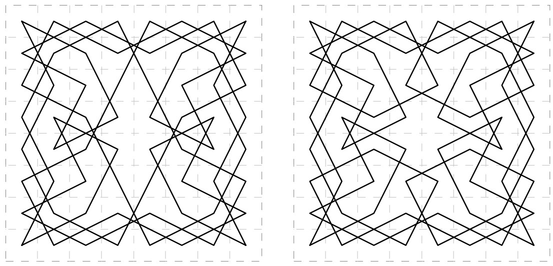

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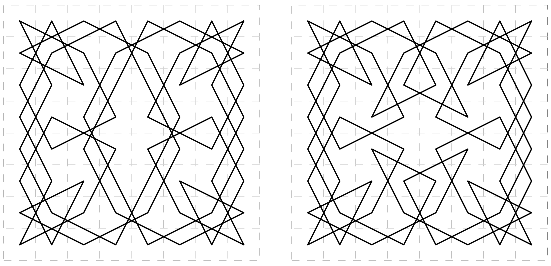

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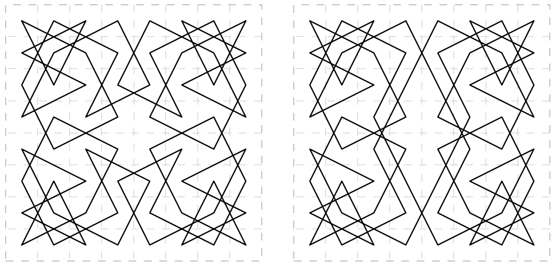

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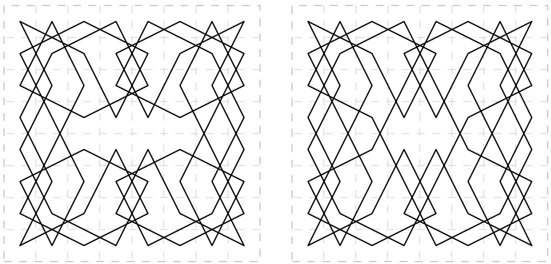

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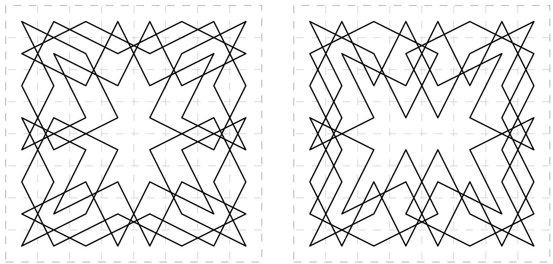

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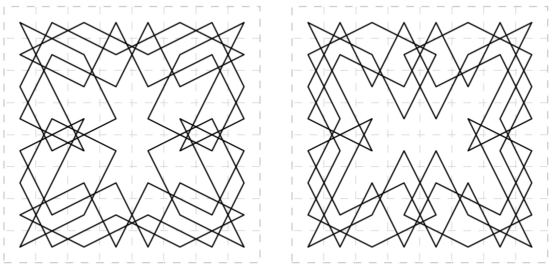

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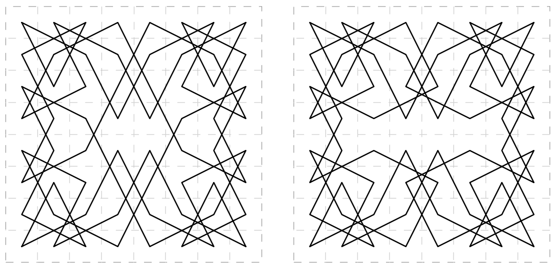

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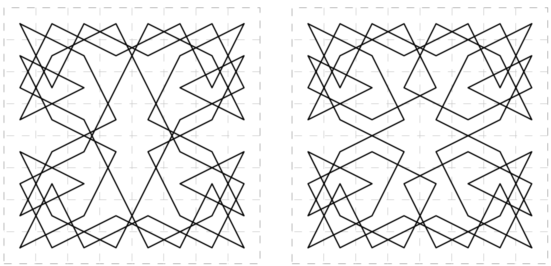

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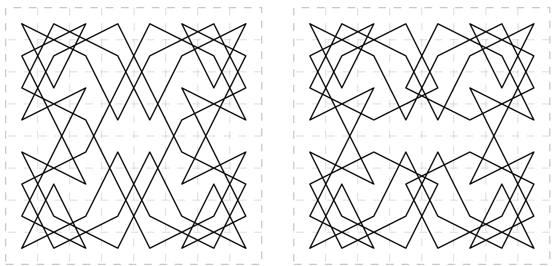

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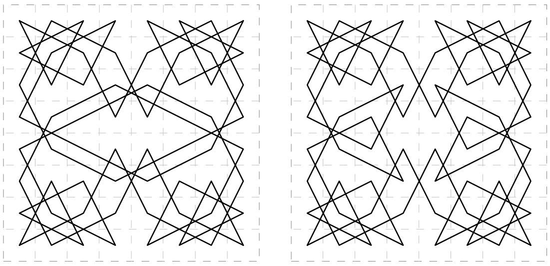

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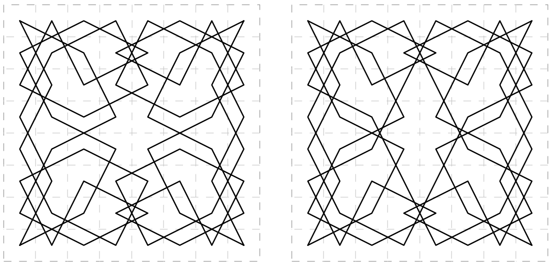

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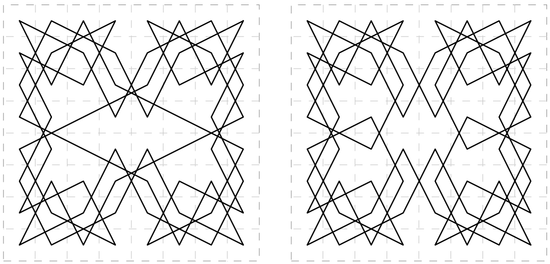

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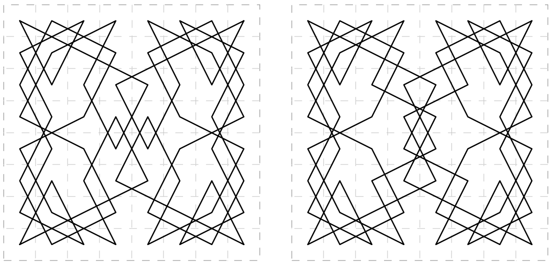

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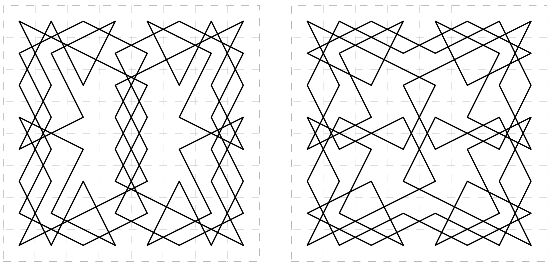

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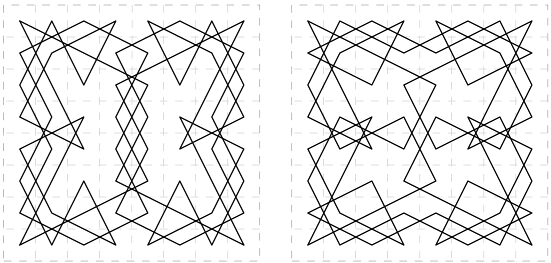

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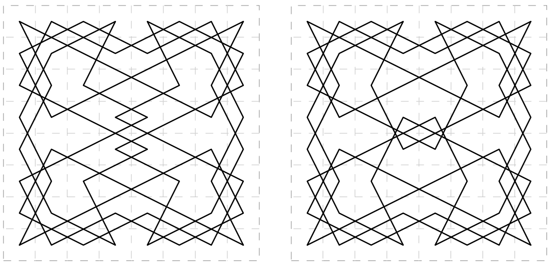

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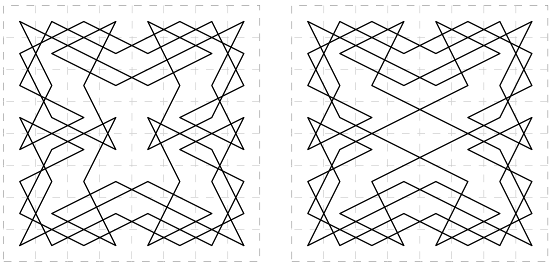

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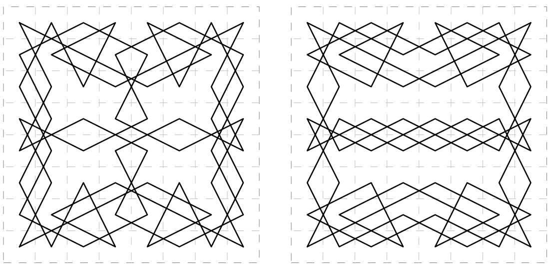

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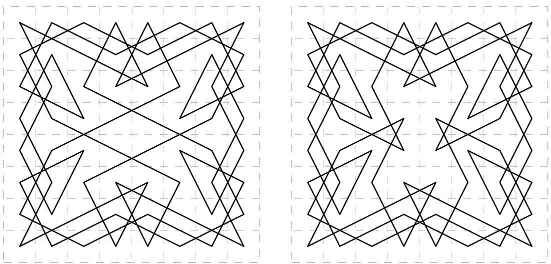

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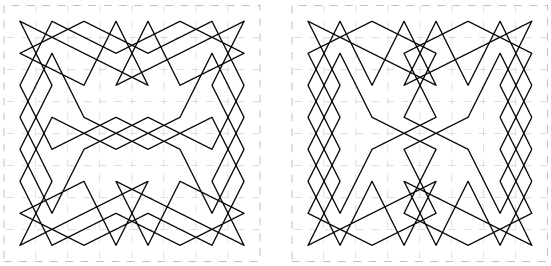

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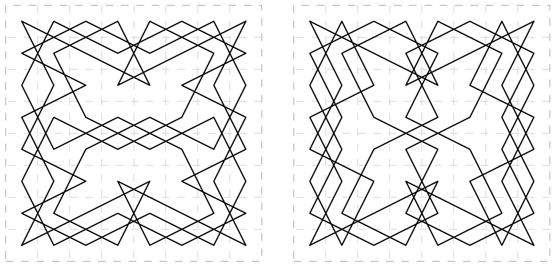

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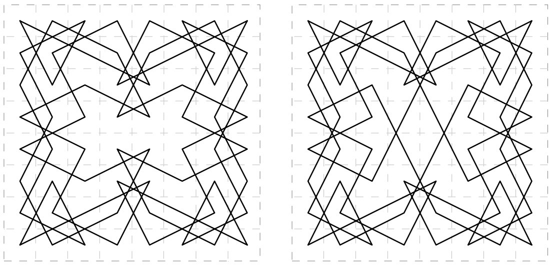

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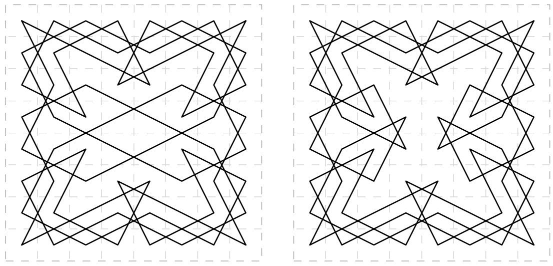

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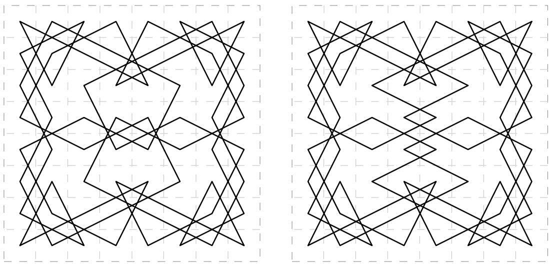

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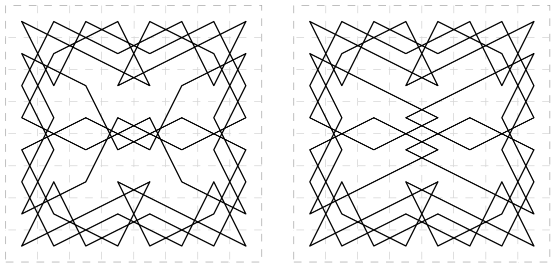

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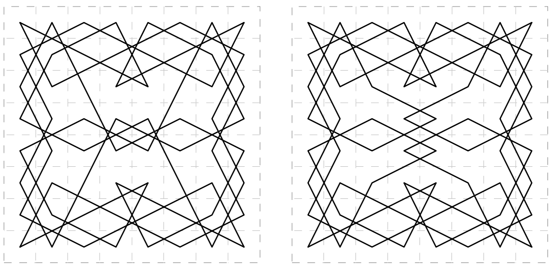

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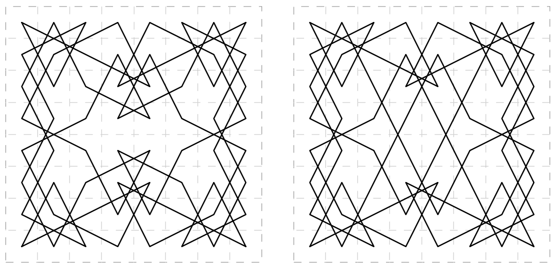

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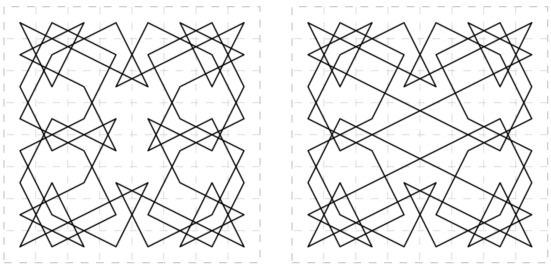

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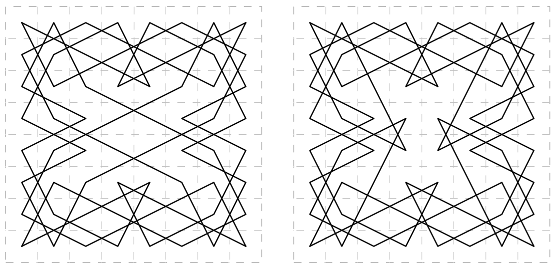

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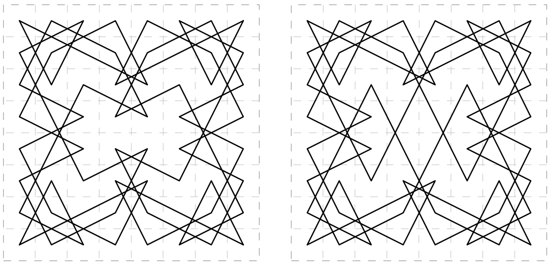

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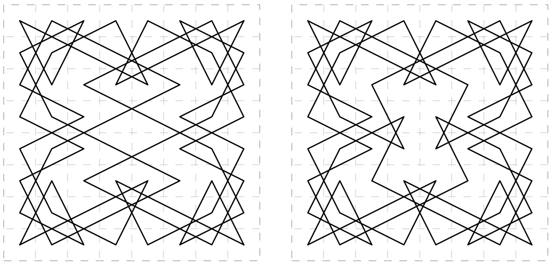

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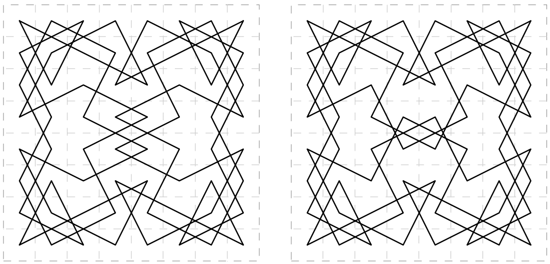

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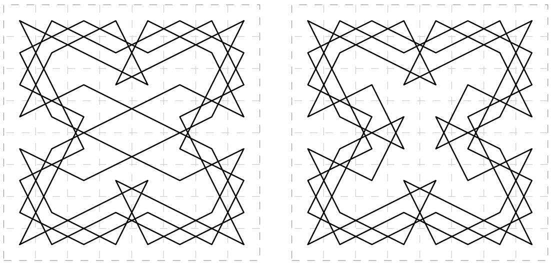

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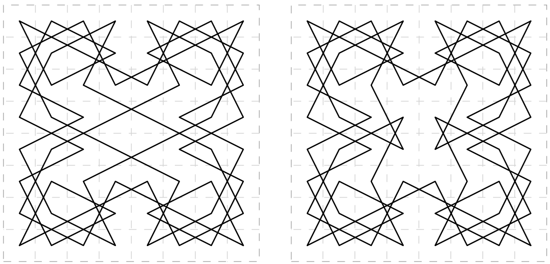

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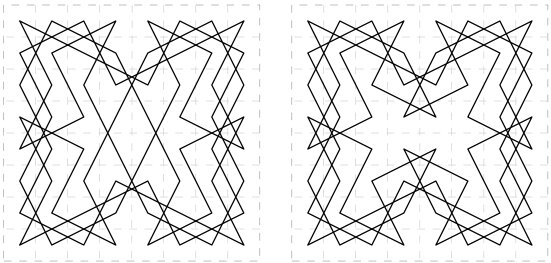

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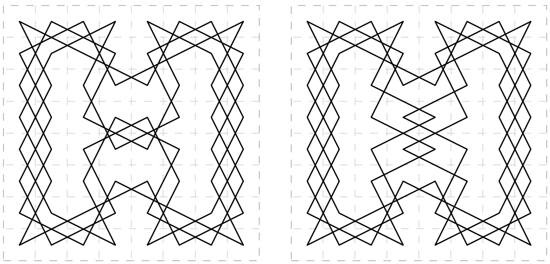

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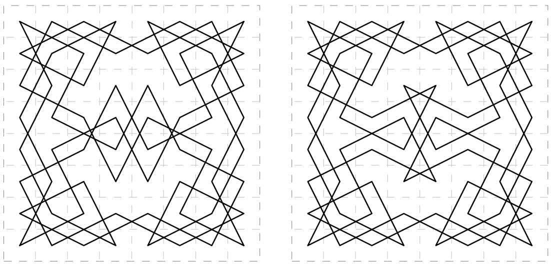

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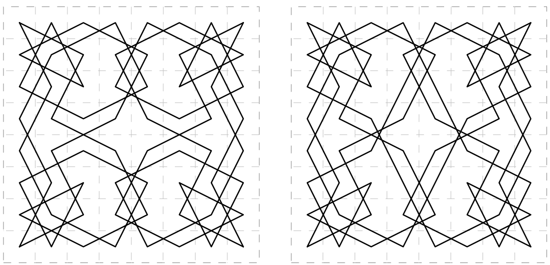

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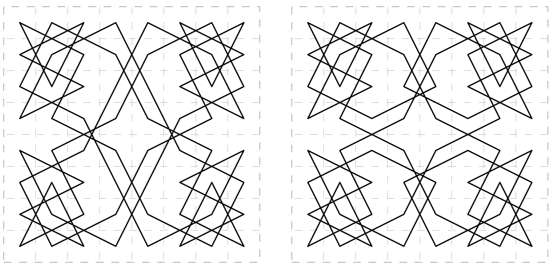

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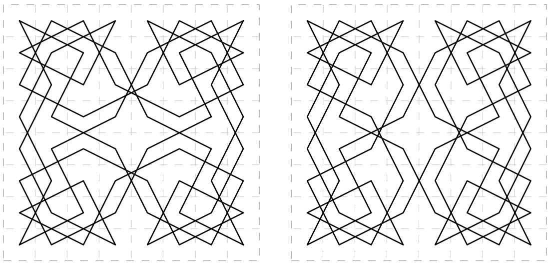

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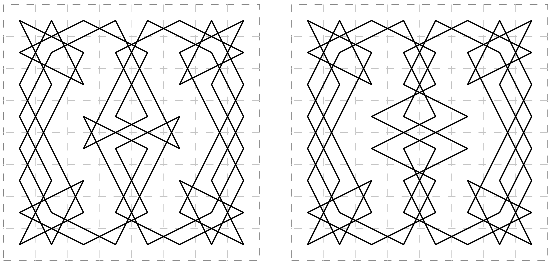

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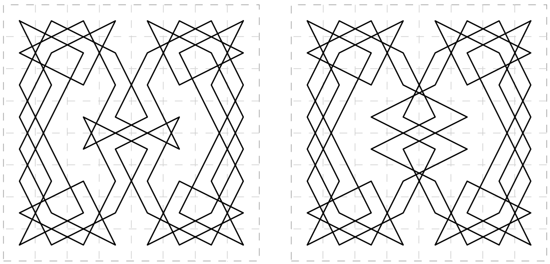

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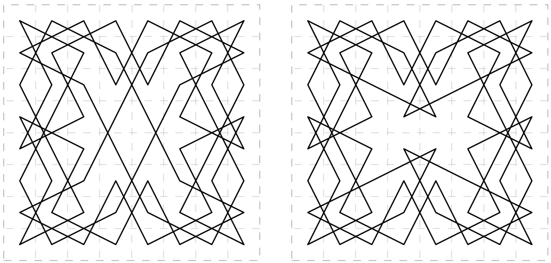

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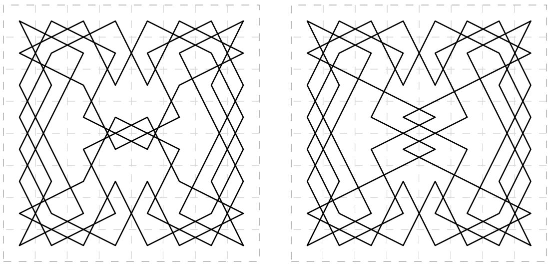

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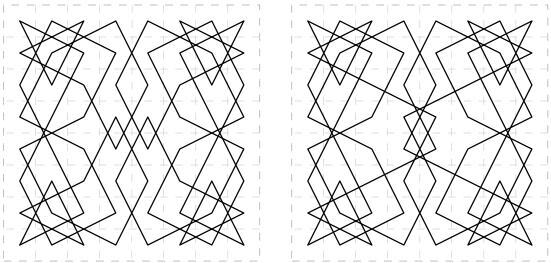

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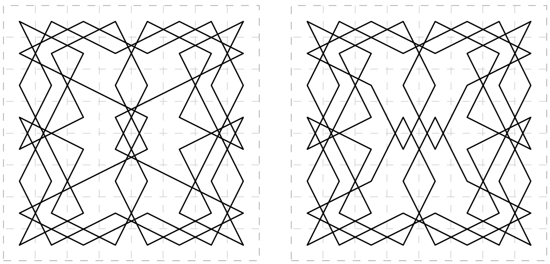

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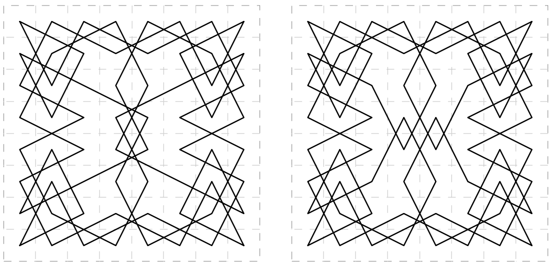

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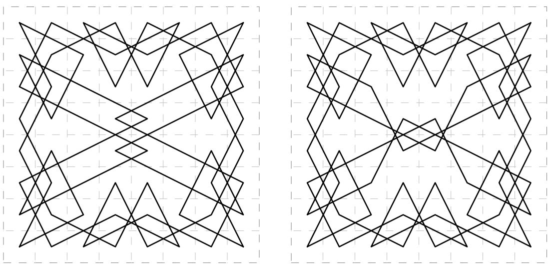

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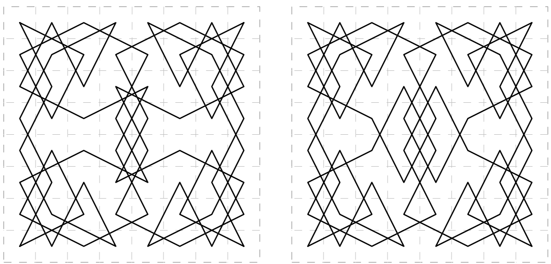

2a formulae 75-84

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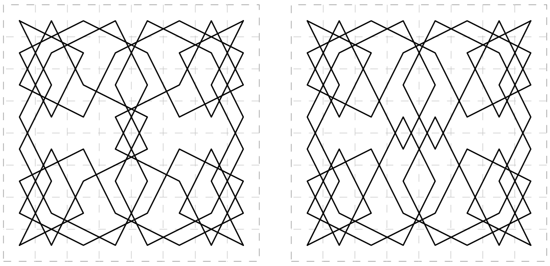

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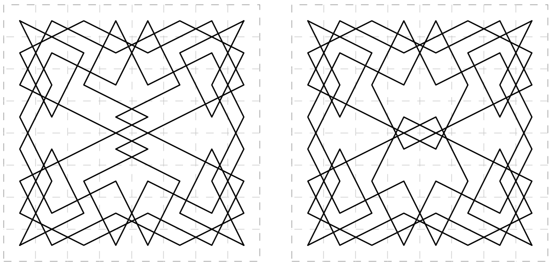

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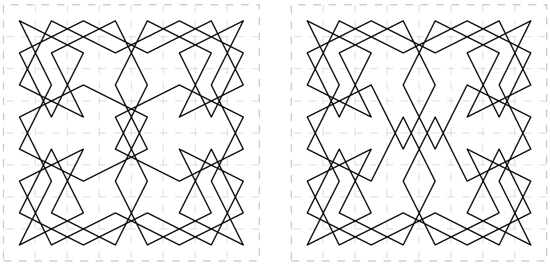

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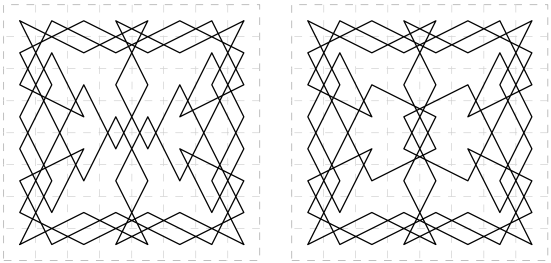

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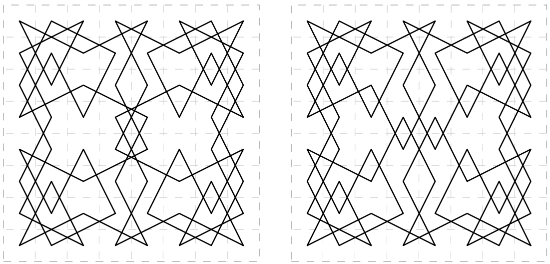

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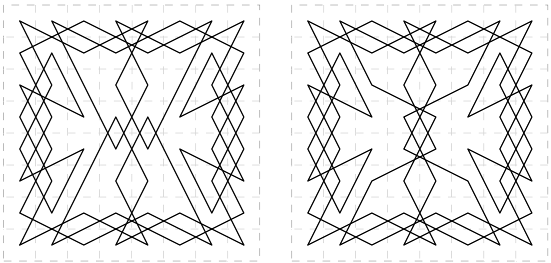

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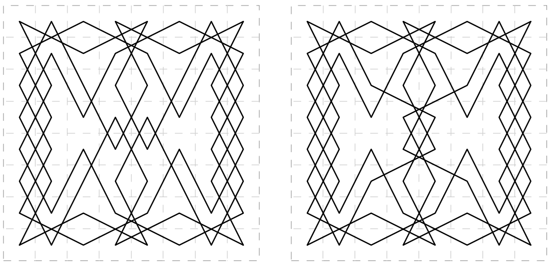

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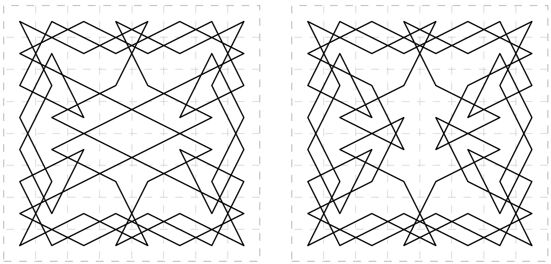

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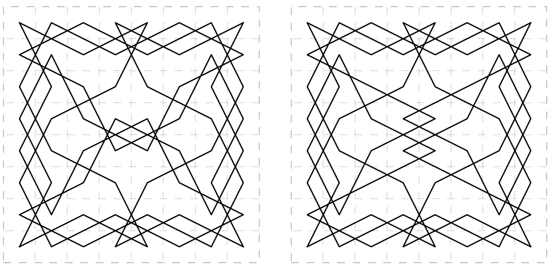

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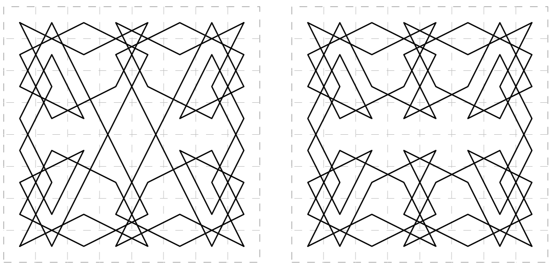

2a formulae 85-94

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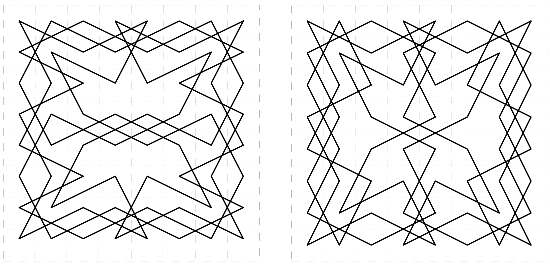

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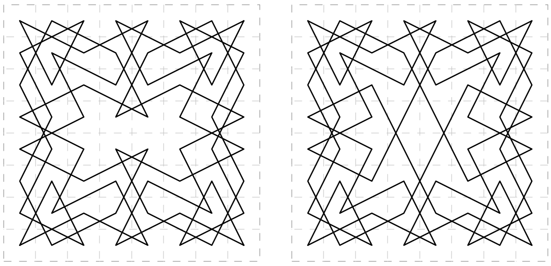

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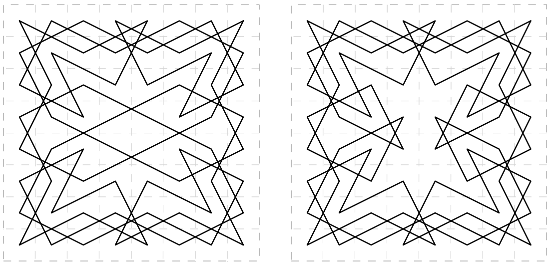

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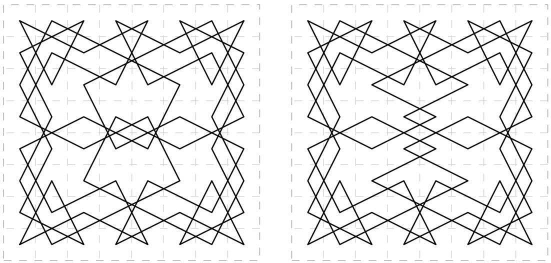

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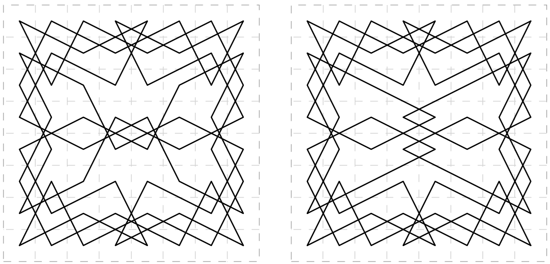

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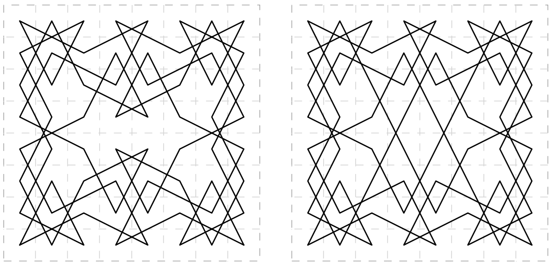

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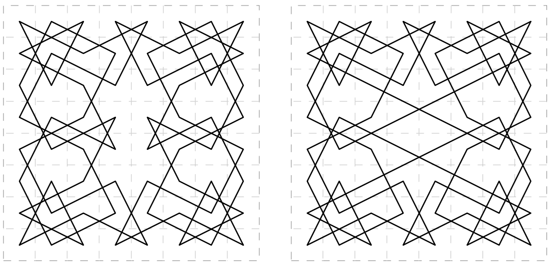

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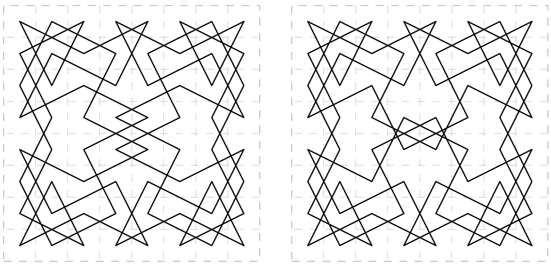

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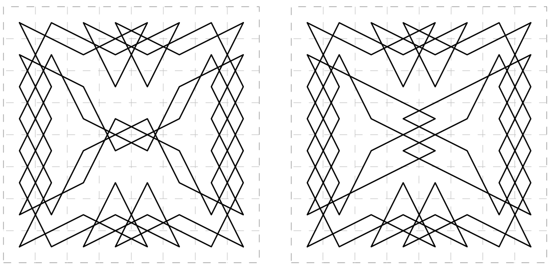

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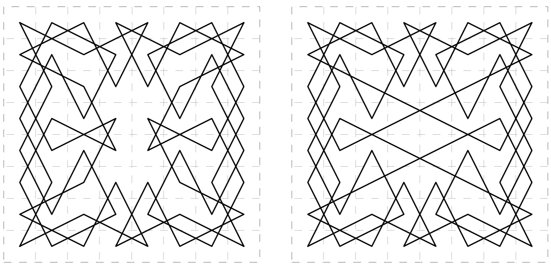

2a formulae 95-104

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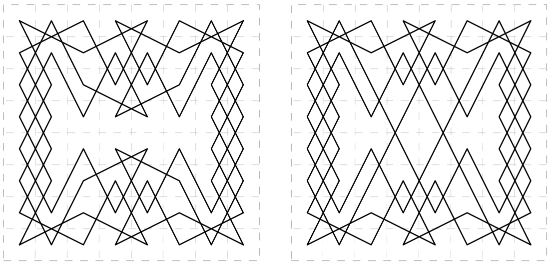

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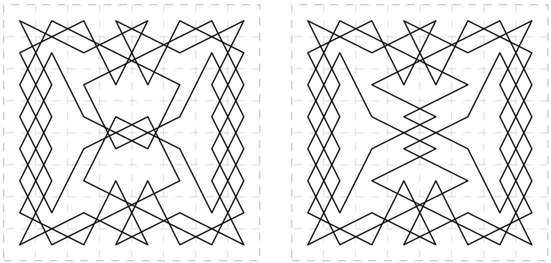

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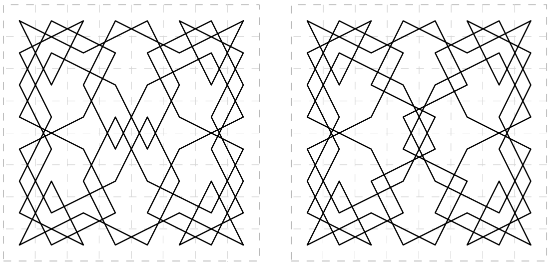

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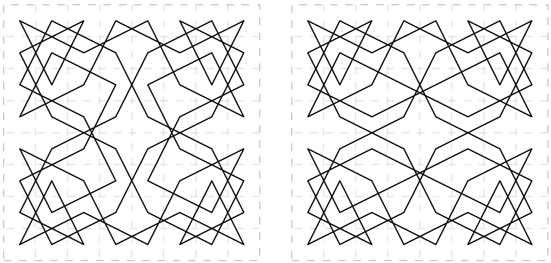

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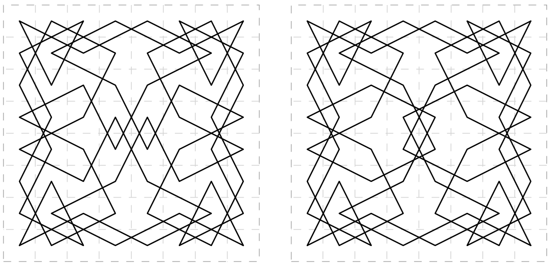

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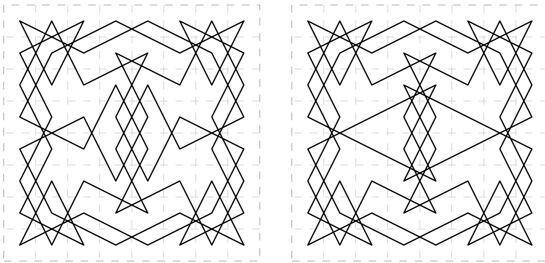

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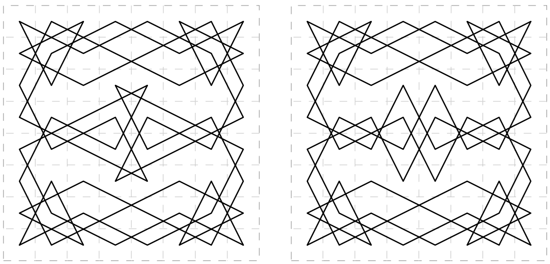

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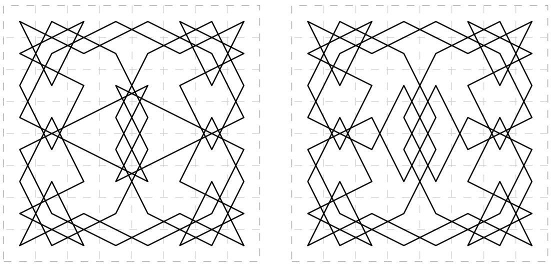

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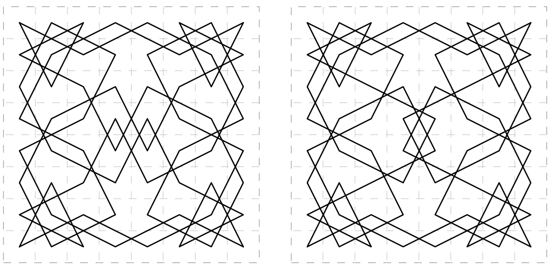

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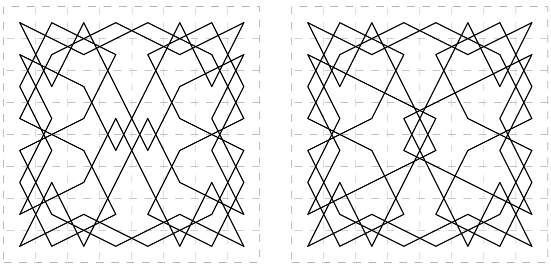

2a formulae 105-111

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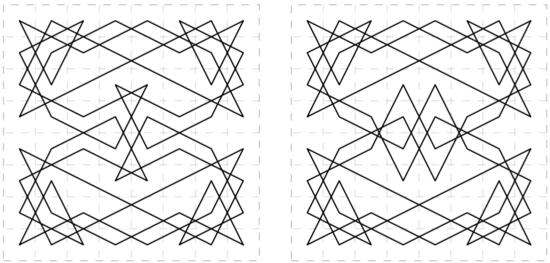

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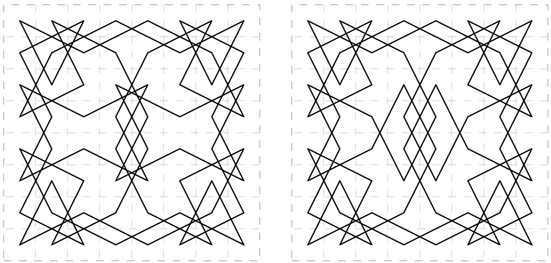

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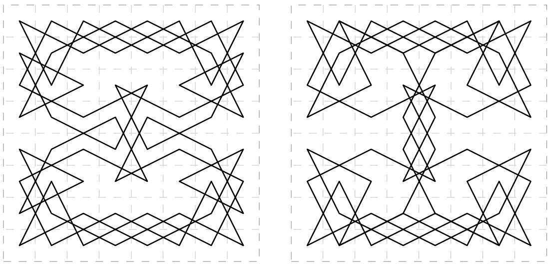

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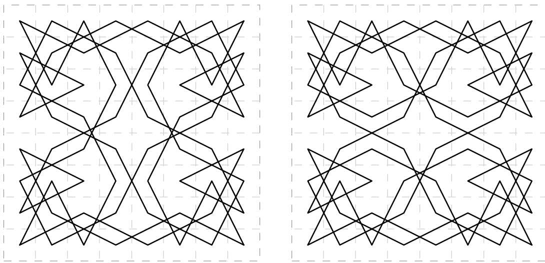

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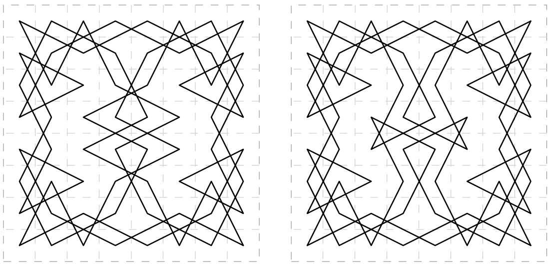

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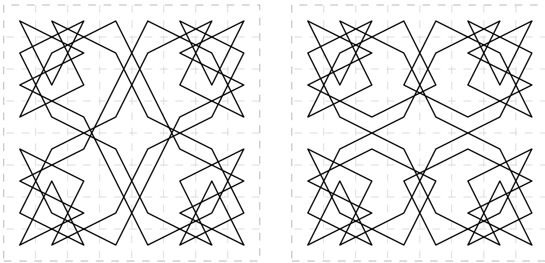

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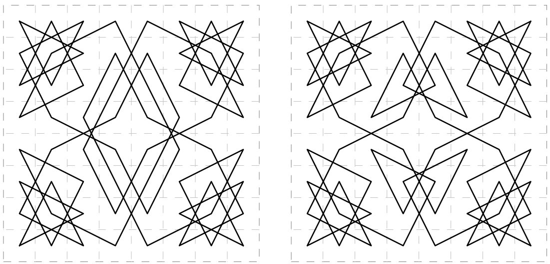

2b: 4 (2Dc): 2b formulae 1-2

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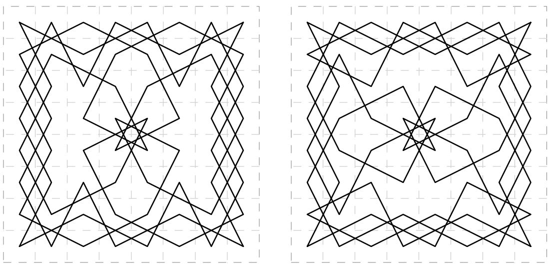

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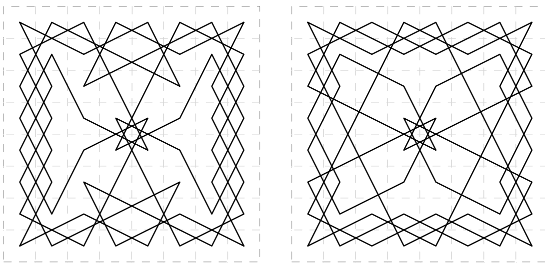
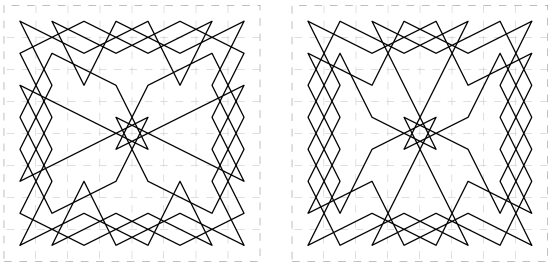


2b formulae 3-4

6283783710156



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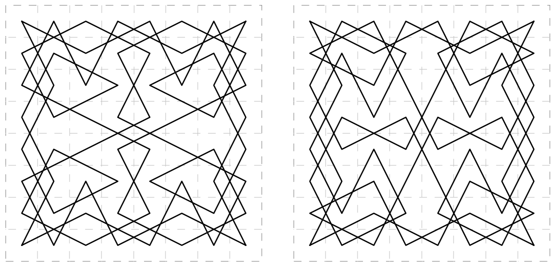

2c: 8 (2Do). 2c formulae 1-8.

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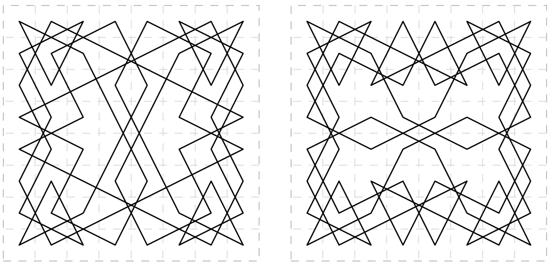

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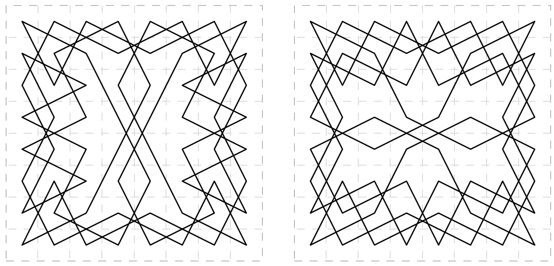

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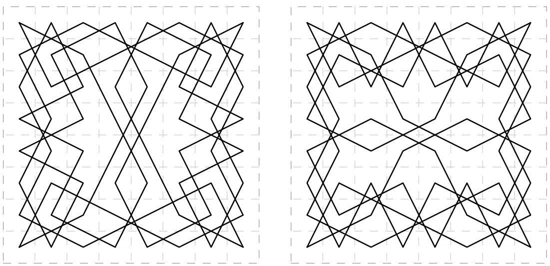

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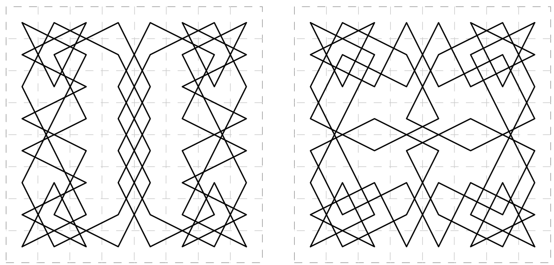

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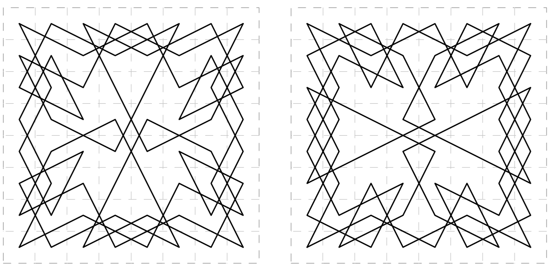

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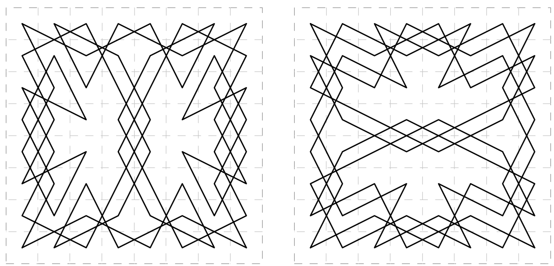

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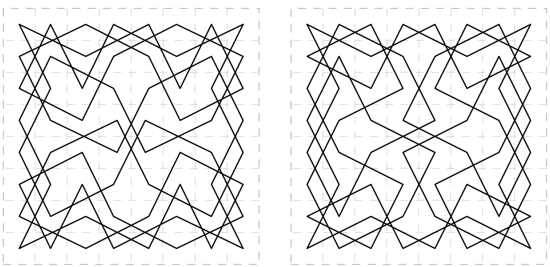

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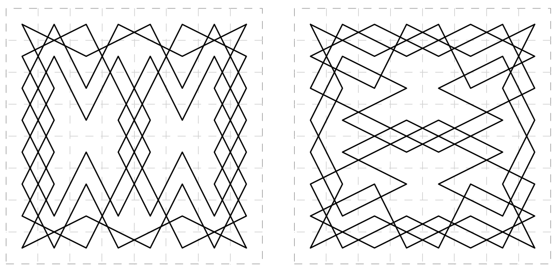

2d: 23 (1Oc, 1Oo).

2d formulae 1-10:

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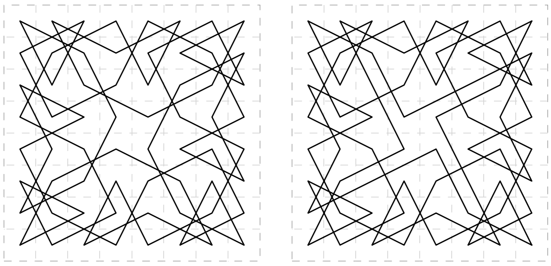

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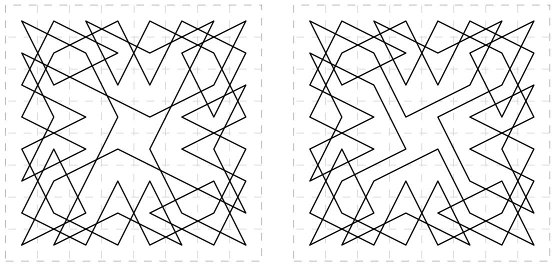

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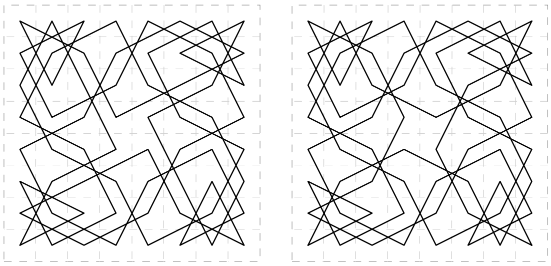

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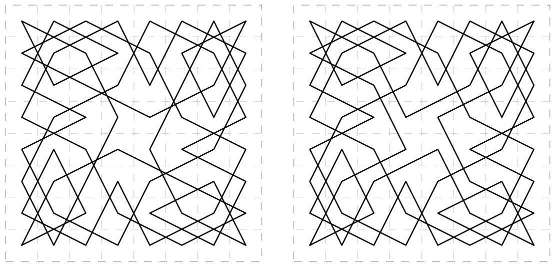

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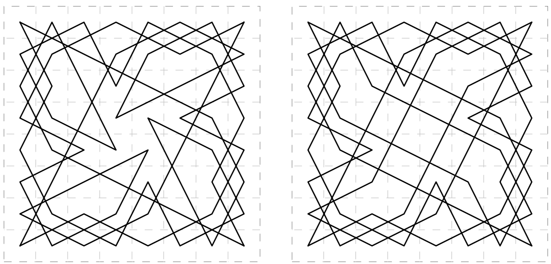

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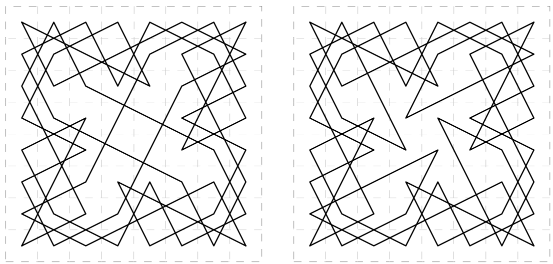

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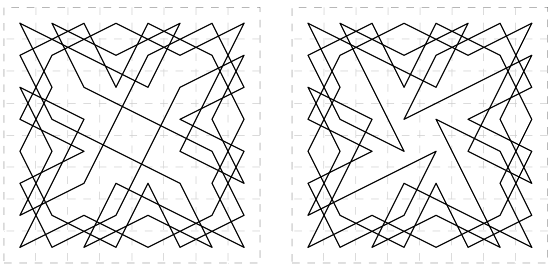

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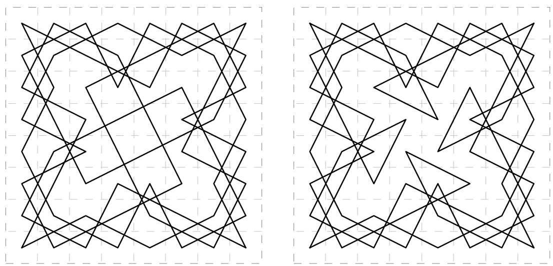

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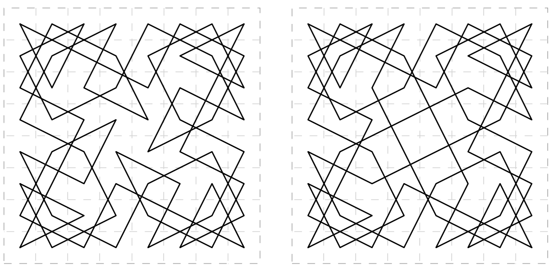

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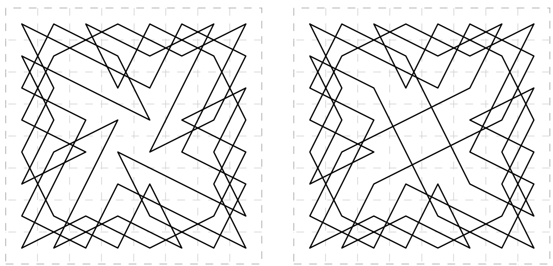

2d formulae 11-20:

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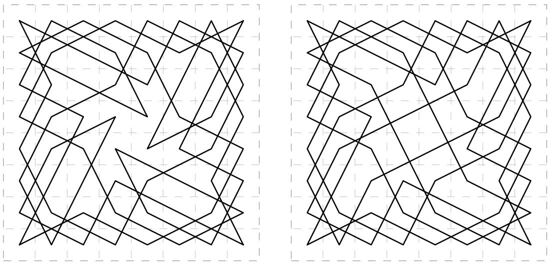

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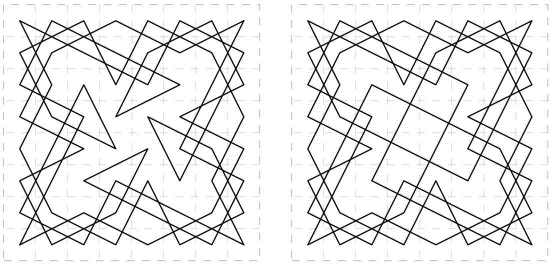

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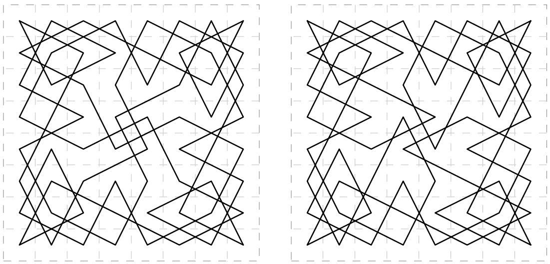

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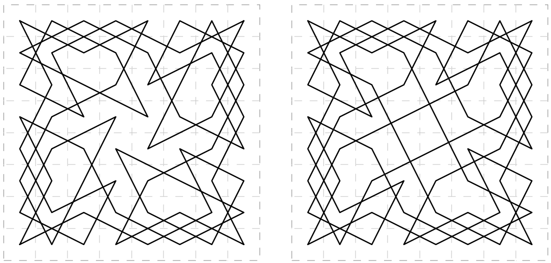

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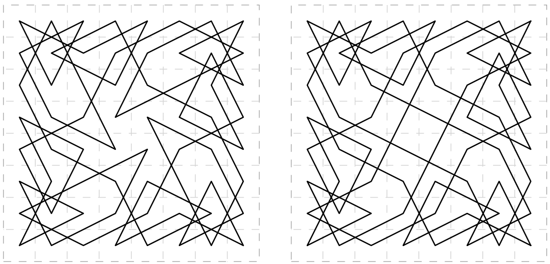

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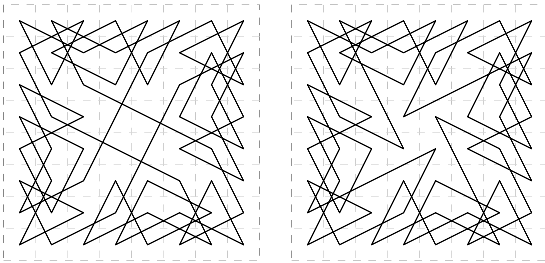

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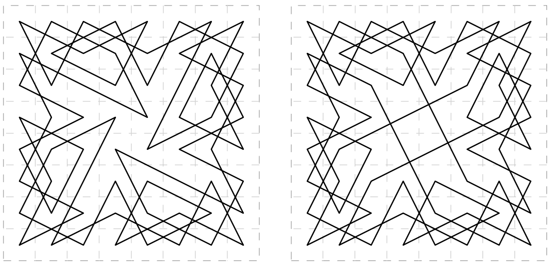

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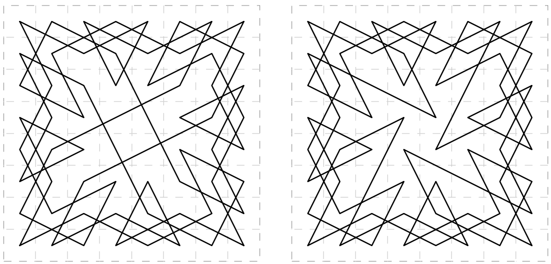

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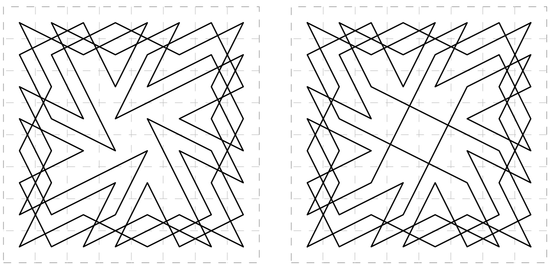

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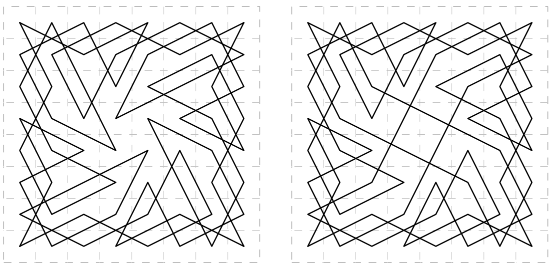

2d formulae 21-23:

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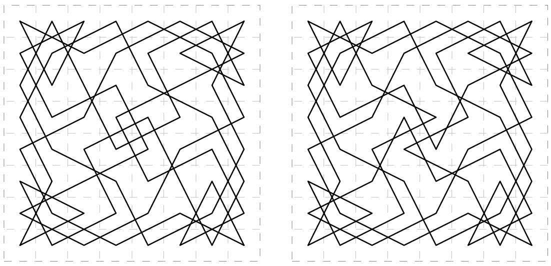

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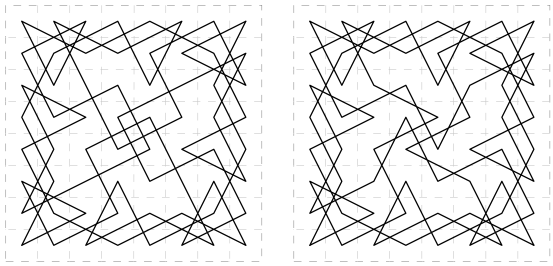
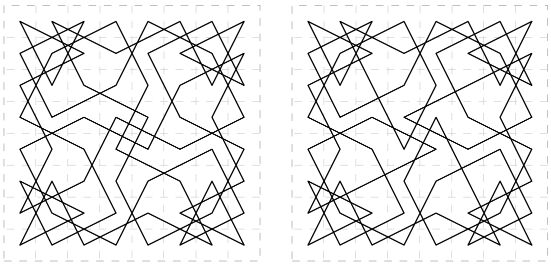
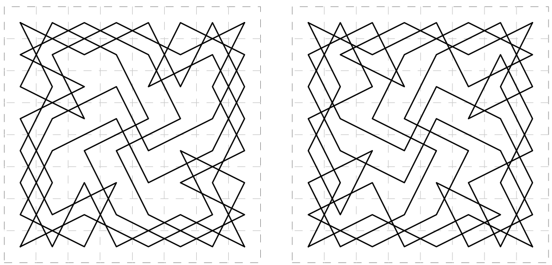


7382016561387



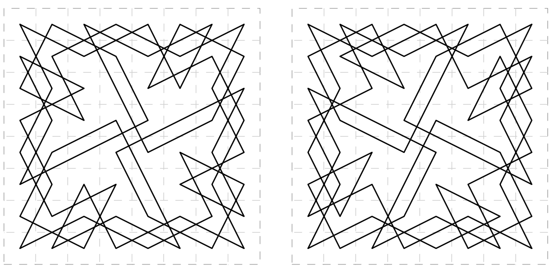
2e: 1 only (2Oc).

6287310387156

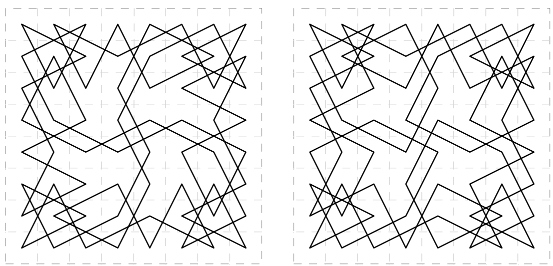


2f: 2 only (2Oo).

6283710387156



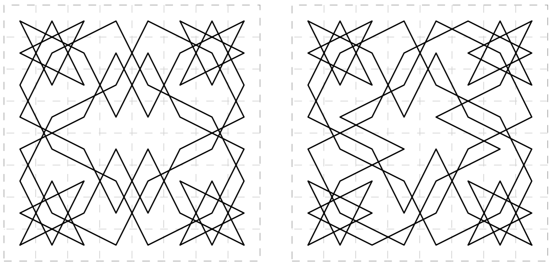
7382615610387



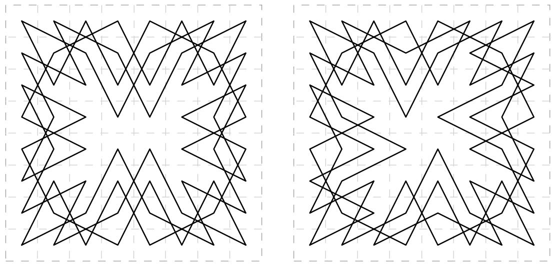
2g: 20 (1Dc, 1Oc).

2g formulae 1-10, cases of one path.

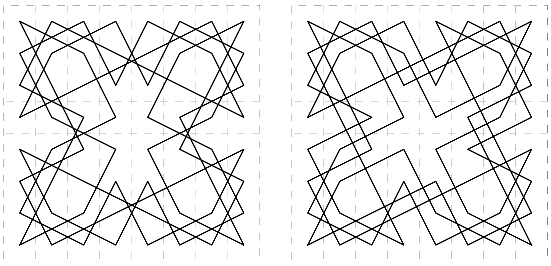
0316561378287



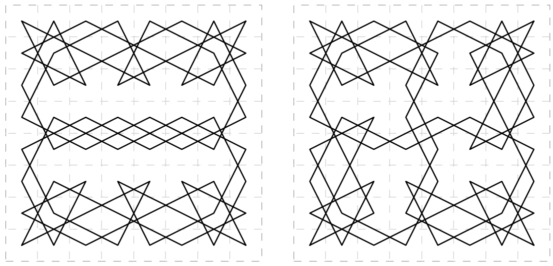
0382837165617



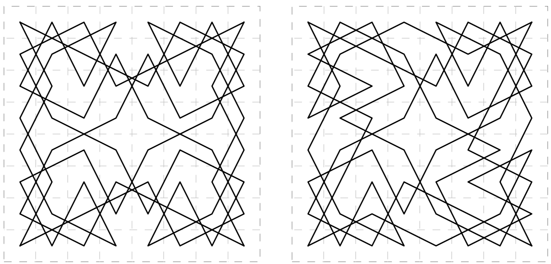
0387165623871



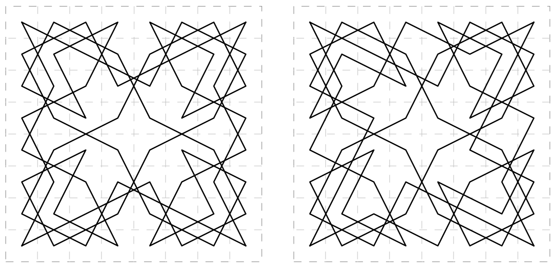
1656103782873



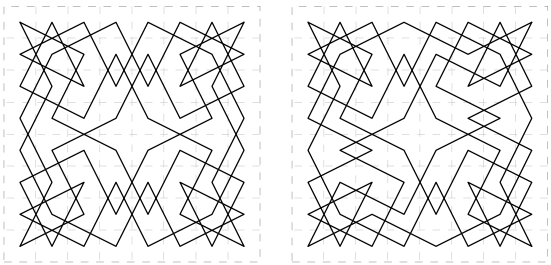
1783031782656



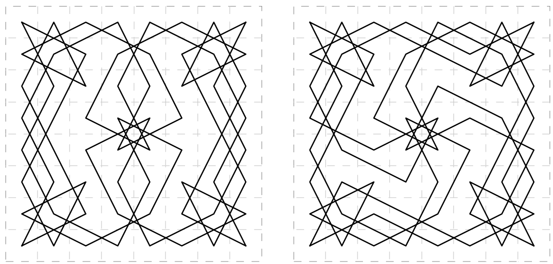
1783038715626



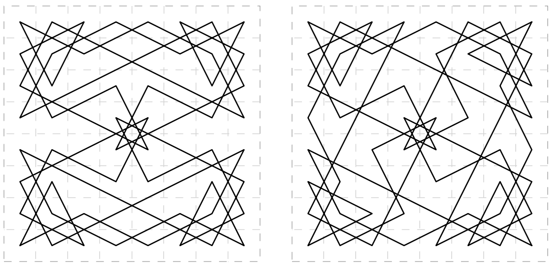
3031782871656



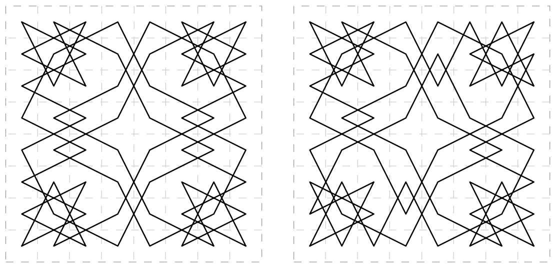
3782873101656



6562387101387

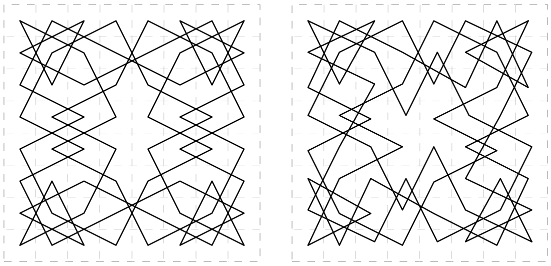


7165613038287

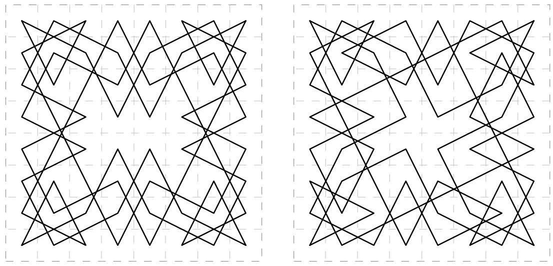


2g formulae 10-20, cases of two different paths:

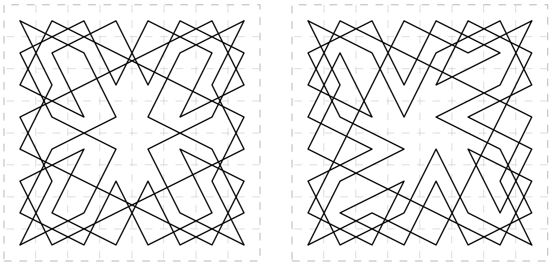
0316561783287



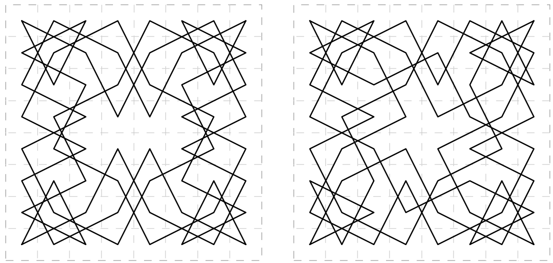
0387165162387



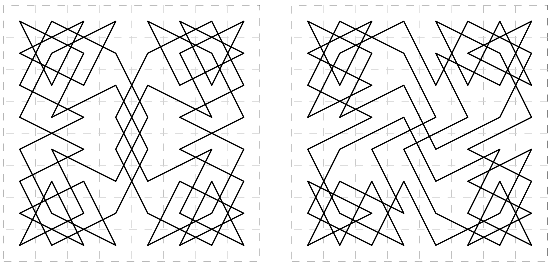
0387165178326



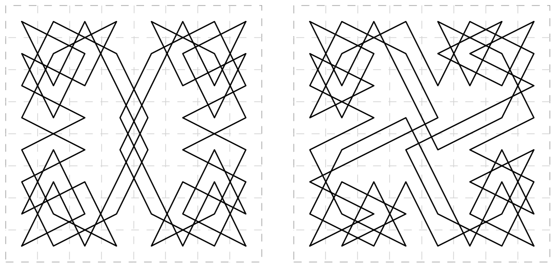
0387165613287



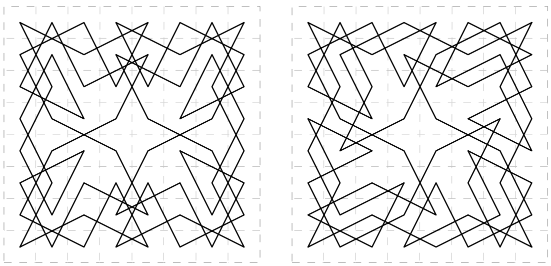
3103871656287



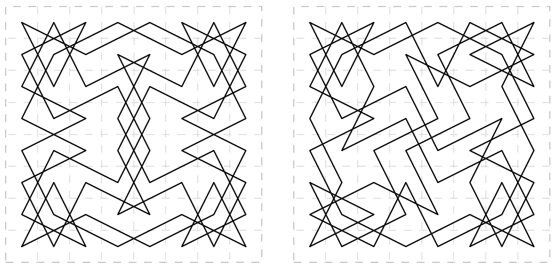
3826561783017



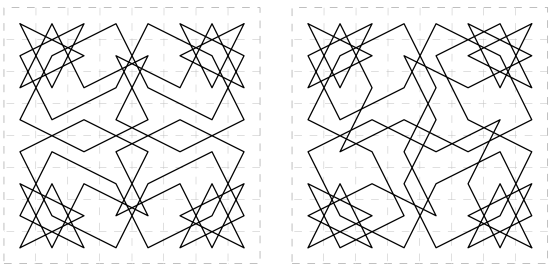
6178303287156



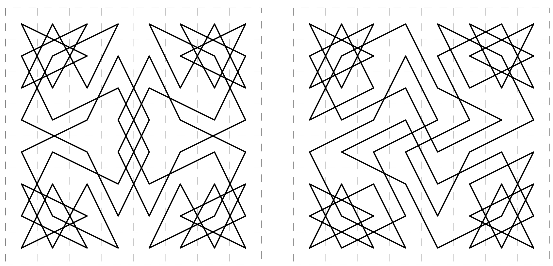
6561782301387



7823016561387



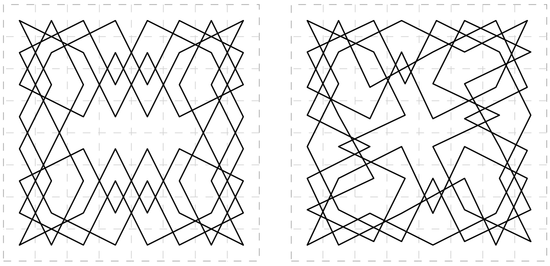
7826561301387



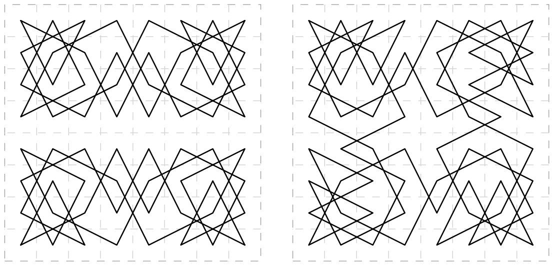
2h: 25 (1Dc, 1Oo).

2h formulae 1-10.

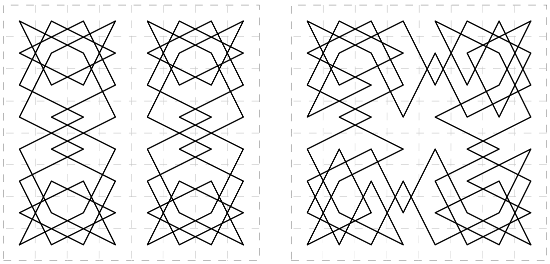
0317823871656



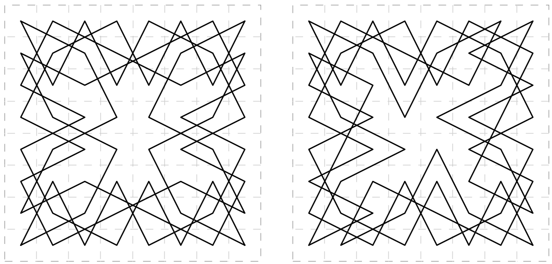
0317826561387



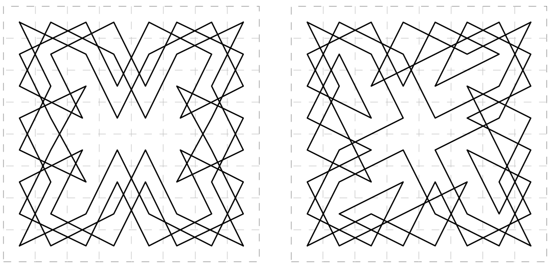
0317826561783



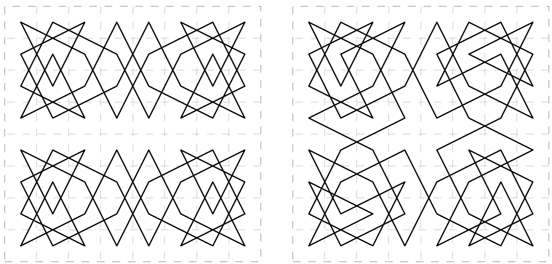
0382387165617



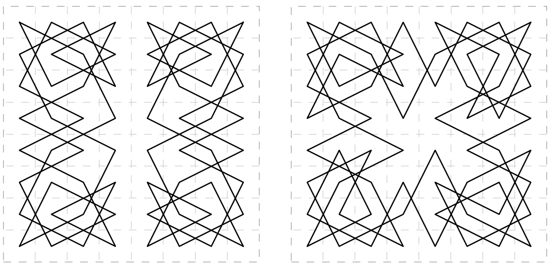
0387156178326



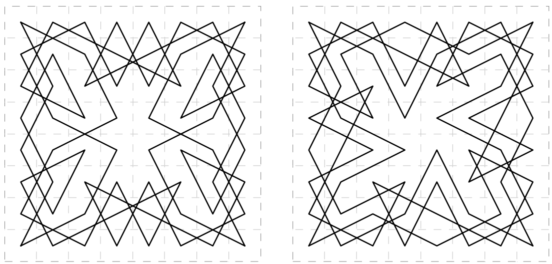
0387156261387



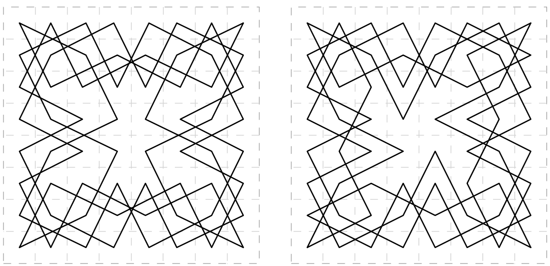
0387156261783



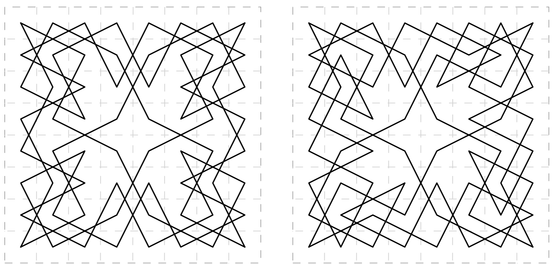
0387162387156



0387165617823

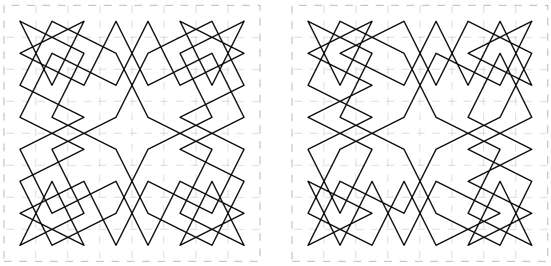


3038715617826

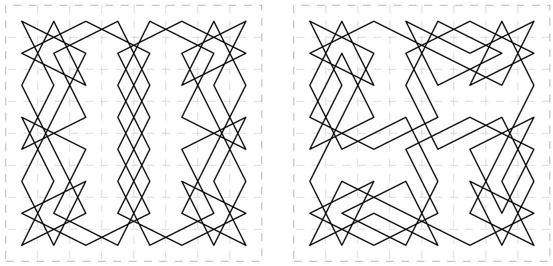


2h formulae 11-20:

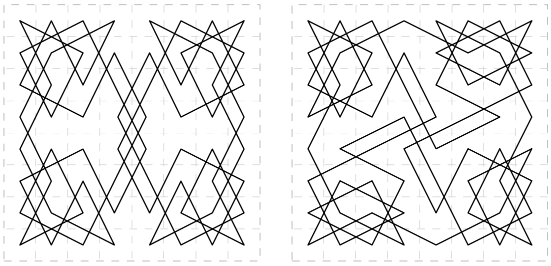
3038716516287



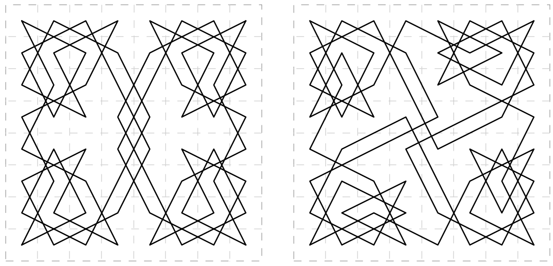
3782873016516



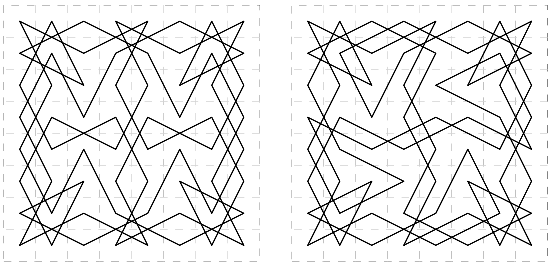
3871031782656



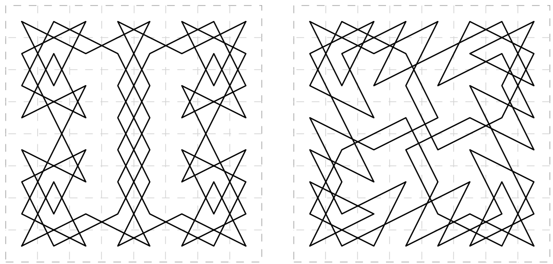
3871038715626



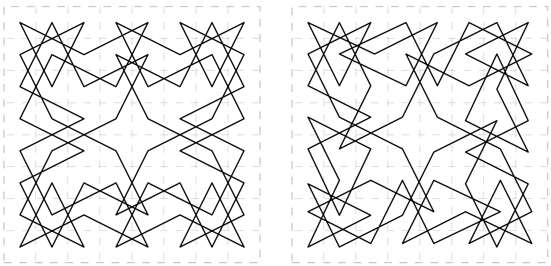
6103782873156



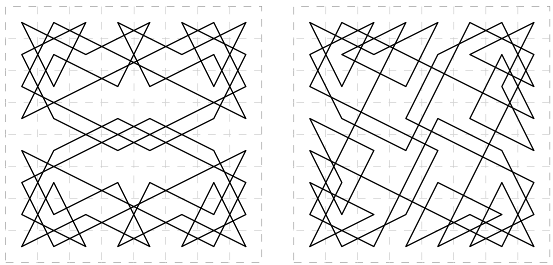
6103871562387



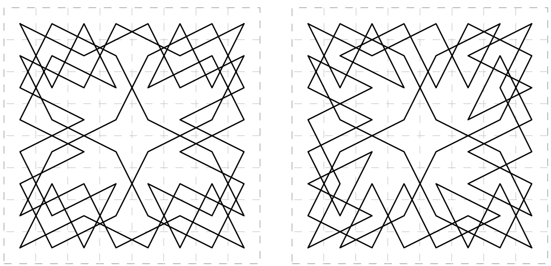
6156178230387



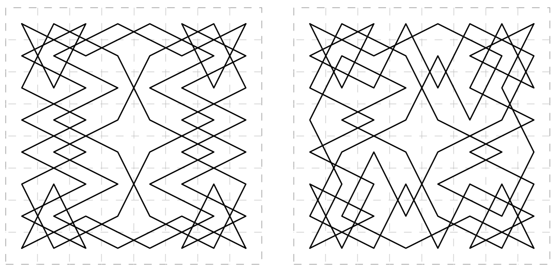
6156238710387



6283038716517

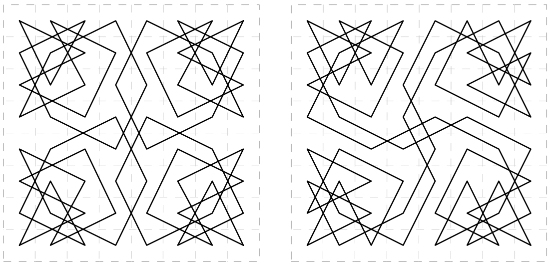


6513038716287

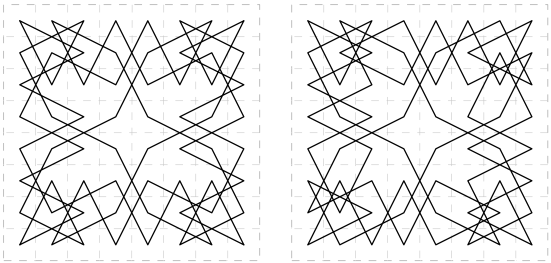


2h formulae 21-25:

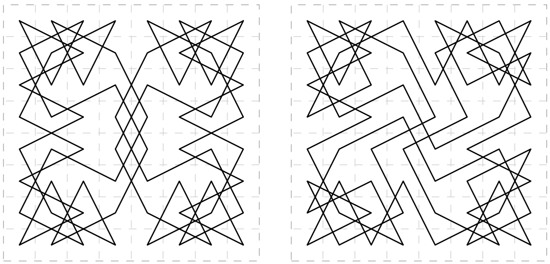
7138265610387



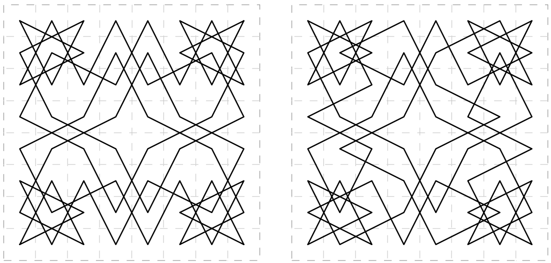
7165162830387



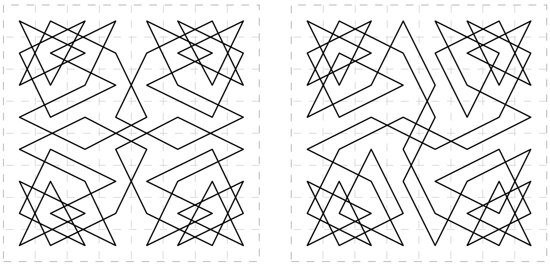
7165628301387



7826156130387



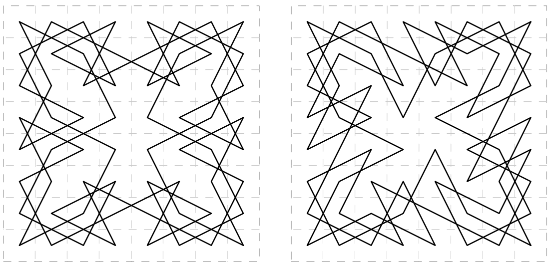
7830162651387



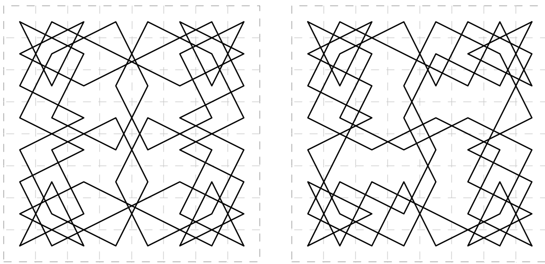
2i: 11 (1Do, 1Oc).

2i formulae 1-4:

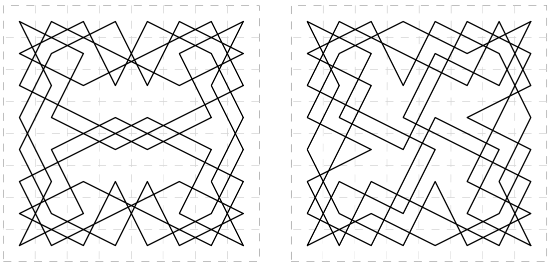
0387156238716



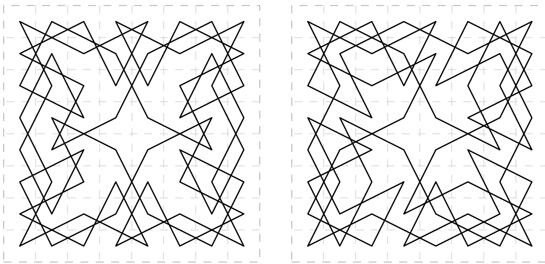
3016561783287



3017832871656

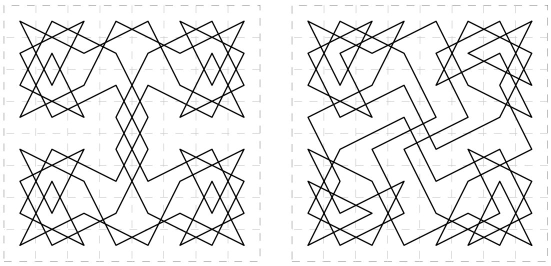


6178230387156

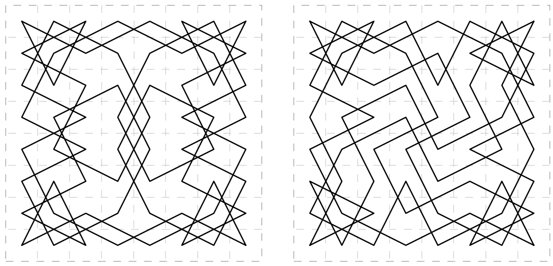


2i formulae 5-11:

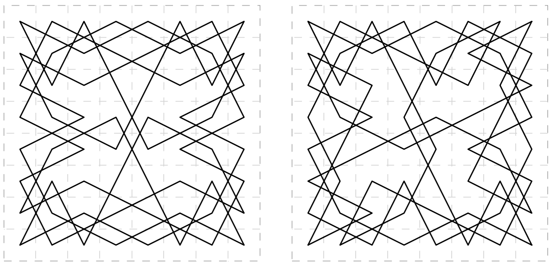
6265178301387



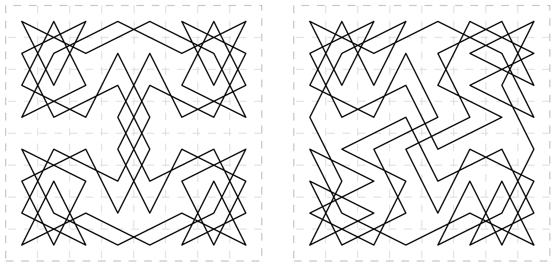
6561783013287



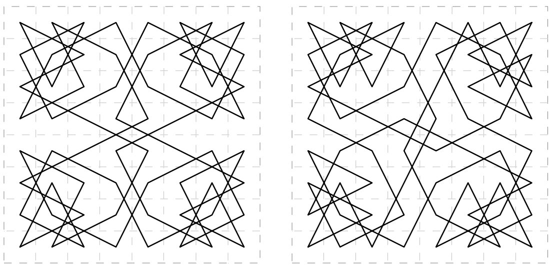
6561783283017



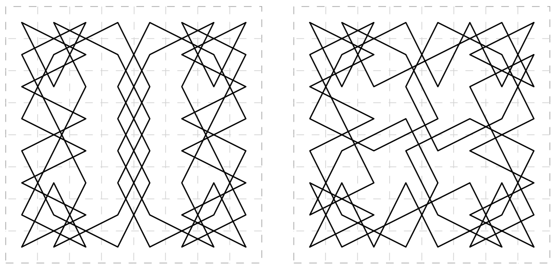
6562871301387



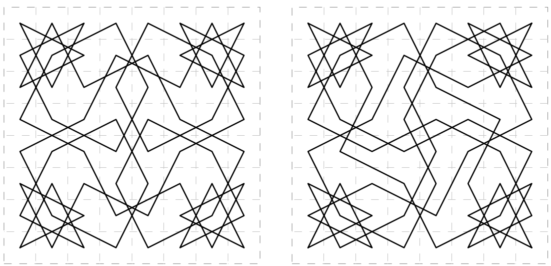
7103826561387



7165610382387



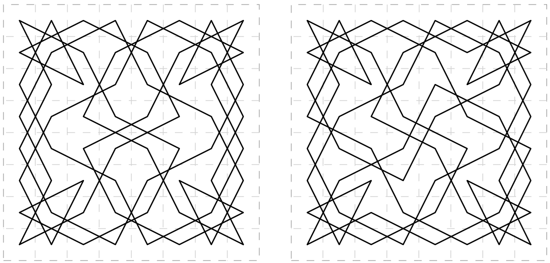
7823165610387



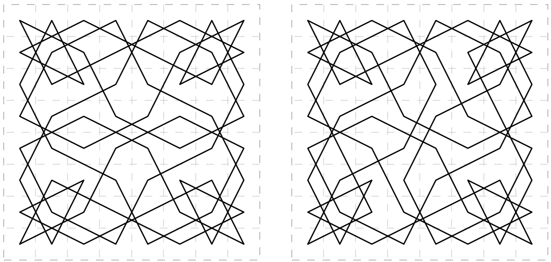
2j: 30 (15 cases of one path and 15, marked *, of two different paths) (1Do, 1Oo).

2j formulae 1-10:

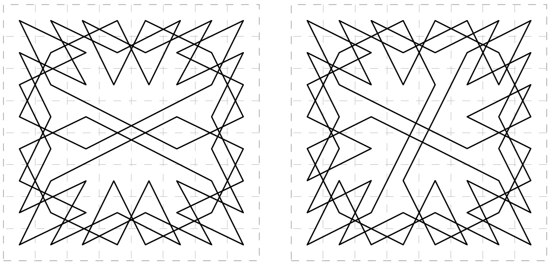
0137828731656



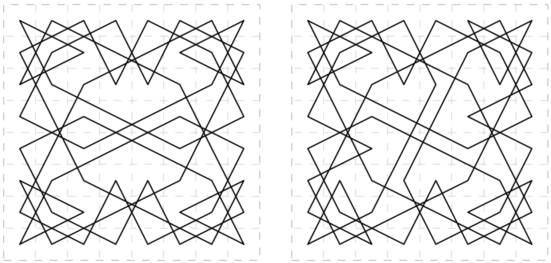
0165613782873



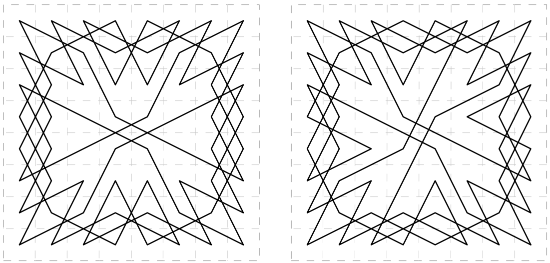
0165617382837



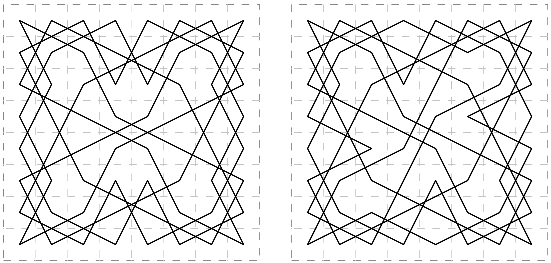
0165617832387



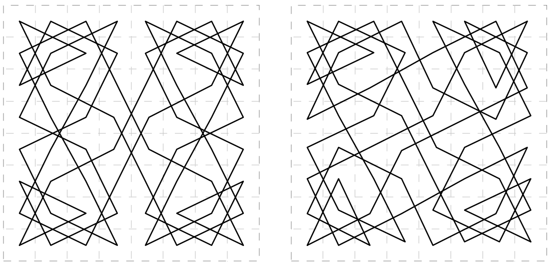
0173828371656



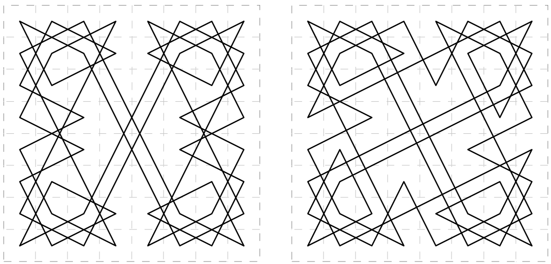
0178323871656



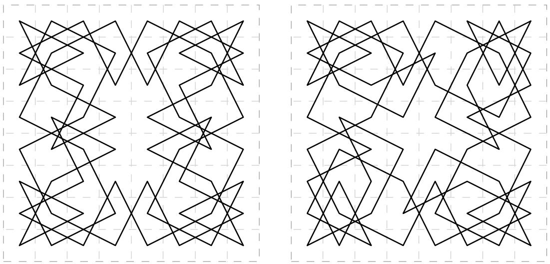
0178326561387



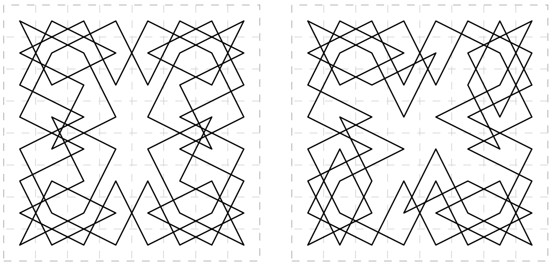
0178326561783



0328716561387 *

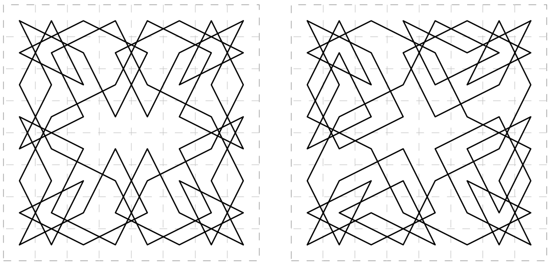


0328716561783 *



2j formulae 11-20:

0378287316516 *



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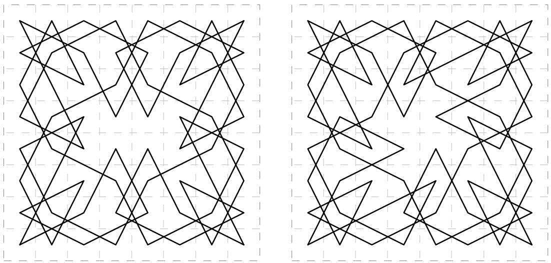

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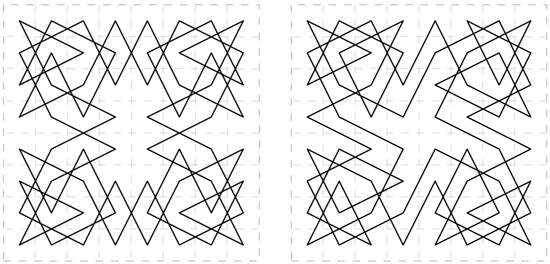

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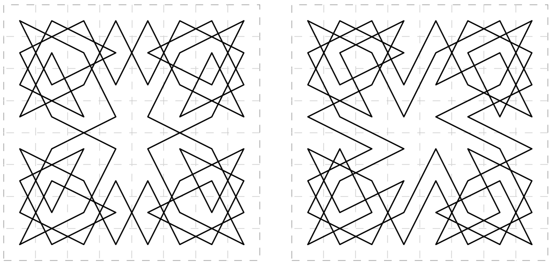

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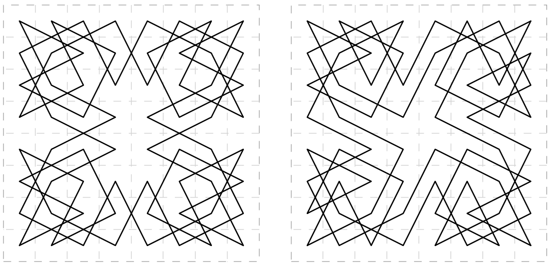

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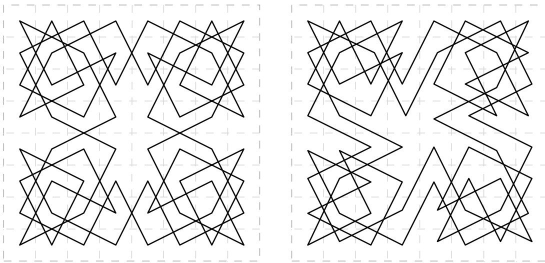

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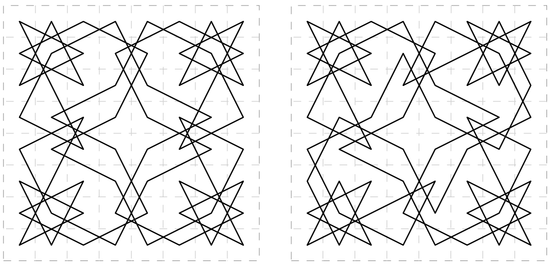

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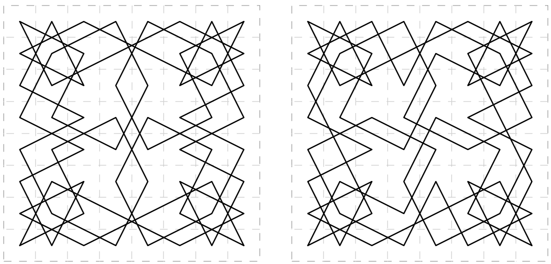

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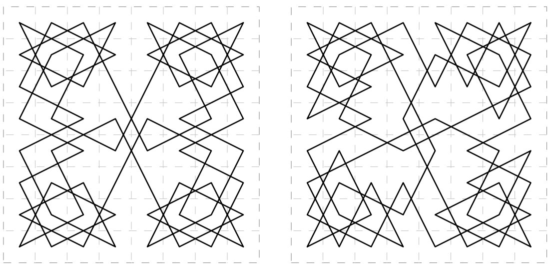

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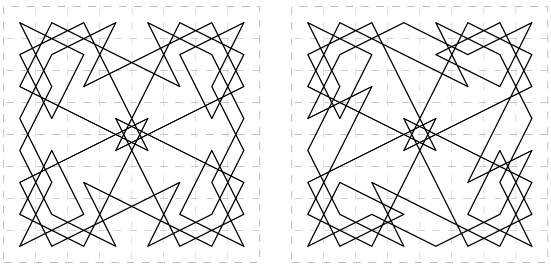

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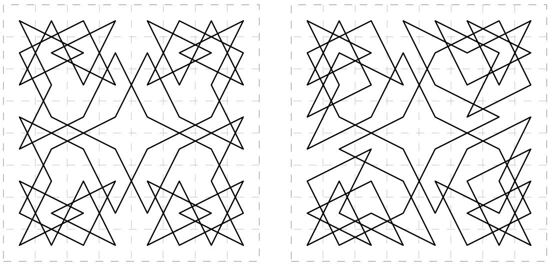

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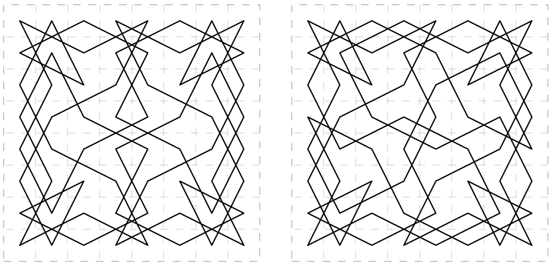

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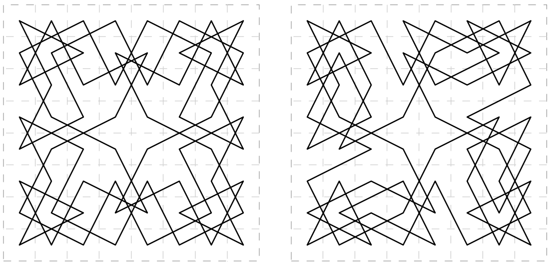

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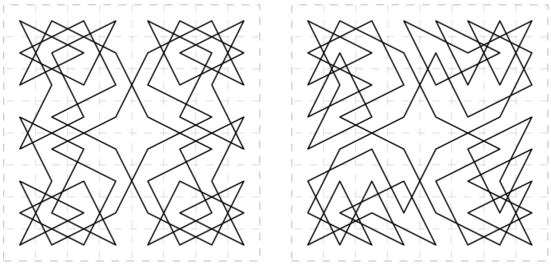

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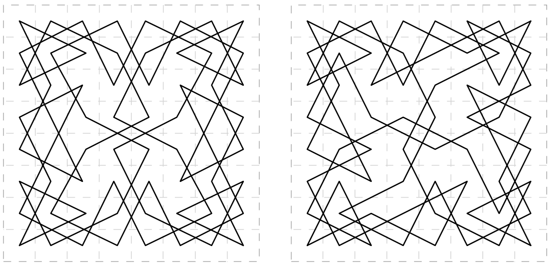

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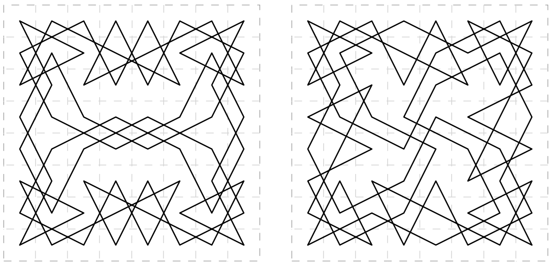

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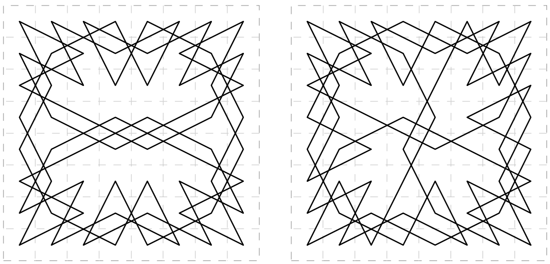

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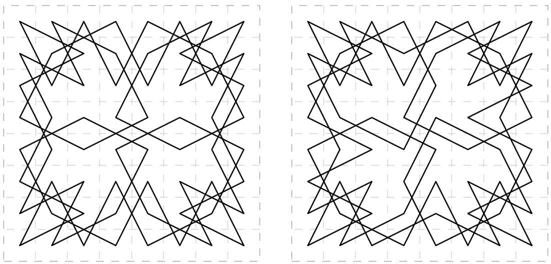

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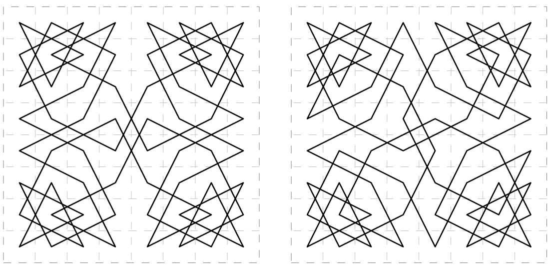

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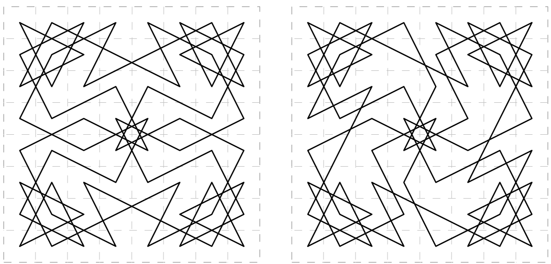

. KNIGHT'S TOUR NOTES

1-fold: There are 302 formulae that yield just one pseudotour. They form four classes 1a 96 Dc, 1b 100 Do, 1c 58 Oc, 1d 48 Oo.

1a: 96 (Dc).
1a formulae 1-20:

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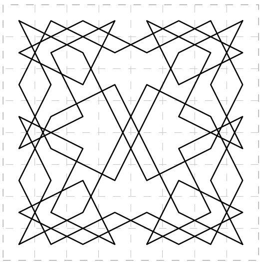

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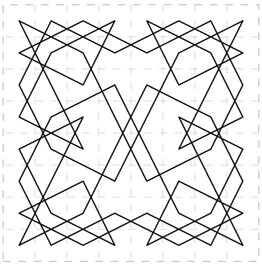

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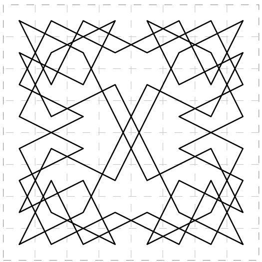

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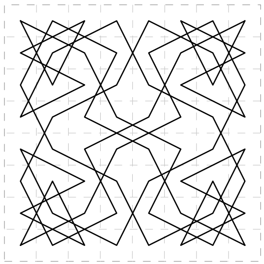

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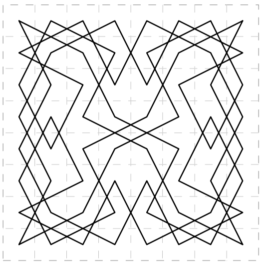

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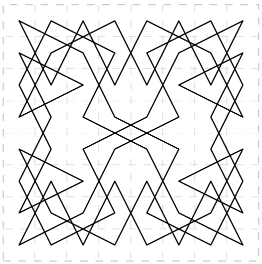

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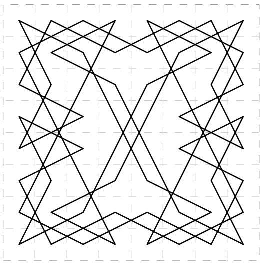

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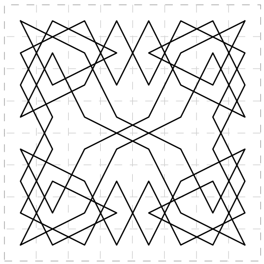

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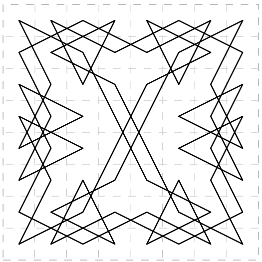

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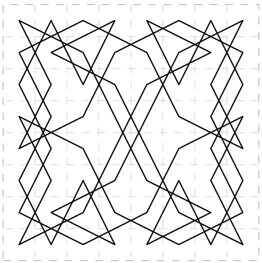

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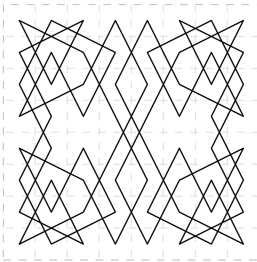

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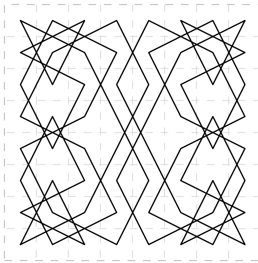

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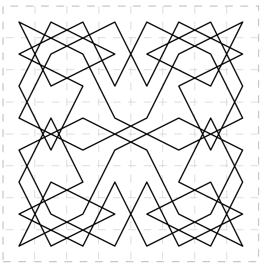

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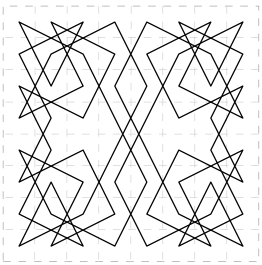

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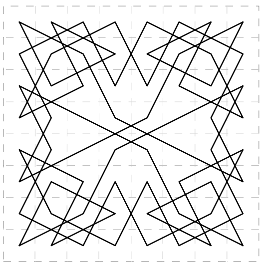

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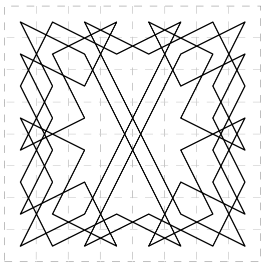

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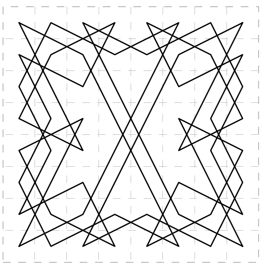

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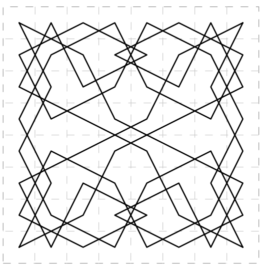

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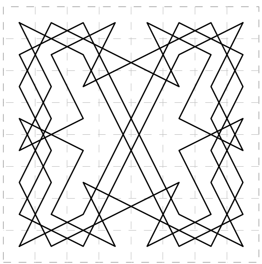

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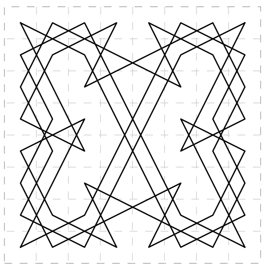

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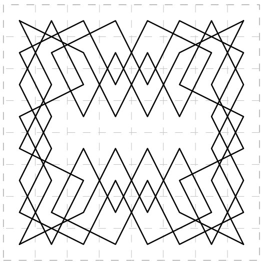

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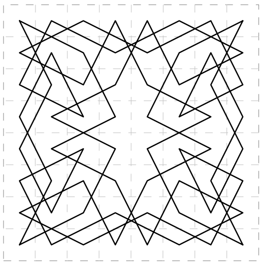

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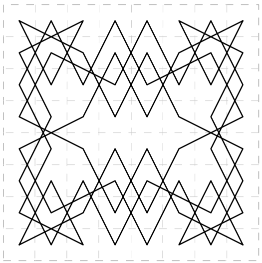

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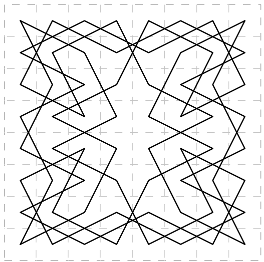

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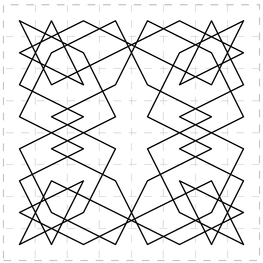

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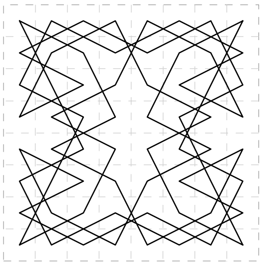

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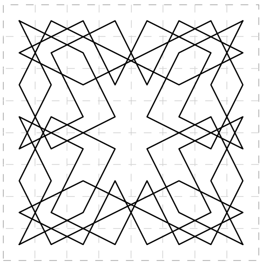

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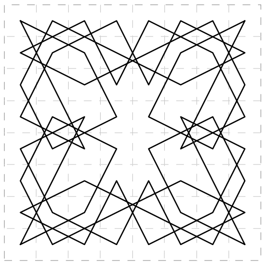

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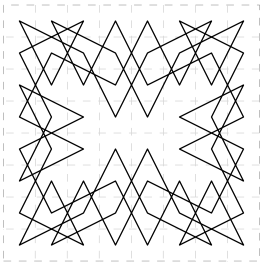

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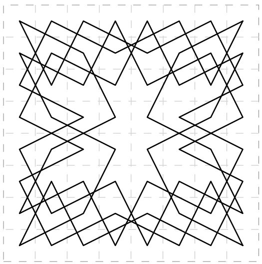

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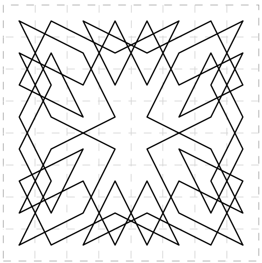

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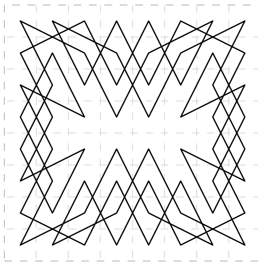

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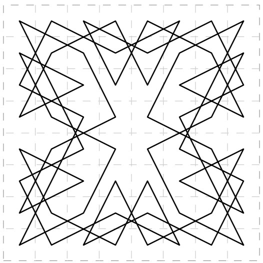

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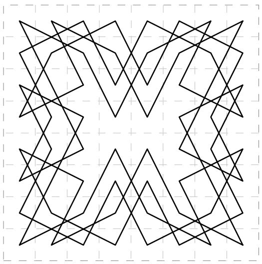

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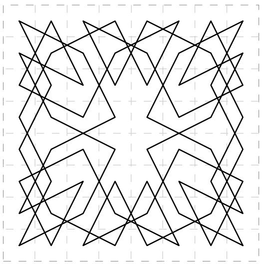

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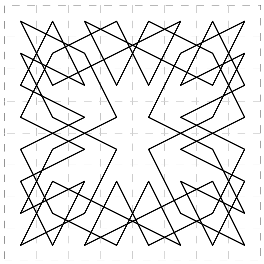

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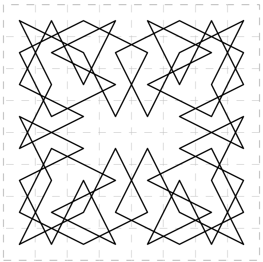

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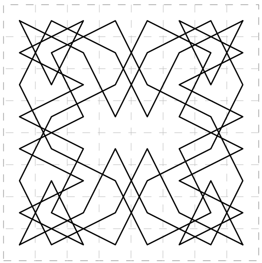

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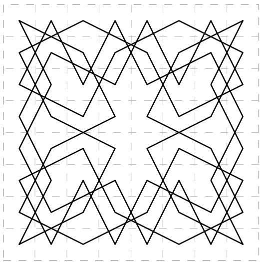

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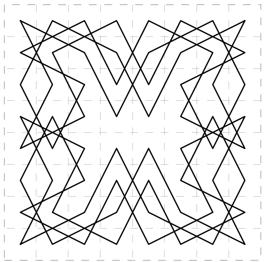

1a formulae 41-60:

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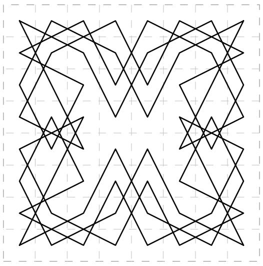

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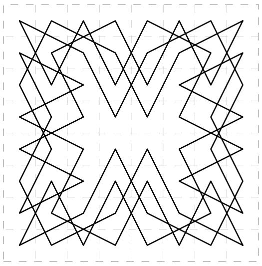

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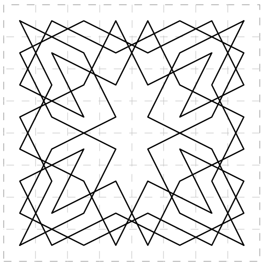

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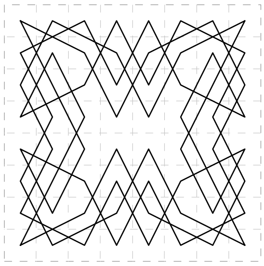

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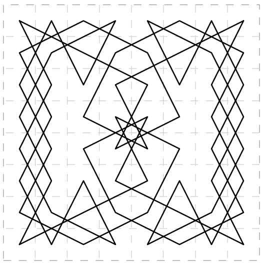

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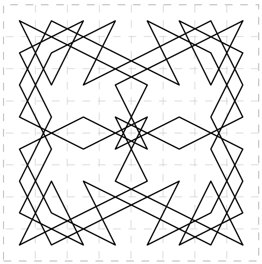

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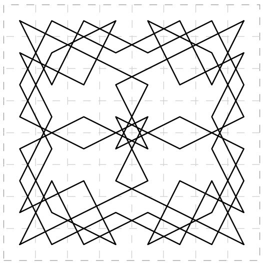

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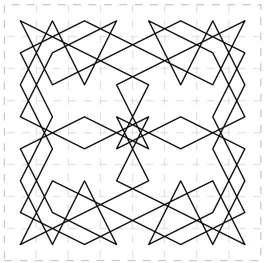

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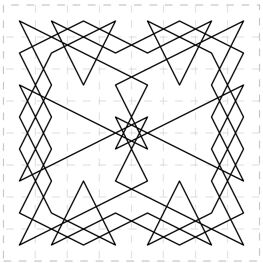

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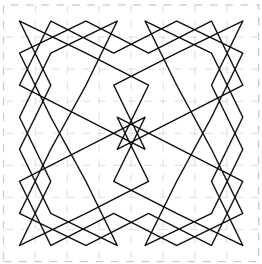

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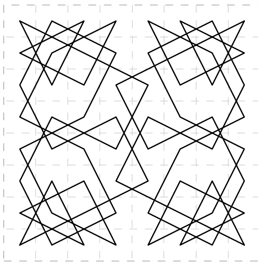

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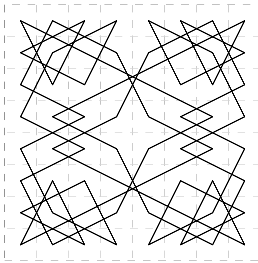

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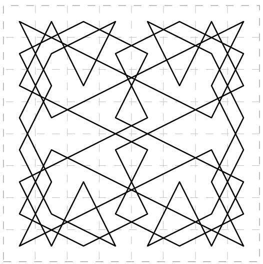

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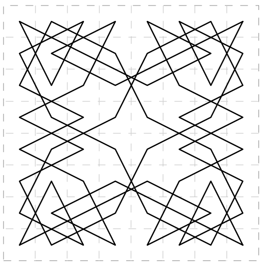

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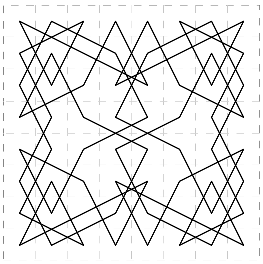

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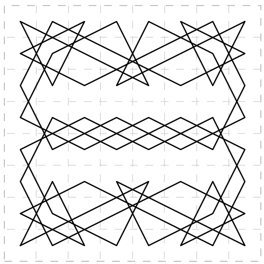

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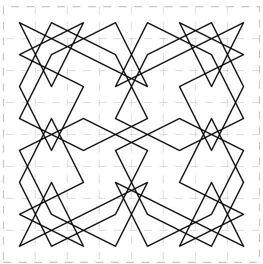

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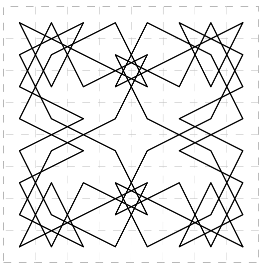

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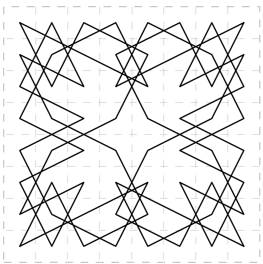

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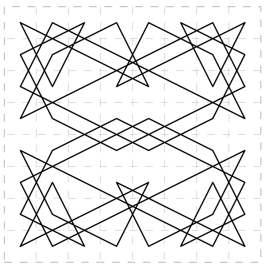

. KNIGHT'S TOUR NOTES

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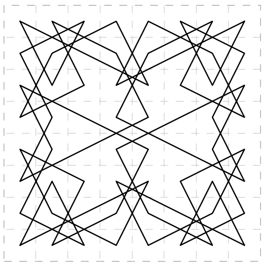

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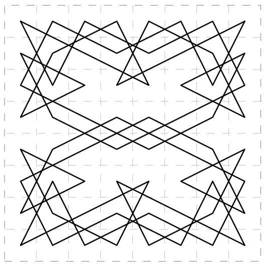

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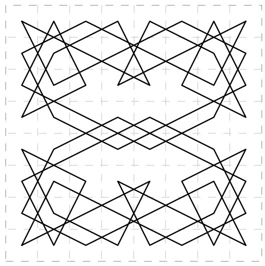

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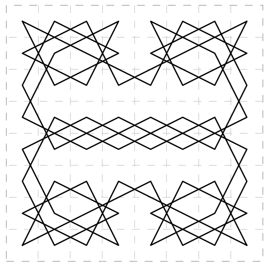

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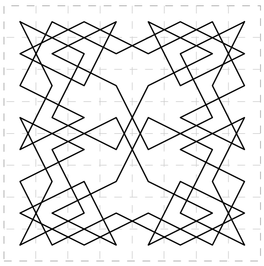

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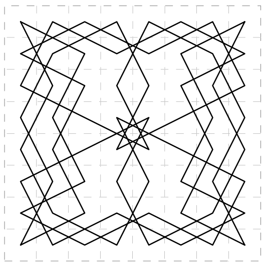

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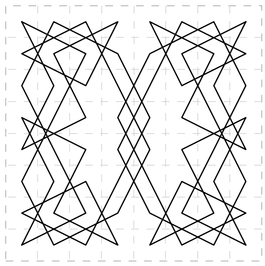

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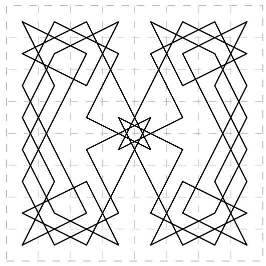

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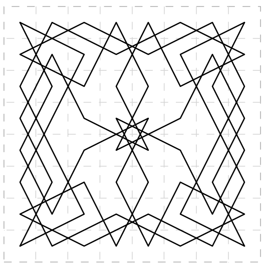

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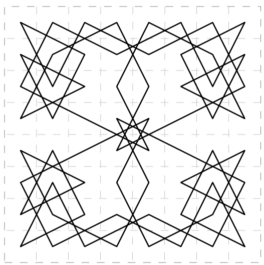

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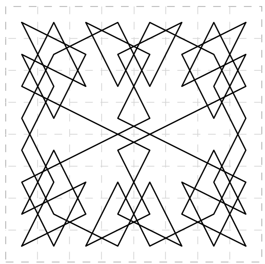

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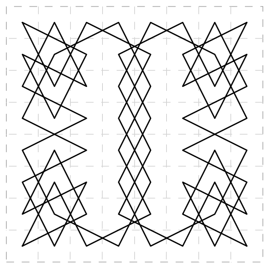

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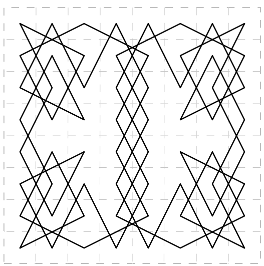

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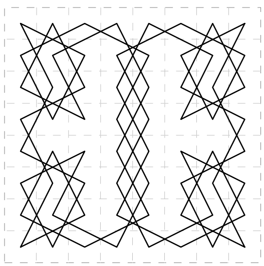

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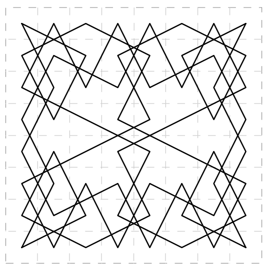

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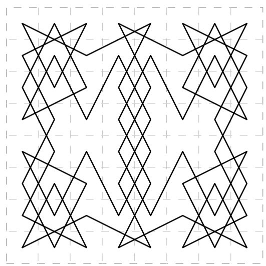

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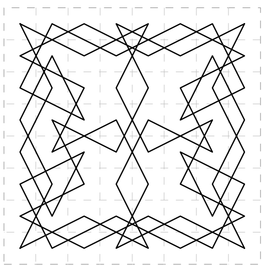

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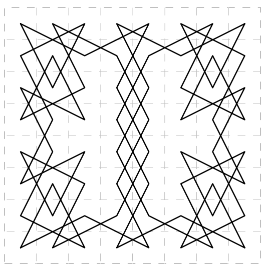

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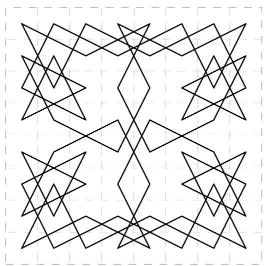

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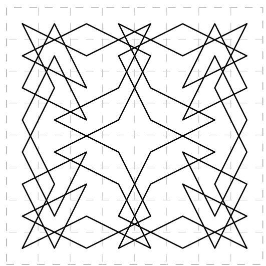

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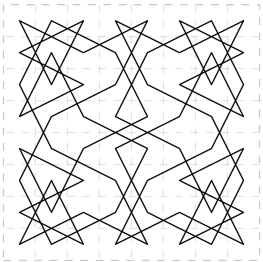

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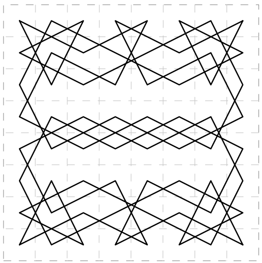

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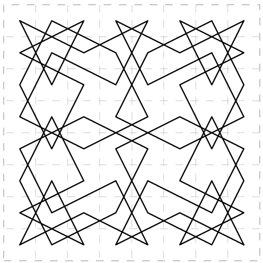

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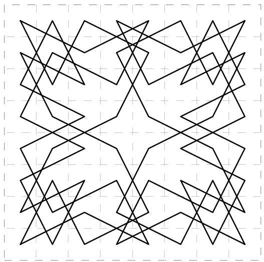

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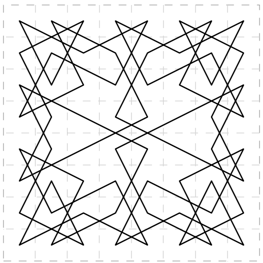

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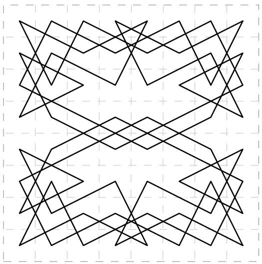

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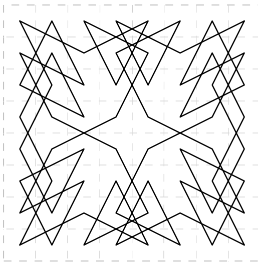

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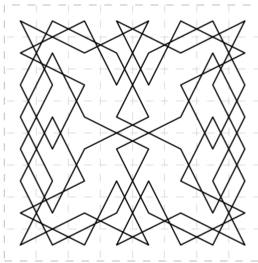

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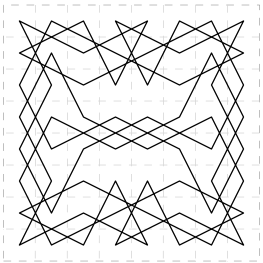

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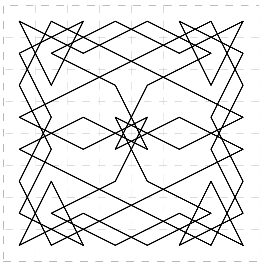

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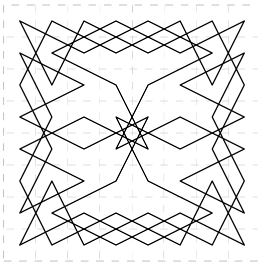

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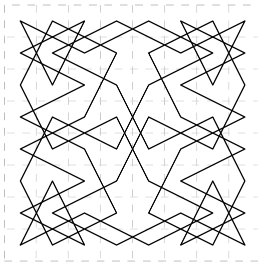

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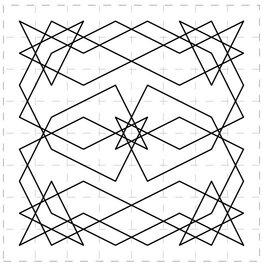

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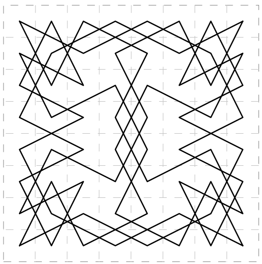

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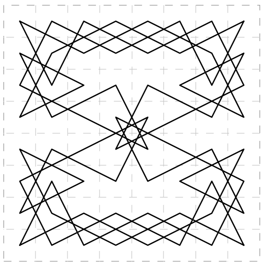

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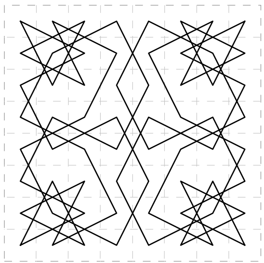

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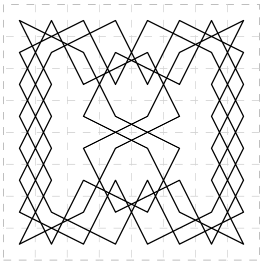

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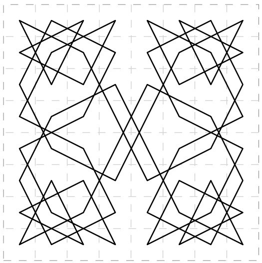

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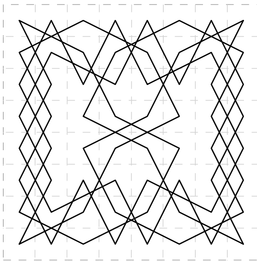

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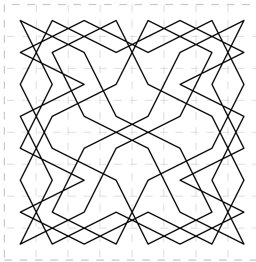

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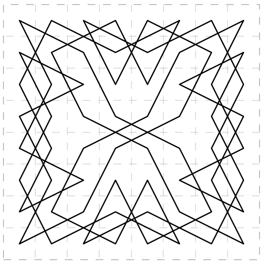

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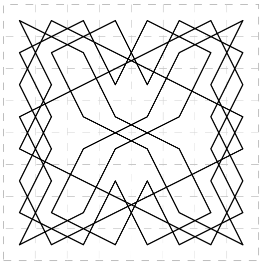

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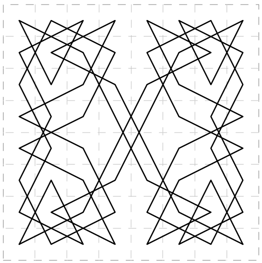

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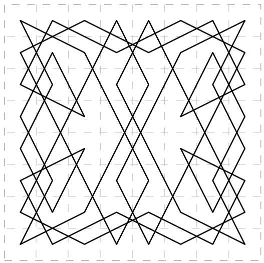

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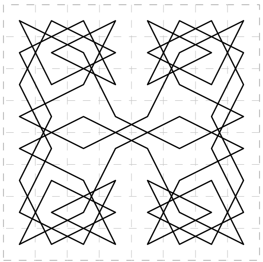

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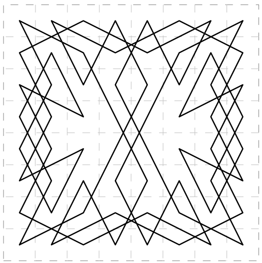

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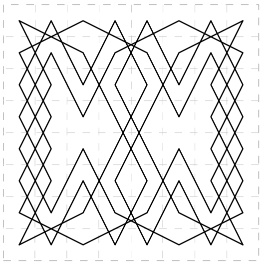

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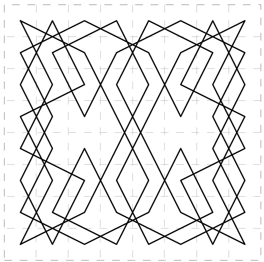

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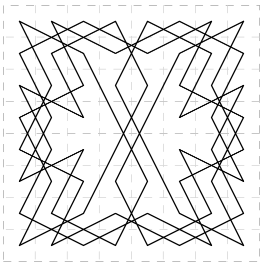

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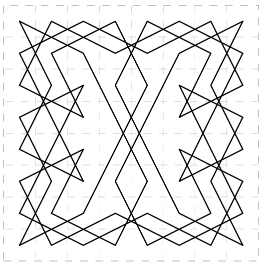

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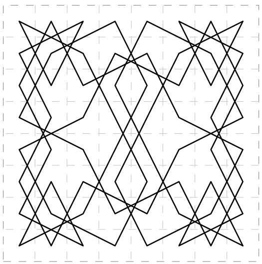

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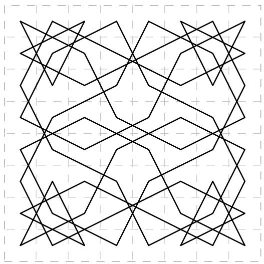

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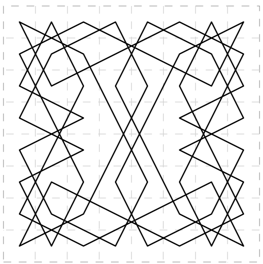

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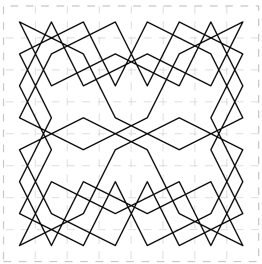

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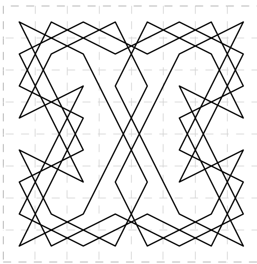

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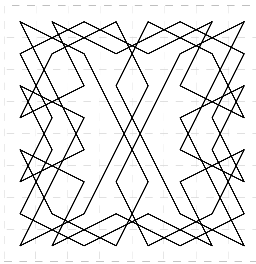

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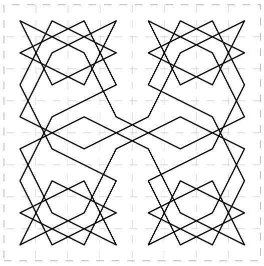

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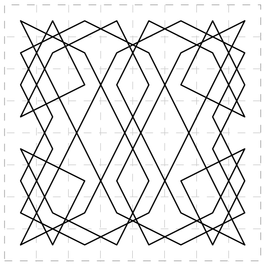

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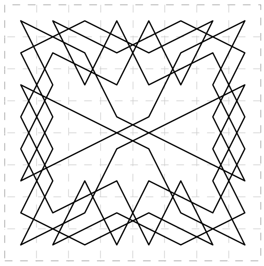

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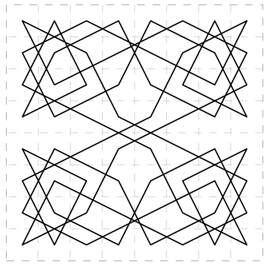

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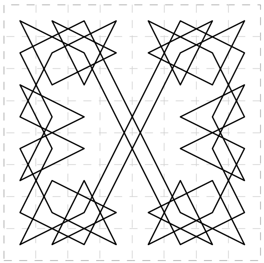

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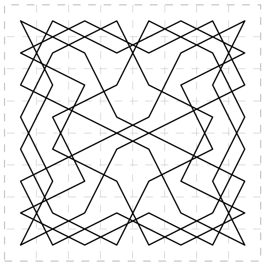

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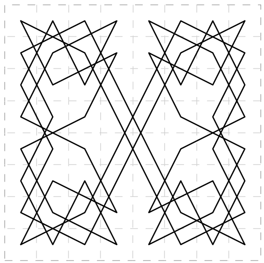

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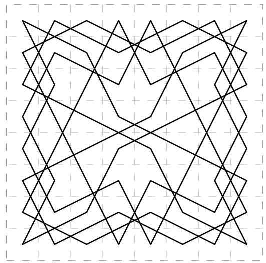

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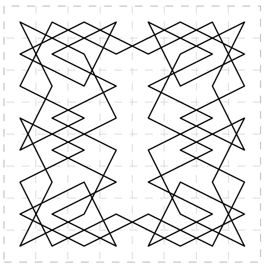

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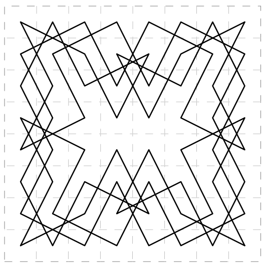

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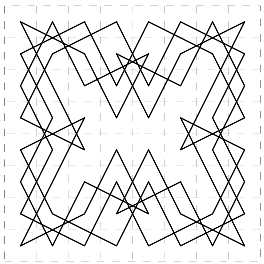

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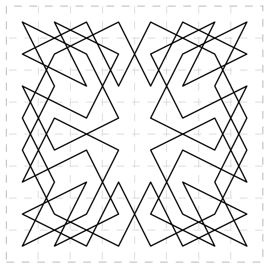

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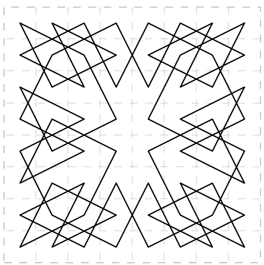

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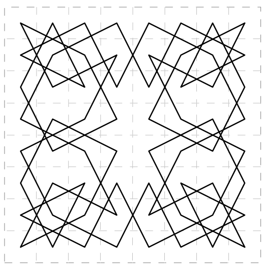

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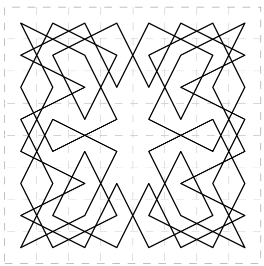

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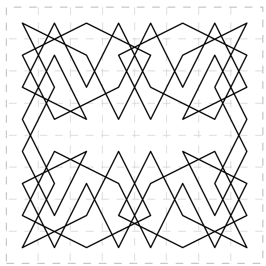

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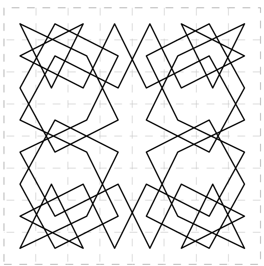

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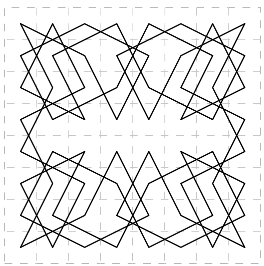

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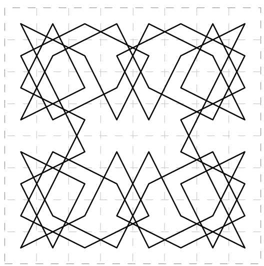

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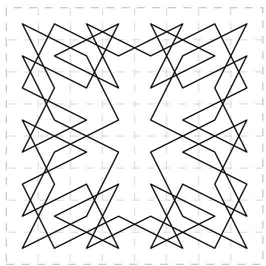

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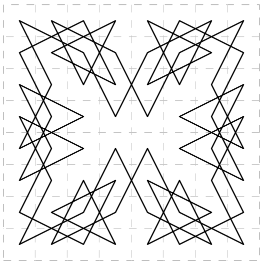

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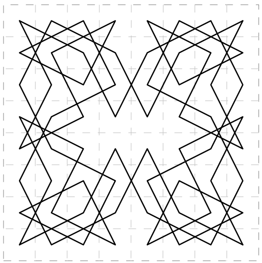

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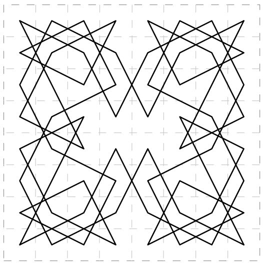

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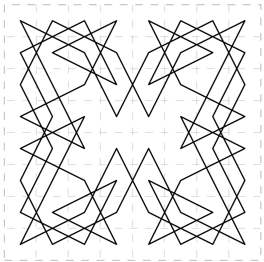

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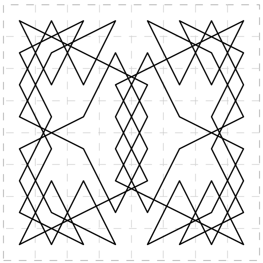

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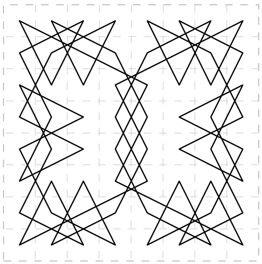

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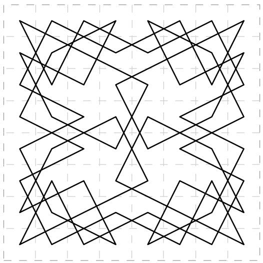

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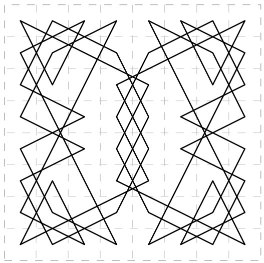

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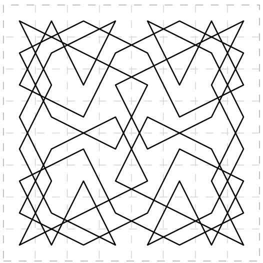

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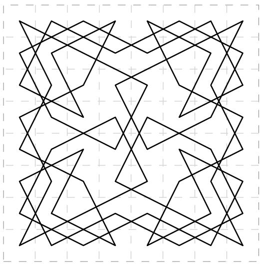

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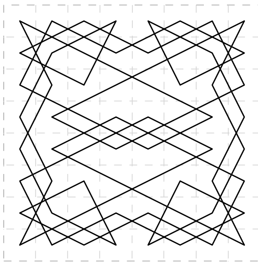

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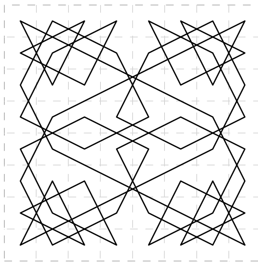

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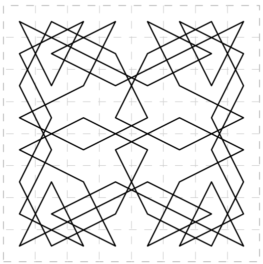

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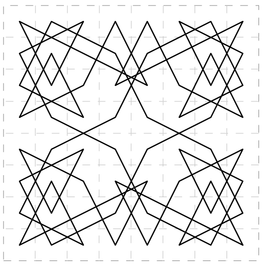

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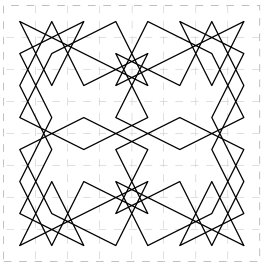

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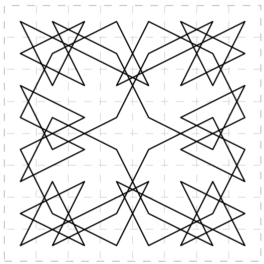

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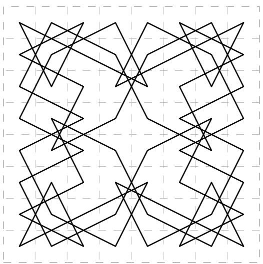

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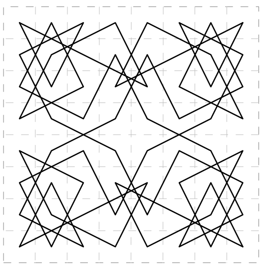

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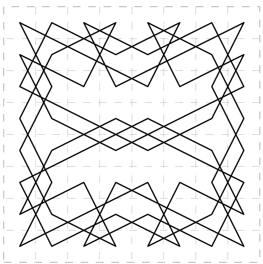

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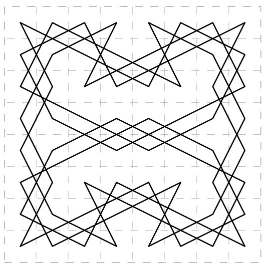

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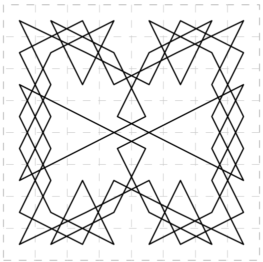

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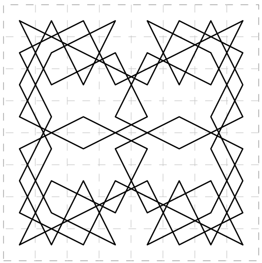

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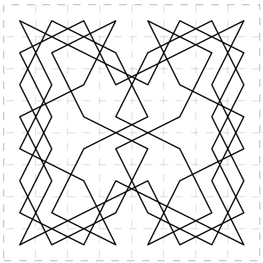

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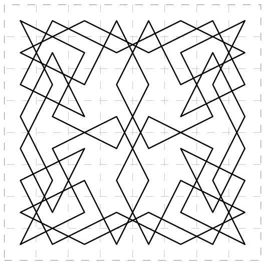

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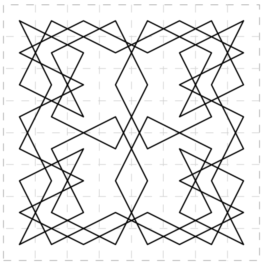

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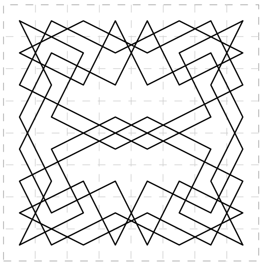

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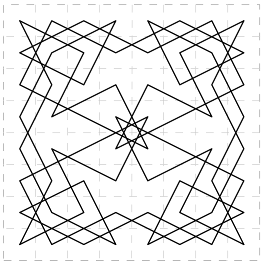

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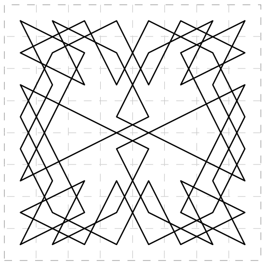

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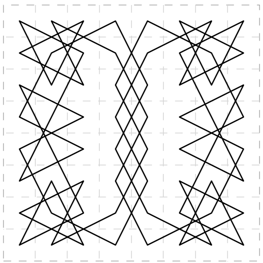

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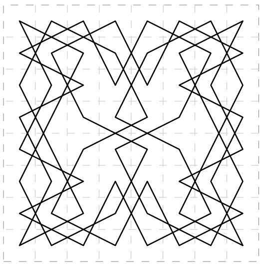

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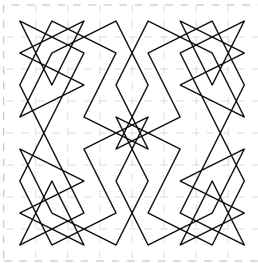

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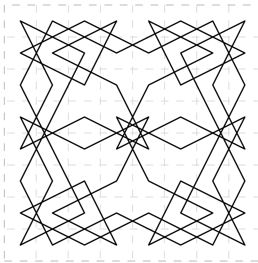

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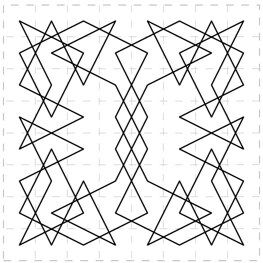

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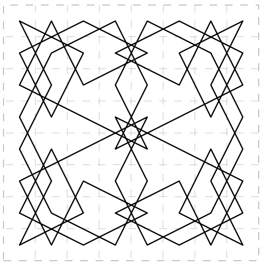

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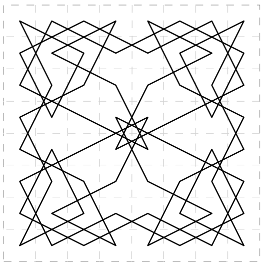

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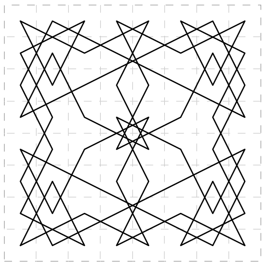

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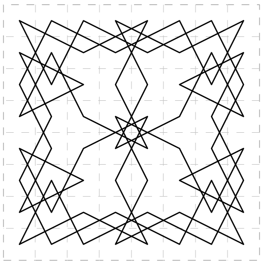

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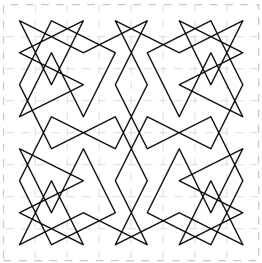

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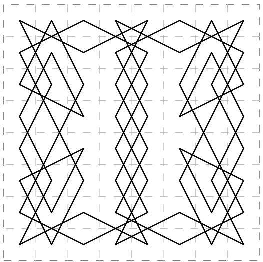

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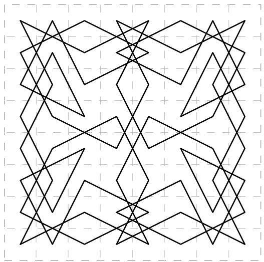

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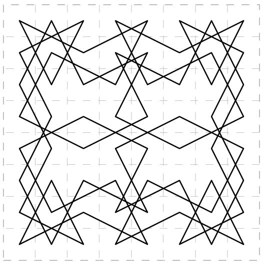

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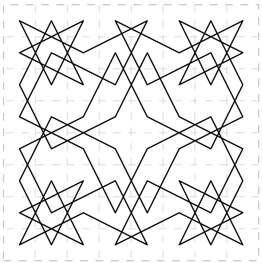

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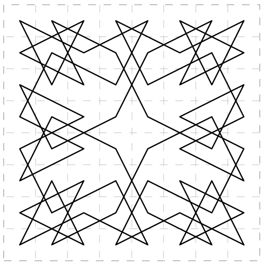

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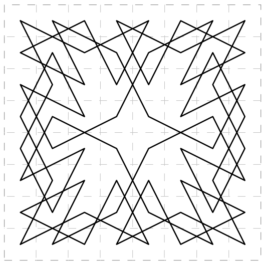

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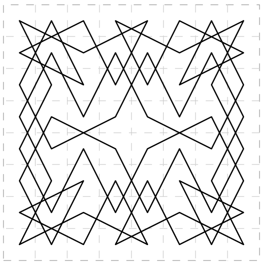

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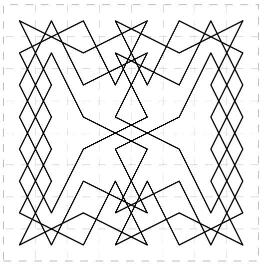

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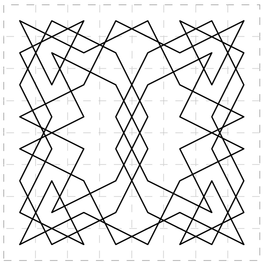

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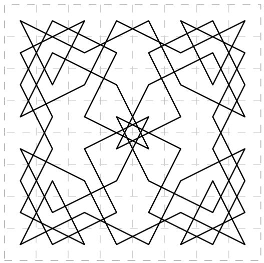

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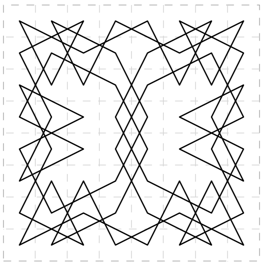

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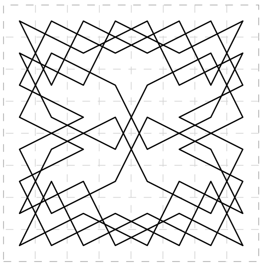

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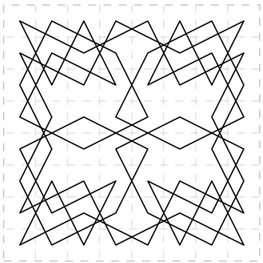

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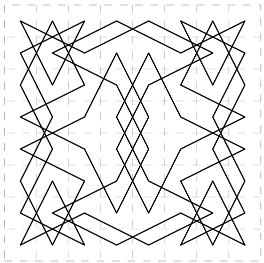

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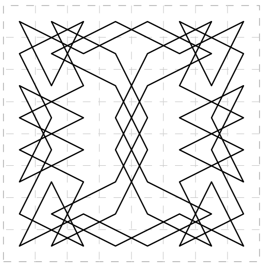

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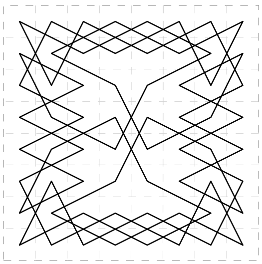

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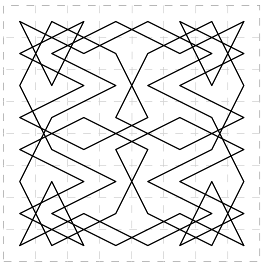

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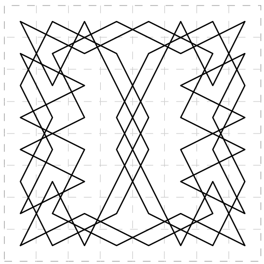

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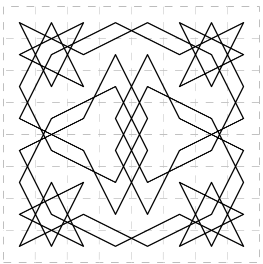

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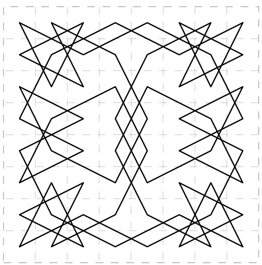

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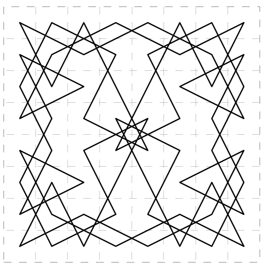

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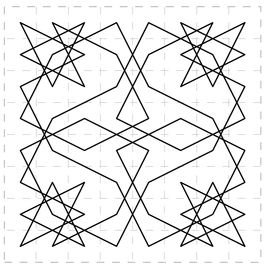

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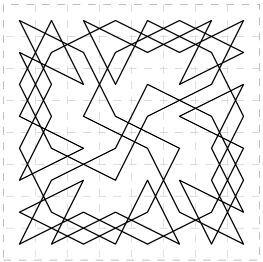

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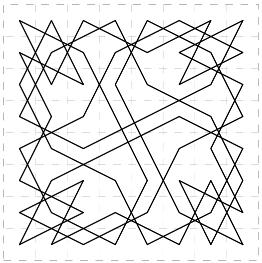

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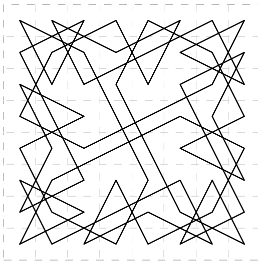

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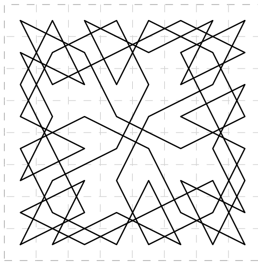

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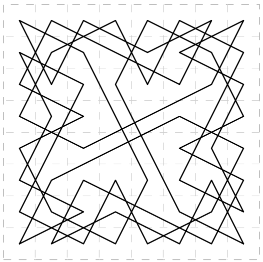

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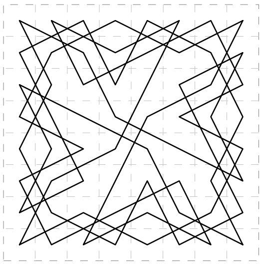

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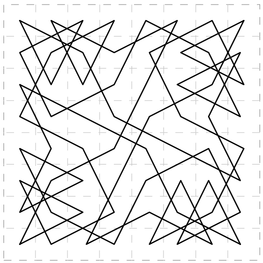

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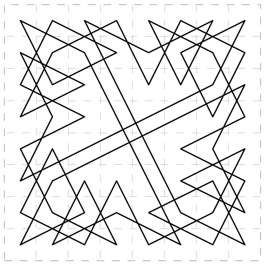

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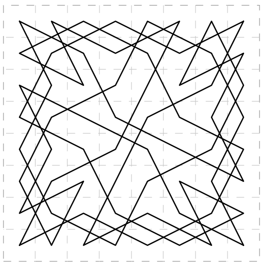

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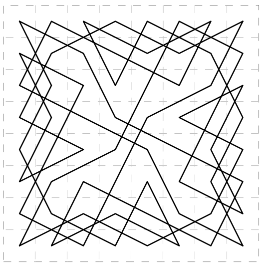

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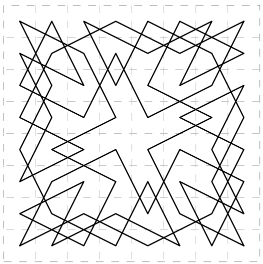

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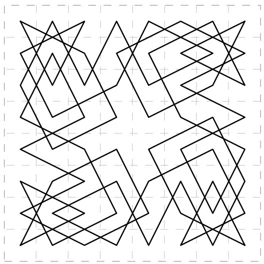

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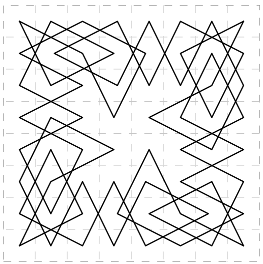

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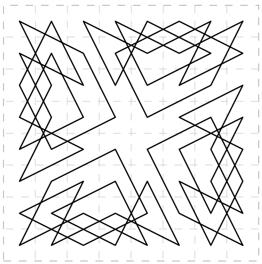

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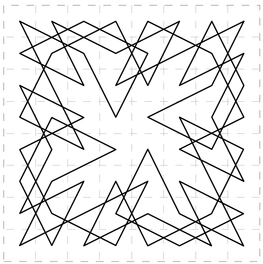

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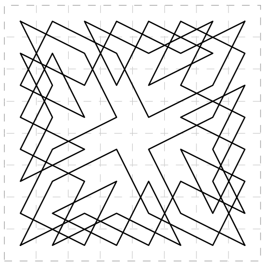

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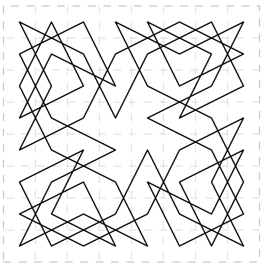

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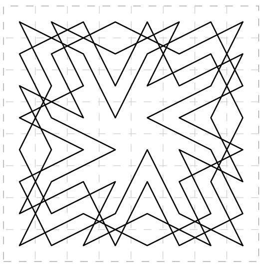

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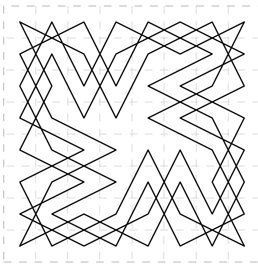

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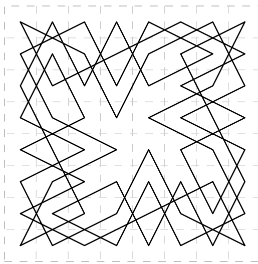

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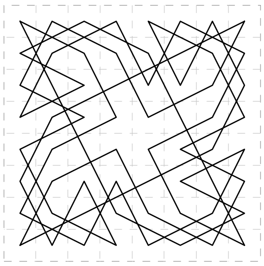

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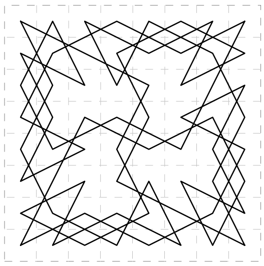

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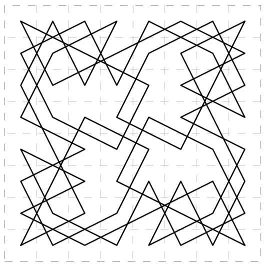

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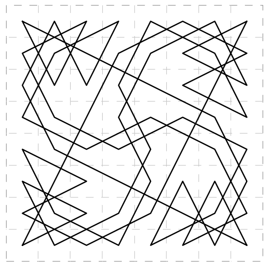

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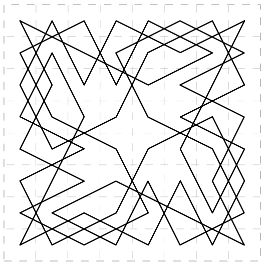

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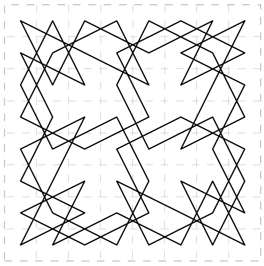

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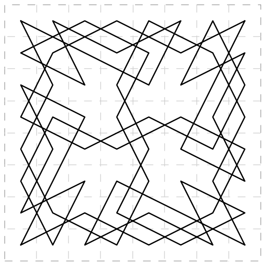

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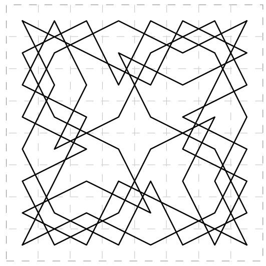

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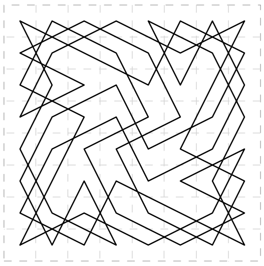

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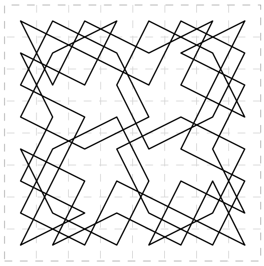

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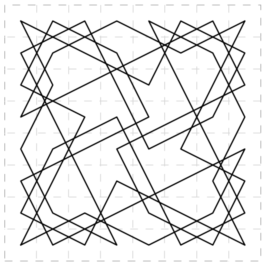

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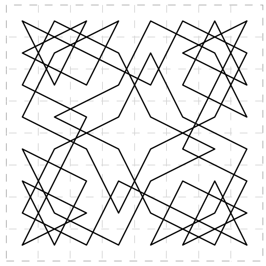

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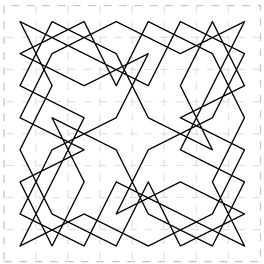

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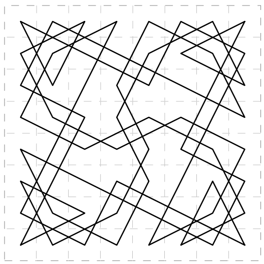

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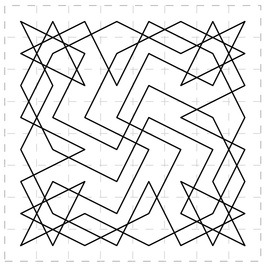

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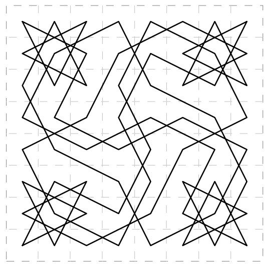

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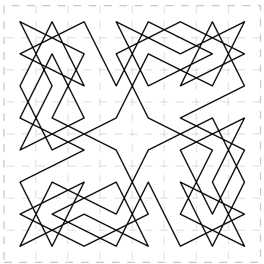

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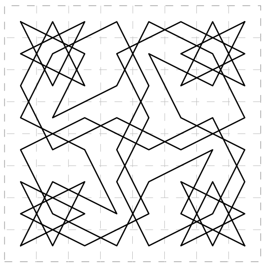

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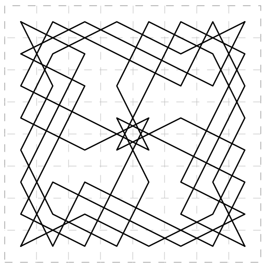

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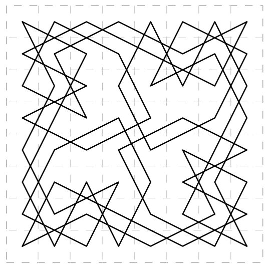

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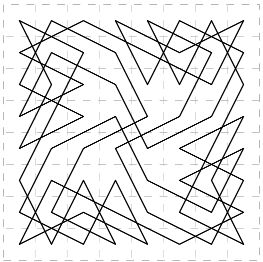

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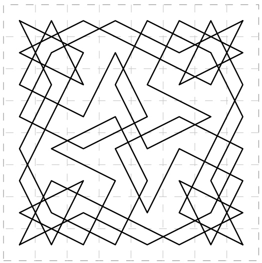

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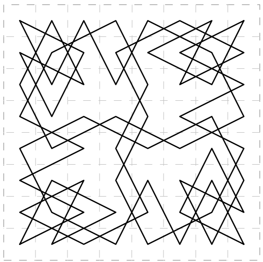

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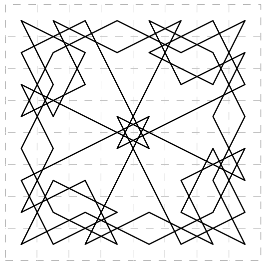

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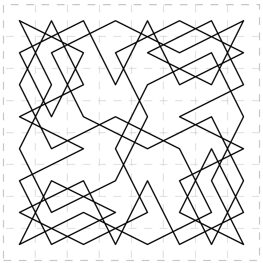

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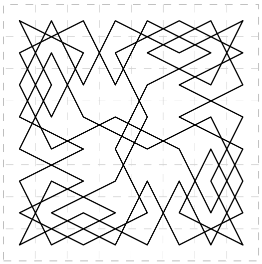

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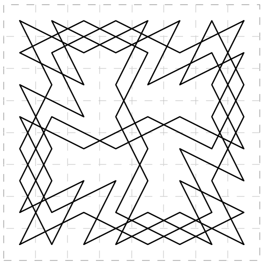

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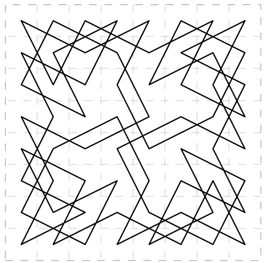

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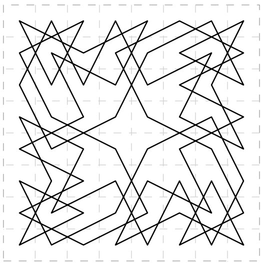

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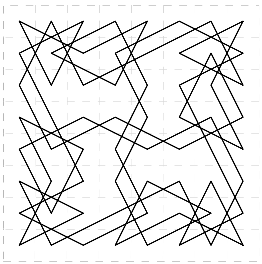

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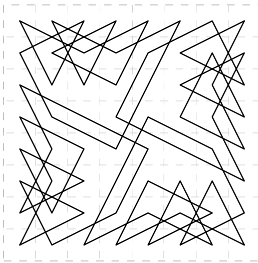

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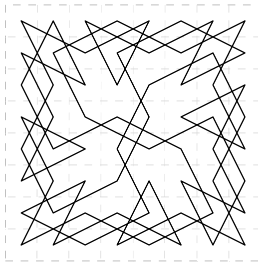

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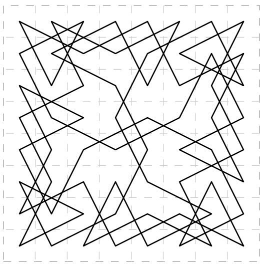

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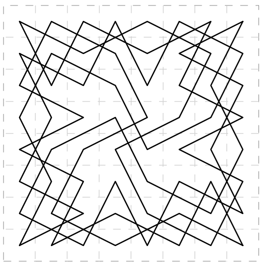

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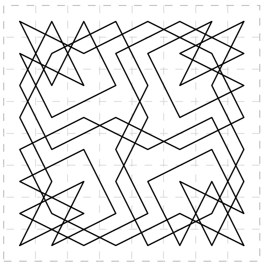

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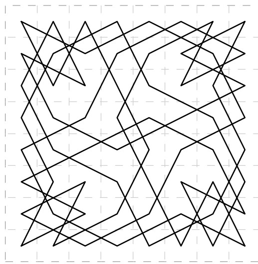

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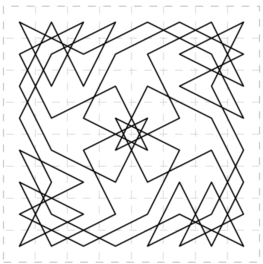

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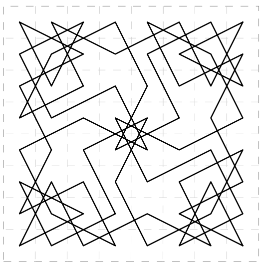

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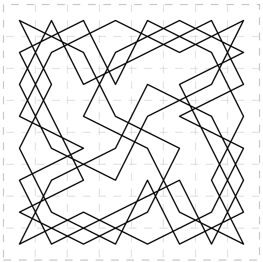

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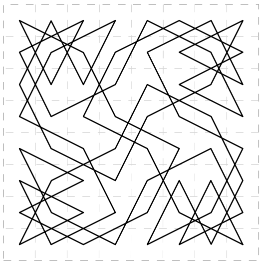

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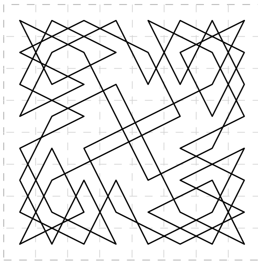

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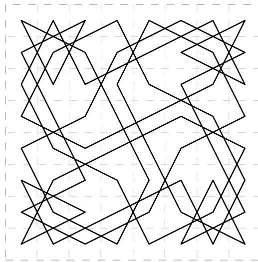

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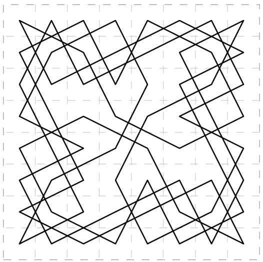

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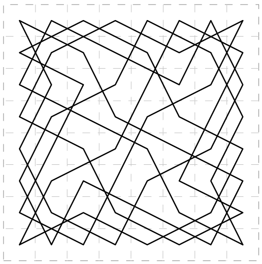

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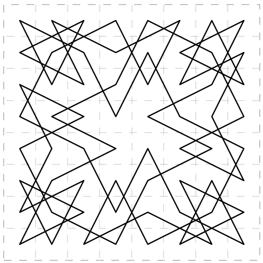

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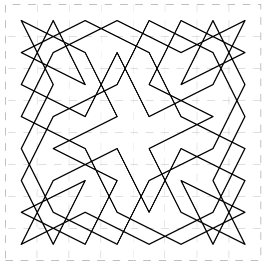

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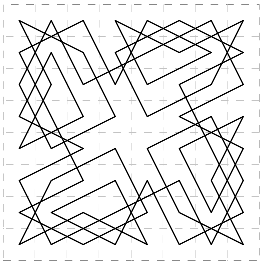

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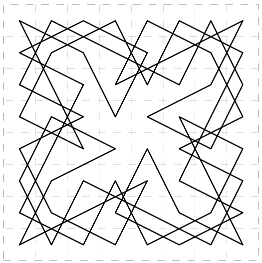

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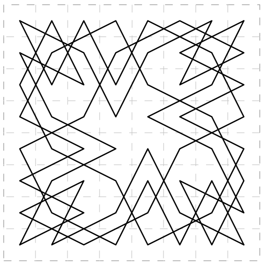

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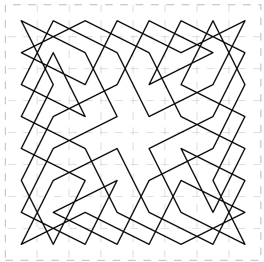

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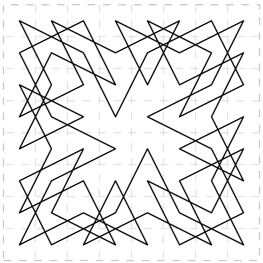

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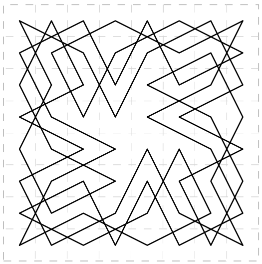

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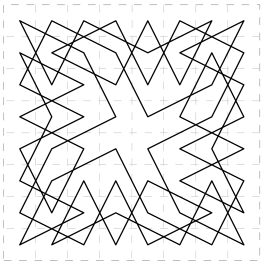

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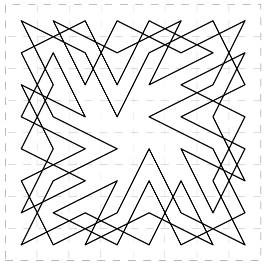

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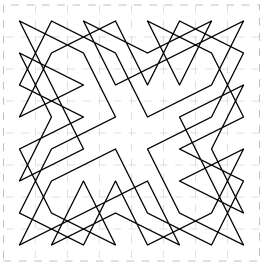

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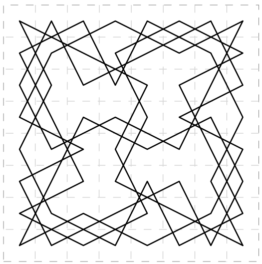

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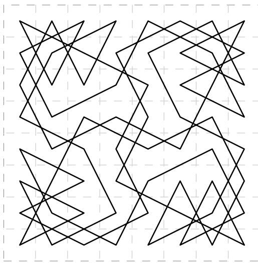

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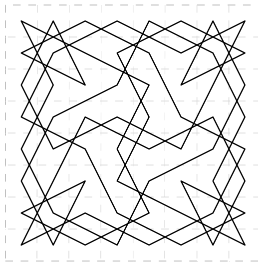

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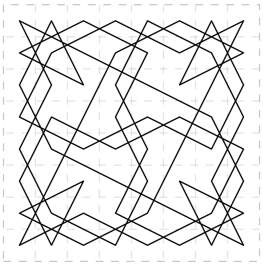

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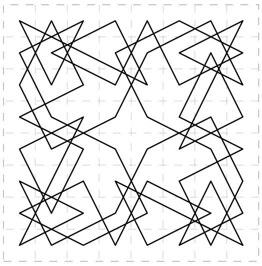

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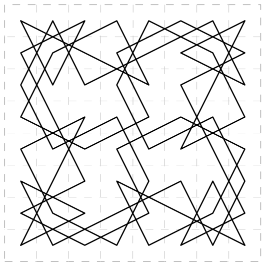

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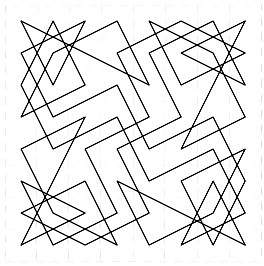

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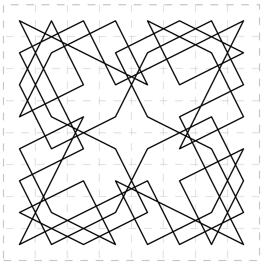

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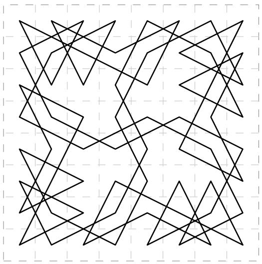

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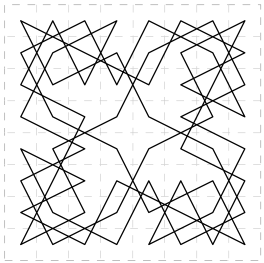

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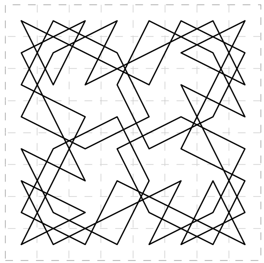

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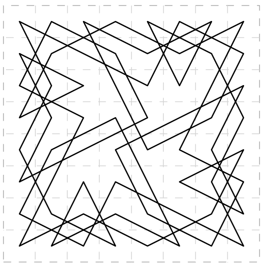

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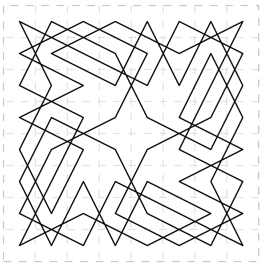

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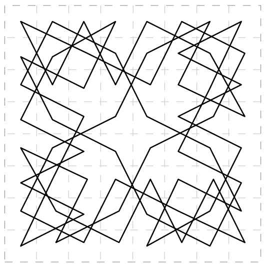

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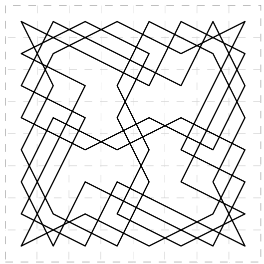

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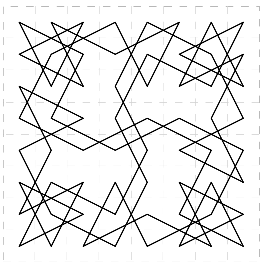

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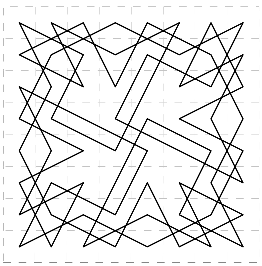

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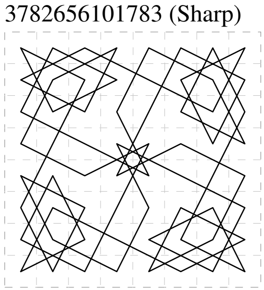

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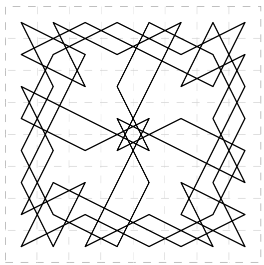

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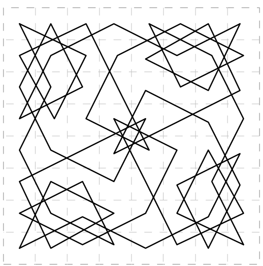

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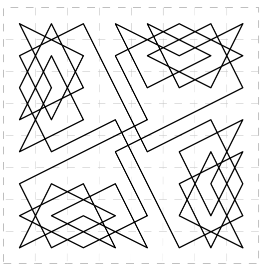

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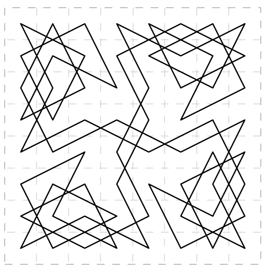

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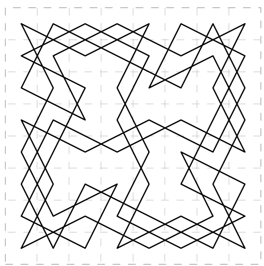

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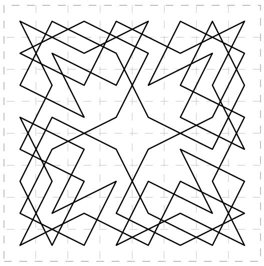

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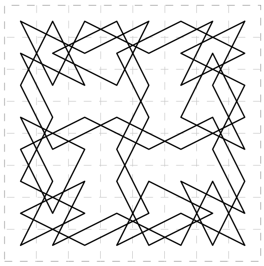

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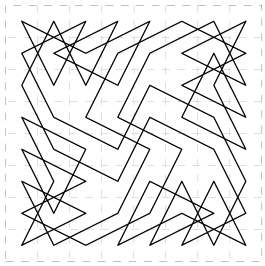

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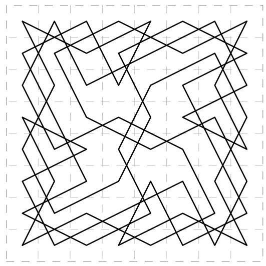

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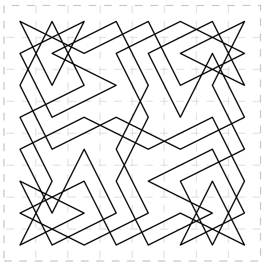

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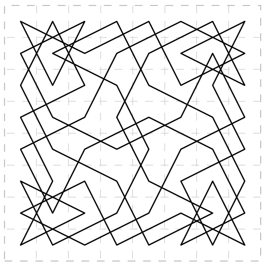

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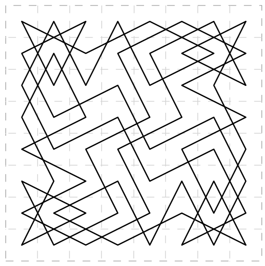

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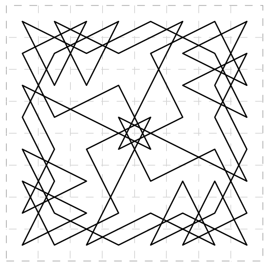


Table of numbers of formulae and pseudotours

Key: D = direct, O = oblique, c = closed, o = open.

	formulae	Dc	Do	Oc	Oo	pseudotours
8-fold	2	8	8	0	0	16
4-fold	62	100	102	26	20	248
2-fold	235	164	168	56	82	470
1-fold	302	96	100	58	48	302
Totals:	601	368	378	140	150	1036

Other Quaternary Pseudotours

Eight-Move Corner to Corner Paths

The following is an enumeration of all sequences of 8-move corner to corner paths. Readers may find it instructive to draw out some of the resulting paths. The sequences in each section are listed in numerical order. My method of enumeration also listed all the reverse paths, which served as a check on the correctness of the count, but the reverse forms are not shown here.

There are 32 that include a 1-1 move. These can only be used to form biaxial patterns, since in rotary formation the 1-1 moves form a short circuit round the centre.

01137, 01156, 01173, 02311, 02611, 03113, 03116, 03117, 03711, 11023
11026, 11037, 11303, 11323, 11326, 11387, 11623, 11626, 11656, 11783,
30113, 30116, 30117, 31137, 31156, 31173, 37116, 37117, 61137, 61156,
65117, 71137,

There are 41 beginning with 0:

01323, 01326, 01371, 01387, 01513, 01516, 01517, 01561, 01623, 01626
01651, 01656, 01731, 01783, 02316, 02317, 02371, 02387, 02613, 02617
02651, 02656, 02831 02837, 02871, 02873, 03151, 03156, 03173, 03231
03237, 03261, 03283, 03287, 03713, 03716, 03783, 03823, 03826, 03871
03873

There are 59 beginning with 1:
including four symmetric: 13031, 13231, 16261, 16561
10137, 10156, 10173, 10231, 10237, 10261, 10283, 10287, 10316, 10317
10323, 10326, 10371, 10387, 13013, 13016, 13017, 13023, 13026, 13037,
13203, 13237, 13261, 13283, 13287. 13713, 13716, 13783, 13823, 13826,
13871, 13873, 15103, 15137, 15173, 15613, 15617, 15623, 15626, 16203,
16237, 16283, 16287, 16513, 16516, 16517, 17303, 17316, 17317, 17323,
17326, 17387, 17823, 17826, 17837

There are 41 beginning with 3:
including three symmetric: 31013, 31513, 38283
30137, 30156, 30173, 30237, 30283, 30287, 30316, 30317, 30326, 30387
31016, 31017. 31023, 31026, 31037, 31516, 31517, 31623, 31626, 31656
31783, 32016. 32017, 32037, 32316, 32317, 32387, 32617, 32656, 32837
32873, 37137, 37156, 37826, 37837, 38237, 38287, 38716

There are 16 beginning with 6:
including two symmetric: 61016, 61516
61017, 61026, 61037, 61326, 61387, 61517, 62017, 62037, 62317, 62387
62617, 62837, 65137, 65617

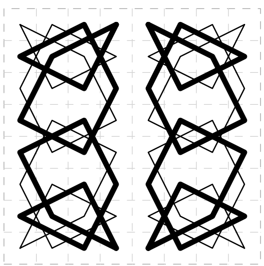
There are 8 beginning with 7:
including five symmetric: 71017, 71517, 73037, 73237, 78287
71037, 71387, 73287

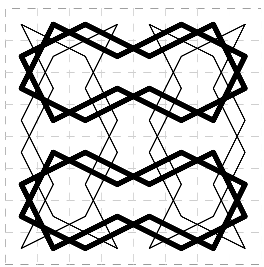
Total of paths 197, including 14 symmetric.
The symmetric circuits will be found as components in the octonary pseudotours.

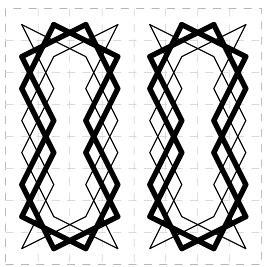
Quaternary pseudotours using 8-move paths

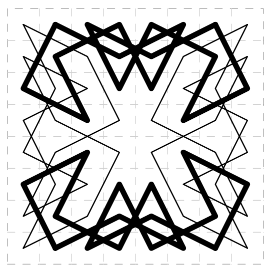
Instead of four 16-move paths all alike, the following quaternary pseudotours use 8-move paths of two types; an 8-move path corner to corner, and another 8-move path to fill the remaining cells. My enumeration, not yet independently checked, found 31 pseudotours of this type as shown below comprising 9 direct closed, 3 oblique closed, 13 direct open, 6 oblique open. (There are also the three octonary examples, generated by the corner circuits 16561, 61516, 78287). Some of the open solutions will be found to be formed of bisatin circuits (two cells in each rank and file). The codes for the 8-move corner-to corner paths are shown (omitting 94...49 as usual).

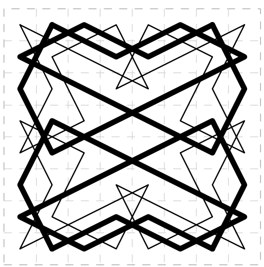
Corner to same corner (12 cases)

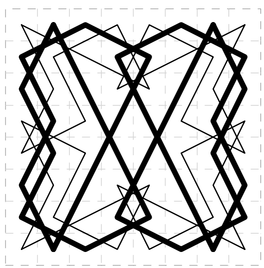

03783


02565


02565 = crosspatch


03826


16516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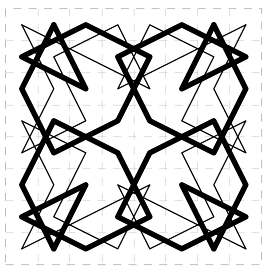
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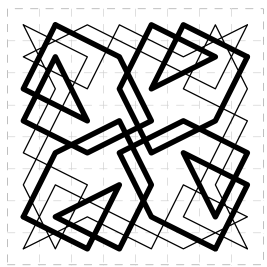
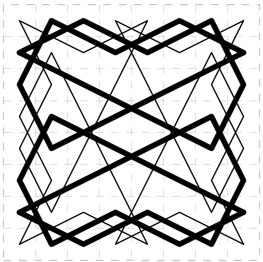

37826

continued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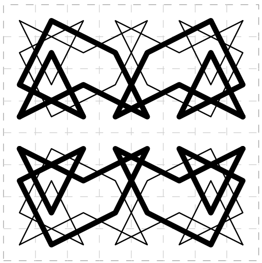
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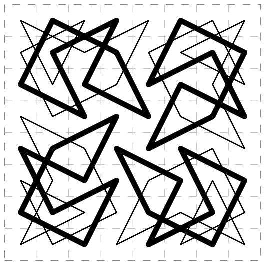
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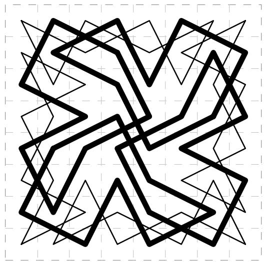
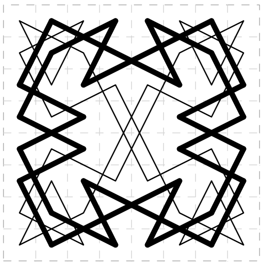

62837

Corner to opposite corner (19 cases)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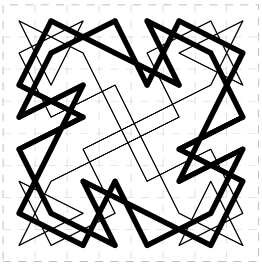
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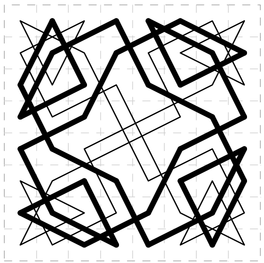
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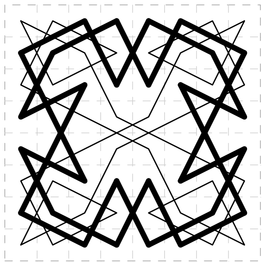
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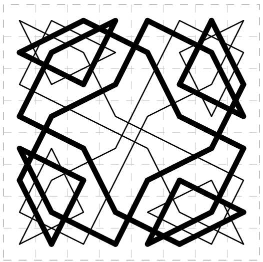
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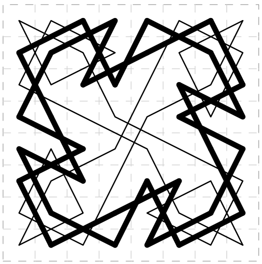
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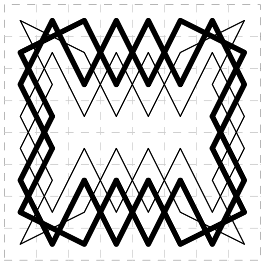
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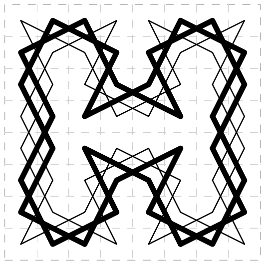
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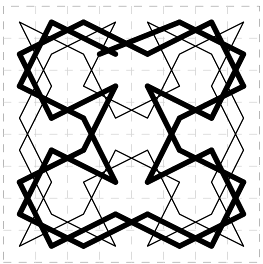
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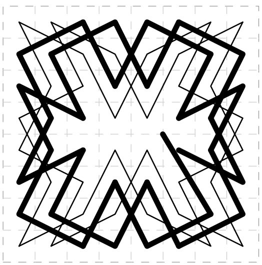
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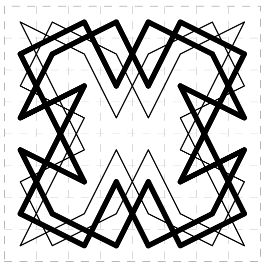
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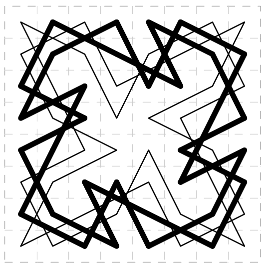
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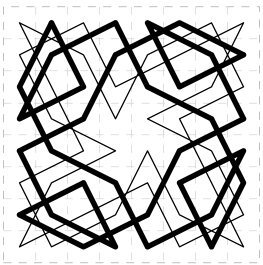
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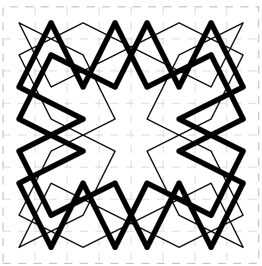
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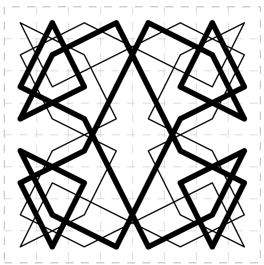
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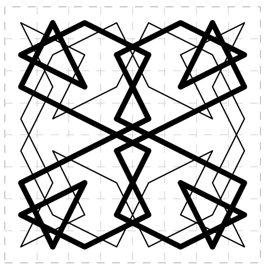
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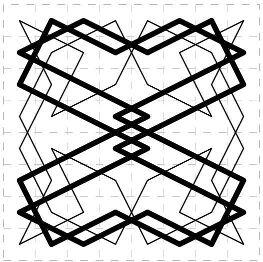
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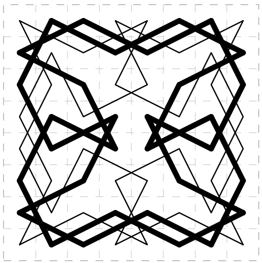
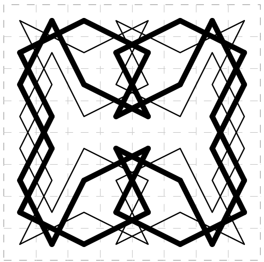
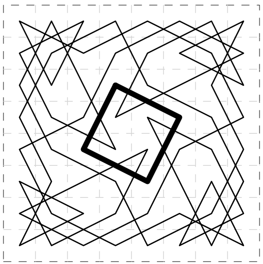
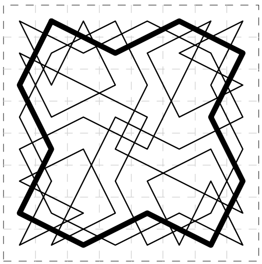

61156

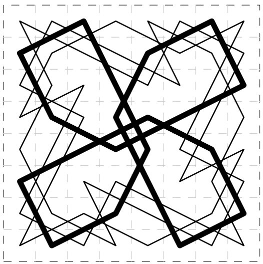

61156

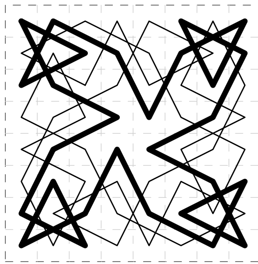
Different Length Circuits

The following four pseudotours were given by “E. H.” in the *Glasgow Weekly Herald* of 11 October 1873 (shown there in numerical, not diagram form). They show pairs of circuits of $8x + 4$ cells in oblique quaternary symmetry which together cover all the cells of the 8×8 board.

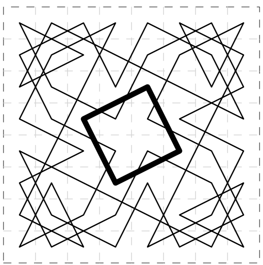

4 + 60

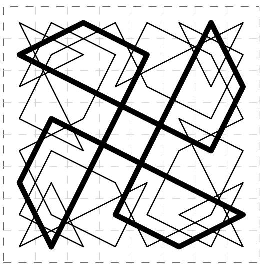

12 + 52

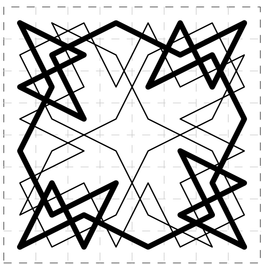

20 + 44

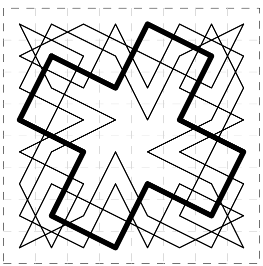

28 + 36

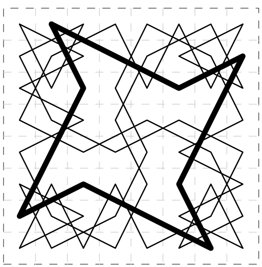
The possibility of forming such patterns was rediscovered by Ernest Bergholt in his memoirs (1917) where he gave examples 4 + 60 and 20 + 44 shown below. I show also one of the 28 + 36 case and three more of the 12 + 52 case. There are six other 12-move circuits with oblique quaternary symmetry but they do not admit combination with a 52-move circuit.

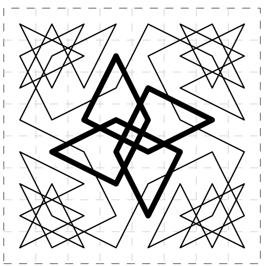

4 + 60


20 + 44


28 + 36


12 + 52

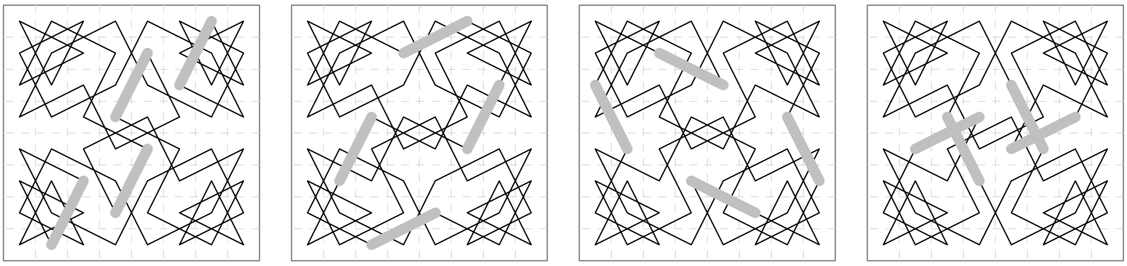

12 + 52


12 + 52

Tours Derived from Quaternary Pseudotours

Vandermondian Tours

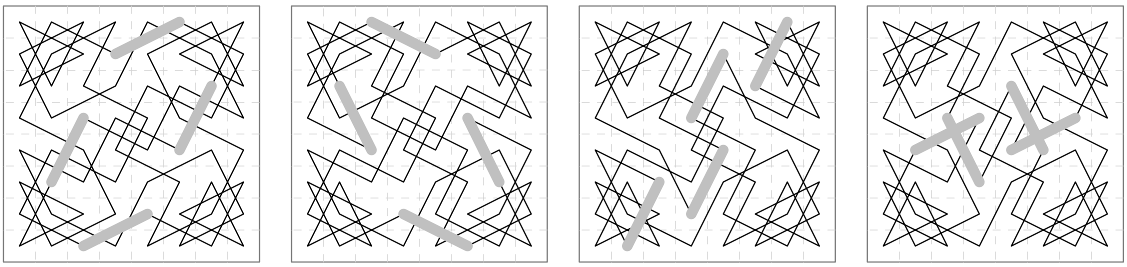
Vandermomde's article and tour of 1771 is described in the History. As noted there it is in fact possible to link Vandermonde's four circuits to form a symmetric tour with only four deletions and insertions in four ways (Wenzelides #48, Jaenisch §86 p.13 and §87 p.16, and Jelliss c.1985). There are also three ways of simple linking the circuits that lead to asymmetric closed tours. The deleted and inserted moves form eight-move linkage polygons as illustrated in the note on simple linking. The inserted moves are emphasised in the following diagrams. (kn=, 2nn=, ln~)



A peculiarity of the circuits chosen by Vandermonde is that by reflecting them in a diagonal they can also be arranged in oblique quaternary symmetry (i.e. birotary), as noted by H. E. Dudeney (*Amusements in Mathematics* 1917), though whether this was intentional is unclear. It may be a consequence of his method of construction using coordinates.

For it to be possible to arrange four identical 16-move paths in both direct and oblique quaternary formation, the pattern of cells occupied by two diametrally paired paths must be reflectible in a diagonal, as well as being superposable by a half-turn. These two properties combine to ensure that it must also be reflectible in the other diagonal, i.e. the pattern of cells (but not the pattern of moves) has both diagonals as axes of symmetry.

The four circuits in birotary symmetry can also be linked together with four deletions and four insertions in seven ways, four of which are symmetric as shown below. They also use the same shapes of linkage polygon, though the two cases forming a hexagon are related by rotation in the biaxial case and by reflection in the birotary case. (2nn~, kn~, ln=)



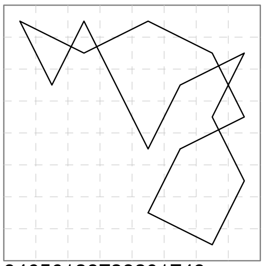
There are ten other formulae that give four pseudotours related in the same way as in Vandermonde's example. The coding sequence for Vandermonde's tour is 94783102656138749. The other 10 cases (omitting the corner links 94...49) are:

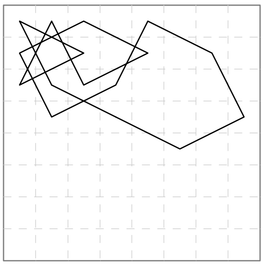
7831656102387, 7820316561387, 6561783201387, 6561783028317, 3871026561783, 3820387165617, 3165617830287, 3028716561783, 0238716561783, 0238716561387.

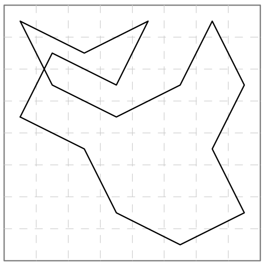
Diagrams of these and all other pseudotours formed of a fourfold repeated 16-move path are given our catalogue that follows, based on the work of de Hijo (1882).

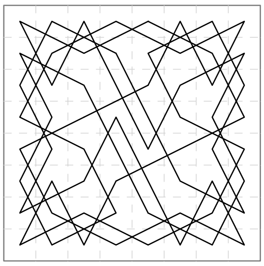
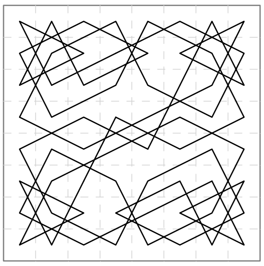
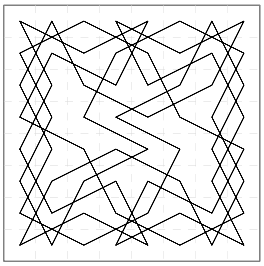
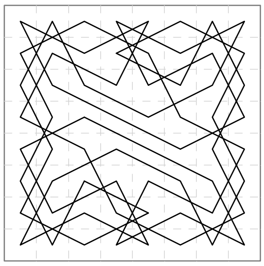
Jaenischian Tours

In his 1862 *Treatise* Jaenisch shows a number of examples of tours formed by joining four closed circuits by deletion of one move from each. Here are some examples he gives using one pattern of circuit repeated four times (§92 p.38 and p.39, and §91 p.33). I show the pattern of quarter circuit used in diagram and in coded form. The third has two different linkages. (io=, dn~, co=, cd=)

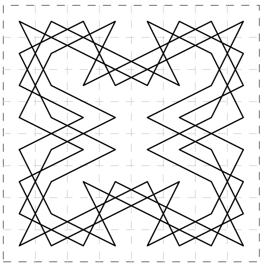

94656138738201749

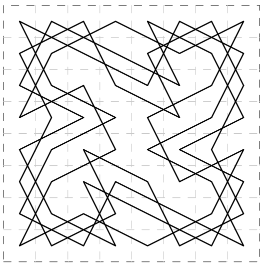

94782378316561049

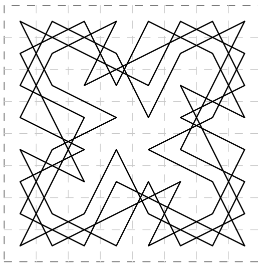

94615613783782049

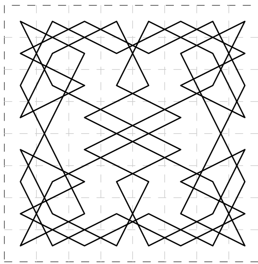





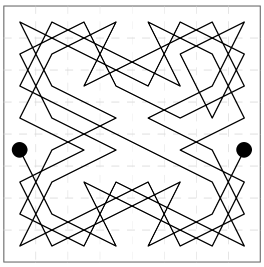
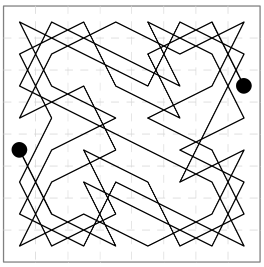
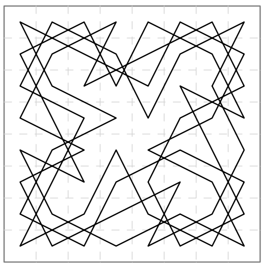
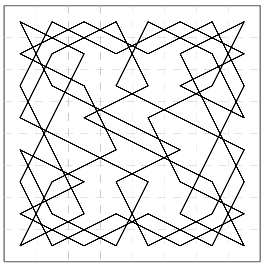
The following tours from Jaenisch use two or more different types of 16-move circuit. The first three are based on squares and diamonds. Jaenisch points out (p.72) that the Troupenas (1842) tour is of this type, formed from four circuits (two of diamonds and two of squares) by deleting one move in each circuit and reconnecting. The first examples below are open tours, with constant diametral difference of 8, of squares and diamonds type, and partially magic. They are (§103 p.71 also Fig 6 in Vol.1, and §103 p.70) which uses four different crcuits. The other two examples are closed tours (§90 p.27, §92 p.36). The first is identical to one by T. Scheidius *Sissa* 1850 but reflected left to right.


four circuits 2 types


four circuits 4 types


four circuits 2 types


four circuits 2 types

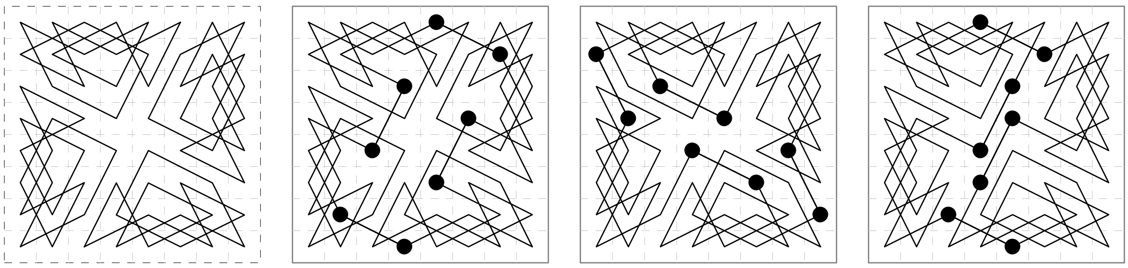





(symmetric centres: gg~, df=)

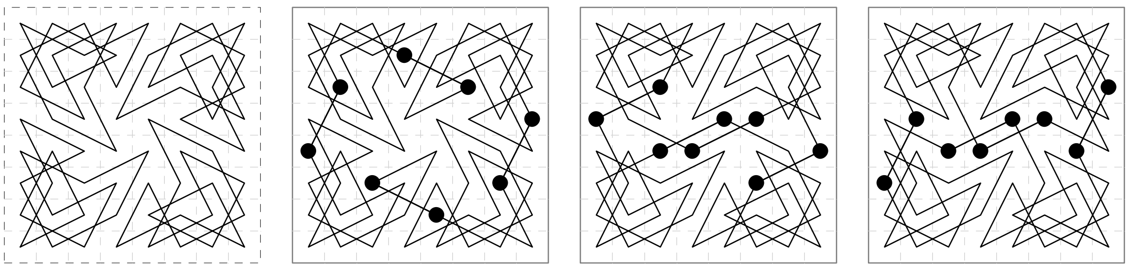
Aladdin's Conundrum

As noted in the History section the impossibility of a knight tour on the 4×4 board was known in mediaeval times, although the construction of a quarter-board tour was sometimes set as a wager problem, in particular in a manuscript that may be based on the work (c.1400) of **Ala'addin Tabrizi** the leading player at the court of Timur (1336-1405). However the fact that there is a unique closed tour of a shaped quarter of the standard board does not seem to have been known until Paul de Hijo included it in his 1882 catalogue of quaternary pseudotours. Whether the author of the ms knew of this solution we may never know, but it makes a good story, appropriate to the name of Aladdin.

Like H. E. Dudeney I rediscovered this result myself (*Chessics* #22 Summer 1985 p.63) and by simple linking we can form five closed tours, three of which are symmetric. C. T. Blanshard (*Chess Amateur* Aug 1923 p.349) had in fact given a ‘four- loop tour’ the first of these three, said to have been found by him before the war (i.e. in 1913). (bb~, bf~, bk=)



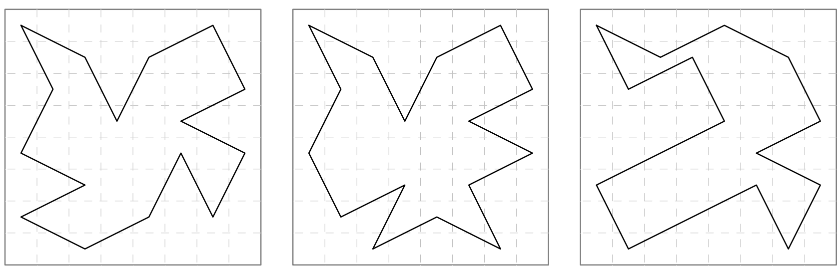
H. E. Dudeney also found another 16-move circuit that covers a disconnected quarter of the board and can be repeated four times in birotary symmetry without intersecting its duplicates. This is also to be found in de Hijo, and I recently found it even earlier in *Le Siécle* ¶118 (16 Mar 1877) in a puzzle by Monsieur Jacquemin-Molez. Like the other quarterboard tour the four circuits can be linked to form three symmetric tours (and six asymmetric). (ll~, jl~, ln=)



The linking moves together with the deleted moves form eight-move circuits. This is a particularly clear case of simple linking of circuits.

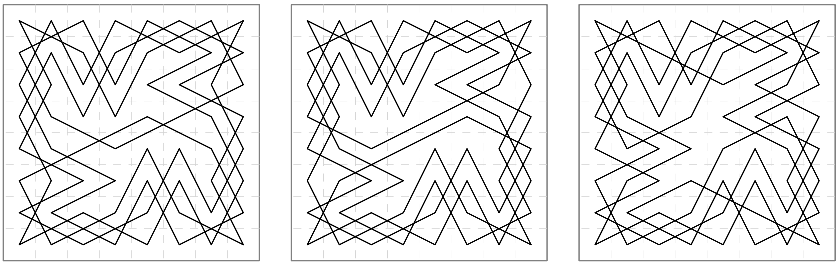
Linking Non-Crossing Circuits

Out of the 140 cases of 16-move circuits in oblique quaternary symmetry there are just three that do not intersect themselves.

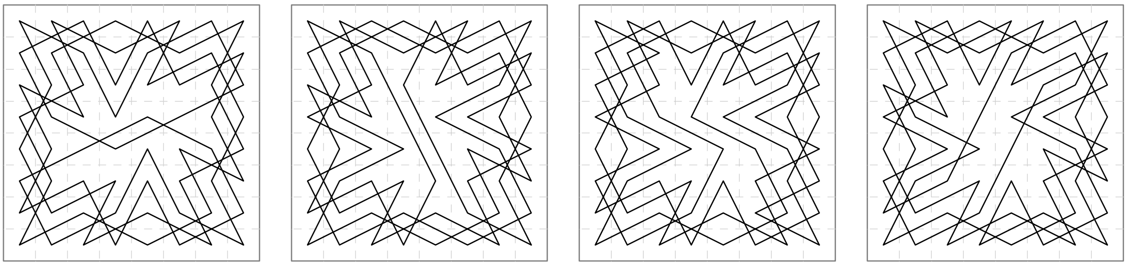


The following diagrams show symmetric tours derived from these by simple linking. Four from the middle case and three from the other two. (gj~, dg~, gh; gj~, dg~, gk~, gi~; kp, bk=, kk~)

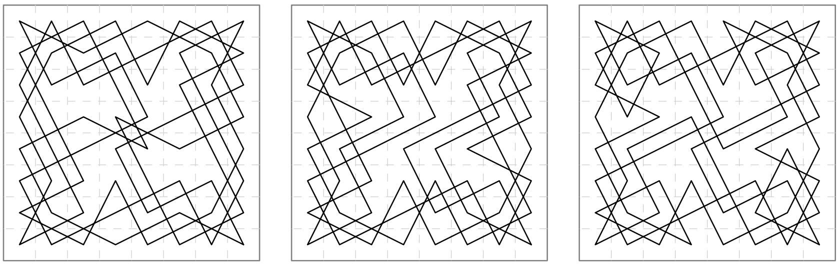
94038716513782649



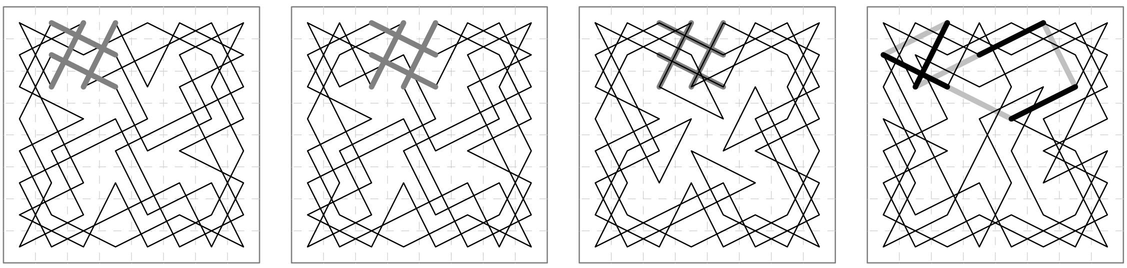
94038716283715649



94656178238710349



This last case is also remarkable as being linkable by a star polygon in two ways to give asymmetric tours. According to my web-page notes only one other pseudotour is linkable by a star polygon, as in the last diagram below, though I have forgotten how I arrived at this conclusion. This pseudotour path intersects itself once.

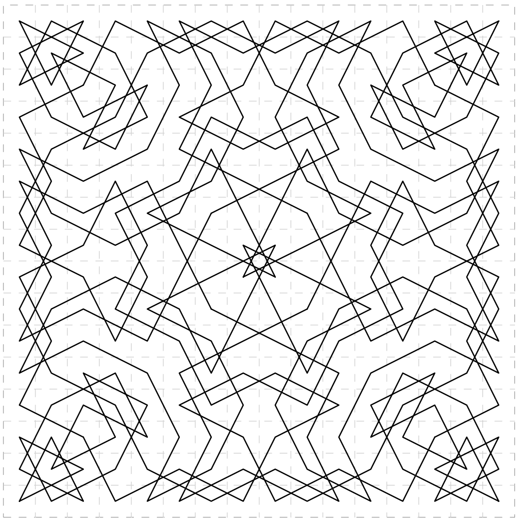


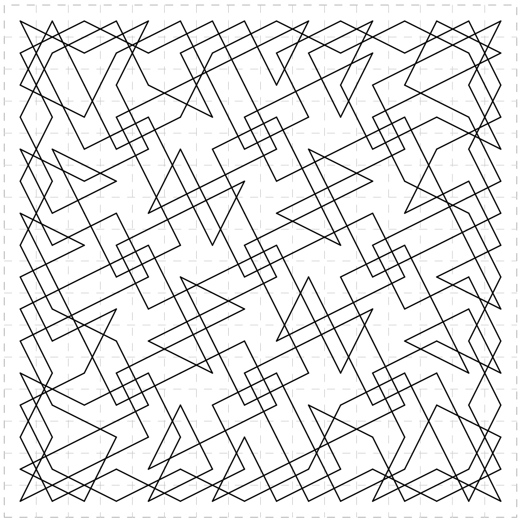
The fourth example here (Jelliss 1992) was composed just to illustrate the general case; an asymmetric linkage polygon joining four asymmetric equal-length circuits (three of which are non-self-intersecting).

Larger Pseudotours

16×16 Pseudotours

From *Linaludo* by Archibald Sharp (1925) I show two 16×16 pseudotours. The broken border line indicates a pseudotour. The first, shown on his page 28, is octonary and consists of four16-move circuits, each repeated 4 times.





The second, derived like the similar 12×12 tour (p.622), from the back cover pattern of crosses and reels is quaternary, consisting of two identical 128-move circuits (or four identical 64-move corner to corner paths). The process can be continued to 20×20, 24×24 and so on.

32×32 Board

Pseudotour. The left-hand and central part of this diagram is shown on page 25 of *Linaludo* (1925) by Archibald Sharp. It is a 32×32 pseudotour with octonary symmetry, composed of the minimum of 8 circuits. Each circuit consists of eight copies of one path, either of 15 or 17 moves, starting at a cell on one diagonal and ending on a cell of the other diagonal.